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Euler's tallest column revisited

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ABSTRACT

In 1757, Leonhard Euler started the study of the tallest column, i.e. the shape of a stable column with the symmetry of revolution, such that it attains the maximum height once the total mass is prescribed, buckling due to the effect of a load supported at its top. A more detailed analysis is due to Keller and Niordson in 1966 who formulated the problem in terms of an eigenvalue type problem under some coefficient constraints and also, by eliminating some of the unknowns, as a nonlocal boundary value problem for a p-Laplacian type operator with a negative exponent and with an infinite normal derivative in some of the boundaries. The main contribution of this work is the study of the existence and qualitative behavior of a weak solution completing the approach made by Keller and Niordson (developed merely by asymptotic analysis techniques). Under a suitable condition on the top load, we show that there exists a shape function $a(x)$ for which the smallest eigenvalue is the largest one when $a(x)$ is taken in a suitable class of shape functions (in contrast with the unload case according to a result due to Cox and McCarthy in 1998). We prove also that the nonlocal problem has a solution u such that $u \in W^{1,p}(0, 1)$ for any $p \in [1, 3)$ but with $u \notin W^{1,3}(0, 1)$. We also give a sufficient condition for the uniqueness of the solution.

1. Introduction

1.1. Statement of the problem

In 1757, in his communication *Sur la force des colonnes* [1], Leonhard Euler started the study of the tallest column, i.e. the shape of a stable column with the symmetry of revolution, such that it attains the maximum height once the total mass is prescribed. Since then, many mathematicians have contributed to the subject, which is not yet completely solved.

One could consider that, at the time of Euler, this was already a historical problem since, more than two centuries earlier, it had been considered by Leonardo da Vinci: in a context of architecture for palaces and temples, Leonardo asked what could be the maximum height for a cylindrical column, so that it would not buckle under its own weight. In his 1757 communication, Euler considered mathematically Leonardo's problem. He also considered and solved the case of a prismatic column, and gave a start to the study of the general shape of revolution which would give rise to the tallest column.

This problem has been developed in various contexts. One of them, usually attributed to Lagrange [2], consists in considering a hollow column, the weight of which is therefore neglected. For Lagrange's problem, one can refer to Cox [3], Egorov [4], Keller and Tadjbakhsh [5] and Overton [6], among others.

In Euler's formulation, on the contrary, the column will eventually buckle under the action of its own weight, and under the effect of a load supported at its top. The top is free while the base is clamped. The main reference on a more detailed

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analysis is the approach made by Keller and Niordson in 1966 (cf. [7]), together with the comments by Cox and McCarthy in [8], who stress some difficulties arising with the Keller and Niordson method. The main contribution of this work is to revisit some of the formulations proposed in Keller and Niordson [7], but considering it as a nonlocal problem for which we search a weak solution and not merely as a problem of asymptotic analysis. We show that under a suitable condition on the parameters the problem is well posed in a certain Sobolev space (in fact we prove that the exponent 3 on the integrability of the derivatives of any solution cannot be improved). We also give some sufficient conditions for the uniqueness of solution. We point out that, as far as we know, no previous result on this question was presented before in the literature.

1.2. Equation at loss of equilibrium

A reformulation of the problem considered by Euler in 1757 could be the following:

Let us consider a vertical column of height H and volume V, made of some material of mass density ρ and Young's modulus E. Let $A(z)$ be the area and $I(z)$ the moment of inertia of the cross section at height z (so that $I(z) = \alpha A^2(z)$, where α is a constant). Let P be a vertical load at the top of the column and let g be the acceleration of gravity. Let us denote by $y(z)$ the lateral deflection of the column from the vertical in some fixed vertical plane, which we call the plane of bending. Making use of the Bernoulli–Euler theory (see, for instance, [9]), which connects the bending moment of the column to its curvature, Euler reached the following equation at loss of equilibrium, i.e. at the first time of bending:

$$[E]\begin{cases}\alpha EA^2 y_{zz} = \displaystyle\int_z^H \rho g A(z')[y(z') - y(z)]\mathrm{d}z' + P[y(H) - y(z)] & 0 \le z \le H,\\ y_z \ge 0, \quad y(0) = y_z(0) = 0, \quad \displaystyle\int_0^H A(z')\mathrm{d}z' = V. \end{cases}$$

It will be shown *a posteriori* that, for the tallest column, A is a continuous function (in fact a C^1-function) on $(0, 1)$.

The fact that the column is clamped at its bottom explains the initial conditions at $z = 0$. The shape of revolution, with vertical axis, is described by the area function $A(z)$. The problem we shall consider is to determine the function $A(z)$ leading to the maximal H. Our final result is that the tallest column does exist, and that its shape is uniquely determined, provided that the load P at its top is large enough.

1.3. Rescaling

We now follow the mentioned work of Keller and Niordson. It is useful to reformulate the problem by introducing dimensionless variables and some new parameters:

$$\begin{aligned} &x = z/H \qquad a(x) = HA(xH)/V \qquad \eta(x) = y(xH)/H\\ &\lambda = \rho g H^4/\alpha EV \qquad k = P/\rho g V \qquad u = \eta_x. \end{aligned}$$

Equation $[E]$ becomes, for $u = \eta_x$:

$$\left[a(x)^2 u'(x)\right]' + \lambda u(x)\left(\int_x^1 a(y)\mathrm{d}y + k\right) = 0. \tag{1}$$

Hence, Euler's tallest column problem can be reformulated in the following terms:

Fix $k > 0$.

1/ For any rescaled shape $a \in L^1(0, 1)$, $a \ge 0$, $\int_0^1 a(x)\mathrm{d}x = 1$, *solve (whenever it is possible) the Problem* $[E(a)]$:

$$[E(a)] \quad \begin{cases}\text{Find } \lambda = \lambda(a) > 0 \text{ and a function } u \ge 0,\ u \not\equiv 0, \text{ satisfying the equation}\\ \left[a(x)^2 u'(x)\right]' + \lambda u(x)\left(\displaystyle\int_x^1 a(y)\mathrm{d}y + k\right) = 0\\ u(0) = 0, \qquad \displaystyle\lim_{x\to 1} a(x)^2 u_x(x) = 0.\end{cases}$$

2/ Determine the rescaled shape function a corresponding to the largest λ (*note that* λ *is proportional to* H^4). (It will be shown later that such an optimal function a belongs to the Sobolev space $W^{1,1}(0, 1)$.)

One could understand problem $[E(a)]$ as the search of some kind of bifurcation. Let us fix a rescaled shape a. If λ is small, then $u \equiv 0$ is the only solution of Eq. (1), which means that no deflexion takes place – in other words, for small values of H, the column is stable. For large values of λ, Eq. (1) will admit a nontrivial solution u and deflexion appears – which means that for too large H, the column is unstable. The smallest possible of such values λ, denoted $\lambda(a)$, corresponds physically to the best value of H for given a, and mathematically corresponds to solving (1) by means of a nontrivial and nonnegative function u.

1.4. The associated problem $[P(A, B)]$

In this spirit, Keller and Niordson [7] reinterpret problem $[E(a)]$ as the search of an eigenfunction u and the determination of the smallest eigenvalue $\lambda(a)$, of an operator associated with a. They propose the two following assumptions:

$$[A1]\quad \lambda(a) = \inf_{\substack{v\in H^1_{loc}([0,1]), v\neq 0\\ v(0)=0,\ \lim_{x\to 1} a^2(x)v'(x)=0}} \frac{\int_0^1 a(x)^2 v_x(x)^2 \mathrm{d}x}{\int_0^1 v(x)^2 \left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x}$$

in problem $[E(a)]$ and the solution u is a minimizer for this Rayleigh quotient: in other words,

$$\lambda_{\max} = \sup_{a\in L^1([0,1]), a\geq 0, \int_0^1 a(x)\mathrm{d}x=1} \inf_{\substack{v\in H^1_{loc}([0,1]), v\neq 0\\ v(0)=0,\ \lim_{x\to 1} a^2(x)v'(x)=0}} \frac{\int_0^1 a(x)^2 v_x(x)^2 \mathrm{d}x}{\int_0^1 v(x)^2 \left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x}; \tag{2}$$

$[A2]\quad \lambda(a)$ depends smoothly on a.

The validity of those two assumptions will depend on the class of functions the shape a is assumed to belong. As pointed out by Cox and McCarthy in [8], they fail to be true in the unloaded case $P = 0$ (the operator associated with the Rayleigh quotient above has not a discrete lower spectrum), and probably neither in the case of a small load. We shall discuss them in detail in Section 3 and show finally (cf. Theorem 3) that, in contrast to the case of $P = 0$, they are satisfied when the load is large enough.

When they are satisfied, one can write the equation $D\lambda(a) = 0$ of a maximum for λ, and get

$$[DE(a)]\quad \exists\mu > 0,\quad a(x)\, u_x(a)^2 = \frac{\lambda(a)}{2}\left(\int_0^x u(y)^2\mathrm{d}y + \mu\right)$$

with μ a Lagrange multiplier provided by the constraint $\int_0^1 a(x)\mathrm{d}x = 1$.

