Convergence to Travelling Waves for Quasilinear Fisher-KPP Type Equations

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Abstract

We consider the Cauchy problem

$$\begin{cases} u_t = \varphi(u)_{xx} + \psi(u) \ , & (t,x) \in \mathbb{R}^+ \times \mathbb{R} \\ u(0,x) = u_0(x) & x \in \mathbb{R} \ , \end{cases}$$

when the increasing function φ satisfies that $\varphi(0)=0$ and the equation may degenerate at $u = 0$ (in the case of $\varphi'(0) = 0$). We consider the case of $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0(x) \le 1$ a.e. $x \in \mathbb{R}$ and the special case of $\psi(u) = u - \varphi(u)$. We prove that the solution approaches the travelling wave solution (with speed $c = 1$), spreading either to the right or to the left, or to the two travelling waves moving in opposite directions.

1 Introduction

In this paper we consider the Cauchy problem for the equation

$$u_t = \varphi(u)_{xx} + \psi(u) \ , \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R} \tag{1.1}$$

with function $\psi(s)$ vanishing only for $s = 0$ and $s = 1$. Here we shall assume that $\psi(u) = u - \varphi(u)$, hence the equation we shall study takes the form

$$u_t = \varphi(u)_{xx} + u - \varphi(u) \tag{1.2}$$

with $\varphi(0)$=0. Without loss generality we can also assume that $\varphi(1)$=1 (see Remark.1.1 below). In consequence, the equation (1.2) admits only two constant stationary states $\overline{u}_\infty(x) \equiv 1$ and $\underline{u}_\infty(x) \equiv 0$. More general functions $\psi(u)$ but vanishing only for $s = 0$ and $s = 1$ will be considered in a separated work ([DiKa]).

We prove that under some conditions the solution of the Cauchy problem for (1.2) converges, as $t \to \infty$, to a travelling wave linking, from $-\infty$ to $+\infty$, these constant values, 1 and 0.

Equation (1.2) for the special case $\varphi(u) = u^m$

$$u_t = (u^m)_{xx} + u - u^m \tag{1.3}$$

was studied in [KaR1],[KaR2] and we use below some of the ideas from these papers. In fact, the main goal of this paper is to extend those results to other quasilinear parabolic equations (non necessarily homogeneous or degenerate).

Though the interest to extend a given result is natural, perhaps some perspective is in order. In a mathematical modelling of a natural phenomena, simplifying assumptions are made not only to overcome complexity, but often due to our partial understanding of the process at hand. It suffices to glance through any of large body of studies dealing with reaction-diffusion processes as they emerge in biological or other complex systems, to realizes the very limited scope of our mathematical models. Acknowledging that our knowledge of the exact process is partial at best, it is thus essential to study the structural robustness of any model of interest, which is to say that one has to examine the extent to which the dynamics is sensitive to the assumptions made in the model. Since in [KaR1],[KaR2] a pretty complete asymptotic characterization of the solutions of the model equation (1.3) is given, it is natural to test the extent to which our results depend on the specifics of the model. Our main effort will be to replace u^m with $\varphi(u)$ covering also cases in which $\varphi(u)$ is not degenerated.

After the celebrated paper by Kolmogorov, Petrovsky and Piscunov [KPP], in 1937, the problem of studying travelling wave solutions for parabolic equations attracted much attention. This is a very rich subject of a great relevance in genetic theory (see Fisher [Fi]). For the "state of art" till 1987 the reader may consult an excellent survey by A. Volpert [Vo] written as some comments to the paper [KPP]. Many papers are devoted to the existence of TW solutions with different sources $\psi(u)$ and, in particular, the possible speeds of these waves. See [Kn], [GKe1], [ArW], [LMH], [MeOHo] [VoVoVo] and the references there. The importance of the TW solutions is the possibility to use them for the study of the behaviour of the general Cauchy problem. It is proved, for some cases, that solutions of the Cauchy problem converge to TW in some sense (by speed or by profile). In this paper we study equation (1.2) with nonlinear diffusion $\varphi(u)$. There are many results in that direction (see the above references). We would like to mention that to the best of our knowledge the only results which is closed to ours appears in the paper [MeOHo] where the authors proved the convergence by speed under some conditions on $\varphi(u)$, $\psi(u)$ and u_0 (see also the recent paper [DuM]). We point out that our assumptions on $\varphi(u)$ are less restrictive than the corresponding ones assumed in [MeOHo].

As to the equation (1.2) it is in fact quite special choice of the source term, but thanks to it quite complete information on the convergence to TW is obtained (see Theorems 2.1- 2.3).

The proof of this convergence is based on *dynamically weighted conservation law* (see below Lemma 2.4). This conservation law plays the role of the "invariant" in time and consequently leads to the identification of the limit function.

These results are also important because they are used in our forthcoming paper [DiKa] for the proof of the convergence by speed for rather large class of terms $\psi(u)$. The tools in [DiKa] are quite different from these used in the present paper. In [DiKa] we use extensively the properties of traveling waves solutions presented in details in the recent book by Gilding and Kersner [GKe1].

Remark 1.1 By the obvious change of variables the results presented below may be "translated" to the equation

$$u_t = \varphi(u)_{xx} + a[u - b\varphi(u)]$$

when a, b are some positive constants, assumed that $b\varphi(s) < s$ for any $s \in (0, \overline{u}^*_\infty)$, with $\overline{u}^*_\infty > 0$ such that $\overline{u}^*_\infty = b\varphi(\overline{u}^*_\infty)$.

