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NON-HOOKEAN BEAMS AND PLATES: VERY WEAK SOLUTIONS AND THEIR NUMERICAL ANALYSIS

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Dedicated to the 65th birthday of Professor F. J. Lisbona

Abstract. We consider very weak solutions of a nonlinear version (non-Hookean materials) of the beam stationary Bernoulli-Euler equation, as well as the similar extension to plates, involving the bi-Laplacian operator, with Navier (hinged) boundary conditions. We are specially interested in the case in which the usual Sobolev space framework cannot be applied due to the singularity of the load density near the boundary. We present some properties of such solutions as well as some numerical experiences illustrating how the behaviour of the very weak solutions near the boundary is quite different to the one of more regular solutions corresponding to non-singular load functions.

Key Words. Beam and plate, non Hookean material, very weak solutions, numerical experiences.

1. Introduction

Given a linear boundary value problem on a bounded regular open set Ω of $\mathbb{R}^N$

$$(P_L)\begin{cases} Lu = f(x) & \text{in } \Omega, \\ + \text{ boundary conditions} \equiv \ (BC) & \text{on } \partial\Omega, \end{cases}$$

where Lu denotes an elliptic differential operator (of order $2m$, $m \in \mathbb{N}$) in divergence form, the usual notion of *weak solution* is defined by introducing the associated "energy space", $V \subset H^m(\Omega)$ (the Sobolev space or order m, i.e. $D^\alpha u \in L^2(\Omega)$ for any $\alpha \in \mathbb{N}^N$, $|\alpha| \leq m$), and then, assumed that

$$f \in V', \tag{1}$$

we introduce the associated bilinear form $a : V \times V \to \mathbb{R}$, and require the condition

$$a(u, \zeta) = \langle f, \zeta \rangle_{V'V}\,, \text{ for any } \zeta \in V$$

(see [17], [1] and their many references).

A weaker notion of solution can be given leading to a correct mathematical treatment for a more general class of data f (i.e. for f not necessarily in V'). For instance, for $f \in L^1_{loc}(\Omega)$ the notion of *very weak solution of problem* (P_L) can be introduced by integrating $2m-$times by parts (and not merely $m-$times as before) and by requiring, merely, that $u \in L^1(\Omega)$ and that

$$\int_\Omega u(x) L^* \zeta(x) dx = \int_\Omega f(x)\zeta(x) dx,$$

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for any $\zeta \in W := \overline{\{\zeta \in C^{2m}(\overline{\Omega}) \colon \zeta \text{ satisfies } (BC)\}}^{W^{2m,\infty}(\Omega)}$, once we assume that

$$\int_\Omega |f(x)\zeta(x)|\, dx < \infty, \text{ for any } \zeta \in W.$$

Here L^* denotes the adjoint operator of L.

Most of the theory on very weak solutions available in the literature deals with second order equations. Recently, sharper results have been obtained, to this case, when $f \in L^1(\Omega : \delta)$, with $\delta = dist\,(x, \partial\Omega)$. That was originally proved by Haim Brezis, in the seventies, in a famous unpublished manuscript concerning Dirichlet boundary conditions (see also a 1996 paper [5]). For more recent references see [29], [20], [21] and [22]).

The main goal of my past lecture at Jaca 2010 (see [18]) was to present some new results proving that in the case of *higher order* equations the class of $L^1_{loc}(\Omega)$ data for which the existence and uniqueness of a very weak solution can be obtained is, in general, larger than $L^1(\Omega : \delta)$ (the optimal class for the case of second order equations). For instance, for the case of the beam equation with Dirichlet boundary conditions ($u = u' = 0$ on the boundary) I proved that *the optimal class of data* is the space $L^1(\Omega : \delta^2)$ but, for instance, for the simply supported beam ($u = u'' = 0$ on the boundary) *the optimal class of data* is again $L^1(\Omega : \delta)$. One of my main arguments was the use of the Green function $G(x, y)$ associated to the corresponding boundary value problem.

An important open problem in our days is the searching of solutions (beyond the class of weak solutions) for the case in which the operator L is *nonlinear*. Obviously, we cannot integrate $2m-$times by parts and, which seems to be more important, we do not have any kind of Green function associated to the problem.

The main goal of this paper is to present some new results concerning very weak solutions for *nonlinear problems*. Moreover, we shall give here some indications about their numerical approximation. We point out that, without loss of generality we can assume that the beam is represented by the interval $(0, L)$ with $L = 1$ (which we shall do in the rest of the lecture). To fix ideas I will concentrate my attention in the *nonlinear beam equation with simply supported boundaries*

$$(B_{SS}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = \phi(u'')(0) = 0, & \\ u(1) = \phi(u'')(1) = 0, & \end{cases}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\phi(0) = 0$. A standard example corresponds to the linear case $\phi(s) = EIs$ for any $s \in \mathbb{R}$ (E, I positive constants) but many other cases arise in the more diverse applications (case of non-Hookean materials such as cat iron, stone, rubber, bioelastic materials, concrete and most of the composite materials). Again, by dimensional analysis we can assume equal to one any constant arising in the constitutive law of the material. So, for instance, a very often treated case in the literature is $\phi(s) = |s|^{\alpha-1} s$ for some $\alpha > 0$ (notice that $\alpha = 1$ corresponds to the linear case: see [1]).

We shall also make some few comments on the case of a *nonlinear cantilever beam*

$$(B_{Cant}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, & \\ \phi(u'')(1) = \phi(u'')'(1) = 0. & \end{cases}$$

In the last section we shall consider a hinged plate (i.e. with the Navier boundary conditions) or more in general, the N-dimensional problem

$$(P_{Nd})\begin{cases} -\Delta\phi(-\Delta u(x)) = f(x) & \text{in } \Omega\subset\mathbb{R}^N,\\ u=\phi(-\Delta u)=0, & \text{on } \partial\Omega.\end{cases}$$

2. Very weak solutions and the optimal class of data for the nonlinear beam equation with simply supported boundaries

Consider the *nonlinear beam equation with simply supported boundaries*

$$(B_{SS})\begin{cases} \phi(u''(x))''=f(x) & \text{in } \Omega=(0,1),\\ u(0)=\phi(u'')(0)=0, & \\ u(1)=\phi(u'')(1)=0, & \end{cases}$$

where

(2) $\quad \phi:\mathbb{R}\to\mathbb{R}$ is a continuous strictly increasing function such that $\phi(0)=0$.

Definition 2.1. Given $f\in L^1_{loc}(0,1)$ a function $u\in W^{2,1}_{loc}(0,1)$ is a distributional solution of the differential equation $\phi(u''(x))''=f(x)$ in $D'(0,1)$ if $\phi(u''(x))\in L^1_{loc}(0,1)$ and

$$\left\langle \phi(u''(x)),\frac{d^2\zeta}{dx^2}\right\rangle_{D'D}=\langle f,\zeta\rangle_{D'D}$$

for any $\zeta\in D(0,1)=C_c^\infty(0,1)$.