One can now, as in [7], eliminate a between the two equations $[E(a)]$ and $[DE(a)]$ to reach the new formulation of the problem: Given $\Lambda > 0$, find a function $u(x) \geq 0, x \in [0, 1]$, solution of

$$[P(A,B)]\begin{cases} -\big(A(u)(x)\phi(u_x)\big)_x + \big(B(u)(x) + \Lambda\big)u = 0 & \text{in } (0,1),\\ u(0) = 0 & \\ u_x(x) \to +\infty & \text{as } x \to 1,\end{cases}$$

with, for a given $\mu > 0$,

$$A(u)(x) = \left(\mu + \int_0^x u(y)^2\mathrm{d}y\right)^2, \qquad B(u)(x) = 2\int_x^1 u_x(\tau)^{-2}\left(\mu + \int_0^x u(y)^2\mathrm{d}y\right)\mathrm{d}\tau$$

and

$$\phi(q) = -\frac{1}{q^3}.$$

Remark 1. Conversely, from a solution u to Problem $[P(A, B)]$, one can recover a and λ through the inversion formulas

$$\frac{2}{\lambda} = \int_0^1 u_x^{-2}\left(\mu + \int_0^x u^2(t)\mathrm{d}t\right)\mathrm{d}x = \frac{1}{2}B(u)(0) \qquad a(x) = \frac{\lambda}{2}u_x^{-2}\left(\mu + \int_0^x u(t)^2\mathrm{d}t\right) \tag{3}$$

so that both equations $[E(a)]$ and $[DE(a)]$ are satisfied. (Note that such a belongs to $W^{1,1}$). Thanks to those expressions, the behavior of u and u_x near $x = 1$ determines how the column is shaped at its top.

Remark 2. This way of solving Euler's problem allows to separate the unknown variables H and A.

Remark 3. One has $\Lambda = 4k/\lambda$. The unloaded case $P = 0$ corresponds to $\Lambda = 0$. In this case, a solution to $P(A, B)$ should satisfy $c_1(1 - x)^{-2} \leq u(x) \leq c_2(1 - x)^2, x \in [0, 1[$, for some $c_1, c_2 > 0$, which leads to a highly degenerate problem. This would lead to a shape $A(z)$ such that $(H - z)^{-3}A(z) \to \alpha > 0$ as $z \to H$. Which means that the tallest column should have a cusp at its top (cf. [7,8]).

We shall focus on the loaded case $P \neq 0$: then Λ is inversely proportional to H^4. We shall see that, for P not too small, the behavior of u near $x = 1$ is such that $(1 - x)^{-2/3}u(x) \to c > 0$. So, in this case, the tallest column has a round top (cf. Theorem 3 below).

We point out that $P(A, B)$ provides an example of equation involving a p-Laplacian operator (see, e.g. [10]) with negative p (here, $p = -2$). Another case of negative exponent p can be found, for instance, in [11] for the Newton best body again friction problem. Other column optimization problems have been also considered in various dierent contexts: see, for instance, [13–20].

1.5. Statement of the results

Theorem 1. *Assume that*

$$\Lambda > \frac{9\pi^2}{128}. \tag{4}$$

Then problem $P(A, B)$ admits a solution u with $u \in W^{1,p}(0, 1)$ for any $p \in [1, 3)$ and such that,

$$u_x(x) = O\left(\frac{1}{(1-x)^{1/3}}\right) \quad \text{near}\, x = 1. \tag{5}$$

In particular, $u \notin W^{1,3}(0, 1)$.

Theorem 2. *There exists a positive Λ_0 such that, for given $\Lambda \geq \Lambda_0$, the solution u of problem $P(A, B)$ is unique.*

Theorem 3. *There exist P_0 such that, for $P \geq P_0$, the following properties are satisfied:*

i/ *There exist one and only one $a \in W^{1,1}(0, 1)$, $a \geq 0$, $\int_0^1 a(x)\mathrm{d}x = 1$, one and only one $\lambda > 0$ and (up to a multiplicative constant) one and only one $u \in L^2(0, 1)$, $u \geq 0$, $u \not\equiv 0$, such that equations $[E(a)]$ and $[DE(a)]$ are satisfied.*

ii/ *One has $\lim(1-x)^{-2/3}a(x) = \gamma$ and $\lim(1-x)^{1/3}a_x(x) = \frac{2\gamma}{3}$ as $x \to 1$, for some $\gamma > 0$.*

iii/ *Euler's tallest column problem admits one and only one solution.*

iv/ *The Keller and Niordson assumptions $[A1]$ and $[A2]$ hold true.*

Note that, by assertion 2/ above, the tallest column has a round top, as foreseen in [7]. For an unloaded column, they conjecture that the column is tapered at its top.

2. The shape of the best column

In this section, we discuss the shape of the best column corresponding to a solution of Euler's problem. In the following we shall denote by u any nonnegative solution of $P(A, B)$ on $[0, 1]$.

Proposition 4. *Assume $\Lambda > 0$. Then*

1/ *u is an increasing and convex function.*

2/ *There exists $c > 0$ such that $\lim_{x\to 1}(1-x)^{1/3}u_x = c$.*

3/ *There exist $c_1, c_2 > 0$ such that $c_1(1-x)^{-1/3} \leq u_x(x) \leq c_2(1-x)^{-1/3}$, $x \in [0, 1]$.*

Proof. 1/ By assumption, u satisfies the following equation

$$\frac{A(u)(x)}{u_x^3(x)} = \int_x^1 \big(B(u)(x) + \Lambda\big)u(x)\mathrm{d}x. \tag{6}$$

Hence, $\frac{A(u)}{u_x^3}$ is decreasing, while $A(u)$ is increasing. So, u_x is an increasing function and u is convex.

As $u(0) = 0$, one has $u_x(0) \geq 0$. As u_x increases, it is a nonnegative function and u is increasing.

2/ When x tend to 1, $A(u)(x)$ and $u(x)$ have finite positive limits while $B(u)(x)$ tends to 0. Hence, $\left[\frac{A(u)}{u_x^3}\right]_x$ has a strictly negative limit $-c_0 = -\Lambda\, u(1)$. As $\frac{A(u)}{u_x^3}$ vanishes at $x = 1$, $(1-x)^{-1}\frac{A(u)}{u_x^3}$ converges to c_0 as x goes to 1, and $(1-x)^{-1/3}u_x$ converges to $\left(\frac{A(u)(1)}{c_0}\right)^{1/3}$.

3/ The function $(1-x)^{1/3}u_x(x)$ extends as a continuous nonvanishing positive function on $[0, 1]$, and consequently is bounded from below by $c_1 > 0$ and it is bounded from above by $c_2 > c_1$. ■

Next result asserts that the column has a cross section area which decreases from its bottom to its top, and presents a cusp at its top.

Proposition 5. *Let a be the rescaled shape associated to u by (3). Then*

1/ *$a(x)$ is a decreasing function of x in $[0, 1]$.*

2/ *There exists $\gamma > 0$ such that $\lim_{x\to 1}(1-x)^{-2/3}a(x) = \gamma$.*

3/ *One has $\lim_{x\to 1}(1-x)^{1/3}a_x(x) = \frac{2\gamma}{3}$.*

Proof. By formula (3), $a(x)$ is proportional to $A(u)^{1/2}\, u_x^{-2}$. By (6), $A(u)^{1/2}\, u_x^{-3/2}$ is decreasing. By Proposition 4, $u_x^{-1/2}$ is decreasing, so that $a(x)$ is decreasing.

By (3), one has $\lim_{x\to 1} a(x)u_x^2 = \frac{\lambda}{2}A(u)(1)$ hence, by Proposition 4, $\lim_{x\to 1}(1-x)^{-2/3}a(x) = \gamma$ for some $\gamma > 0$.

3/ Differentiating (6), one finds $u_{xx} > 0$ on $[0, 1]$ and the existence of $L > 0$,

$$L = \lim_{x\to 1} \frac{A(u)(x)}{u_x^4} u_{xx}.$$

Hence, $u_{xx}(1-x)^{4/3} \to_{x\to 1} c_0$ for some $c_0 > 0$.

Differentiating in Eq. (3), one gets $a_x(1-x)^{1/3} \to_{x\to 1} \gamma_0$ for some γ_0, which in virtue of 2/ must be equal to $2\gamma/3$. ■

3. On an auxiliary eigenvalue problem

In this section, we assume $k \geq k_{min} > 0$

Given a function $a \in W^{1,1}((0, 1))$, $a \geq 0$, $\int_0^1 a(x)\mathrm{d}x = 1$, we consider the associated problem

$$[E(a)] \quad \begin{cases} \text{Find } \lambda > 0 \text{ and a function } u \geq 0 \text{ satisfying the equation} \\ \left[a(x)^2 u'(x)\right]' + \lambda u(x) \left(\int_x^1 a(y)\mathrm{d}y + k\right) = 0 \\ u(0) = 0, \qquad \lim_{x\to 1} a(x)^2 u_x(x) = 0. \end{cases}$$

Notice that any solution u of $[E(a)]$ must satisfy

$$a(x)^2 u'(x) = \lambda \int_x^1 u(y) \left(\int_y^1 a(z)\mathrm{d}z + k\right) \mathrm{d}y. \tag{7}$$

In order to determine the shape and the height of Euler's tallest column, one must find the function a corresponding to the largest possible λ.