2 The main result

Below, we assume the following hypothesis on the function $\varphi(u)$

$$(H_1) \begin{cases} \varphi \in C^0([0, +\infty)) \cap C^1([0, +\infty)) \cap C^2((0, +\infty)), \\ \varphi(0)=0, \ \varphi(1)=1, \\ \varphi'(s) > 0 \text{ for } s \in (0, +\infty). \end{cases}$$

It is obvious that if $\varphi(s)$ satisfies (H_1) then

$$\varphi'(0) \geq 0. \tag{2.1}$$

If $\varphi'(0) = 0$ then the equation degenerates at the points $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ where $u = 0$ and a notion of weak solution should be defined.

Below we study the solutions of the Cauchy problem for equation (1.2) with the initial data

$$u(0,x) = u_0(x) \quad x \in \mathbb{R}\ . \tag{2.2}$$

We assume that

$$0 \leq u_0(x) \leq 1 \ \text{ a.e. } \ x \in \mathbb{R}. \tag{2.3}$$

Additionally, a suitable decay $u_0(x) \to 0$ as $x \to +\infty$ will be implicitly assumed later (see condition (2.13) below and Remark 2.3).

We shall also use the additional assumption:

$$(H_2): \quad \varphi(s) < s \quad \text{for all} \quad s \in (0,1).$$

Let $S = \mathbb{R}^+ \times \mathbb{R}$ and, for a given $T > 0$, $S_T = [0,T] \times \mathbb{R}$.

Definition 2.1 By a weak solution of the Cauchy problem (1.2), (2.2) we mean a nonnegative function u such that $u \in C([0,T] : L^1(K)) \cap C((0,T] \times \mathbb{R})$, for any compact interval K and any $T > 0$, which satisfies the identity

$$\begin{aligned} &\iint_{S_T} [\zeta_t u + \zeta_{xx}\varphi(u) + \zeta(u - \varphi(u))]dxdt + \\ &+ \int_{\mathbb{R}} \zeta(0,x)u_0(x)dx = \int_{\mathbb{R}} \zeta(T,x)u(T,x)dx \end{aligned} \tag{2.4}$$

for any $\zeta \in C^{2,1}(S_T)$ which vanishes for large $|x|$ (see [OlKalCh], [Kal4]).

Although the existence and uniqueness of the weak solution of (1.2), (2.2) under hypothesis (H_1), (H_2) is not far from many results in the literature, for the sake of the completeness we have collected in Section 3 several comments on useful references to this respect. For nondegenerate case ($\varphi'(s) > 0$ for any $s \geq 0$) we refer to the book [LSU]. We shall also need a technical assumption on function $\varphi(s)$ which will allow us to guarantee the spatial continuity of the solutions of equation (1.2), even if the initial datum u_0 is possibly discontinuous. We shall also assume

$$(H_3): \quad \left[(\varphi' - [\varphi']^2)'\right]_- \in L^1(0,1),$$

where, in general, $[h]_- := \max(0, -h)$ for any $h \in \mathbb{R}$. Notice that if $\varphi''(s) \geq 0$ for any $s \in (0,1)$ then this condition is automatically verified.

We begin from the study of the travelling waves solutions of the equation (1.2).

Definition 2.2 Function $U(t,x) = f(x-t)$ is called a $(1,0)$-travelling wave with speed 1 *(in short* $(1,0)$-TW) of (1.2), if U is a solution of (1.2) and $f(\eta)$ links the constant values, 1 and 0 in the sense that

$$f(\eta) \to 1 \quad \text{as} \quad \eta \to -\infty \quad \text{and} \quad f(\eta) \to 0 \quad \text{as} \quad \eta \to \infty.$$

In the book [GKe1] the detailed study of TW solutions is performed for the equations which are more general than (1.2). Nevertheless we present the proof of existence of $(1,0)$-TW with $c = 1$ which is quite simple for our special case. The solution may be just calculated and therefore the properties of the solution which we need in this paper are easily obtained. We prove below that this solution is the attractor of the solutions of the Cauchy problem (1.2), (2.2) for some class of initial data (see Theorem 2.1). As a corollary we obtain that $c = 1$ is the "minimal speed" of the $(1,0)$-TW) of (1.2).

Let

$$J(y) = \int_{\frac{1}{2}}^{y} \frac{ds}{s - \varphi^{-1}(s)}. \tag{2.5}$$

Note that by (H_2) we have $\varphi^{-1}(s) > s$ for $s \neq 0$, $s \neq 1$, therefore $J(y)$ is a decreasing function of y. Moreover, the integral in (2.5) may diverge at the points where $s = \varphi^{-1}(s)$, that means for $s = 0$ and for $s = 1$. Let

$$\eta_0 = J(0). \tag{2.6}$$

Then we have $\eta_0 > 0$.