Let us denote the boundary conditions by

$$(BC)\equiv\begin{cases} u(0)=\phi(u'')(0)=0,\\ u(1)=\phi(u'')(1)=0.\end{cases}$$

Definition 2.2. Given $f\in L^1(0,1:\delta)\equiv L^1(\Omega:\delta)$, with $\delta=dist\,(x,\partial\Omega)$, a function $u\in W^{2,1}_{loc}(0,1)$ is a "very weak solution" of (B_{SS}) if $u\in W^{2,1}(0,1)\cap W^{1,1}_0(0,1)$, $\phi(u''(x))\in L^1(0,1)$ and for any $\zeta\in W^{2,\infty}(0,1)\cap W^{1,\infty}_0(0,1)$ we have

$$\int_0^1\phi(u''(x))\frac{d^2\zeta}{dx^2}(x)dx=\int_0^1 f(x)\zeta(x)dx.$$

The main result of this section is the following:

Theorem 2.1. *a)* ***Sufficiency****. Assumed (2), for any $f\in L^1(\Omega:\delta)$ there exists a unique very weak solution of (B_{SS}). Moreover, the (nonlocal) operator $D:L^1(\Omega:\delta)\to L^1(\Omega)$ defined by $D(f)=u$ satisfies that if $D(g)=v$ then the weak maximum principle holds:*

$$\begin{aligned} f(x) &\leq g(x) \text{ implies that}\\ -u''(x) &\leq -v''(x) \text{ and so that } u(x)\leq v(x) \text{ a.e. } x\in\Omega.\end{aligned}$$

Moreover, if we assume additionally that ϕ is locally Lipschitz continuous, i.e., for any $K>0$ there exists a constant $L(K)>0$ such that

(3) $$|\phi(r_1)-\phi(r_2)|\leq L(K)\,|r_1-r_2| \text{ for any } r_1,r_2\in[-K,K],$$

then we have the estimate

(4) $$\begin{aligned}&\textstyle\int_0^1[u(x)-v(x)]_+\,dx\\ &\leq C(\widehat{K})\textstyle\int_0^1\left[\int_0^1[f(\sigma)-g(\sigma)]_+\,G(x,\sigma)d\sigma\right]dx\end{aligned}$$

for some positive constant $C(\widehat{K})$ *depending on* $\widehat{K} = max\left\{\|f\|_{L^1(\Omega:\delta)}, \|g\|_{L^1(\Omega:\delta)}\right\}$, *where, in general,* $h_+ = \max(0,h)$ *and* $G(s,\sigma)$ *is the Green function for the operator* $-\frac{d^2}{dx^2}$ *with homogeneous Dirichlet boundary conditions on* $(0,1)$*: i.e.*

$$G(t,\sigma) = \begin{cases} t(1-\sigma) & 0 \le t \le \sigma \le 1, \\ \sigma(1-t) & 0 \le \sigma \le t \le 1. \end{cases}$$

Finally,u is smoother than said at Definition 2.2 since, at least, $u \in C^2([0,1])$ and $\phi(u''(x)) \in C^1((0,1)) \cap C^0([0,1])$.
b) ***Strong maximum principle****. Let* $f \in L^1(\Omega:\delta)$ *with* $f \ge 0$ *a.e.* $x \in (0,1)$, $f \ne 0$*. Then the very weak solution satisfies that*

$$\phi(u'')(x) \le C\left(\int_0^1 \left[\int_0^1 f(\sigma)\left(-G(s,\sigma)\right)d\sigma\right]\delta(s)ds\right)\delta(x) < 0, \tag{5}$$

for any $x \in (0,1)$*, and*
(6)

$$u(x) \ge -C(\int_0^1 \phi^{-1}\left\{C\left(\int_0^1 \left[\int_0^1 f(\sigma)\left(-G(s,\sigma)\right)d\sigma\right]\delta(s)ds\right)\delta(t)dt\right\}\delta(x) > 0,$$

for any $x \in (0,1)$ *and for some positive constant* C *independent of* f*.*
c) **Necessity.** *Assume that* $f \in L^1_{loc}(0,1)$*, such that* $f \ge 0$ *a.e.* $x \in (0,1)$*. Then if* $\int_0^1 f(x)\delta(x)dx = +\infty$ *it cannot exist any* $u \in C^2([0,1])$ *with* $\phi(u''(x)) \in C^0([0,1])$ *satisfying the boundary conditions (BC) and being also solution in* $D'(0,1)$ *of the differential equation.*
Remark. 2.1. Theorem 2.1 improves some previous results in the literature on the nonlinear formulation (see, e.g. [28] and its references) and contains also some slight improvements with respect to the mentioned results in our previous paper [18] for the linear case for which there is a great amount of previous results in the literature: (Gupta (1988), Agarwal (1989), Bernis (1996), Yao (2008),...). We also point out that the integral $\int_0^1 f(x)\delta(x)dx$ is "equivalent " to the integral $\int_0^1 f(x)x(1-x)dx$ since $x(1-x) \le \delta(x) \le 2x(1-x)$. The optimal growth condition on the data can be easily understood in terms of the physical modelling. For instance, the assumption $f \in L^1(\Omega:\delta)$ is equivalent to the global integrability of the stress function $m(x)$ of the beam ($m \in L^1(0,1)$): see the proof of Theorem 3.1).
Remark. 2.2. Estimate (4) is new in the literature. Notice this estimate implies not only the usual *weak maximum principle* (for instance, if $f(x) \ge 0$ *a.e.* $x \in (0,1)$ then, necessarily $u(x) \ge 0$ for any $x \in (0,1)$) but also a new stronger conclusion: functions u and u'' have constant sign once we merely know that the function satisfies

$$x \to \int_0^1 f(\sigma)G(x,\sigma)d\sigma \text{ is } \ge 0 \text{ a.e. on } \Omega. \tag{7}$$

This generalization with respect to the usual version of the *weak maximum principle* is quite surprising since (it is not difficult to show that) there is a continuum of changing sign functions $f(x)$ satisfying property (7). Indeed: we have

$$\int_0^1 f(\sigma)G(x,\sigma)d\sigma = (1-x)\int_0^x f(\sigma)\sigma d\sigma + x\int_x^1 f(\sigma)(1-\sigma)d\sigma. \tag{8}$$

Then, if we define

$$g(x) = \int_0^x f(\sigma)\sigma d\sigma,$$

we get that $f \in L^1(\Omega : \delta)$ implies that $g \in W^{1,1}(0,1)$ with $g(0) = 0$ and $g'(s) = f(s)s$ for a.e. $s \in (0,1)$. So, since there is a continuum of functions $g \in W^{1,1}(0,1)$ (non necessarily being increasing), such that $g(0) = 0$ and $g(s) \geq 0$ on $(0,1)$, we arrive to our conclusion by taking $f(s) := g'(s)/s$ for such a given function g (note that, of course, f can be discontinuous). A similar argument applies to the second term of (8).