One way to solve problem $[E(a)]$ consists to start by associating to a as above, the quadratic form

$$Q_a(u) = \int_0^1 a(x)^2 u_x(x)^2 \mathrm{d}x, \qquad u \in \mathcal{D}(Q_a) = \left\{ u \in L^2(0, 1) \cap H^1_{loc}([0, 1)), u(0) = 0, a u_x \in L^2(0, 1)\right\}, \tag{8}$$

then the Rayleigh quotient

$$R_a(u) = \frac{Q_a(u)}{\int_0^1 u(x)^2 \left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x}, \quad u \in \mathcal{D}(Q_a), u \neq 0, \tag{9}$$

and finally the spectrum lower bound

$$\lambda(a) = \inf\left\{R_a(u) \,/\, u \in \mathcal{D}(Q_a)\right\}. \tag{10}$$

Then, one can reformulate the problem as

$$[E'(a)] \quad \text{Find } u \neq 0 \text{ such that } R_a(u) = \lambda(a).$$

It is clear that any solution u to $[E'(a)]$ provides a solution $(\lambda = \lambda(a), u)$ to $[E(a)]$. Indeed, the boundary condition at $x = 1$ holds, since u satisfies the integrodifferential equation (7).

Lemma 6. *Let $a \in W^{1,1}(0, 1)$ satisfying the following assumptions:*

(c1) $a \geq 0$, $\int_0^1 a(x)\mathrm{d}x = 1$;
(c2) $\|a\, a_x\|_\infty \leq +\infty$;
(c3) $\exists\, \delta > 0, p < 1$ *such that* $a(x) \geq \delta(1-x)^p$, $x \in (0, 1)$.

Then, if $u_n \to u$ weakly in $L^2(0, 1)$ and $\sup_n Q_a(u_n) < +\infty$, then $au_n' \to au'$ weakly and $u_n \to u$ strongly in $L^2(0, 1)$.

Proof. The sequence $\{au_n'\}$ is bounded in $L^2(0, 1)$. Let w be any weak accumulation point and let us show that $w = au'$. Assume that $u_n \to u$ and $au_n' \to w$ weakly in $L^2(0, 1)$, then, as $aa'u_n \to aa'u$ and $a^2u_n' \to a^2u'$ weakly in $L^2(0, 1)$ (note that a is bounded since a^2 is bounded), $(a^2u_n)' \to (a^2u)'$ weakly in $L^2(0, 1)$. This implies first that $a^2u_n \to a^2u$ strongly and uniformly, and then that $aw = a^2u'$, so that $au_n' \to au'$ weakly. From the strong convergence of $a^2(u_n - u)$ to 0 we deduce that $\int_0^1 a(x)^2(u_n - u)^2 \to 0$, so that we have

$$(1-x)^p(u_n - u) \to 0 \text{ strongly and } (1-x)^p(u_n' - u') \to 0 \text{ weakly in } L^2(0, 1).$$

Suppose now that $(1-x)^q(u_n-u)$ converges strongly to 0 in $L^2(0,1)$ for some $q \le p$. Then

$$\int_0^1 (1-x)^{p+q}(u_n-u)(u_n'-u') \to 0$$

and, integrating by parts, $\left[\frac{1}{2}(1-x)^{p+q}(u_n-u)^2\right]_0^1 + \frac{p+q}{2}\int_0^1 (1-x)^{p+q-1}(u_n-u)^2 \to 0$. We have a sum of two positive terms, so that $(1-x)^{(p+q-1)/2}(u_n-u) \to 0$ strongly.

So, we get a sequence p_n such that $p_0 = p$, $p_{n+1} = \frac{p+p_n-1}{2}$ and $(1-x)^{p_n}(u_n-u) \to 0$ strongly. Then, there is some $n > 0$ such that $p_{n-1} > 0$ and $p_n \le 0$, so that finally $u_n \to u$ strongly in $L^2(0,1)$. ■

Proposition 7. *Let $a \in W^{1,1}(0,1)$ satisfying the assumptions* (c1), (c2) *and* (c3) *of Lemma 6. Then*

1/ $\exists u \in \mathcal{D}(Q_a)$, $u \ne 0$, such that $R_a(u) = \lambda(a)$.

2/ The quadratic form Q_a, as a densely defined quadratic form on $L^2(0,1)$, is closed.

3/ Consider the Hilbert space $L^2(0,1)$ imbedded with the new Hilbert norm

$$\|u\|_a = \left(\int_0^1 u(x)^2 \left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x\right)^{1/2} \tag{11}$$

and, on it, consider the closed selfadjoint nonnegative operator T_a associated with Q_a. Then the operator $T_a - \lambda(a)I$ has

- *a one dimensional kernel $\mathbb{R}\,u(a)$ with $u(a)$ nonnegative nondecreasing, $\|u(a)\|_a = 1$;*
- *a closed codimension 1 image $\{\mathbb{R}\,u(a)\}^\perp$.*

Proof. 1/ Let u_n be a minimizing sequence for R_a, with $\|u_n\|_a = 1, \forall n$. As R_a decreases when one replaces $u(x)$ by $\int_0^x |u'(y)|\mathrm{d}y$, one can suppose $u_n' \ge 0, \forall n$. Replacing the u_n by a subsequence, one can suppose that $u_n \to u$ weakly. By Lemma 6, au_n' converges to au' weakly and u_n converges u strongly in $L^2(0,1)$. Hence, $\|u\|_a = 1$ and $R_a(u) \le \lambda(a)$: i.e. u is a minimizer for R_a.

2/ It is proved in Lemma 6 that $u_n \to u$ and $au_n' \to w$ imply that u is locally in $H^1(0,1)$ and that $w = au'$.

3/ Any nonzero element u in the kernel of $T_a - \lambda(a)I$ satisfies $R_a(u) = \lambda(a)$. Hence, u' has then a constant sign : otherwise, R_a would strictly decrease when replacing $u(x)$ by $\int_0^x |u'(y)|\mathrm{d}y$. This shows that no nonzero element in the kernel of $T_a - \lambda(a)I$ can be orthogonal to $u(a)$, and consequently that $\mathrm{Ker}(T_a - \lambda(a)I) = \mathbb{R}\,u(a)$.

By selfadjointness of T_a, $\{\mathbb{R}u(a)\}^\perp$ is the closure of the image of $T_a - \lambda(a)I$. Conversely, let $w \ne 0$ be orthogonal to $u(a)$, and let us show that w belongs to this image.

Define the conditional minimum

$$\mu(w) = \inf Q_a(u) - \lambda(a)\|u\|_a^2, \quad u \in \mathcal{D}(Q_a) \int_0^1 u(x)w(x)\left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x = 1$$

an let u_n be a minimizing sequence. Note that $\|u_n\|_a \ge 1/\|w\|_a$. Note also that we can suppose that the u_n are orthogonal to $u(a)$ for the scalar product $\langle .,.\rangle_a$ associated to the $\|.\|_a$-norm.

We claim first that $\mu(w) \ne 0$. If this was not true, then $R_a(u_n)$ would tend to $\lambda(a)$ and, as in the proof of item 1/, $u_n/\|u_n\|_a$ would tend to a minimizer u of R_a, i.e. to $\pm u(a)$. One would have $\|u\|_a = 1$, with $\langle u, u(a)\rangle_a = 0$ and $u \in \mathbb{R}u(a)$. Hence we get a contradiction.

We claim then that $\|u_n\|_a$ is a bounded sequence. If not, one can suppose $\|u_n\|_a \to +\infty$ and then $u_n/\|u_n\|_a$ is a minimizing sequence for R_a, which leads to the same contradiction.

As a consequence, the sequence $Q_a(u_n)$ is bounded, and one can suppose that u_n and au_n' have weak limits in $L^2(0,1)$. By Lemma 6 the u_n have a strong limit u, which satisfies $\int_0^1 u(x)w(x)\left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x = 1$ and $Q_a(u) - \lambda(a)\|u\|_a^2 \le \mu(w)$, hence $Q_a(u) - \lambda(a)\|u\|_a^2 = \mu(w)$. Euler's equation for such a minimum is $T_a(u) - \lambda(a)u = \mu(w)w$, which ends the proof. ■

Definition 8. We shall denote by $u(a)$ the only minimizer for $R_a(.)$ – and the only eigenfunction of the operator T_a for the eigenvalue $\lambda(a)$ – which satisfies $u(a) \ge 0$ and $\|u(a)\|_a = 1$. Note that $u(a)$ is increasing.

4. Estimates on a critical point for $\lambda(a)$

Let us associate to $p \in]0,1[$ the following spaces and convex sets

$$\mathcal{E}(p) = \left\{a \in W^{1,1}(0,1) \,/\, (1-x)^{1-p}a' \in L^\infty(0,1), a(1) = 0\right\} \tag{12}$$

with the norm $\|a\|_{\mathcal{E}(p)} = \|(1-x)^{1-p}a'\|_\infty$;

$$\mathcal{E}(p)^0 = \left\{ a \in \mathcal{E}(p) \,/\, \int_0^1 a(x)\mathrm{d}x = 0 \right\}; \tag{13}$$

$$\mathcal{E}(p)^1_+ = \left\{ a \in \mathcal{E}(p) \,/\, a \geq 0, \int_0^1 a(x)\mathrm{d}x = 1 \right\}; \tag{14}$$

$$\widehat{\mathcal{E}(p)^1_+} = \left\{ a \in \mathcal{E}(p)^1_+ \,/\, \exists \delta > 0 \text{ s.t. } a(x) \geq \delta(1-x)^p, \forall x \right\}. \tag{15}$$

It is not difficult to show that the first two spaces become a Banach space with the mentioned norm.

Note that for $a \in \widehat{\mathcal{E}(p)^1_+}$ and $\alpha \in \mathcal{E}(p)^0$ in a neighborhood of 0, one has $a + \alpha \in \widehat{\mathcal{E}(p)^1_+}$; hence $a + \alpha$ satisfies the assumptions of Proposition 7, so that $\lambda(a+\alpha)$ and $u(a+\alpha)$ are well defined.

