

# Large Solutions for a System of Elliptic Equations Arising from Fluid Dynamics\*

J. I. Díaz<sup>†</sup>, M. Lazzo<sup>‡</sup> and P. G. Schmidt<sup>§</sup>

**Abstract.** This paper is concerned with the elliptic system

$$\Delta v = \phi, \quad \Delta \phi = |\nabla v|^2, \quad (0.1)$$

posed in a bounded domain  $\Omega \subset \mathbb{R}^N$ ,  $N \in \mathbb{N}$ . Specifically, we are interested in the existence and uniqueness or multiplicity of “large solutions,” that is, classical solutions of (0.1) that approach infinity at the boundary of  $\Omega$ .

Assuming that  $\Omega$  is a ball, we prove that the system (0.1) has a unique radially symmetric and nonnegative large solution with  $v(0) = 0$  (obviously,  $v$  is determined only up to an additive constant). Moreover, if the space dimension  $N$  is sufficiently small, there exists exactly one additional radially symmetric large solution with  $v(0) = 0$  (which, of course, fails to be nonnegative). We also study the asymptotic behavior of these solutions near the boundary of  $\Omega$  and determine the exact blow-up rates; those are the same for all radial large solutions and independent of the space dimension.

Our investigation is motivated by a problem in fluid dynamics. Under certain assumptions, the unidirectional flow of a viscous, heat-conducting fluid is governed by a pair of parabolic equations of the form

$$v_t - \Delta v = \theta, \quad \theta_t - \Delta \theta = |\nabla v|^2, \quad (0.2)$$

supplemented by suitable initial and boundary conditions;  $v$  and  $\theta$  represent the fluid velocity and temperature, respectively. We conjecture that the solutions of (0.2) may blow up in finite time and that this behavior is related to the existence of large solutions of the associated elliptic system (that is, the system (0.1) with  $\phi = -\theta$ ).

**Key words.** Elliptic system, boundary blow-up, large solutions, radial solutions, existence and multiplicity, asymptotic behavior.

**AMS subject classifications.** 35J60, 35J55, 35Q35.

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# 1 Introduction and Main Results

This paper is a contribution to the study of “explosive behavior” in certain systems of elliptic and parabolic PDEs. Our investigation is motivated by a long-standing question regarding the dynamics of a viscous, heat-conducting fluid.

In general, the flow of a viscous, heat-conducting fluid is governed by a system of balance equations for momentum, mass, and energy. In the so-called Boussinesq approximation, this system reduces to the Navier-Stokes equations for an incompressible fluid, along with a heat equation; these equations are nonlinearly coupled through the buoyancy force and viscous heating. If viscous heating (that is, the production of heat due to internal friction) is neglected, the resulting initial and boundary-value problems are well posed in the same sense as for the classical Navier-Stokes equations without thermal coupling; but if viscous heating is taken into account, well-posedness is an open question. In fact, we conjecture that the solutions, in this case, may exhibit “explosive behavior.” Such behavior would have serious implications for the viability of the Boussinesq approximation in situations where viscous heating cannot be neglected.

To address this issue, we study a simple prototype problem, which can be physically justified by considering a unidirectional flow, independent of distance in the flow direction:

$$v_t - \Delta v = \theta, \quad \theta_t - \Delta \theta = |\nabla v|^2. \quad (1.1)$$

Here,  $v$  (the velocity) and  $\theta$  (the temperature) are scalar functions of time  $t$  and position  $x$ ; the spatial variable  $x$  varies over a bounded domain  $\Omega \subset \mathbb{R}^N$  with  $N \in \mathbb{N}$  ( $N = 2$  in the physically relevant case, where  $\Omega$  is the cross-section of the flow channel). The source terms  $\theta$  and  $|\nabla v|^2$  represent the buoyancy force and viscous heating, respectively. The system (1.1) must be supplemented by suitable initial conditions at time  $t = 0$  and boundary conditions on the boundary  $\partial\Omega$  of the domain  $\Omega$  (for example, a homogeneous Dirichlet condition for  $v$  and a homogeneous Neumann condition for  $\theta$  if the walls of the flow channel are impermeable and thermally insulated).