Remark 2.3. The estimates (5) and (6) implying the strong maximum are new in the literature (even for the linear case). We also point out that the nonexistence result can be also obtained by showing that if $m \in C^0([0,1]) \cap C^1(0,1)$, $m'' \in L^1_{loc}(0,1)$ and $m'' \geq 0$ *a.e.* $x \in (0,1)$, then necessarily $\int_0^1 m''(x)x(1-x)dx < +\infty$. Indeed, by Taylor formula applied to $x = 1/2$ we have that

$$m(x) = m\left(\frac{1}{2}\right) - \left(\frac{1}{2} - x\right) m'\left(\frac{1}{2}\right) + \int_x^{1/2} (\sigma - x) m''(\sigma) d\sigma.$$

Since the integral has a constant sign, letting $x \to 0$ we get that

$$\int_0^{1/2} m''(\sigma)\sigma d\sigma < +\infty.$$

Analogously we get that the convergence of the other integral

$$\int_{1/2}^1 m''(\sigma)(1-\sigma)d\sigma < +\infty$$

is similar.

Remark 2.4. The mathematical results for the case of the cantilever beam (B_{Cant}) can be derived by adapting the results of ([18]) to the nonlinear framework in a similar way to the proof of Theorem 2.1. In that case, the optimal growth condition on $f(x)$ becomes now

$$\phi^{-1}\left[\int_x^1 \int_s^1 f(\sigma)d\sigma ds\right] \in L^1(0,1),$$

which obviously depends of the constitutive law function ϕ (in contrast with the case of the simply supported beam problem (B_{SS}) for which, unexpectedly, the optimal class of data is independent of ϕ). For instance, if $f(x) = cx^a$ and $\phi(s) = |s|^{\alpha-1} s$ for some $\alpha > 0$, then the optimal solvability condition for problem (B_{SS}) is $a > -2$ (for any value of α !!) but for problem (B_{Cant}) the optimal solvability condition is $a > -(\alpha + 1)$ (which depends on α). Notice that, in both cases, the solvability of the boundary value problem is possible beyond the condition $f \in L^1(0,1)$ (which would require to assume $a > -1$).

Remark 2.5. The case of a nonlinear beam equation with clamped boundaries

$$(B_{Clam})\begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0, \end{cases}$$

seems to be more delicate. When ϕ is linear, it was shown in [18] that the optimal set of data is $L^1(\Omega : \delta^2)$, i.e.

$$\int_0^1 |f(x)|\, x^2(1-x)^2 dx < +\infty.$$

We conjecture that in the case of a nonlinear constitutive equation the solvability requires two kinds of conditions: one independent of ϕ, $f \in L^1(\Omega : \delta)$, and another one depending on ϕ,

$$\phi^{-1}\left[\int_0^x \int_s^1 f(\sigma)d\sigma ds\right] \in L^1(\Omega:\delta).$$

3. Main idea of the proofs: a key representation formula

The proof of the part a) (sufficiency) of Theorem 2.1 will made use of the following representation formula:

Theorem 3.1. *Assume $f \in L^1(\Omega:\delta)$. Then the unique very weak solution u of (B_{SS}) is given through the (nonlocal) operator $D : L^1(\Omega,\delta) \to L^1(\Omega)$, defined by the representation formula $D(f) = u$,*

$$u(x) = -\int_0^1 \phi^{-1}\left(\int_0^1 -f(\sigma)G(s,\sigma)d\sigma\right)G(x,s)ds \text{ for any } x \in [0,1], \tag{9}$$

where

$$G(t,\sigma) = \begin{cases} t(1-\sigma) & 0 \le t \le \sigma \le 1, \\ \sigma(1-t) & 0 \le \sigma \le t \le 1. \end{cases}$$

Proof. If we denote by $u(x), w(x), m(x), q(x)$ and $f(x)$ the deflection, the rotation of the beam, the curvature, the shear and the load density at point x, respectively, then, from the beam differential equation we get the system of four first order differential equations

$$\begin{cases} q'(x) &=& f(x), \\ m'(x) &=& q(x), \\ w'(x) &=& \phi^{-1}(m(x)), \\ u'(x) &=& w(x). \end{cases} \tag{10}$$

In fact, this is the mathematical model in terms of differential equations, when we are interested in q, m, w and u. It is well known that the system (10) of four first order differential equations is equivalent to the fourth order differential equation of (B_{SS}). But since

$$\begin{cases} m''(x) = f(x) & \text{in } \Omega = (0,1), \\ m(0) = m(1) = 0, \end{cases}$$

and $f \in L^1(\Omega:\delta)$, we know that

$$m(x) = -\int_0^1 f(\sigma)G(x,\sigma)d\sigma \text{ for any } x \in [0,1]. \tag{11}$$

Thus

$$\begin{cases} u''(x) = \phi^{-1}(-\int_0^1 f(\sigma)G(x,\sigma)d\sigma) & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{12}$$

and since $\phi^{-1}(\int_0^1 f(\sigma)G(x,\sigma)d\sigma) \in C([0,1]) \subset L^1(\Omega:\delta)$ we get formula (9).

Remark 3.1. In [8] other different representation formulae will be obtained. For instance, if we assume $f \in L^1(\Omega)$ then the unique very weak solution u of (B_{SS}) is given trough the representation formula

$$\begin{aligned} u(x) = \int_0^x \{ \int_0^\theta \phi^{-1}(\int_0^t (t-r)f(r)dr - t\int_0^1 (1-\sigma)f(\sigma)d\sigma)dt \\ - \int_0^1 \int_0^s \phi^{-1}\left(\int_0^t (t-r)f(r)dr - t\int_0^1 (1-\sigma)f(\sigma)d\sigma\right) dtds\} d\theta. \end{aligned} \tag{13}$$

Proof of the part a) (sufficiency) of Theorem 2.1. The existence of a very weak solution *of* (B_{SS}) and the definition of the (nonlocal) operator $D : L^1(\Omega : \delta) \to L^1(\Omega)$ defined by $D(f) = u$ is a consequence of (9) given in Theorem 3.1. In order to prove the uniqueness let $D(g) = v$. Then, since

$$\begin{cases} (m_f - m_g)''(x) = f(x) - g(x) & \text{in } \Omega = (0,1), \\ (m_f - m_g)(0) = (m_f - m_g)(1) = 0, \end{cases}$$

and $f, g \in L^1(\Omega : \delta)$ we know that

$$(m_f - m_g)(x) = -\int_0^1 (f(\sigma) - g(\sigma))G(x,\sigma)d\sigma \text{ for any } x \in [0,1].$$

Thus

$$[-\phi(u'')(x) + \phi(v'')(x)]_+ = \left[\int_0^1 (f(\sigma) - g(\sigma))G(x,\sigma)d\sigma\right]_+ .$$

In particular, $f(x) \leq g(x)$ implies that $\phi(u'')(x) \geq \phi(v'')(x)$ and, as ϕ is strictly increasing, $-(u-v)''(x) \leq 0$ on $\Omega = (0,1)$. But, since $u - v = 0$ on $\partial\Omega$ we deduce the comparison $u(x) \leq v(x)$ on Ω. Obviously this implies the uniqueness of very weak solution.