Lemma 2.1 *Suppose hypothesis* (H_1) *and* (H_2) *are satisfied. Then there exist a* $(1,0)$-TW *of (1.2),* $U(t,x) = f(x-t)$ *where* $f \in C(\mathbb{R})$ *is given by*

$$\eta = J[\varphi(f(\eta))] \quad \textit{for} \quad \eta < \eta_0, \tag{2.7}$$

$$f(\eta) = 0 \quad \textit{for} \quad \eta \geq \eta_0 \tag{2.8}$$

and $J(y)$ *and* η_0 *are defined in (2.5),(2.6). Any other* $(1,0)$-TW *with velocity equal 1 is given by* $U(t,x) = f(x - x_0 - t)$ *where* x_0 *is an arbitrary point of* $\mathbb{R}$*. Moreover* $f(\eta)$ *is monotone decreasing.*

The proof of this lemma is presented in Section 3.

We say that the $(1,0)$-TW *has a sharp front at* $\eta=\eta_0$ if $\eta_0<\infty$.

Clearly there are several possibilities for the behavior of TW solution. It may have a sharp front or not. It depends on the behaviour of $\varphi(s)$ near $s=0$.

Examples 1) $\varphi(s)=s^m$, $m>1$ a sharp front arises at $f=0$. 2) $\varphi(s)=\frac{1}{3}s+\frac{2}{3}s^2$, $f(\eta)>0$ for all η, does not have sharp front.

Remark 2.1 The $(1,0)$-TW constructed above is propagating to the right. Changing x to $-x$ leads to a $(0,1)$-TW propagating to the left.

In order to formulate the main result of this section we have to add the next hypothesis about the *weighted global integrability* of the profile of the travelling wave

$$(H_4) \qquad I=\int_{\mathbb{R}} f(\eta)e^{\eta}d\eta<\infty \tag{2.9}$$

where $f(\eta)$ was defined in Lemma 2.1. In the following, we shall say that the $(1,0)$-TW is *global weighted integrable* if (2.9) is satisfied.

Remark 2.2 It is clear that if $\eta_0<\infty$ then $I<\infty$, but assumption (2.9) is satisfied also for profiles having $\eta_0=\infty$. For the general case we have to use the expression for $f(\eta)$ given by (2.7), (2.8). Lemma 2.2 below, which is also proved in the Appendix, provides some criteria under which the integral in (2.9) converges:

Lemma 2.2 *Suppose that for some* $\delta>0$

$$\varphi(s)=\alpha s \quad \textit{for all} \quad s\in[0,\delta],$$

then $I<\infty$ *if* $\alpha<\frac{1}{2}$ *but* $I=\infty$ *if* $\alpha\geq\frac{1}{2}$.

The unique determination of a *weighted global integrable* $(1,0)$-TW travelling wave can be attained by associating a special point $x_0\in\mathbb{R}$ to it. Indeed, if f satisfies (2.9) then, for any $x\in\mathbb{R}$ we can define the function

$$I(x)=\int_{\mathbb{R}} f(\eta-x)e^{\eta}d\eta \tag{2.10}$$

transforming increasingly $\mathbb{R}$ onto $(0,+\infty)$. Thus, given $q>0$ there exists a unique $x_0=x_0(q)$ such that

$$\int_{\mathbb{R}} f(\eta-x_0)e^{\eta}d\eta=q\ . \tag{2.11}$$

In what follows, given $q > 0$, we shall denote by $U_q(t,x)$ the *weighted global integrable* $(1,0)$-TW travelling wave

$$U_q(t,x) = f(x - t - x_0), \tag{2.12}$$

were the dependence of q on x_0 is defined by (2.11).

Our main result for the Cauchy problem (1.2), (2.2) is the following:

Theorem 2.1 *Suppose that the assumptions $(H_1) - (H_4)$ are satisfied. Suppose also that $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0(x) \le 1$, $u_0(x) \not\equiv 0$,*

$$\int_{\mathbb{R}} u_0(x) e^x dx = q_1 < \infty. \tag{2.13}$$

Then $u(t,x) - U_{q_1}(t,x) \to 0$ uniformly inside any strip $\alpha \le x - t \le \beta$, as $t \to \infty$, where U_{q_1} is defined by (2.11),(2.12). Moreover

$$e^{-t} \int_{\mathbb{R}} |u(t,x) - U_{q_1}(t,x)| e^x dx \to 0 \quad \text{as } t \to \infty. \tag{2.14}$$

For the proof of this theorem we shall use next properties of the solutions of (1.2).

(P_1) There exists a weak solution of the Cauchy problem (1.2),(2.2).

(P_2) *Comparison principle*: if u_1 and u_2 are weak solutions of (1.2) and $u_1(0,x) \le u_2(0,x)$ a.e. $x \in \mathbb{R}$ then $u_1(t,x) \le u_2(t,x)$ for all $x \in \mathbb{R}$, $t \in \mathbb{R}^+$.

(P_3) This solution is classical at the points (t,x) where $0 < u(t,x) < 1$.

(P_4) *Positivity property*: if $u(t_0,x_0) > 0$ then $u(t,x_0) > 0$ for all $t > t_0$.

(P_5) For every x_0 there exists $T = T(x_0)$ such that $u(t,x_0) > 0$ for every $t \ge T(x_0)$.

Notice that the comparison principle implies the uniqueness of weak solution. In fact we shall use some peculiar form of the construction of the weak solution

$(P_6 = P_1')$ Every bounded weak solution may be obtained as the limit of the classical solutions u_ϵ, $0 < u_\epsilon(t,x) < 1$, $u_\epsilon \to u$ uniformly on any bounded set.