Note that we cannot hope to find weak solutions of the resulting initial-boundary value problem in the usual Hilbert-space setting: if  $v$  takes values in  $H^1(\Omega)$ , then the right-hand side of the second equation in (1.1) maps, a priori, only into  $L^1(\Omega)$ . However, local-in-time existence and uniqueness of a strong solution can be established by means of semigroup theory in a suitable  $L^p$ -space setting (details will appear in a forthcoming publication). We conjecture that the solution may blow up in finite time (in the sense that

a suitable norm of  $(v, \theta)$  approaches infinity as  $t \rightarrow T^-$ , for some  $T > 0$ ) and that this behavior is linked to the existence of “large solutions” of the associated elliptic system

$$-\Delta v = \theta, \quad -\Delta \theta = |\nabla v|^2, \quad (1.2)$$

posed in the domain  $\Omega$ . By a “large solution” of (1.2) we mean a classical solution that blows up at the boundary of  $\Omega$ , that is,  $|(v(x), \theta(x))| \rightarrow \infty$  as  $\text{dist}(x, \partial\Omega) \rightarrow 0$ . Note that the  $\theta$ -component of any solution of (1.2) is superharmonic in  $\Omega$  and thus, cannot approach  $\infty$  at the boundary (by the maximum principle); for a similar reason,  $v$  and  $\theta$  cannot simultaneously approach  $-\infty$  at the boundary. We therefore expect any large solution  $(v, \theta)$  of (1.2) to satisfy  $v(x) \rightarrow \infty$  and  $\theta(x) \rightarrow -\infty$  as  $\text{dist}(x, \partial\Omega) \rightarrow 0$ .

Henceforth, we assume that  $\Omega$  is a ball in  $\mathbb{R}^N$ , centered at the origin; that is,  $\Omega = B_R^N(0)$  for some  $R > 0$ . For convenience, we introduce the function  $\phi = -\theta$  and seek radially symmetric large solutions of the system

$$\begin{cases} \Delta v = \phi \\ \Delta \phi = |\nabla v|^2 \end{cases} \quad \text{in } B_R^N(0), \quad (1.3)$$

that is, radial solutions with  $|(v(x), \phi(x))| \rightarrow \infty$  as  $|x| \rightarrow R^-$ .

**Remark 1.1** The system (1.3) has a scaling property that we will exploit repeatedly. Suppose  $(v_1, \phi_1)$  is a (large) solution of (1.3) in  $B_{R_1}^N(0)$ . For  $\lambda \in (0, \infty)$ , let  $R_\lambda := \lambda^{-1}R_1$ . For  $x \in B_{R_\lambda}^N(0)$ , define

$$v_\lambda(x) := \lambda^2 v_1(\lambda x), \quad \phi_\lambda(x) := \lambda^4 \phi_1(\lambda x).$$

Then  $(v_\lambda, \phi_\lambda)$  is a (large) solution of (1.3) in  $B_{R_\lambda}^N(0)$ .

**Remark 1.2** If  $(v, \phi)$  is a (large) solution of (1.3), then so is  $(v + c, \phi)$ , for any constant  $c \in \mathbb{R}$ . Thus, we may restrict attention to solutions with  $v(0) = 0$ .

We will now state our main results, the first of which guarantees the existence of a unique (up to a shift in  $v$ ) radially symmetric and nonnegative large solution for any space dimension.

**Theorem 1.3** *For every  $N \in \mathbb{N}$  and  $R > 0$ , the system (1.3) has a unique radially symmetric large solution  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) > 0$ . Both components of this solution are increasing functions of the radial variable  $r$ .*

If the space dimension is sufficiently small, the system (1.3) has exactly one additional radially symmetric large solution with  $v(0) = 0$ , which, of course, fails to be nonnegative.

**Theorem 1.4** *For every  $N \in \mathbb{N}$  with  $N \leq 10$  and every  $R > 0$ , the system (1.3) has a unique radially symmetric large solution  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) < 0$ . The  $\phi$ -component of this solution is an increasing function of the radial variable  $r$ , while the  $v$ -component is decreasing to a negative minimum and increasing thereafter.*

Let us note that the bound on  $N$  in the above result is not sharp. In fact, based on numerical evidence (see Remarks 3.4 and 4.4), we conjecture that the solution of Theorem 1.4 exists if, and only if,  $N \leq 14$ .