To get the quantitative estimate (4) we can adapt the argument already used in [18] for the linear case by arguing in two steps. Indeed, from the representation formula for $m(x)$ we get that

$$\int_0^1 m(x)dx = -\int_0^1 \left[\int_0^1 f(\sigma)G(x,\sigma)d\sigma\right] dx. \tag{14}$$

Now we shall apply the following result:

Lemma 3.1 (Crandall-Tartar [15]). *Let X, Y two vector lattices and λ_X, λ_Y be nonnegative linear functionals on X and Y respectively. Let $C \subseteq X$ and $f, g \in C$ imply $f \vee g \in C$. Let $T : C \to Y$ satisfy $\lambda_X(f) = \lambda_Y(T(f))$ for $f \in C$. Then $(a) \Rightarrow (b) \Rightarrow (c)$ where $(a), (b), (c)$ are the properties:(a) $f, g \in C$ and $f \leq g$ imply $T(f) \leq T(g)$,(b) $\lambda_Y((T(f) - T(g))_+) \leq \lambda_X((f-g)_+)$ for $f, g \in C$, (c) $\lambda_Y(|T(f) - T(g)|) \leq \lambda_X(|f-g|)$. Moreover, if $\lambda_Y(F) > 0$ for any $F > 0$, then $(a), (b), (c)$ are equivalent.*

Then, by taking $C = X = L^1(\Omega : \delta)$, $Y = L^1(0,1)$, $\lambda_Y(e) = \int_0^1 e(x)dx$, $T(f) = D(f)$ and

$$\lambda_X(f) = -\int_0^1 \left[\int_0^1 f(\sigma)G(x,\sigma)d\sigma\right] dx,$$

thanks to (14) and the weak maximum principle we get (b) of Lemma 3.1 which implies that

$$\int_0^1 [-m_f(x) + m_g(x)]_+ dx \leq \int_0^1 \left[\int_0^1 [f(\sigma) - g(\sigma)]_+ G(x,\sigma)d\sigma\right] dx.$$

But we know that $u''(x) = \phi^{-1}(m_f(x))$ and $v''(x) = \phi^{-1}(m_g(x))$. Then, since $\phi^{-1}(m_f(x))$ and $\phi^{-1}(m_g(x))$ are in $C([0,1])$ the same happens with u and v. Taking $K = \max\left\{\|u\|_{L^\infty(0,1)}, \|v\|_{L^\infty(0,1)}\right\}$ we can apply the locally Lipschitz assumption on ϕ to conclude that

$$\int_0^1 [-u''(x) + v''(x)]_+ dx \leq L(K)\int_0^1 \left[\int_0^1 [f(\sigma) - g(\sigma)]_+ G(x,\sigma)d\sigma\right] dx.$$

Finally, applying the same arguments than before but now for u and v instead m_f and m_g we get estimate (4) for some positive constant $C(\widehat{K})$ depending on $\widehat{K} = max\left\{\|f\|_{L^1(\Omega:\delta)}, \|g\|_{L^1(\Omega:\delta)}\right\}$.

With respect to the additional regularity of the very weak solution it is enough to use that, from (11) $m = \phi(u'') \in C^1((0,1)) \cap C^0([0,1])$ and thus using (12) and the above regularity we get that $u \in C^2([0,1])$.

The proof of the strong maximum principle uses the following estimate: if

$$\begin{cases} -U''(x) = F(x) & \text{in } \Omega = (0,1), \\ U(0) = U(1) = 0, \end{cases}$$

with $F \in L^1(\Omega : \delta)$, $F \geq 0$ then there exists a positive constant C such that

$$U(x) \geq C(\int_0^1 F(s)\delta(s)ds)\delta(x) > 0 \text{ for any } x \in (0,1).$$

That was proved first by J.M. Morel and L. Oswald (in an unpublished manuscript communicated to this author in 1985) and later developed in [4]. Thus, applying it to function $m(x)$ we get the strong negativeness for $\phi(u'')$, estimate (5), and applying it again, now to (12), we conclude the strict positivity of u, i.e. estimate (6).

To prove part c), and more specifically the complete blow up (in the whole interval $(0,1)$) when $f \notin L^1(\Omega : \delta)$ we truncate f generating the sequence $f_n(x) = \min(f(x), n)$. Now, if u_n is the associated solution (notice that $f_n \in L^\infty(0,1) \subset L^1(\Omega : \delta)$) then $u_n(x) \geq \alpha(\|f_n\|_{L^1(0,1:\delta)})\delta(x)$, for a suitable increasing function α such that $\alpha(\|f_n\|_{L^1(0,1:\delta)}) \nearrow +\infty$ as $n \nearrow +\infty$, which implies that $u_n(x) \nearrow +\infty$ for any $x \in (0,1)$. The proof of Theorem 2.1 is now completed.

4. Remarks on the numerical analysis of very weak solutions

Remark 4.1. Given $N \in \mathbb{N}$, it takes sense to search for some discrete approximations $D^N[f]$ of the nonlocal operator $D[f]$) given in Theorem 3.1 (but without approximating the function f) allowing to compute an approximation u^N of the solution u in a faster way. So $D^N : L^1(\Omega : \delta) \to S^N$, where S^N is a *specialized* (finite dimensional) subspace of $L^1(\Omega)$. In the linear case ($\phi(s) = s$) the nonlocal operator is given by the direct Green function G_L associated to the (linear) operator L on the open interval $\Omega = (0,1)$ and with the corresponding boundary conditions

$$u(x) = D[f](x) = \int_0^1 f(s)G_L(x,s)ds,$$

and so the approximation $D^N[f]$ can be now searched through the notion of "Discrete Green Function" $G_L^N(x,s)$ leading to

$$u^N(x) = D^N[f](x) = \int_0^1 f(s)G_L^N(x,s)ds,$$

where we impose now that $u^N \in S^N$.