$(P_7 = P_1'')$ If $u_0^\epsilon \to u_0$ in $L^1(K)$ on any compact interval of $\mathbb{R}$, then $u^\epsilon(t,x) \to u(t,x)$ uniformly on any compact $\mathcal{K} \subset \mathbb{R}^+ \times \mathbb{R}$

(P_8) Every sequence of uniformly bounded solutions is equicontinuous on every compact set $\mathcal{K}$ of S_T for any $T > 0$.

Properties (P_1) to (P_8) are quite well-known in the context of degenerate parabolic equations although not always explicitely proven under a general framework which allows their application to our special formulation. Some exposition which were motivated by the present work are [Di1] and [Di2]. The reader can find there a complementary exposition on properties (P_1) to (P_8) except on (P_4) and (P_5) which will be commented in Section 3. As we shall illustrate there, the above list of properties can be checked for other equations different that (1.2) which could allow the application of the arguments used in the proof of Theorem 2.1 to get different generalizations of it.

To prove Theorem 2.1 we need several lemmas, which are similar to those used in [KaR1] for the case $\varphi(u) = u^m$. Some parts of the proofs are the same as in [KaR1], therefore we omit it.

We suppose below that all the assumptions of Theorem 2.1 are satisfied and $u(t,x)$ be the solution of the Cauchy problem (1.2),(2.2).

Lemma 2.3 *Suppose that $u_0(x)$ has compact support at the right (i.e. support $u_0 \subset (-\infty, x_0]$ for some $x_0 \in (-\infty, +\infty)$) and $u_0(x) \leq 1-\delta$, $\delta > 0$. Then there exists $q^* > 0$ and $C > 0$, independent on $t \geq 0$, such that*

$$u(t,x) \leq U_{q^*}(t,x) \tag{2.15}$$

and

$$\int_{\mathbb{R}} u(t,x)e^x dx \leq Ce^t. \tag{2.16}$$

Proof. It follows from the assumptions of the lemma that there exist some q^* such that $u_0(x) \leq U_{q^*}(0,x)$ and therefore by the comparison principle $u(t,x) \leq U_{q^*}(t,x)$. Hence we have

$$\int_{\mathbb{R}} u(t,x)e^{x-t}dx \leq \int_{\mathbb{R}} U_{q^*}(t,x)e^{x-t}dx = q^*. \tag{2.17}$$

and the conclusion holds.■

Lemma 2.4 *(Dynamically weighted conservation law). For all $t \geq 0$*

$$\int_{\mathbb{R}} u(t,x)e^x dx = e^t \int_{\mathbb{R}} u_0(x)e^x dx \ . \tag{2.18}$$

Proof. Suppose first that $u_0(x)$ satisfies the assumptions of Lemma 2.3. Then (2.16) holds and we proceed as in the proof of Lemma 2.2 in [KaR1] by substituting $\zeta = e^{x-t}\eta_\ell(x)$ in

the integral identity (2.4). Here $\eta_\ell(x)$ is the corresponding cut function. The passage to the limit as $\ell \to \infty$ is possible due to the assumption (H_2), and (2.16).
It remains to remove the restriction that $u_0(x)$ satisfies the conditions of Lemma 2.3. Let $u_0^\epsilon(x)$ have compact support from the right $u_0^\epsilon(x) \le 1 - \delta_\epsilon$, $\delta_\epsilon > 0$, and $u_0^\epsilon(x) \nearrow u_0(x)$ in $L^\infty(K)$ on any compact interval K of $\mathbb{R}$ as $\epsilon \to 0$. Let $u^\epsilon(t,x)$ be the solution of (1.1) with $u^\epsilon(0,x) = u_0^\epsilon(x)$ and

$$\int_{\mathbb{R}} u_0^\epsilon(x) e^x dx = q_\epsilon \le q.$$

Using the dynamically weighted conservation property for $u^\epsilon(t,x)$ and passing to the limit when $\epsilon \to 0$, we get (2.18) for u.$\blacksquare$

Remark 2.3 The presence of the dynamic weight e^{x-t} in the above conservation law is essential. Notice, for instance, that in fact, if $\int_{\mathbb{R}} u_0(x)dx < \infty$, it can be shown (by taking a sequence of test functions ζ_n such that $\zeta_n \to 1$) that

$$\int_{\mathbb{R}} u(t,x)dx > \int_{\mathbb{R}} u_0(x)dx$$

for any $t > 0$. We also point out that assumption (2.13) (respectively conclusion (2.18)) implies, implicitly, a suitable rate of convergence of $u_0(x) \to 0$ when $x \to +\infty$ (respectively of the convergence $u(t,x) \to 0$ when $x \to +\infty$, for any fixed $t > 0$).

Lemma 2.5 *(Dynamically weighted contraction principle) Let u and v be weak solutions of (1.2),*(2.2) *and*

$$\int_{\mathbb{R}} u_0(x) e^x dx < \infty \ , \quad \int_{\mathbb{R}} v_0(x) e^x dx < \infty \ . \tag{2.19}$$

Then

$$\int_{\mathbb{R}} |u(T,x) - v(T,x)| e^x dx \le e^{T-\tau} \int_{\mathbb{R}} |u(\tau,x) - v(\tau,x)| e^x dx \ . \tag{2.20}$$

for $0 \le \tau \le T$.

First notice that by Lemma 2.4 all integrals in (2.20) converge.