Proposition 9. *Let $a \in \widehat{\mathcal{E}(p)^1_+}$. Then*

1/ There exists a neighborhood $\mathcal{V}_0$ of 0 in the Banach space $\mathcal{E}(p)^0$ such that, for $\alpha \in \mathcal{V}_0$, one has $a+\alpha \in \widehat{\mathcal{E}(p)^1_+}$ and $\mathcal{D}(Q_{a+\alpha}) = \mathcal{D}(Q_a)$.

2.i/ The map $\alpha \to u(a+\alpha)$ is smooth from $\mathcal{V}_0$ into $\mathcal{D}(Q_a)$, equipped with the graph norm $\|u\|_{Q_a} = \left(\|u\|_2^2 + Q_a(u)\right)^{1/2}$.

2.ii/ The map $\alpha \to \lambda(a+\alpha)$ is a smooth map from $\mathcal{V}_0$ into $\mathbb{R}$.

3/ One has $D\big(\alpha \to \lambda(a+\alpha)\big) = 0$ at $\alpha = 0$ if and only if

$$\exists \mu > 0 \text{ such that } a\,u(a)_x^2 = \frac{\lambda(a)}{2}\left(\int_0^x u(a)(y)^2\mathrm{d}y + \mu\right). \tag{16}$$

Proof. 1/ By definition of $\widehat{\mathcal{E}(p)^1_+}$, one has $a(x) \geq \delta(1-x)^{1-p}$ $(x \in (0,1))$ for some $\delta > 0$. Set $\eta = \delta(1-p)/2$. If $\|\alpha\|_{\mathcal{E}(p)} \leq \eta$, then $|\alpha'(x)| \leq \frac{\delta}{2}(1-x)^p$ $(x \in [0,1])$ and $a+\alpha \in \widehat{\mathcal{E}(p)^1_+}$. Take as $\mathcal{V}_0$ the ball with radius η in $\mathcal{E}(p)^0$.

If $u \in \mathcal{D}(Q_a)$, then au_x is square integrable, then $(1-x)^p u_x$ is square integrable, and αu_x is square integrable: u belongs to $\mathcal{D}(Q_{a+\alpha})$. The same argument shows the converse inclusion.

2/ For $\alpha \in \mathcal{V}_0$, define $T^{1/2}_{a+\alpha}$ as the positive square root of $T_{a+\alpha}$ in $L^2(0,1)$ with the $\|.\|_{a+\alpha}$-norm. Its domain is $\mathcal{D}(Q_{a+\alpha}) = \mathcal{D}(Q_a) = \mathcal{D}(T_a^{1/2})$.

$T_a^{1/2} - \lambda(a)^{1/2}I$ is a closed selfadjoint operator with kernel $\mathbb{R}u(a)$. Its image is contained in $\mathbb{R}u(a)^\perp$ (the orthogonal for the scalar product $\langle .,. \rangle_a$ associated with the $\|.\|_a$-norm) and contains the image of $T_a - \lambda(a)I = \big(T_a^{1/2} - \lambda(a)^{1/2}I\big)\big(T_a^{1/2} + \lambda(a)^{1/2}I\big)$. Hence the image of $T_a^{1/2} - \lambda(a)^{1/2}I$ is closed and equal to $\mathbb{R}u(a)^\perp$.

Consider the subspace W of $L^2(0,1)$ defined as

$$W = \left\{ w \in \mathcal{D}(Q_a) \,/\, \langle w, u(a)\rangle_a = 0 \right\}.$$

It is a closed hyperplane of Q_a, equipped with the graph norm, and hence a Banach space.

Consider now the following map

$$\mathcal{V}_0 \times W \times \mathbb{R}^*_+ \ni (\alpha, w, \mu) \mapsto \varphi(\alpha, w, \mu) = T^{1/2}_{a+\alpha}(u(a)+w) - \mu\,(u(a)+w) \in L^2(0,1).$$

As a straightforward consequence of item 3/ of Proposition 7, the partial differential $D_{w,\lambda}\varphi(0,0,\lambda(a)^{1/2})$ is an isomorphism from $W \times \mathbb{R}$ onto $L^2(0,1)$. The implicit function theorem provides a neighborhood $\mathcal{V}_1 \subset \mathcal{V}_0$ of O in $\mathcal{E}(p)^0$, and the smooth maps $\alpha \to \mu(\alpha)$ from $\mathcal{V}_1$ into $\mathbb{R}^*_+$, and $\alpha \to w(\alpha)$ from $\mathcal{V}_1$ into W, such that $\varphi(\alpha, w(\alpha), \mu(\alpha)) = 0, \forall \alpha \in \mathcal{V}^1$. Moreover, $w(0) = 0$ and $\mu(0) = \lambda(a)^{1/2}$.

This means $T^{1/2}_{a+\alpha}(u(a)+w(\alpha)) = \mu(\alpha)(u(a)+w)$, i.e. $\mu(\alpha)^2$ is an eigenvalue of $T_{a+\alpha}$. In particular, $\lambda(a+\alpha) \leq \mu(\alpha)^2 \to_{\alpha\to 0} \lambda(a)$. From which we deduce that $\lambda(a+\alpha)$ is bounded on $\mathcal{V}_1$.

We claim now that the map $\alpha \mapsto u(a+\alpha)$ is continuous at $\alpha = 0$. Suppose that this is not true. Let $\alpha_n \to 0$ in $\mathcal{V}^0$ such that $u(a+\alpha_n)$ does not converge to $u(a)$: the sequences $u(a+\alpha_n)$ and $(a+\alpha_n)u(a+\alpha_n)'$ are bounded in $L^2(0,1)$, and one can suppose that they have weak limits u and au' respectively. $au(a+\alpha_n)'$ converges also weakly in $L^2(0,1)$ since $\frac{a}{a+\alpha} \to 1$ uniformly as $\alpha \to 0$. By Lemma 6, $u(a+\alpha_n)$ will converge strongly to u, so that one has $\|u\|_a = 1$ and $Q_a(u) \leq \liminf Q_{a+\alpha}(u(a+\alpha))$, which provides $R_a(u) \leq \liminf \lambda(a+\alpha_n) \leq \lim \mu(\alpha_n)^2 = \lambda(a)$. As $u \geq 0$, we have $u = u(a)$ and the claim is proved.

We claim now that $v(a+\alpha) = u(\alpha) + w(a)$ is proportional to $u(a+\alpha)$ for α close to 0. If not, let α_n such that $v(a+\alpha_n) \notin \mathbb{R}u(a+\alpha_n)$. As an eigenfunction of $T_{a+\alpha_n}$, $v(a+\alpha_n)$ is orthogonal to $u(a+\alpha_n)$, which leads to a contradiction as $\alpha_n \to 0$:

$$0 = \langle v(a+\alpha_n), u(a+\alpha_n)\rangle_{a+\alpha_n} \to \langle u(a), u(a)\rangle_a = 1.$$

We have proved that, for α close to 0, one has $\mu(\alpha)^2 = \lambda(a+\alpha)$ and $v(a+\alpha) = \rho(a+\alpha)u(a+\alpha)$. As $\rho(a+\alpha) \to 1$, it is > 0 for α close to 0; consequently $u(a+\alpha) = \frac{v(a+\alpha)}{\|v(a+\alpha)\|_{a+\alpha}}$, which is a smooth function.

3/ One has $D\lambda(a) = 0$ if and only if $\frac{d}{ds}R_{a+s\alpha}(u(a+s\alpha)) = 0 \Leftrightarrow \frac{d}{ds}R_{a+s\alpha}(u(a)) = 0$ at $s = 0$, for any $\alpha \in \mathcal{E}(p)^0$ (we use the fact that $u(a)$, as a minimizer, is a critical point for $u \mapsto R_a(u)$). This can be written – with $\lambda = \lambda(a)$ and $u = u(a)$ – as

$$\left.\begin{aligned} 2\int_0^1 a(x)\alpha(x)u'^2\mathrm{d}x &= \lambda\int_0^1 u(x)^2\int_x^1 \alpha(y)\mathrm{d}y\mathrm{d}x \\ &= \lambda\int_0^1 \alpha(x)\int_0^x u(y)^2\mathrm{d}y\mathrm{d}x \end{aligned}\right\} \quad \forall\alpha \in \mathcal{E}(p),\ \int_0^1 \alpha(x)\mathrm{d}x = 0.$$

Hence the result. ■

In the remainder of this section, we shall assume that $a \in \widehat{\mathcal{E}(p)^1_+}$ is a critical point for $\lambda(.)$. For sake of simplicity, we set $\lambda = \lambda(a)$ and $u = u(a)$.