Regarding the asymptotic behavior of the solutions, we find that, as expected, both components approach infinity, and we determine the exact blow-up rates; those are the same for all radially symmetric large solutions and independent of the space dimension. Here and in the sequel, we write  $f(x) \sim g(x)$  if  $f, g : B_R^N(0) \rightarrow \mathbb{R}$  satisfy  $f(x)/g(x) \rightarrow 1$  as  $|x| \rightarrow R^-$ .

**Theorem 1.5** *Let  $(v, \phi)$  be a radially symmetric large solution of (1.3), for a given  $N \in \mathbb{N}$  and  $R > 0$ . Then, as  $|x| \rightarrow R^-$ ,*

$$v(x) \sim \frac{30}{(R - |x|)^2} \quad \text{and} \quad \phi(x) \sim \frac{180}{(R - |x|)^4}.$$

The study of “explosive behavior,” be it finite-time blow-up in evolutionary problems or boundary blow-up in stationary problems, has a long history, going back to seminal work by Keller [10] and Osserman [15] in the 1950’s; we refer to the papers [2, 3, 5, 18] and the extensive references therein. However, virtually all of the existing literature is concerned with scalar equations. Coupled systems of equations have been attacked only recently; see for example [4, 6, 7, 11]. Due to the lack of variational structure and comparison principles, methods that have proven successful for scalar equations will, in general, fail to be useful for systems, even if the expected results are analogous. For example, our existence and multiplicity result for the elliptic system (1.3) (existence of one large nonnegative solution for any space dimension, existence of a second large solution for sufficiently small space dimension) is analogous to a result by McKenna, Reichel and Walter [12] for a class of scalar equations with variational structure. However, our method of proof is entirely different, and our result appears to be the first of its kind for an elliptic system. We expect that our work, while currently

focused on a very specific problem, will lead to general insights and new methods with potential applications to a much wider class of elliptic and parabolic systems.

The paper is organized as follows. In Section 2 we reduce the problem to the study of a system of first-order ODEs, establish some basic properties of its solutions, and prove the existence and uniqueness of nonnegative large radial solutions for the system (1.3); Theorem 1.3 is an immediate consequence of Proposition 2.5. Section 3 is devoted to the proof of Theorem 1.4 (existence of a second large radial solution for sufficiently small space dimension), which follows from Lemma 3.1(a) and Proposition 3.3. This section also includes a discussion of numerical experiments, suggesting a sharper version of Theorem 1.4, and remarks on a related Dirichlet problem. In Section 4 we analyze the asymptotic behavior of large radial solutions; Theorem 1.5 follows from Proposition 4.1, whose proof relies on dynamical-systems theory, applied to an asymptotically autonomous and cooperative ODE system in  $\mathbb{R}^3$ .

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## 2 Preliminaries and Nonnegative Large Solutions

Given  $N \in \mathbb{N}$  and  $R > 0$ , radially symmetric solutions of the system (1.3) correspond to solutions of the ODE system

$$\begin{cases} v'' + \frac{N-1}{r} v' = \phi \\ \phi'' + \frac{N-1}{r} \phi' = |v'|^2 \end{cases} \quad \text{in } (0, R) \quad (2.1)$$

with  $v'(0) = \phi'(0) = 0$ ; large solutions are those with  $|(v(r), \phi(r))| \rightarrow \infty$  as  $r \rightarrow R^-$ . In view of Remark 1.2, we may impose the initial condition  $v(0) = 0$ . Finding radially symmetric large solutions of the system (1.3) is therefore equivalent to finding initial conditions  $\phi(0) = p$  such that the

solution of the Cauchy problem

$$\begin{cases} v'' + \frac{N-1}{r} v' = \phi, & v(0) = 0, v'(0) = 0 \\ \phi'' + \frac{N-1}{r} \phi' = |v'|^2, & \phi(0) = p, \phi'(0) = 0 \end{cases} \quad (2.2)$$

exists on the interval  $[0, R)$  and “blows up” at  $R$ .