The proof of this lemma also follows the line of the proof of the contraction principle in [KaR1]. We first prove this for classical solutions u and v. Multiply the difference of two equations by $e^{x-t}\eta_e(x)p[\varphi(u) - \varphi(v)]$, where η_ℓ is the cut function. Then integrate by parts and let $p(s)$ tend the $\text{sign}_+ s$. Passing to the limit as $\ell \to \infty$ we obtain (2.20) for classical solutions.
Next as in [KaR1] we approximate solutions u and v by the sequences u^ϵ and v^ϵ of classical solutions and pass to the limit as $\varepsilon \to 0$.$\blacksquare$

Proof of Theorem 2.1 Let

$$u_h(t,x) = u(t+h, x+h), h > 0 \tag{2.21}$$

and instead of study the behaviour of $u(t,x)$ as $t \to \infty$ we consider the behaviour of the sequence $\{u_h\}$ on bounded sets of S_T as $h \to \infty$. Such shifting transformation plays here the same role as a scaling transformation for the proof of attractivity properties of self-similar solutions (see, e.g. the results on the pure diffusively case quoted in [V] and its many references on it). Note that $U_q(t,x)$ is invariant with respect to the shifting (2.21) for any q that means that $U_q(t+h, x+h) = U_q(t,x)$. It follows from (2.18) that

$$\int_{\mathbb{R}} u_h(t,x)e^{x-t}dx = \int_{\mathbb{R}} u(t+h,y)e^{y-t-h}dy = q\ . \tag{2.22}$$

Sequence $\{u_h(t,x)\}$ is uniformly bounded, and thus,by (P_8), is equicontinuous on any bounded set in $\mathbb{R}^+ \times \mathbb{R}$ ([Di1]). Therefore there exists a subsequence $h_i \to \infty$ such that

$$u_{h_i}(t,x) \to w(t,x)\ , \tag{2.23}$$

and the convergence is uniform on any bounded set. The limit function w is defined for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and is a weak solution of (1.1). Assume first that $u_0(x)$ satisfies the assumptions of Lemma 2.3. Then, by (2.14)

$$u_h(t,x) = u(t+h, x+h) \le U_{q^*}(t+h, x+h) = U_{q^*}(t,x). \tag{2.24}$$

It follows from (2.24) that

$$w(t,x) \le U_{q^*}(t,x)\ . \tag{2.25}$$

In view of (2.24), (2.25) and (2.23)

$$\int_{\mathbb{R}} |u_{h_i}(t,x) - w(t,x)|e^{x-t}dx \to 0 \text{ as } h_i \to \infty \tag{2.26}$$

and by (2.22)

$$\int_{\mathbb{R}} w(t,x)e^{x-t}dx = q\ . \tag{2.27}$$

Let $\hat{q}$ be some fixed number. Define

$$I^h(t;\hat{q}) = \int_{\mathbb{R}} |u_h(t,x) - U_{\hat{q}}(t,x)|e^{x-t}dx\ . \tag{2.28}$$

Note that it follows from (2.25) and (2.27) that $w \neq 0$ and $w \neq 1$.

Obviously

$$
\begin{aligned}
I^h(t;\hat{q}) &= \int_{\mathbb{R}} |u(t+h,x+h) - U_{\hat{q}}(t+h,x+h)|e^{x-t}dx \\
&= \int_{\mathbb{R}} |u(t+h,y) - U_q(t+h,y)|e^{y-(t+h)}dy = I^0(t+h);\hat{q})
\end{aligned}
$$

By the contraction principle $I^0(t+h;\hat{q})$ is a nonincreasing function of t and h, therefore, there exists

$$
\lim_{h\to\infty} I^h(t;\hat{q}) = I^\infty(\hat{q}) \geq 0 \quad \text{for all} \quad t\ . \tag{2.29}
$$

It follows from (2.26),(2.28) and (2.29) that for all $t \geq 0$

$$
\int_{\mathbb{R}} |w(t,x) - U_{\hat{q}}(t,x)|e^{x-t}dx = I^\infty(\hat{q})\ . \tag{2.30}
$$

We now use (2.27) and (2.30) to prove the next statement.

Lemma 2.6 *Suppose that*

$$
w(0,x_1) = U_{\tilde{q}}(0,x_1) \in (0,1) \tag{2.31}
$$

for some $\tilde{q}$ *and some* x_1*. Then*

$$
\frac{\partial w}{\partial x}(0,x_1) = \frac{\partial U_{\tilde{q}}}{\partial x}(0,x_1) \tag{2.32}
$$

Proof. The proof of (2.32) is based on the strong maximum principle which holds for nondegenerate equations. By the assumption (2.31) and property (P_3) solution $w(x,t)$ is a classical one for $|x-x_1| < a$, $0 < t < b$, for some values a and b.