This point of view (which is very closely related with the study of the discrete maximum principle) was adopted by many authors: Ciarlet [11], Ciarlet and Varga [13], Deeter and Gray [16], Mugler [26], Chung and Yau [10] and Vejchodský and Solín [32], among others.

For instance, we can take as S^N the L-spline subspace $Sp_0(L, \Pi^N, z)$ satisfying the homogeneous boundary conditions (see [13]), but many other finite dimensional subspaces can be considered as well.

The continuous dependence estimate (4) obtained in Theorem 2.1 allows to extend *to the nonlinear case* (corresponding to constitutive laws ϕ non necessarily linear) the theory on the Discrete Variational Green's Function (usually restricted to functions f in $L^2(\Omega)$) *and on the optimal solvability space* $L^1(\Omega:\delta)$:

Corollary 4.2 *Let $\phi:\mathbb{R}\to\mathbb{R}$ be a continuous strictly increasing function such that $\phi(0)=0$ and let $f\in L^1(\Omega:\delta)$. For any $k>0$ let $T_k:L^1(\Omega:\delta)\to L^\infty(\Omega)$ be the truncation operator*

$$T_k(f)(x)=\begin{cases}\min(f(x),k), & \text{if } f(x)\geq 0,\\ \max(f(x),-k), & \text{if } f(x)\leq 0.\end{cases}$$

Let S^N be a finite dimensional subspace of $H_0^1(\Omega)$ and let G_Δ^N be the discrete variational Green function (in the sense of [13]*) associated to the problem*

$$\begin{cases}-U''(x)=F(x) & \text{in } \Omega=(0,1),\\ U(0)=U(1)=0.\end{cases}$$

For any $k>0$ and $N\in\mathbb{N}$ consider the discrete version of the nonlocal operator $D^N:L^\infty(\Omega)\to S^N$ given by

$$D^N[f](x)=-\int_0^1\phi^{-1}\left(\int_0^1 -f(\sigma)G_\Delta^N(s,\sigma)d\sigma\right)G_\Delta^N(x,s)ds.$$

Let u_k^N and u be the discrete and continuous very weak solutions corresponding to $T_k(f)$ and f respectively (i.e. $u_k^N=D^N[T_k(f)]$ and $u=D[f]$). Then $u_k^N\to u$, in $L^1(\Omega:\delta)$, as k and $N\to+\infty$.

Proof. We have

$$\begin{aligned}\left\|D^N[T_k(f)]-D[f]\right\|_{L^1(\Omega:\delta)} &\leq \left\|D^N[T_k(f)]-D[T_k(f)]\right\|_{L^1(\Omega:\delta)}\\ &\quad+\left\|D[T_k(f)]-D[f]\right\|_{L^1(\Omega:\delta)}.\end{aligned}$$

Then, the first term goes to zero (and even when the norm is replaced by the L^∞ norm) as $N\to+\infty$ thanks to [13] and the continuity of the function ϕ^{-1}. Moreover, the second term goes to zero as $k\to+\infty$ thanks to the estimate (4) obtained in Theorem 2.1 (because $T_k(f)\to f$ in $L^1(\Omega:\delta)$ as $k\to+\infty$).

Some additional results on the discrete maximum principle for the discrete fourth order problem are presently in progress.

Remark 4.2. Another general subject on the numerical approach of problem (B_{SS}) have a different nature and, in fact, it is the main goal of the paper [8]. One of the main points in the approximation of the solution u of a boundary value problem by a finite differences algorithm giving u_h is the study of the convergence $u_h\to u$ when the step h of the discretization goes to zero. It is well know (see, for instance the book by Ciarlet [12] page 41) that, at least, in the case of linear problems as, for instance,

$$\begin{cases}m''(x)=f(x) & \text{in } \Omega=(0,1),\\ m(0)=m(1)=0,\end{cases}\tag{15}$$

if $\varphi_h=(\varphi_i)_{i=0}^{i=N+1}$, $h=\frac{1}{N+1}$ is the solution of the approximate problem

$$\frac{\varphi_{i+1}-2\varphi_i+\varphi_{i-1}}{h^2}=b_{h,i}$$

with $b_{h,i}=f(x_i)$, $x_i=ih$, then, in fact, we have

$$m(x_i)=\varphi_i,$$

if we assume that $f(x)$ is a constant. Indeed, in that case $m'''(x)=0$ and then it is enough to apply Taylor formula. Obviously, in general $m(x_i)\neq\varphi_i$ since in the discretization of the second derivative operator there is a non zero remaining term. A curious fact, which seems to be very few analyzed in the literature (except in references [6] and [7]), is that we can produce other finite differences algorithms $\{u_h\}$ for which $u_h(x_i) \equiv u(x_i)$ for any i (and any h small enough) even when f is merely integrable (and in fact for very weak solutions of the problem). The prize we must pay to that is to replace the discrete values $b_{h,i} = f(x_i)$ of the data $f(x)$ on the points x_i by the values of the nonlocal operator $D(f)$ (given in Theorem 3.1) in these points $D(f)(x_i)$. Obviously, since we know a formula for the own solution, $u = D[f]$, this only has a marginal interest. Nevertheless, for the sake of the curiosity of the reader we state this direct consequence:

Corollary 4.2 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $\phi(0) = 0$ and let $f \in L^1(\Omega : \delta)$ and $h = \frac{1}{N+1}$. Consider the finite difference algorithm associated to problem (B_{SS}) :*

$$\Delta_h^* \phi(\Delta_h^* u) = H_h,$$

where Δ_h^* denotes the progressive difference operator ($\Delta_h^* u(x) = u(x+h) - u(x)$) and

$$H_h(x) = \Delta_h^* \phi(\Delta_h^* D[f])(x).$$

Here $D : L^1(\Omega : \delta) \to L^1(\Omega)$ is the nonlocal operator given in Theorem 3.1. Then

$$u_h(x_i) \equiv u(x_i) \text{ for any } i \text{ (and any } h \in (0,1)).$$

In the linear case we get the "fourth order" functional equation

$$\begin{aligned} u(x+4h) = \; & 4u(x+h) - 6u(x+2h) + 4u(x+3h) - u(x) + \\ & -4D(x+h) + 6D(x+2h) - 4D(x+3h) + D(x+4h), \end{aligned} \tag{16}$$

where $D(y) = D[f])(y)$. In fact, if we use the notation (10) for $u(x), w(x), m(x)$ and $q(x)$ then we can prove similar results for the associate finite differences schemes of "third", "second" and "first" order for $w(x)$, $m(x)$ and $q(x)$, respectively. For instance, in the linear case we get

$$\begin{aligned} w(x+3h) = \; & w(x) - 3w(x+h) + 3w(x+2h) \\ & +3C(x,h) - 3C(x,2h) + C(x,3h), \end{aligned}$$

$$m(x+2h) = 2m(x+h) - m(x) - 2B(x,h) + B(x,2h), \tag{17}$$

$$q(x+h) = q(x) + A(x,h), \tag{18}$$

for suitable functions $A(x,h)$, $B(x,h)$ and $C(x,h)$ (see [6] and [7]). Such functional equations give the exact solution at the interpolating points for any $h = \Delta x$.