Lemma 10. *Let $a \in \widehat{\mathcal{E}(p)^1_+}$. One has*

$$\lambda\left(1-\frac{1}{2k}\right) \le a(x)u'^2 \le \lambda\left(1+\frac{1}{2k}\right), \quad x \in [0,1[. \tag{17}$$

Proof. Since $\int_0^1 a(x)\mathrm{d}x = 1$, one can write, invoking (16)

$$\begin{aligned} \lambda = \int_0^1 a(x)^2u'^2 &= \frac{\lambda}{2}\int_0^1 a(x)\left(\int_0^x u(y)^2\mathrm{d}y + \mu\right)\mathrm{d}x \\ &= \frac{\lambda}{2}\left(\int_0^1 u(x)^2\int_x^1 a(y)\mathrm{d}y\mathrm{d}x + \mu\right) \end{aligned} \tag{18}$$

which implies

$$2 - \int_0^1 u(x)^2\mathrm{d}x \le \mu \le 2. \tag{19}$$

Since $\|u\|_a = 1$ and $\int_0^1 a(x)\mathrm{d}x = 1$, one obviously has

$$\frac{1}{k+1} \le \int_0^1 u(x)^2\mathrm{d}x \le \frac{1}{k}, \tag{20}$$

from which one deduces

$$2 - \frac{1}{k} \le \mu \le 2.$$

As (16) implies for any $x \in [0,1[$

$$\frac{\lambda}{2}\mu \le a(x)u'^2 \le \frac{\lambda}{2}\left(\int_0^1 u(x)^2\mathrm{d}x + \mu\right),$$

one gets the result. ■

Lemma 11. *Assume $a \in \widehat{\mathcal{E}(p)^1_+}$. One has, for $x \in [0,1]$*

$$u(x) \le 2^{3/4}\lambda^{1/4}k^{-1/4}\left(1+\frac{1}{2k}\right)^{1/2}\left(1-(1-x)^{2/3}\right)^{3/4}. \tag{21}$$

Proof. According to Lemma 10, one has

$$a(x)^2u'^4 \le \lambda^2\left(1+\frac{1}{2k}\right)^2. \tag{22}$$

From (7), one deduces, since u is nondecreasing,

$$a(x)^2u'(x) \ge \lambda k u(x)(1-x). \tag{23}$$

Hence $u(x)u'^3 \le \lambda k^{-1}\left(1+\frac{1}{2k}\right)^2(1-x)^{-1}$, $u(x)^{1/3}u'^{1/3}k^{-1/3}\left(1+\frac{1}{2k}\right)^{2/3}(1-x)^{-1/3}$ and we get the result, by integration. ■

Lemma 12. *There exist a constant C (independent on a, p and k) such that, for any $a \in \widehat{\mathcal{E}(p)^1_+}$, critical point of $\lambda(.)$, one has*

$$\int_0^1 a(x)^4(1-x)^{-2}\mathrm{d}x \le C\,\lambda(a)^2\,k^2\left(1+O\left(\frac{1}{k}\right)\right) \quad (k\to+\infty). \tag{24}$$

Proof. From Eq. (7) and Lemma 11, one deduces

$$\begin{aligned} a(x)^2u'(x) &= \lambda\int_x^1 u(y)\left(\int_y^1 a(z)\mathrm{d}z+k\right)\mathrm{d}y \\ &\le \lambda(k+1)\int_x^1 u(y)\mathrm{d}y \\ &\le 2^{3/4}\lambda^{5/4}k^{3/4}\left(1+O\left(\frac{1}{k}\right)\right)\int_x^1\left(1-(1-y)^{2/3}\right)^{3/4}\mathrm{d}y \\ &= 2^{3/4}\lambda^{5/4}k^{3/4}\left(1+O\left(\frac{1}{k}\right)\right)\int_0^{1-x}(1-y^{2/3})^{3/4}\mathrm{d}y \\ &\le 2^{3/4}\lambda^{5/4}k^{3/4}\left(1+O\left(\frac{1}{k}\right)\right)\int_0^{1-x}(1-y)^{1/2}\mathrm{d}y \\ &\le \frac{2}{3}2^{3/4}\lambda^{5/4}k^{3/4}\left(1+O\left(\frac{1}{k}\right)\right)(1-x^{3/2}) \end{aligned} \tag{25}$$

and

$$a^4(x)u'^2 \le C_0\lambda^{5/2}k^{3/2}\left(1+O\left(\frac{1}{k}\right)\right)(1-x^{3/2})^2.$$

On the other side, by Lemma 10, one has

$$a(x)u'^2 \ge \lambda\left(1-\frac{1}{2k}\right).$$

The two inequalities imply

$$a^3(x) \le C_0\lambda^{3/2}k^{3/2}\left(1+O\left(\frac{1}{k}\right)\right)(1-x^{3/2})^2$$

and

$$a^4(x) \le C_0^{4/3}\lambda^2k^2\left(1+O\left(\frac{1}{k}\right)\right)(1-x^{3/2})^{8/3},$$

hence the result. ∎

5. A 'climbing the top' principle

The minimum principle will assert that a critical point for $\lambda(a)$ must be a local maximum:

Theorem 13. *There exists $k_0 > 0$ such that, for $k \ge k_0$, for $p \in]0,1[$ and $a \in \widehat{\mathcal{E}(p)^1_+}$ solution of $D\lambda(a) = 0$, one has*

$$\frac{\mathrm{d}^2\lambda(a+s\alpha)}{\mathrm{d}s^2}_{|s=0} < 0, \quad \forall\alpha \in \mathcal{E}(p)^0, \alpha \neq 0. \tag{26}$$

The proof of this theorem requires some preliminary lemmas. The first one, which is a key for the next estimates, states a Poincaré type (or a Poincaré–Hardy type inequality: cf. [12]):

Lemma 14. *Let $u, v \in W^{1,1}(]0,1[)$, $u'v'^2(0,1)$, $u(0)=0$, $v(1)=0$, $u \ge 0$ and convex. Then, one has*

$$\int_0^1 u(x)^2v(x)^2 \le \frac{1}{12}\int_0^1 u'^2v'^2. \tag{27}$$

Proof. One has, since u is convex

$$u(x) = \int_0^x u_x(y)\mathrm{d}y \le xu'(x)$$

then

$$|u(x)v(x)| \le u(x)\int_x^1 |v'(y)|\mathrm{d}y \le xu'(x)\int_x^1 |v'(y)|\mathrm{d}y \le x\int_x^1 u'(y)|v'(y)|\mathrm{d}y.$$

By Cauchy–Schwartz, we get $u(x)^2v(x)^2 \le x^2(1-x)\int_0^1 u'^2v'^2\mathrm{d}y$ and we get the result, by integration. ■

Lemma 15. *For any $a \ge 0$, $\int_0^1 a(x)\mathrm{d}x = 1$, one has*

$$\lambda(a) \le \frac{16^2}{k}. \tag{28}$$

Proof. Set $\Omega = \{x \in [0, 3/4] \,/\, a(x) \le 4\}$. Note that Ω has a Lebesgue measure $|\Omega| \ge 1/2$.

Choose $u(x)$ such that $u_x = 1_\Omega$, i.e. $u_x(x) = 1$ if $x \in \Omega$, $u_x(x) = 0$ if $x \notin \Omega$. Note that $u(x) \ge 1/2$ for $x \ge 3/4$, so that $\|u\|_2 \ge 1/4$ and $\|u\|_a^2 \ge k/16$.

On the other hand, one has $0 \le a(x)u_x(x) \le 4$ for any $x \in [0, 1]$, so that $Q_a(u) \le 16$. Hence, $R_a(u) \le 16^2/k$, and we get the result for $\lambda(a) \le R_a(u)$. ■

Lemma 16. *Let $a \in \widehat{\mathcal{E}(p)^1_+}$ and $\alpha \in \mathcal{E}(p)^0$ such that $\frac{\mathrm{d}\lambda(a+s\alpha)}{\mathrm{d}s} = 0$ at $s = 0$. Then one has*

$$\frac{\mathrm{d}^2\lambda(a+s\alpha)}{\mathrm{d}s^2}_{|s=0} \le -4\int_0^1 \alpha^2u'^2 + \frac{2\lambda}{k}\int_0^1 u(x)^2\left(\int_x^1 \alpha(y)\mathrm{d}y\right)^2\mathrm{d}x. \tag{29}$$

Proof. Set $\lambda = \lambda(a)$ and $u = u(a)$.

Define $v \in \mathcal{D}(Q_a)$ by

$$v(x) = -\int_0^x \frac{\alpha(y)}{a(y)}u'(y)\mathrm{d}y \tag{30}$$

so that $av' = -\alpha u'$, and, for s close to 0, define $\mu(s)$ as

$$\mu(s) = \frac{N(s)}{D(s)} \quad \text{with } N(s) = Q_{a+s\alpha}(u+sv), \qquad D(s) = \|u+sv\|^2_{a+s\alpha}. \tag{31}$$

One has $\mu(s) = R_{a+s\alpha}(u+sv) \ge \lambda(a+s\alpha)$, with equality at $s = 0$. One has also, at $s = 0$

$$\begin{aligned}\frac{\mathrm{d}\mu(s)}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}R_{a+s\alpha}(u) + \frac{\mathrm{d}}{\mathrm{d}s}R_a(u+sv)\\ &= \frac{\mathrm{d}}{\mathrm{d}s}R_{a+s\alpha}(u)\\ &= \frac{\mathrm{d}}{\mathrm{d}s}R_{a+s\alpha}(u(a+s\alpha))\\ &= 0\end{aligned} \tag{32}$$

using twice the fact that u is a minimum for $R_a(.)$. From which we conclude, first that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\lambda(a+s\alpha) \le \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mu(s); \tag{33}$$

and then that, at $s = 0$, $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mu(s)$ is equal to $\frac{\mathrm{d}^2}{\mathrm{d}s^2}N(s) - \lambda\frac{\mathrm{d}^2}{\mathrm{d}s^2}D(s)$.