Despite the singularity at  $r = 0$  for  $N > 1$ , the Cauchy problem (2.2) is well posed. Indeed, for every  $p \in \mathbb{R}$ , there exists a unique maximal solution, which depends continuously on  $p$  (in the usual sense); see Lemma 2.3 below for details.

**Remark 2.1** The scaling property of the system (1.3), as described in Remark 1.1, and the well-posedness of (2.2) imply that all solutions of the Cauchy problem with  $p > 0$  ( $p < 0$ ) are “rescalings” of the solution with  $p = 1$  ( $p = -1$ ). Indeed, if  $(v_1, \phi_1)$  is the maximal solution with initial value  $p = 1$  ( $p = -1$ ), then the maximal solution with initial value  $p > 0$  ( $p < 0$ ) is given by  $(v_\lambda, \phi_\lambda)$  with  $\lambda = |p|^{1/4}$ . Consequently, if the maximal solution with initial value  $p = 1$  ( $p = -1$ ) blows up at  $R_1$ , then the maximal solution with initial value  $p > 0$  ( $p < 0$ ) blows up at  $R_p = |p|^{-1/4} R_1$ .

**Remark 2.2** In light of the preceding remark, it is clear that the elliptic system (1.3) has large radial solutions, for any given  $R > 0$ , if and only if the solutions of the Cauchy problem (2.2) with  $p = \pm 1$  exhibit finite-time blow-up. More precisely, (1.3) has exactly one large radial solution with  $v(0) = 0$  and  $\phi(0) > 0$  if and only if the solution of (2.2) with  $p = 1$  blows up in finite time; (1.3) has exactly one large radial solution with  $v(0) = 0$  and  $\phi(0) < 0$  if and only if the solution of (2.2) with  $p = -1$  blows up in finite time. In particular, (1.3) cannot have more than two large radial solutions.

The Cauchy problem (2.2) is equivalent to the first-order system

$$\begin{cases} v' = w, & v(0) = 0, \\ w' + \frac{N-1}{r} w = \phi, & w(0) = 0, \\ \phi' = \psi, & \phi(0) = p, \\ \psi' + \frac{N-1}{r} \psi = w^2, & \psi(0) = 0. \end{cases}$$

Obviously, we can eliminate  $v$  and drop the first equation and initial condition;  $v$  is recovered from  $w$  via anti-differentiation. Furthermore, we may replace  $N - 1$  ( $N \in \mathbb{N}$ ) with a continuous parameter  $\mu \in \mathbb{R}_+$ . Thus, we are led to the Cauchy problem

$$\begin{cases} w' + \frac{\mu}{r} w = \phi, & w(0) = 0, \\ \phi' = \psi, & \phi(0) = p, \\ \psi' + \frac{\mu}{r} \psi = w^2, & \psi(0) = 0. \end{cases} \quad (2.3)$$

**Lemma 2.3** *For every  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$ , the Cauchy problem (2.3) has a unique maximal, that is, noncontinuable, solution  $(w, \phi, \psi) \in C^1([0, R], \mathbb{R}^3)$ , for some  $R \in (0, \infty]$ . If  $R < \infty$ , then  $|(w(r), \phi(r), \psi(r))| \rightarrow \infty$  as  $r \rightarrow R^-$ . Moreover,  $(w, \phi, \psi)$  depends continuously on  $\mu$  and  $p$ .*

*Proof.* What we claim is that, despite the singularity at  $r = 0$  in the case  $\mu > 0$ , the Cauchy problem (2.3) has the usual, well-known properties of a regular initial-value problem in  $\mathbb{R}^3$ . Since we could not find a general result in the literature that would cover our problem, we provide a few remarks on the proof.