Remark 4.3. Some numerical experiences on very weak solution for the problem (B_{SS}) corresponding to different constitutive laws functions ϕ are presented in Figure 1 (see [8] for more details).

5. Further remarks

Remark 5.1. *Some other boundary conditions.* In the cantilever case we have:

Theorem 5.1. *If we assume*

$$\phi^{-1}[\int_x^1 \int_s^1 f(\sigma) d\sigma ds] \in L^1(0,1), \tag{19}$$

then the unique very weak solution u of (B_{Cant}) is given by the representation formula

$$u(x)=\int_0^x\left(\int_0^\sigma\left\{\phi^{-1}[\int_s^1\int_t^1 f(r)drdt]\right\}ds\right)d\sigma. \tag{20}$$

Proof. As in Theorem 3.1, since

$$\begin{cases} m''(x)=f(x) & \text{in } \Omega=(0,1), \\ m(1)=m'(1)=0, \end{cases}$$

and f satisfies (19) (and in particular f is integrable near $x=1$) we know, by simply integration, that

$$m(x)=\int_x^1\left(\int_t^1 f(r)dr\right)dt \text{ for any } x\in[0,1].$$

Thus

$$\begin{cases} u''(x)=\phi^{-1}[\int_x^1\left(\int_t^1 f(r)dr\right)dt] & \text{in } \Omega=(0,1), \\ u(0)=u'(0)=0, \end{cases}$$

and so, since $\phi^{-1}(\int_x^1\left(\int_t^1 f(r)dr\right)dt)$ is integrable near $x=0$ (thanks to (19)), formula (20) is well justified (notice that the function

$$x\to\int_0^x\left\{\phi^{-1}[\int_s^1\int_t^1 f(r)drdt]\right\}ds$$

is automatically integrable near $x=0$ once we impose that (19) holds).

We send the reader to [8] for other different representation formulae concerning problem (B_{Cant}). We also point out that in the linear case we have that $\frac{u(x)}{\delta(x)}\in W_0^{1,1}(\Omega)$, thanks to the results of [9], which gives a very rich information near the boundary.

Remark 5.2. *Concentrated charges: measures as right hand side.* The existence result holds also in the more general class of Radon measures $f\in M(\Omega:\delta)$ (something very useful to justify the engineers study when the load is concentrated in isolated points). Notice that although the usual Radon measure space (without weight) $M(0,1)$ is a subset of the dual space $H^{-2}(0,1)$ it is not always true that the duality $\langle f,\zeta\rangle_{H^{-2}(0,1),H_0^2(0,1)}$ coincides with the $\langle f,\zeta\rangle_{M(0,1),C^0([0,1])}=\int_0^1\zeta(x)df$ duality. See [21] for some results related to the second order case.

Remark 5.3. *The associated parabolic problems.* Like in the linear case ([18]), the above results lead to interesting results on the existence and asymptotic behaviour of solutions of parabolic problems of the type

$$(HP)\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial^2}{\partial x^2}\phi(\dfrac{\partial^2}{\partial x^2}u)=f(t,x) & t\in(0,T),x\in(0,L), \\ +\text{ boundary conditions}, & t\in(0,T), \\ u(0,x)=u_0(x) & x\in(0,L). \end{cases}$$

We shall develop it elsewhere.

Remark 5.4. *General higher (2mth-order) equations.* Like in Bernis [3], the above results remain valid for other 2mth-order equations with similar nonlinearities (for instance on the derivatives of the mth-order). Such type of problems appear in many different applications (lubrication, semiconductors, ...).

Remark 5.5. *Perturbed nonlinear problems.* Many applications can be obtained to nonlinear perturbed problems of the type

$$(NLSP)\begin{cases} \phi(u''(x))''+\beta(u)=f(x) & x\in\Omega=(0,1),\\ u(0)=\phi(u'')(0)=0, & \\ u(1)=\phi(u'')(1)=0, & \end{cases}$$

in the same spirit than in the linear case (see [18]). Some other references (always with $\beta(0)=0$) concerning the linear case can be found in [25], and for the nonlinear case in [28]. Nevertheless, it is also possible to consider the case in which the nonlinear term $\beta(u)$ becomes singular at $u=0$ as for instance

$$P(a,b;g)\begin{cases} \phi(u''(x))''=\dfrac{h(x)}{u^a} & x\in\Omega=(0,1),\\ u(0)=\phi(u'')(0)=0, & \\ u(1)=\phi(u'')(1)=0, & \end{cases}$$

$$h(x)=\frac{g(x)}{\delta(x)^b}$$

with $g\in L^\infty(\Omega)$ such that

$$0<C_g\leq g(x),$$

and a, $b\geq 0$. We can prove the existence of the (positive) *very weak solutions* for problem $P(a,b;g)$ in the sense that we ask for $u\in W^{2,1}_{loc}(0,1)$, with $u(x)>0$ a.e. $x\in\Omega$, such that $\delta^{-b}u^{-a}\in L^1(\Omega,\delta)$, $u\in W^{2,1}(0,1)\cap W^{1,1}_0(0,1)$, $\phi(u''(x))\in L^1(0,1)$ and satisfying, for any $\zeta\in W^{2,\infty}(0,1)\cap W^{1,\infty}_0(0,1)$,

$$\int_0^1\phi(u''(x))\frac{d^2\zeta}{dx^2}(x)dx=\int_0^1\frac{g(x)}{d^bu^a}(x)\zeta dx.$$

By applying the techniques of [19] we easily get the following result:

Corollary 5.1. *i) Let $a+b>1$ with $b\in[0,2)$. Then there exists a very weak solution u of $P(a,b;g)$. Moreover $\phi(u''(x))\in C(\overline{\Omega})\cap W^{2,p}_{loc}(\Omega)$ for any $p\in[1,+\infty)$. ii) Let $a+b<1$. Then there exists a very weak solution u of $P(a,b;g)$. Moreover $\phi(u''(x))\in W^1_0(\Omega,|\cdot|_{N(\gamma),\infty})\cap W^{2,p}_{loc}(\Omega)$, for any $\gamma\in]0,1[$ and for any $p\in[1,+\infty)$.*

We point out that if $b\geq 2$ no very weak solution of $P(a,b;g)$ may exists (since any very weak solution of problem (15), for a certain f associated to $\frac{g(x)}{\delta^bu^a}$, can not exists). Moreover, once we have existence of very weak solutions they are unique since the uniqueness result of Crandall, Rabinowitz and Tartar [14] (for second order singular problems) can be easily adapted to our framework (see [19] for the adaptation to very weak solutions of other second order singular equations).