One computes at $s = 0$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}D(s) &= \int_0^1 v(x)^2\left(\int_x^1 a(y)\mathrm{d}y + k\right)\mathrm{d}x + 2\int_0^1 u(x)v(x)\left(\int_x^1 \alpha(y)\mathrm{d}y\right)\mathrm{d}x\\ &\ge k\int_0^1 v(x)^2\mathrm{d}x + 2\int_0^1 u(x)v(x)\left(\int_x^1 \alpha(y)\mathrm{d}y\right)\mathrm{d}x\\ &\ge -\frac{1}{k}\int_0^1 u(x)^2\left(\int_x^1 \alpha(y)\mathrm{d}y\right)^2\mathrm{d}x;\end{aligned} \tag{34}$$

then

$$\frac{1}{2}\frac{d^2}{ds^2}N(s) = \int_0^1 \Big(a(x)^2 v'^2 + \alpha(x)^2 u'^2 + 4a(x)\alpha(x)u'(x)v'(x)\Big)dx$$
$$= -2\int_0^1 \alpha(x)^2 u'^2 dx\,. \tag{35}$$

Hence the result, invoking Lemma 14. ∎

Proof of Theorem 13. According to [7], if a is a critical point then $u = u(a)$ is a solution of the problem $P(A, B)$. It is shown in Proposition 4 that u must be convex. So the Poincaré inequality (27) of Lemma 14 holds, and one can use it in the inequality of Lemma 16, with $v(x) = \int_x^1 \alpha(y)dy$, One gets

$$\frac{d^2\lambda(a+s\alpha)}{ds^2}_{|s=0} \le \left(-4 + \frac{\lambda}{6k}\right)\int_0^1 \alpha^2 u'^2.$$

By Lemma 15, one will have

$$\frac{d^2\lambda(a+s\alpha)}{ds^2}_{|s=0} < 0$$

as soon as $k^2 > 32/3$. Which ends the proof. ∎

6. Uniqueness of a critical point

The purpose of this section is to prove the following uniqueness theorem:

Theorem 17. *There exists $k_1 > 0$ such that, for $p \in]0, 1[$ and $k \ge k_1$, there exists at most one solution in $\widehat{\mathcal{E}(p)}^1_+$ for the equation $D\lambda(a) = 0$.*

The proof of this theorem requires a preliminary Poincaré inequality, more general than Lemma 14:

Lemma 18. *Let $a \in \widehat{\mathcal{E}(p)}^1_+$ with associated u and λ. Then one has*

$$\int_0^1 u(x)^2 v(x)^2 \le k^{-2}\lambda^{-2}C(a)\int_0^1 u'^2 v'^2, \quad \forall v \in L^2(0,1),\ u'v' \in L^2(0,1),\ v(1) = 0, \tag{36}$$

with $C(a) = \int_0^1 a(x)^4(1-x)^2 dx$.

Proof. Replacing $v(x)$ by $\int_x^1 |v'(y)|dy$, one can suppose $v \ge 0$ and decreasing.

As u is increasing, one can write

$$\int_0^1 u(x)^2 v(x)^2 dx = \int_0^1 u(x)^2\left(\int_x^1 v'(y)dy\right)^2 dx$$
$$\le \int_0^1\left(\int_x^1 u(y)v'(y)dy\right)^2 dx. \tag{37}$$

Inequality (23) implies

$$\left(\int_x^1 u(y)v'(y)dy\right)^2 \le \lambda^{-2}k^{-2}\left(\int_x^1 v'(y)u'^2(1-y)^{-1}dy\right)^2$$
$$\le \lambda^{-2}k^{-2}C(a)\int_x^1 u'^2 v'^2 dy \tag{38}$$

and the result. ∎

Proof of Theorem 17. We suppose that there are two different critical points a_0 and a_1 in $\widehat{\mathcal{E}(p)}^1_+$, and we set $a(s) = sa_1 + (1-s)a_0$. By Theorem 13, the map $s \to \lambda(a(s))$ is decreasing for s close to 0, increasing for s close to 1, so that $\min_{s\in[0,1]}\lambda(a(s))$ is reached at some $\sigma \in]0, 1[$. The purpose is to combine Lemmas 16, 12 and 18 for showing that, for k large enough, one has $\frac{d^2\lambda(a(s))}{ds^2} < 0$ at $s = \sigma$; which provides a contradiction.

Step 1. *Estimate for* $C(a(\sigma))$. By convexity of $C(a)$, Lemma 12 provides, with $\lambda_0 = \lambda(a_0)$ and $\lambda_1 = \lambda(a_1)$,

$$\begin{aligned} C(a(\sigma)) &\le \sigma C(a_0) + (1-\sigma)C(a_1) \\ &\le C\big(\sigma\lambda_1^2 + (1-\sigma)\lambda_0^2\big)k^2\left(1 + O\left(\frac{1}{k}\right)\right). \end{aligned} \tag{39}$$

Step 2. *Estimate for* $\lambda(\sigma)$. Let $u = u(\lambda(\sigma))$. One has $\|u\|_{a(\sigma)} = 1$, which by (20) implies that $\|u\|^2_{a_0}$ and $\|u\|^2_{a_1}$ lie between $\frac{k}{k+1}$ and $\frac{k+1}{k}$.

With $u = u(a(\sigma))$, one has

$$\begin{aligned} \lambda(a(\sigma)) &= R_{a(\sigma)}(u) = Q_a(u) \\ &\ge \sigma^2 Q_{a_1}(u) + (1-\sigma)^2 Q_{a_0}(u) \\ &\ge \left(1 - O\left(\frac{1}{k}\right)\right)\big(\sigma^2 R_{a_1}(u) + (1-\sigma)^2 R_{a_0}(u)\big) \\ &\ge \left(1 - O\left(\frac{1}{k}\right)\right)\big(\sigma^2\lambda_1 + (1-\sigma)^2\lambda_0\big). \end{aligned} \tag{40}$$

Step 3. *Estimate for* $\frac{C(a(\sigma))}{k^3\,\lambda(a(\sigma))}$. From (39) and (40), one deduces (with $\lambda_{\max}(0)$ as defined in the proof of Theorem 13)

$$\begin{aligned} \frac{C(a(\sigma))}{k^3\lambda} &\le \frac{C}{k}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{\sigma\lambda_1^2 + (1-\sigma)\lambda_0^2}{\sigma^2\lambda_1 + (1-\sigma)^2\lambda_0} \\ &\le \frac{C\,\lambda_{\max}(0)}{k}\left(1 + O\left(\frac{1}{k}\right)\right)\frac{\sigma\lambda_1 + (1-\sigma)\lambda_0}{\sigma^2\lambda_1 + (1-\sigma)^2\lambda_0} \\ &\le \frac{C\,\lambda_{\max}(0)}{k}\left(1 + O\left(\frac{1}{k}\right)\right)\left(\sqrt{\frac{\lambda_0}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_0}}\right) \end{aligned} \tag{41}$$

since the function $\sigma \to \frac{\sigma\lambda_1 + (1-\sigma)\lambda_0}{\sigma^2\lambda_1 + (1-\sigma)^2\lambda_0}$ attains its maximum at $s = \frac{\sqrt{\lambda_0}}{\sqrt{\lambda_0}+\sqrt{\lambda_1}}$.

Step 4. *Estimate for* λ_0. By Lemma 10, we have

$$\sqrt{a_0(x)}u_0'(x) \ge \sqrt{\lambda_0\left(1 - \frac{1}{2k}\right)}$$

which implies

$$\int_0^1 a_0(x)^2 u_0'(x)\mathrm{d}x \ge \sqrt{\lambda_0\left(1 - \frac{1}{2k}\right)}\int_0^1 a_0(x)^{3/2}\mathrm{d}x \ge \sqrt{\lambda_0\left(1 - \frac{1}{2k}\right)}.$$

On the other hand, (7) will provide

$$\int_0^1 a_0(x)^2 u_0'(x\mathrm{d}x) \le \lambda_0\int_0^1 u_0(x)\left(\int_x^1 a(y) + k\right)\mathrm{d}x \le \lambda_0\sqrt{k+1};$$

hence $\sqrt{\lambda_0(1 - \frac{1}{2k})} \le \lambda_0\sqrt{k+1}$ and

$$\sqrt{\lambda_0} \ge \frac{1}{\sqrt{k}}\left(1 - O\left(\frac{1}{k}\right)\right). \tag{42}$$

This implies, supposing $\lambda_0 \le \lambda_1$,

$$\sqrt{\frac{\lambda_0}{\lambda_1}} + \sqrt{\frac{\lambda_1}{\lambda_0}} \le 1 + \sqrt{\lambda_{\max}(0)}\,\sqrt{k}\left(1 + O\left(\frac{1}{k}\right)\right). \tag{43}$$

Finally, we get the existence of an absolute constant C' such that

$$\frac{C(a(\sigma))}{k^3\lambda} \le \frac{C'}{\sqrt{k}}, \quad k \ge k_0. \tag{44}$$

Step 5. *End of the proof.* Let us write, with $a = a(\sigma)$, $u = u(a)$, $\lambda = \lambda(a(\sigma))$ and $\alpha = a_1 - a_0$:

$$\begin{aligned}\frac{\mathrm{d}^2\lambda(a+s\alpha)}{\mathrm{d}s^2}_{|s=0} &\le -4\int_0^1 \alpha(x)^2 u'^2 \mathrm{d}x + \frac{2\lambda}{k}\int_0^1 u(x)^2\left(\int_x^1 \alpha(y)\mathrm{d}y\right)^2 \mathrm{d}x \quad \text{by (29)}\\ &\le -4\int_0^1 \alpha(x)^2 u'^2 \mathrm{d}x + \frac{2C(a)}{k^3\lambda}\int_0^1 \alpha(x)^2 u'^2 \mathrm{d}x \quad \text{by Lemma 18}\\ &\le \left(-4+\frac{2C'}{\sqrt{k}}\right)\int_0^1 \alpha(x)^2 u'^2 \mathrm{d}x \quad \text{by (44)}\end{aligned}$$

so that one has $\frac{\mathrm{d}^2\lambda(a(s))}{\mathrm{d}s^2}_{|s=\sigma} < 0$ as soon as $k > 4/C'^2$. Which ends the proof. ■

7. Proof of Theorems 2 and 3

Proof of Theorem 2. Let $u \in L^2(0,1) \cap H^1_{loc}([0,1[)$, be a solution of Problem $[P(A,B)]$. Then, one has $u > 0$ on $]0,1]$, u increasing and convex. By Proposition 5, the associated rescaled shape a will belong to $\widehat{\mathcal{E}(p)}^1_+$ with $p = 2/3$. By applying Theorem 17, we get the uniqueness of a.