Define $\overline{u}(t,x)$ $(\underline{u}(t,x))$ as a solution of (1.1) with $\overline{u}(0,x) = \max\{w(0,x), U_{\tilde{q}}(0,x)\}(\underline{u}(0,x) = \min(w(0,x), U_{\tilde{q}}(0,x)\}$. By the comparison principle

$$
\overline{u}(t,x) \geq \max\{w(t,x), U_{\tilde{q}}(t,x)\}, \quad \underline{u}(t,x) \leq \min\{w(t,x), U_{\hat{q}}(t,x)\}\ . \tag{2.33}
$$

By the assumption (2.31) and continuity of w and $U_{\tilde{q}}$ for τ and Δx small enough $0 < w(t,x) < 1$ and $0 < U_{\tilde{q}}(t,x) < 1$ for $0 \leq t \leq \tau$, $|x - x_1| \leq \Delta x$. This implies that $\overline{u} - w$ is a solution of some nondegenerate parabolic equation with smooth coefficients in the cylinder $(x_1 - \Delta x, x_1 + \Delta x) \times (0,\tau)$. Assume that (2.32) does not hold. It would imply that

$$
\frac{\partial w}{\partial x}(0,x_1) \neq \frac{\partial U_{\tilde{q}}}{\partial x}(0,x_1) \tag{2.34}
$$

Consequently for $x \in (x_1 - \Delta x, x_1 + \Delta x)$

$$\overline{u}(0,x) - w(0,x) \not\equiv 0 \ , \ \overline{u}(0,x) - U_{\tilde{q}}(0,x) \not\equiv 0 \ .$$

Hence using (2.33), by the strong maximum principle we obtain, that for some $\tau > 0$ and $\Delta x > 0$, $\overline{u}(\tau,x) - w(\tau,x) > 0$. $\overline{u}(\tau,x) - U_{\tilde{q}}(\tau,x) > 0$ on $(x_1 - \Delta x, x_1 + \Delta x)$ and hence

$$\overline{u}(\tau,x) > \max\{w(\tau,x), U_{\tilde{q}}(\tau,x)\}$$

for all $x \in (x_1 - \Delta x, x_1 + \Delta x)$.
Therefore, in view of ((2.30))

$$\begin{aligned}
&\int_{\mathbb{R}} [\overline{u}(1,x) - \underline{u}(1,x)] e^{x-\tau} dx \\
&> \int_{\mathbb{R}} [\max\{w(1,x), U_{\tilde{q}}(1,x)\} - \min\{w(1,x), U_{\tilde{q}}(1,x)\}] e^{x-\tau} dx \\
&= \int_{\mathbb{R}} |w(1,x) - U_{\tilde{q}}(1,x)| e^{x-\tau} dx = I^\infty(\hat{q}).
\end{aligned}$$

On the other hand, by the contraction principle and (2.30)

$$\begin{aligned}
\int_{\mathbb{R}} [\overline{u}(1,x) - \underline{u}(1,x)] e^{x-1} dx &\leq \int_{\mathbb{R}} [\overline{u}(0,x) - \underline{u}(0,x)] e^{x} dx \\
&= \int_{\mathbb{R}} |w(0,x) - U_{\tilde{q}}(0,x)| e^{x} dx = I^\infty(\widetilde{q}) \ ,
\end{aligned}$$

a contradiction. Thus the assumption (2.34) is false.■

Proof of Theorem 2.1 (continuation). In order to proceed with the proof of Theorem 2.1, we suppose that for all q

$$w(0,x) \neq U_q(0,x) \tag{2.35}$$

Then, for any point $x \in \mathbb{R}$ such that $w(0,x) \in (0,1)$ there exists some $\overline{q} = \overline{q}(x)$ such that $w(0,x) = U_{\overline{q}}(0,x)$. By Lemma 2.6 we have $\frac{\partial w}{\partial x}(0,x) = \frac{\partial U_{\overline{q}}}{\partial x}(0,x)$ and by (2.35) $w(0,x) \not\equiv U_{\overline{q}}(0,x)$. This means that $w(0,x)$ is the envelope of the set of curves $U_{\overline{q}}(0,x)$ and this is impossible. Hence (2.35) is wrong.
Therefore, we finally proved that for some $\tilde{q}$

$$w(0,x) \equiv U_{\tilde{q}}(0,x) \ .$$

But because of (2.12) and (2.27) we obtain that $\tilde{q} = q$, therefore

$$w(t,x) = U_q(t,x) \ . \tag{2.36}$$

Hence $\lim_{h_i \to \infty} u_{h_i}$ does not depend on the subsequence and therefore the whole sequence u_h converges to U_q. This convergence is as defined by (2.14) and, as follows from the proof, is uniform on any compact set. Thus we have

$$\int_{\mathbb{R}} |u(t+h, x+h) - U_q(t+h, x+h)| e^{x-t} dx$$
$$= \int_{\mathbb{R}} |u(\tau, y) - U_q(\tau, y)| e^{y-\tau} dy \to 0 \ ,$$

as $\tau \to \infty$ and

$$|u(\tau, y) - U_q(\tau, y)| \to 0 \ ,$$

uniformly on every set

$$\alpha < y - \tau < \beta \ ,$$

for any fixed α, β.