6. On the N-dimensional formulation

For the N-dimensional problem on a bounded open set Ω of $\mathbb{R}^N$

$$(P_{Nd})\begin{cases} -\Delta\phi(-\Delta u(x))=f(x) & \text{in }\Omega\subset\mathbb{R}^N,\\ u=\Delta u=0, & \text{on }\partial\Omega,\end{cases}$$

the notion of very weak solution can be stated in the following terms:

Definition 6.1. Given $f\in L^1(\Omega:\delta)$, with $\delta=dist\,(x,\partial\Omega)$, a function $u\in W^{2,1}_{loc}(\Omega)$ is a "very weak solution" of (P_{Nd}) if $u\in W^{2,1}(\Omega)\cap W^{1,1}_0(\Omega)$, $\phi(-\Delta u))\in L^1(\Omega)$ and for any $\zeta\in W^{2,\infty}(\Omega)\cap W^{1,\infty}_0(\Omega)$ we have

$$\int_\Omega\phi(-\Delta u(x))(-\Delta\zeta(x))dx=\int_\Omega f(x)\zeta(x)dx.$$

This time the representation formula (similar to the one given in Theorem 3.1) becomes

$$u(x) = \int_\Omega \phi^{-1}\left(\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right)G_\Omega(x,s)ds \text{ for a.e. } x\in\Omega, \tag{21}$$

once we know the existence (and positivity) of the Green function $G_\Omega(x,\xi)$ (see, e. g. the books by Stakgold [31],(1998) and Friedman [23]).

Theorem 6.1. *a)* ***Sufficiency****. Assume (2) as well as*

$$|r| \le C_1 |\phi(r)| + C_2 \text{ for any } r\in\mathbb{R}. \tag{22}$$

Then, for any $f\in L^1(\Omega:\delta)$ there exists a unique very weak solution of (P_{Nd}). Moreover, the (nonlocal) operator $D: L^1(\Omega:\delta)\to L^1(\Omega)$ defined by $D(f)=u$ in (21) *satisfies that if $D(g)=v$ then the weak maximum principle holds:*

$$f(x)\le g(x) \text{ a.e. in } \Omega$$

implies

$$-\Delta u(x)\le -\Delta v(x)$$

and so

$$u(x)\le v(x) \quad a.e.\ x\in\Omega.$$

Moreover, we have the estimate

$$\int_\Omega [u(x)-v(x)]_+\,dx \tag{23}$$

$$\le \int\left[\int_\Omega \phi^{-1}\left(\int_\Omega [f(\sigma)-g(\sigma)]_+ G_\Omega(s,\sigma)d\sigma\right)G_\Omega(x,s)ds\right]dx \tag{24}$$

where, in general, $h_+=\max(0,h)$, and $G_\Omega(x,\xi)$ is the Green function associated to the operator $-\Delta$ with homogeneous boundary conditions on $\partial\Omega$. Moreover u is smoother than said at Definition 6.1 since, at least, $u\in W_0^{1,s}(\Omega)$ for any $1\le s<(N-1)$ and if $f\in L^1(\Omega,\delta^\alpha)$ for some $0\le\alpha<1$ then $|\nabla\phi(-\Delta u(x))|$ belongs to the space $L^{\frac{N}{N-1+\alpha}}(\Omega)$.

b) ***Strong maximum principle****. Let $f\in L^1(\Omega:\delta)$ with $f\ge 0$ a.e. $x\in\Omega$, $f\ne 0$. Then the very weak solution satisfies that*

$$\phi(-\Delta u)(x)\ge C\left(\int_\Omega\left[\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(x)>0, \tag{25}$$

a.e. $x\in\Omega$, and

(26)

$$u(x)\ge C\left(\int_\Omega \phi^{-1}\left\{C\left(\int_\Omega\left[\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(y)dy\right\}\right)\delta(x)>0,$$

a.e. $x\in\Omega$, for some positive constant C independent of f.

c) **Necessity.** *Assume that $f\in L^1_{loc}(\Omega)$, such that $f\ge 0$ a.e. $x\in\Omega$. Then if $\int_\Omega f(x)\delta(x)dx=+\infty$ it can not exists any very weak solution of (P_{Nd}).*

The proof follows the same type of arguments than the proof of Theorem 2.1. Which is now much harder than in the one-dimensional case is the question of the regularity of the very weak solution. Nevertheless, since the function $m(x):=\phi(-\Delta u(x))$ is a very weak solution of the second order problem

$$(P_2)\begin{cases} -\Delta m = f(x) & \text{in } \Omega\subset\mathbb{R}^N,\\ m=0, & \text{on } \partial\Omega,\end{cases}$$

we can apply the results by [20] and [21], which justify the regularity stated $\phi(-\Delta u(x))$. Thanks to the assumption (22) we know that $-\Delta u(x) = F$ with $F \in L^1(\Omega)$ given by $F := \phi^{-1}(m)$ and (in fact it is enough to know that $F \in L^1(\Omega : \delta)$) so we can apply well known results in the literature to end the proof of the regularity stated in Theorem 6.1 (in particular the regularity in the space $L^{\frac{N}{N-1+\alpha}}(\Omega)$ is a consequence of the results of [22]). The rest of the arguments (the strong maximum and its consequences) remains valid in the N-dimensional case (see details and improvements in [2]).

Remark 6.1. It was kindly communicated by J.M. Rakotoson to the author that an alternative proof of the necessity part of Theorems 2.1 and 6.1 can be obtained more directly by using the first eigenvalue φ_1 of the Dirichlet problem

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0, & \text{on } \partial\Omega. \end{cases}$$

Indeed, if, for instance, we consider the one-dimensional case, then

$$\int_0^1 \phi(u''(x))\varphi_1(x)dx = \int_0^1 f(x)\varphi_1(x)dx.$$

But in dimension one we know, explicitly, that $\varphi_1(x) = \sin(\pi x)$, $x \in (0,1)$, and one can see that $\varphi_1(x)$ is equivalent to πx as $x \to 0$ and to $\pi(1-x)$ as $x \to 1$. Therefore $\varphi_1(x) \sim \delta(x)$ (something which is also true in higher dimensions). Thus, under the condition $\phi(u'') \in C[0,1]$, for $f \geq 0$, we get that necessarily

$$0 \leq \int_0^1 f(x)\delta(x)dx \leq C\,|\phi(u'')|_\infty \int_0^1 \varphi_1(x)dx < \infty.$$

Remark 6.2. We can avoid the additional growth condition (22) getting the existence for any ϕ satisfying (2) if we know that $m \in L^\infty(\Omega)$ since then $F := \phi^{-1}(m) \in L^\infty(\Omega)$. Obviously this requires some additional information on $f(x)$. For instance, it is enough to know that $f \in L^p(\Omega : \delta)$ for $p > (N-1)$ (see [29]) or, that

$$0 \leq f(x) \leq \delta(x)^{-\beta} \quad \text{for some } \beta < 2, \text{ a.e. } x \in \Omega,$$

because then $0 \leq m(x) \leq \delta(x)^\theta$ for some $\theta > 0$ ([19]). Of course that, in that case u becomes much more regular than stated in Theorem 6.1. We also point that some additional regularity can be obtained by applying some results in [27] (see [2]). For some results on a singular perturbation problem, but for the case of Dirichlet boundary conditions, see [24].