Note that the first equation in (2.3) can be written as  $(r^\mu w)' = r^\mu \phi$ . Together with the initial condition  $w(0) = 0$ , this is equivalent to the integral equation

$$w(r) = \int_0^r \left(\frac{s}{r}\right)^\mu \phi(s) ds. \quad (2.4)$$

Similarly, the remaining differential equations and initial conditions in (2.3) are equivalent to the integral equations

$$\phi(r) = p + \int_0^r \psi(s) ds \quad (2.5)$$

and

$$\psi(r) = \int_0^r \left(\frac{s}{r}\right)^\mu w^2(s) ds. \quad (2.6)$$

Since we have  $0 < s/r < 1$  for  $0 < s < r$ , the “singular term”  $(s/r)^\mu$  does not cause any difficulties in proving the existence and uniqueness of a solution  $(w, \phi, \psi) \in C([0, \varepsilon], \mathbb{R}^3)$  of Equations (2.4)–(2.6), for some  $\varepsilon > 0$ , by means of the contraction mapping principle. Clearly,  $w, \phi$  and  $\psi$  are continuously differentiable on  $(0, \varepsilon]$  and satisfy the differential equations and initial conditions in (2.3). In fact, all three components are continuously differentiable on the *closed* interval  $[0, \varepsilon]$ . This is obvious for  $\phi$ , but less so

for  $w$  and  $\psi$ . Note, however, that

$$\begin{aligned} \lim_{r \rightarrow 0^+} w'(r) &= \lim_{r \rightarrow 0^+} \left( \phi(r) - \frac{\mu}{r} w(r) \right) = p - \mu \lim_{r \rightarrow 0^+} \frac{1}{r} \int_0^r \left( \frac{s}{r} \right)^\mu \phi(s) ds \\ &= p - \mu \lim_{r \rightarrow 0^+} \frac{1}{r^{\mu+1}} \int_0^r s^\mu \phi(s) ds = p - \mu \lim_{r \rightarrow 0^+} \frac{r^\mu \phi(r)}{(\mu+1)r^\mu} \\ &= p - \mu \lim_{r \rightarrow 0^+} \frac{\phi(r)}{\mu+1} = p - \mu \frac{p}{\mu+1} = \frac{p}{\mu+1}, \end{aligned}$$

where we used l'Hospital's rule to get the fourth equality. This implies that  $w \in C^1([0, \varepsilon], \mathbb{R})$  with  $w'(0) = p/(\mu+1)$ . Similarly, one shows that  $\psi \in C^1([0, \varepsilon], \mathbb{R})$  with  $\psi'(0) = 0$ .

Once existence and uniqueness of a local  $C^1$ -solution are established, the remaining claims about maximal continuation and continuous dependence on parameters and initial data can be proved in the same way as for regular initial-value problems.  $\square$

**Lemma 2.4** *Let  $(w, \phi, \psi) \in C^1([0, R], \mathbb{R}^3)$  be the maximal solution of (2.3), for some  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$ ,  $p \neq 0$ . Then the function  $\phi$  is strictly increasing on  $[0, R)$ , and  $L := \lim_{r \rightarrow R^-} \phi(r)$  is either 0 or  $\infty$ . In fact,*

- (a) *if  $L < \infty$ , then  $R = \infty$  and  $L = 0$ ;*
- (b) *if  $L = \infty$ , then  $R < \infty$  and  $w(R^-) = \phi(R^-) = \psi(R^-) = \infty$ .*

*Proof.* Taking into account the equations and initial conditions in (2.3), it is easy to see that the function  $\psi_\mu(r) := r^\mu \psi(r)$  is strictly increasing on  $[0, R)$ . As a consequence,  $\psi_\mu$  (and thus,  $\psi$ ) is positive on  $(0, R)$ , and this implies that  $\phi$  is strictly increasing on  $[0, R)$ , with  $L := \lim_{r \rightarrow R^-} \phi(r) \in (p, \infty]$ .

(a) Assume  $L < \infty$ , that is,  $\phi$  is bounded. By (2.4),  $w(r)$  grows at most linearly with  $r$ , and by (2.6),  $\psi(r)$  grows no faster than  $r^3$ . In particular,  $|(w(r), \phi(r), \psi(r))|$  cannot go to infinity in finite time. Thus,  $R = \infty$ .