By the results in Section 3, u is then uniquely determined up to a multiplicative constant. One checks easily that two proportional u cannot be solution of Problem $[P(A,B)]$, with the same constants μ and Λ, unless they are equal. For this, note that $\lambda = \lambda(a)$ is also uniquely determined, while changing u in νu $(\nu > 0, \nu \ne 1)$ would change $\frac{2}{\lambda}$ in $\frac{2}{\lambda'} = \frac{2}{\lambda} + (\nu^{-2}-1)\mu\int_0^1 u_x^{-2}\mathrm{d}x$, according to Formula (3). □

Proof of Theorem 3. 1/ Let (a, λ, u) satisfy the assumptions of the theorem. Then eliminating a between Equations $[E(a)]$ and $[DE(a)]$, one sees that u must be a solution of Problem $[P(A,B)]$. Hence, by Proposition 5, a belongs to $\widehat{\mathcal{E}(p)}^1_+$ with $p = 2/3$. And then, we conclude as above.

2/ is proved in Proposition 5.

3/ is nothing but a reformulation of 1/.

4/ comes from Propositions 7 and 9.

8. A variational formulation to problem $[P(A,B)]$

We shall consider the problem to minimize a functional $J : \mathcal{K} \to]0, +\infty[$ of the type

$$J(u) = \int_0^1 \frac{a(x)}{u_x(x)^2}\mathrm{d}x + \int_0^1 b(x)u(x)^2\mathrm{d}x$$

with $a(x) > \mu > 0$ and $b(x) > 0$ a.e., $x \in (0,1)$, on the convex cone

$$\mathcal{K} = \left\{u \in C([0,1]) \cap W^{1,1}_{loc}(0,1),\, u \ge 0,\, u_x \ge 0,\, u(0) = 0,\, \frac{\sqrt{a}}{u_x} \in L^2(0,1)\right\}$$

This is an example of minimization on a convex set.

Remark 19. We do not suppose a priori that u is convex $(u_{xx} \ge 0)$ but from the Eq. (48) below we can show that any minimizer u is such that u_x increases, whenever $a(x)$ is assumed to be nondecreasing.

Let us denote $\eta = \inf_{u\in\mathcal{K}} J(u)$. It is easy to construct special functions $v \in \mathcal{K}$ such that $J(v) < \infty$. So, we have that $\eta < +\infty$.

Proposition 20. *There exists a unique minimizer v of J on $\mathcal{K}$.*

Proof. The uniqueness of the minimizer comes from the strict convexity of J. The proof of the existence of the minimizer will be divided in several steps.

Step 1. Let $\{u_n\}$ be a minimizing sequence in $\mathcal{K}$ such that $J(u_n) \to \eta$. Then $\{u_n\}$ is bounded in the weighted space $L^2_b(0,1) = \{u : (0,1) \to \mathbb{R} : \int_0^1 b(x)u(x)^2\mathrm{d}x < \infty\}$. Thus, replacing it by a subsequence, one can suppose that it has a weak limit $u \in L^2_b(0,1)$. On the other hand, the sequence $\{\frac{1}{u_{nx}}\}$ is bounded in $L^2_a(0,1)$ and so, there exists $\psi \in L^2_a(0,1)$ such that $\{\frac{1}{u_{nx}}\} \rightharpoonup \psi$ weakly in $L^2_a(0,1)$ and

$$\int_0^1 a(x)\psi(x)^2\mathrm{d}x + \int_0^1 b(x)u(x)^2\mathrm{d}x \le \eta. \tag{45}$$

Step 2. Define $\varepsilon_N = \sup_{n\geq N} J(u_n) - \eta$, so that $\varepsilon_N \to 0$ as $N \to \infty$. Since ψ belongs to the weak closure of the set $\{\frac{1}{u_{nx}}, n \geq N\}$, for each N there exists a convex combination $\psi_N = \sum_{k=1}^{K} \lambda_k \frac{1}{u'_{n_k}}$ (with $n_k \geq N \forall k$) such that $\|\psi_N - \psi\|_{L^2_a(0,1)} \leq \varepsilon_N$. Let $v_N = \sum_{k=1}^{K} \lambda_k u_{n_k}$ be the corresponding convex combination of the u_n. Then $v_N \to u$ strongly in $L^2_b(0, 1)$. In fact, as $x \to 1/x$ is a convex function on $\mathbb{R}^*_+$, then $\{u_{nx}, n \geq N\}$ converges weakly in $L^2_a(0, 1)$, and so, by replacing $\{u_n\}$ by a sequence, one can suppose that $\{u_n\} \to u$ strongly in $L^2_b(0, 1)$. Moreover $\frac{1}{v_{Nx}} \leq \psi_N$ and consequently

$$\limsup_{N\to\infty} \int_0^1 \frac{a(x)}{v_{Nx}(x)^2} \mathrm{d}x \leq \lim_N \int_0^1 a(x)\psi_N^2(x)\mathrm{d}x = \int_0^1 a(x)\psi(x)^2 \mathrm{d}x. \tag{46}$$

Step 3. Since $J(v_N) \geq \eta$, using (45), we get

$$\begin{aligned}
&\liminf_N \int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}\mathrm{d}x + \int_0^1 b(x)u(x)^2\mathrm{d}x \\
&= \liminf_N \left(\int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}\mathrm{d}x + \int_0^1 b(x)v_N(x)^2\mathrm{d}x\right) = \liminf_N J(v_N) \\
&\geq \eta \geq \int_0^1 a(x)\psi(x)^2\mathrm{d}x + \int_0^1 b(x)u(x)^2\mathrm{d}x.
\end{aligned}$$

Hence

$$\lim \int_0^1 a(x)\frac{1}{v_{Nx}(x)^2}\mathrm{d}x = \int_0^1 a(x)\psi(x)^2\mathrm{d}x. \tag{47}$$

Step 4. We have that $0 \leq \frac{1}{v_{Nx}} \leq \psi_N$ and $\lim_N \int_0^1 \left(\psi_N^2 - \frac{1}{{v'_N}^2}\right) a(x)\mathrm{d}x = 0$. Then $\lim_N \frac{1}{v_N x} = \psi$ strongly in $L^2_a(0, 1)$. Substituting the v_N by a suitable subsequence, one can suppose that $v_N \to u$ and $\frac{1}{v_{Nx}} \to \psi$ a.e. in $(0, 1)$. Then $v_{Nx} \to \frac{1}{\psi}$a.e. and by Fatou's Lemma $\int_0^x \frac{1}{\psi(t)}\mathrm{d}t \leq \liminf_N \int_0^x v_{Nx}(t)\mathrm{d}t = \liminf_N v_N(x) = u(x)$. Defining $v(x) = \int_0^x \frac{1}{\psi(t)}\mathrm{d}t$, then $v \in \mathcal{K}$, $J(v) \leq \int_0^1 a(x)\psi(x)^2\mathrm{d}x + \int_0^1 b(x)u(x)^2\mathrm{d}x \leq \eta$ and so v is a minimizer for J. ■

Proposition 21. *Let v be the minimizer of J on $\mathcal{K}$. Then*

$$\frac{a(x)}{v_x(x)^3} = \int_x^1 b(t)v(t)\mathrm{d}t \tag{48}$$

and hence the boundary flux blowing-up condition $\lim_{x\to 1} v'(x) = +\infty$ holds. Moreover v is a weak solution of

$$\left(\frac{a(x)}{v_x^3}\right)_x + b(x)v = 0. \tag{49}$$

Proof. Choose $\sigma \neq 0$ in $L^\infty(0, 1)$ and set $w(x) = \int_0^x \sigma(t)v_x(t)\mathrm{d}t$ and $v_s = v + sw$ for $s \in \mathbb{R}$. For s in $\mathbb{R}$, $|s| < \frac{1}{\|\sigma\|_\infty}$, one has $v_{sx}(x) = u_x(x)(1 + s\sigma(x)) \geq (1 - s\|\sigma\|_\infty)u_x(x) \geq 0$ and $v_s(x) \leq (1 + s\|\sigma\|_\infty)u(x)$. So $v_s \in \mathcal{K}$ with

$$\begin{aligned}
J(v_s) &= \int_0^1 a(x)\frac{1}{(1+s\sigma(x))^2 v_x(x)^2}\mathrm{d}x + \int_0^1 b(x)v_s(x)^2\mathrm{d}x \\
&\leq \frac{1}{(1-s\|\sigma\|_\infty)^2}\int_0^1 a(x)\frac{1}{v_x(x)^2}\mathrm{d}x + (1+s\|\sigma\|_\infty)^2\int_0^1 b(x)v(x)^2\mathrm{d}x \\
&\leq \max\left(\frac{1}{(1-s\|\sigma\|_\infty)^2}, (1+s\|\sigma\|_\infty)^2\right)J(v) < +\infty.
\end{aligned}$$