Remark 6.3. *Some numerical experiences.* Figures 2-9 (taken from [2]) illustrate the behaviour near the boundary of the very weak solution u, and Δu, for a linear problem on $\Omega = (0,1) \times (0,1)$ corresponding to the load function

$$f(x,y) = \frac{1}{|x+\varepsilon|^k\,|x+\varepsilon-1|^k\,|y+\varepsilon|^k\,|y+\varepsilon-1|^k}.$$

Notice that if $\varepsilon = 0$ and $k = 1$ then $f \notin L^1(\Omega)$ but $f \in L^1(\Omega : \delta)$.

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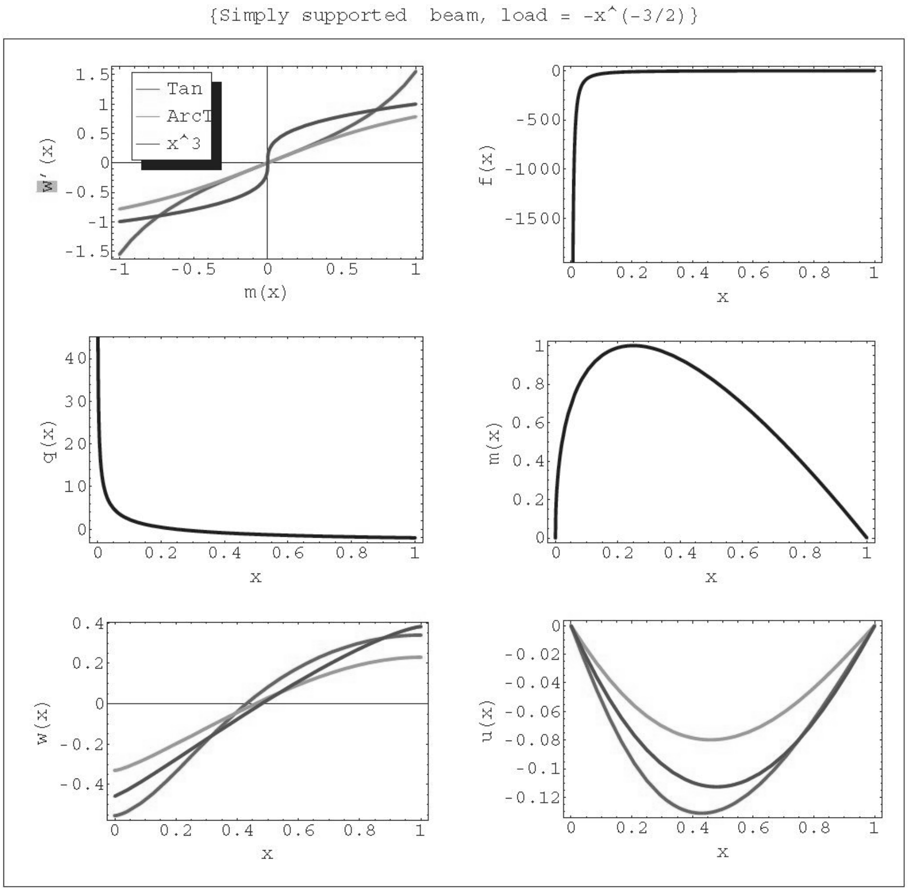

FIGURE 1

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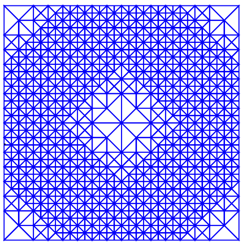
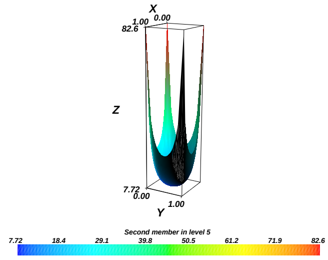


Figure 2



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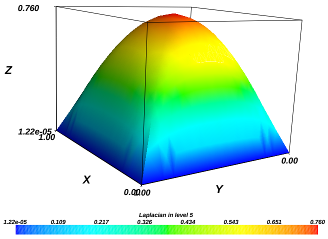

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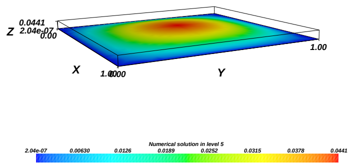

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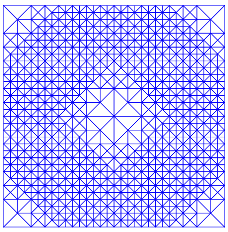

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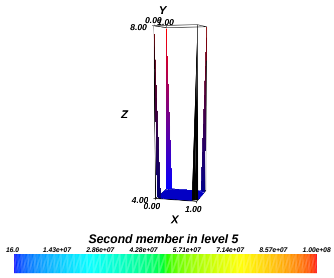

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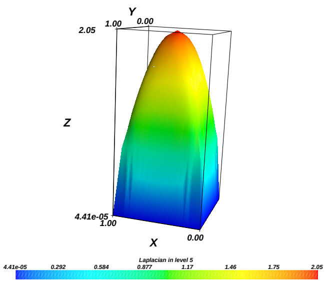

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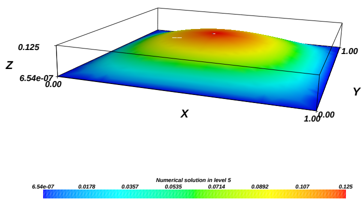

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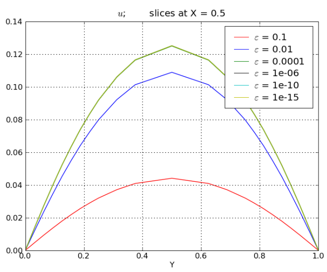

Figure 10

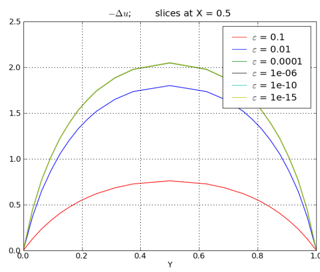

Figure 11

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