Now suppose that  $L \neq 0$ . If  $L > 0$ , choose a number  $r_0 > 0$  such that  $\phi(r) \geq L/2$  for every  $r \geq r_0$ . It follows that

$$w(r) \geq \int_0^{r_0} \left( \frac{s}{r} \right)^\mu \phi(s) ds + \frac{L}{2} \int_{r_0}^r \left( \frac{s}{r} \right)^\mu ds$$

for every  $r \geq r_0$ , and we conclude that  $\lim_{r \rightarrow \infty} w(r) = \infty$  (note that the last integral is of order  $r$ ). If  $L < 0$ , we infer in a similar way that  $\lim_{r \rightarrow \infty} w(r) = -\infty$ . In any case, we can choose a number  $r_1 > 0$  such that  $w^2(r) \geq 1$  for every  $r \geq r_1$ . As a consequence,

$$\psi(r) \geq \int_0^{r_1} \left( \frac{s}{r} \right)^\mu w^2(s) ds + \int_{r_1}^r \left( \frac{s}{r} \right)^\mu ds$$



for every  $r \geq r_1$ , and so,  $\lim_{r \rightarrow \infty} \psi(r) = \infty$ . But this implies

$$L = \lim_{r \rightarrow \infty} \phi(r) = p + \lim_{r \rightarrow \infty} \int_0^r \psi(s) ds = \infty,$$

a contradiction. It follows that  $L = 0$ .

(b) Assume  $L = \infty$  and, by way of contradiction, suppose that  $R = \infty$ . Then there exist  $c_0, r_0 > 0$  such that  $\phi(r) \geq c_0$  for every  $r \geq r_0$ , and as in the proof of Part (a), it follows that  $w(r) \rightarrow \infty$  and  $\psi(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . In particular, we can choose  $r_* > 0$  such that  $w(r), \phi(r), \psi(r) > 0$  for every  $r \geq r_*$ . Define  $\eta := w\phi\psi$ . Then we have

$$\begin{aligned} \eta' &= \phi^2\psi + w\psi^2 + w^3\phi - \frac{2\mu}{r}w\phi\psi \\ &= Q(w, \phi, \psi)\eta^{13/12} - \frac{2\mu}{r}\eta \quad \text{in } [r_*, \infty), \end{aligned} \quad (2.7)$$

where  $Q$  is defined by

$$Q(x, y, z) := \frac{y^2z + xz^2 + x^3y}{(xyz)^{13/12}},$$

for  $x, y, z > 0$ . Note that  $Q = Q_1 + Q_2 + Q_3$ , with

$$Q_1 := \frac{y^2z}{(xyz)^{13/12}}, \quad Q_2 := \frac{xz^2}{(xyz)^{13/12}}, \quad Q_3 := \frac{x^3y}{(xyz)^{13/12}}.$$

It is easy to see that  $Q_1^5 Q_2^4 Q_3^3 \equiv 1$ , which implies that  $\max(Q_1, Q_2, Q_3) \geq 1$ . Hence, we have  $Q(x, y, z) \geq 1$  for all  $x, y, z > 0$ , and (2.7) yields

$$\eta' \geq \eta \left( \eta^{1/12} - \frac{2\mu}{r_*} \right) \quad \text{in } [r_*, \infty). \quad (2.8)$$

Recall that  $w(r), \phi(r), \psi(r) \rightarrow \infty$  as  $r \rightarrow \infty$  and choose  $r^* \geq r_*$  such that  $\eta(r^*) > (2\mu/r_*)^{12}$ . Then the maximal solution  $\zeta$  of the initial-value problem

$$\zeta' = \zeta \left( \zeta^{1/12} - \frac{2\mu}{r_*} \right), \quad \zeta(r^*) = \eta(r^*)$$

approaches infinity in finite time. But due to (2.8),  $\zeta$  is bounded from above by  $\eta$  on  $[r^*, \infty)$ . This is a contradiction, and it follows that  $R$  is finite.