Then, $\frac{\mathrm{d}}{\mathrm{d}s}J(v_s) = 0$ at $s = 0$ provides

$$-2\int_0^x a(x)\frac{1}{v_x(x)^2}\sigma(x)\mathrm{d}x + 2\int_0^1 b(x)v(x)w(x)\mathrm{d}x = 0. \tag{50}$$

The second term in this equation is

$$\int_0^1 b(x)v(x)w(x)\mathrm{d}x = \int_0^1 b(x)v(x)\left(\int_0^x \sigma(t)v_x(t)\mathrm{d}t\right)\mathrm{d}x = \int_0^1 \sigma(t)v_x(t)\left(\int_t^1 b(x)v(x)\mathrm{d}x\right)\mathrm{d}t$$
$$= \int_0^1 \sigma(x)v_x(x)\left(\int_x^1 b(t)v(t)\mathrm{d}t\right)\mathrm{d}x,$$

so, (50) becomes

$$\int_0^1 \left(-a(x)\frac{1}{v_x(x)^2} + v_x(x)\left(\int_x^1 b(t)v(t)\mathrm{d}t\right)\right)\sigma(x)\mathrm{d}x = 0.$$

This holds for any σ in $L^\infty(0,1)$, and so we get $a(x)\frac{1}{v_x(x)^2} = v_x(x)\left(\int_x^1 b(t)v(t)\mathrm{d}t\right)$, i.e.

$$a(x)\frac{1}{v_x(x)^3} = \int_x^1 b(t)v(t)\mathrm{d}t. \tag{51}$$

This provides also the boundary condition

$$\lim_{x\to 1} a(x)\frac{1}{v_x(x)^3} = 0 \tag{52}$$

and

$$\left(a(x)\frac{1}{v_x(x)^3}\right)_x = -b(x)v(x). \quad \blacksquare$$

9. Proof of Theorem 1

This theorem is proved by iteration. Set (for instance) $u_0(x) = x$ and define the sequence u_n in $W^{1,1}_{loc}(0,1)$ by setting, for $n \geq 1$,

$$A_n = A(u_{n-1}) \qquad B_n = B(u_{n-1})$$

and u_n as the minimizer in $\mathcal{K}$ of the functional

$$J_n(u) = \int_0^1 \frac{A_n(x)}{u_x(x)^2}\mathrm{d}x + \int_0^1 \big(B_n(x) + \Lambda\big)u(x)^2\mathrm{d}x \tag{53}$$

provided by Proposition 20.

It is clear that, for each n, $B_n(x) \to 0$ as $x \to 1$. Moreover we have

Lemma 22. *Assume (4). Then $\{u_n\}$ is bounded in $L^2(0,1)$.*

Proof. Since A_n and u_n are nonnegative and increasing functions we have

$$\frac{A_n(1)}{u_{nx}(x)^3} \geq \frac{A_n(x)}{u_{nx}(x)^3} = \int_x^1 (B_n(t) + \Lambda)u_n(t)\mathrm{d}t \geq \Lambda(1-x)u_n(x),$$

which provides

$$u_n(x)^{1/3}u_{nx}(x) \leq \frac{A_n(1)^{1/3}}{\Lambda^{1/3}}(1-x)^{-1/3}. \tag{54}$$

Integrating between 0 and x

$$u_n(x)^{4/3} \leq 2\frac{A_n(1)^{1/3}}{\Lambda^{1/3}}\big[1-(1-x)^{2/3}\big]. \tag{55}$$

Notice that $A_n(1) = (\mu + \|u_{n-1}\|_2^2)^2$. Then, since $\int_0^1 \big[1-(1-x)^{2/3}\big]^{3/2}\mathrm{d}x = \frac{3\pi}{32}$, integrating between 0 and 1, we get that

$$\|u_n\|_2^2 \leq \frac{3\pi\sqrt{2}}{16\sqrt{\Lambda}}\left(\mu + \|u_{n-1}\|_2^2\right). \tag{56}$$

Consequently, due to (4) the sequence $\|u_n\|_2^2$ is bounded. $\blacksquare$

We point out that $\|A_n\|_\infty \le \left(\mu + C_2^2\right)^2$ with $C_2 = \sup_n \|u_n\|_2$. More precisely we have

Proposition 23. *The sequence $\{u_n\}$ is bounded and equicontinuous in $C([0,1])$.*

Proof. By (54) and (55), the $u_n^{4/3}$ are bounded and their derivative are equi-integrable, so that the sequence $u_n^{4/3}$ is bounded and equicontinuous. The same property holds then for the u_n. ■

In order to prove the equicontinuity of the B_n we shall prove previously the following result:

Lemma 24. *Assume (4). Then there exists $\gamma > 0$ such that $u_{nx}(x) \ge \gamma(1-x)^{-1/3}$ for any n and a.e. $x \in (0,1)$. In particular $u_n(x) \ge \frac{3\gamma}{2}[1-(1-x)^{2/3}]$ for any n and any $x \in [0,1)$.*

Proof. We have $\lim_{x\to 1}(1-x)^{1/3}u_{nx}(x) = c_n$ with $c_n^3 = A_1(1)/\Lambda u_n(1) > 0$. In particular, $(1-x)^{1/3}u_1(x) > 0$ on $[0,1[$ and has a strictly positive limit as $x \to 1$. So, there exists $\gamma_1 > 0$ such that $(1-x)^{1/3}u_{1x}(x) \ge \gamma_1$ a.e. $x \in (0,1)$. By induction, suppose $(1-x)^{1/3}u_{(n-1)x}(x) \ge \gamma > 0$ with γ small enough. In particular, $\gamma \le 1$ and $\gamma \le \gamma_1$. Setting $C_\infty = \sup_m \|u_m\|_\infty$ we have

$$
\begin{aligned}
\frac{\mu^2}{u_{nx}(x)^3} \le \frac{A_n(x)}{u_{nx}(x)^3} &= \int_x^1 (B_n(x) + \Lambda)u_n(x)\mathrm{d}x \\
&\le C_\infty\left[(1-x)\Lambda + 3\gamma^{-2}(\mu + C_\infty^2)\int_x^1 (1-t)^{2/3}\mathrm{d}t\right] \\
&\le C_\infty\left[(1-x)\Lambda + \frac{9}{5}\gamma^{-2}(\mu + C_\infty^2)(1-x)^{5/3}\right] \\
&\le C_\infty\,\gamma^{-2}(1-x)\left[\Lambda + 2(\mu + C_\infty^2)\right].
\end{aligned}
$$

So, if we take γ small enough, and in particular such that

$$
\mu^2 \ge \gamma\, C_\infty\left[\Lambda + 2(\mu + C_\infty^2)\right]
$$

we get that $u_{nx}(x)^3 \ge \gamma^3(1-x)^{-1}$. ■

Proposition 25. *Assume (4). Then the sequences $\{A_n\}$ and $\{B_n\}$ are bounded and equicontinuous in $C([0,1])$.*

Proof. The uniform boundedness of $\{A_n\}$ and $\{A_{nx}\}$ results from previous results. The uniform boundedness of $\{B_n\}$ and $\{B_n'\}$ results from the boundedness of $\{u_n\}$ and the previous lemma. ■

Proof of Theorem 1. To end the proof of Theorem 1 we apply Ascoli–Arzela lemma to get the uniform convergence of $\{u_n\}$, $\{A_n\}$ and $\{B_n\}$ to some functions u, A and B in $C([0,1])$. Obviously $A(x) = A(u)(x) = \left(\mu + \int_0^x u(t)^2\mathrm{d}t\right)^2$. Moreover, from (54) and Lemma 24 we get that

$$
\frac{d_1}{(1-x)^{1/3}} \le u_{nx}(x) \le \frac{d_2}{(1-x)^{1/3}}, \quad \text{a.e. } x \in (0,1) \tag{57}
$$

for some uniform constants $d_1, d_2 > 0$ (use the strong maximum principle near $x = 0$). Then $\{u_{nx}\}$ converges weakly to u_x in $W^{1,p}(0,1)$ for any $p \in [1,3)$ and the estimate (5) holds.

Moreover, from (57) we get that $\left\{\frac{A_n(x)}{u_{nx}(x)^3}\right\}$ is uniformly bounded in any $L^p(0,1)$ for any $p \in [1,3)$ and so, there exists a function $\xi \in L^p(0,1)$ for any $p \in [1,3)$ such that $\left\{\frac{A_n}{u_{nx}^3}\right\} \rightharpoonup \xi$ weakly in $L^p(0,1)$. In particular $\left\{\frac{1}{u_{nx}^3}\right\} \rightharpoonup \frac{\xi}{A(u)}$ and as the function $r \to -r^{-3}$ generates a maximal monotone graph in $L^2(0,1)$, we get that necessarily $\frac{\xi}{A(u)} = \frac{1}{u_x^3}$. Then, multiplying by a test function in the corresponding equation (49) for u_n we can pass to the limit and so u is a weak solution of $P(A,B)$. ■

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