Thus Theorem 2.1 is proved for the case where $u_0(x)$ satisfies the assumptions of Lemma 2.3. Now suppose that $0 \leq u_0(x) \leq 1$. Let $u_0^\epsilon(x)$ be a sequence of functions, each compactly supported from the right, $u_0^\epsilon \leq 1 - \epsilon$ and

$$\int_{\mathbb{R}} |u_0^\epsilon(x) - u_0(x)| e^x dx \leq \varepsilon \ . \tag{2.37}$$

The sequence u_0^ϵ may be chosen such that

$$\int_{\mathbb{R}} u_0^\epsilon(x) e^x dx = q.$$

Let $u^\epsilon(t,x)$ be the solution of (1.1) with initial data

$$u^\epsilon(0,x) = u_0^\epsilon(x).$$

By (2.37) and the contraction principle

$$\int_{\mathbb{R}} |u^\epsilon(t,x) - u(t,x)| e^{x-t} dx \leq \varepsilon \ . \tag{2.38}$$

Moreover, as we proved above,

$$\int_{\mathbb{R}} |u^\epsilon(t,x) - U_q(t,x)| e^{x-1} dx \to 0 \ , \tag{2.39}$$

as $t \to \infty$. Because ε is arbitrarily small it follows from (2.38) and (2.39) that (2.14) holds. It follows from (2.23) that the convergence is uniform on the set $\alpha \le x+t \le \beta$ and Theorem 2.1 is proved. ■

Corollary 2.1. *The speed $c = 1$ is the minimal one, that means that there is no solution of the form $f(x-ct)$ with $c < 1$.*

Proof. It follows from Theorem 2.1 by comparison. Suppose that there exists some TW solution of (1.2) which has the form $f(x-ct)$ with $c < 1$.Let $u_0(x)$ be some initial data satisfying $u_0(x) \le f(x)$ a.e. $x \in \mathbb{R}$. Then the corresponding solution satisfies $u(t,x) \le f(x-ct)$ which is impossible by (2.14).■

The change of variable x to $-x$ leads to the next result.

Theorem 2.2 *Suppose that the assumptions $(H_1)-(H_4)$ are satisfied. Suppose also that $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0(x) \le 1$, $u_0(x) \not\equiv 0$,*

$$\int_{\mathbb{R}} u_0(x) e^{-x} dx = q_1^* < \infty. \tag{2.40}$$

Then $u(t,x) - U_{q_1^*}(t,-x) \to 0$ uniformly inside any strip $\alpha \le x+t \le \beta$, as $t \to \infty$, where $U_{q_1^*}$ is defined by (2.11),(2.12). Moreover

$$e^{-t} \int_{\mathbb{R}} |u(t,x) - U_{q_1^*}(t,-x)| e^{-x} dx \to 0 \quad \text{as } t \to \infty. \tag{2.41}$$

The next result concerns the case when both conditions (2.13) and (2.40) are satisfied. It follows mainly from Theorems 2.1 and 2.2. We formulate it below and refer for details to [KaR1]

Theorem 2.3 *Suppose that the assumptions $(H_1)-(H_4)$ are satisfied. Suppose also that $u_0 \in L^\infty(\mathbb{R})$, $0 \le u_0(x) \le 1$, $u_0(x) \not\equiv 0$,*

$$\int_{\mathbb{R}} u_0(x) e^{x} dx = q_1 < \infty \ \text{and} \ \int_{\mathbb{R}} u_0(x) e^{-x} dx = q_1^* < \infty. \tag{2.42}$$

Then $u(t,x) - U_{q_1}(t,x) \to 0$ and $u(t,x) - U_{q_1^*}(t,-x) \to 0$ uniformly inside any strip of the form $\alpha \le x-t \le \beta$ and $\alpha \le x+t \le \beta$, respectively, as $t \to \infty$, where $U_{q_1^*}$, $U_{q_1^*}$ are defined by (2.11),(2.12). Moreover

$$e^{-t} \int_{\mathbb{R}} |u(t,x) - U_{q_1}(t,x)| e^{x} dx \to 0 \ \text{and} \ e^{-t} \int_{\mathbb{R}} |u(t,x) - U_{q_1^*}(t,-x)| e^{-x} dx \to 0 \quad \text{as } t \to \infty.$$

Remark 2.4 The regularity of $\varphi \in C^1[0,1]$ assumed in H_1 is not important but it was made here for the sake of simplicity in the exposition. If, for instance, we consider a function $\varphi \in C^1[0,1)$ such that $\varphi'(s) \to +\infty$ as $s \to 1$ (and satisfying the rest of conditions) then the change of variables $w = 1-u$ leads to the Cauchy problem

$$\begin{cases} w_t = \Lambda(w)_{xx} + \Lambda(w) - w & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ w(0,x) = w_0(x) & x \in \mathbb{R}, \end{cases}$$

where $\Lambda(w) = 1 - \varphi(1-w)$ and $w_0(x) = 1 - u_0(x)$. The fact that now $\Lambda'(s) \to +\infty$ as $s \to 0$ explain that the diffusion operator becomes a fast diffusion (at the level $w = 0$). Nevertheless, the proof of the existence of a (0,1)-TW remains true and the same for the convergence result. Something peculiar to this special case is that if the initial datum $w_0(x)$ does not link 0 and 1 but is a local perturbation of 0, i.e. $w_0(x) \in [0,1]$ and support w_0 is compact then it can be shown that there exists a finite time T_e such that $w(t,x) \equiv 0$ for any $x \in \mathbb{R}$ and any $t \geq T_e$ (use the fact that $\Lambda(w) - w \leq 0$ and apply the results in Chapter 2 of [AnDiSh]). As a consequence, $u(t,x) \equiv 1$ for any $x \in \mathbb{R}$ and any $t \geq T_e$

3 On the list of properties (P_1)-(P_7) and proofs of the auxiliary lemmas.

Properties (P_1)-(P_7) used in the proof of Theorem 2.1 are satisfied in the framework of our assumptions on function φ (conditions $(H_1), (H_2)$ and (H_3)) and also, under some slight modifications, for other type of nonlinear diffusion operators. Let us start by referring to the case of the problem under consideration. As commented before, the reader can find in the papers [Di1] and [Di2] a detailed exposition about different qualitative properties satisfied by the weak solution of a general class of unidimensional quasilinear equations (including, in particular, equation (1.2)). Among such qualitative properties the reader will find properties (P_1) to (P_8) except (P_4) and (P_5). For instance, the continuity of the solution obtained as limit of classical solutions of the regularized problem, and properties (P_7) to (P_8) was proved assured, thanks to the results of [Di1], once we suppose the conditions (H_1) and (H_3).

Assumptions (2.3) and (H_1) imply that the weak solution of (1.2) and (2.2) satisfies $0 \leq u(t,x) \leq 1$ for any $t > 0$ and any $x \in \mathbb{R}$. Thus $u(t,x) - \varphi(u(t,x)) \geq 0$ for any $t > 0$ and any $x \in \mathbb{R}$ and, in consequence,

$$\underline{u}(t,x) \leq u(t,x) \text{ for any } t > 0 \text{ and any } x \in \mathbb{R}, \tag{3.1}$$

where $\underline{u}$ is the (unique) weak solution of the pure diffusion equation

$$\begin{cases} u_t = \varphi(u)_{xx} & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

In this way we can now conclude from (3.1) properties (P_4) and (P_5) for the solution of our problem (1.2) and (2.2). Indeed, for small times $t > 0$. If $\varphi'(0) > 0$, then (P_4) and (P_5) follow from the strong maximum principle. For $\varphi'(0) = 0$ properties (P_4) and (P_5) are proved, for problem (3.2) in [OlKalCh], [Kal2], [Kne].

Before passing to the proofs of the auxiliary lemmas presented in the above Section we mention that with some slight modifications the list of properties remain true for other pure diffusion equations as, for instance, the case of doubly nonlinear diffusion

$$\underline{u}_t = \psi(\varphi(\underline{u})_x)_x \tag{3.3}$$

once we assume some additional properties to functions ψ and φ (see [Kal4] and [V]) and so the main conclusion of this paper could be extended to some suitable perturbations of equation (3.3). Nevertheless we shall not pursuit here such a generalization.

Proof of Lemma 2.1. Substitute $U(t,x) = f(x-t)$ in (1.2). We obtain the ordinary differential equation for $f(\eta)$

$$-f' = [\varphi(f)]'' + f - \varphi(f) \ . \tag{3.4}$$

Multiplying (3.4) by e^{η} and integrating, we arrive to the first order equation

$$-f = [\varphi(f)]' - \varphi(f) + Ce^{-\eta} \ .$$

The standard analysis shows that in order to have a bounded solution for all $\eta \in \mathbb{R}$ one has to put $C = 0$. Therefore, we have

$$\frac{d\varphi(f)}{d\eta} = \varphi(f) - f \ .$$

Let $y(\eta) = \varphi[f(\eta)]$. Then

$$\frac{dy}{d\eta} = y - \varphi^{-1}(y) \tag{3.5}$$

where φ^{-1} is the inverse to φ. We find first the solution of (3.5) which is equal to 1/2 at $\eta = 0$. It can be expressed in the form

$$\eta = J(y) = \int_{1/2}^{y} \frac{ds}{s - \varphi^{-1}(s)} \ . \tag{3.6}$$

Let

$$0 < \eta_0 = J(0) \le \infty$$

and define the function $f(\eta)$ by

$$\eta = J[\varphi(f(\eta)] \quad \text{for} \quad \eta \le \eta_0$$

$$f(\eta) = 0 \quad \text{for} \quad \eta \ge \eta_0.$$

It follows from the analysis of the integral that if η_0 is bounded from above then $\varphi'(0) = 0$, and

$$U(t, x) = f(x - t)$$

is a weak solution of (1.2). Thus Lemma 2.1 is proved. □

Proof of Lemma 2.2 Let η^* be large enough so that $y(\eta^*) = \delta$ where $y(\eta)$ is the solution of (3.5). Then for all $\eta \ge \eta^*$ one has $\varphi(f(\eta)) = \alpha f(\eta)$ and hence it follows from the equation (3.5) that

$$y' = y - \frac{1}{\alpha} y \quad \text{for} \quad \eta > \eta^* \quad y(\eta^*) = \delta \ .$$

Therefore

$$y(\eta) = C_0 e^{(1-\frac{1}{\alpha})\eta} \quad \text{for} \quad \eta > \eta^*$$

where $C_0 = \delta e^{(\frac{1}{\alpha}-1)\eta^*}$ and thus

$$f(\eta) = \frac{1}{\alpha}\varphi(f(\eta)) = \frac{1}{\alpha} y(\eta) = \frac{C_0}{\alpha} e^{\frac{\alpha-1}{\alpha}\eta} \quad \text{for} \quad \eta > \eta^* \ .$$

Substituting the last expression in (2.10) we see that

$$I < \infty \quad \text{if} \quad \frac{\alpha - 1}{\alpha} + 1 < 0$$

which means that

$$\alpha - 1 + \alpha < 0, \quad \alpha < \frac{1}{2}$$

and $I = \infty$ if $\alpha \ge \frac{1}{2}$.

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