In order to prove our last claim, we first note that both  $w$  and  $\psi$  have (proper or improper) limits as  $r \rightarrow R^-$ . Indeed, since  $\phi$  is eventually positive, the function  $w_\mu(r) := r^\mu w(r)$  is eventually increasing and so, has a

limit as  $r \rightarrow R^-$ . As we observed earlier, the same holds for the function  $\psi_\mu := r^\mu \psi(r)$ . Since  $R$  is finite, it follows that  $w(r)$  and  $\psi(r)$ , too, have limits as  $r \rightarrow R^-$ . Moreover, since  $R$  is finite, all three of the functions  $w, \phi, \psi$  would be bounded if one of them were. But  $\phi$  is unbounded (by assumption) and so,  $w$  and  $\psi$  are unbounded as well. Clearly, this implies that  $w(R^-) = \phi(R^-) = \psi(R^-) = \infty$ .  $\square$

**Proposition 2.5** *For every  $\mu \in \mathbb{R}_+$ , the maximal solution of (2.3) with  $p = 1$  blows up in finite time.*

*Proof.* Fix  $\mu \in \mathbb{R}_+$  and let  $(w, \phi, \psi) \in C^1([0, R], \mathbb{R}^3)$  be the maximal solution of (2.3) with  $p = 1$ . According to Lemma 2.4,  $\phi$  is increasing with  $L := \lim_{r \rightarrow R^-} \phi(r) \in \{0, \infty\}$ . Since  $\phi(0) > 0$ , we have  $L = \infty$ , and Part (b) of the lemma implies that  $R$  is finite.  $\square$

*Proof of Theorem 1.3.* Thanks to Remark 2.2, the preceding proposition guarantees that the system (1.3), for arbitrary  $N \in \mathbb{N}$  and  $R > 0$ , has exactly one large radial solution  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) > 0$ . By Lemma 2.4,  $\phi$  is a strictly increasing function of the radial variable  $r$ , and the same then holds for  $v$ . (Note that, by Lemma 2.4,  $\phi(r)$  approaches infinity as  $r \rightarrow R^-$ , and so do  $v'(r)$  and  $\phi'(r)$ . That the same holds for  $v(r)$  is not clear yet, but will be shown in Section 4.)  $\square$

The figures below show computed profiles of the nonnegative large radial solutions  $(v, \phi)$  of the system (1.3), with  $R = 1$ , for two values of the space dimension  $N$  (see Remark 4.4 for comments on the numerical method).

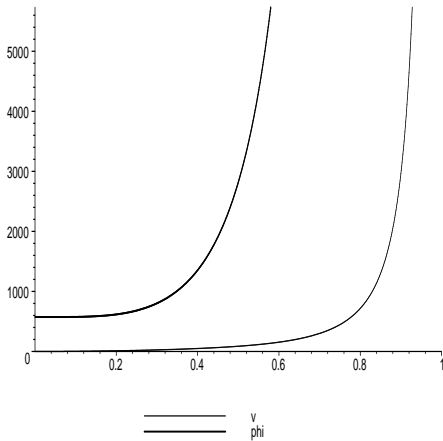


Fig. 1:  $N=1$

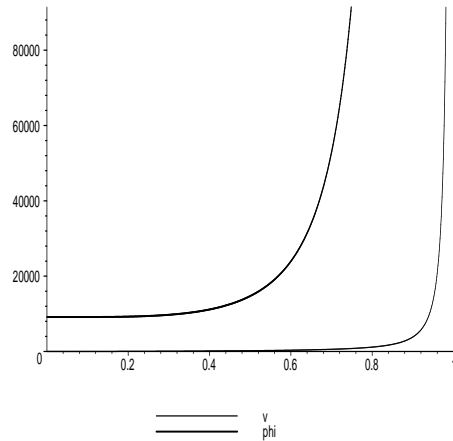


Fig. 2:  $N=6$

### 3 Existence of a Second Large Solution

To prove Theorem 1.4, we need to know for which values of  $\mu$  (if any) the maximal solution of (2.3) with  $p = -1$  blows up in finite time.

**Lemma 3.1** *For  $\mu \in \mathbb{R}_+$ , let  $R_\mu$  denote the exit time of the maximal solution of (2.3) with  $p = -1$ . Define  $M := \{\mu \in \mathbb{R}_+ : R_\mu < \infty\}$  and  $\mu^* := \sup\{\mu \in \mathbb{R}_+ : [0, \mu] \subset M\}$  (with the understanding that  $\sup \emptyset = 0$  and  $\sup A = \infty$  if  $A \subset \mathbb{R}_+$  is unbounded).*

- (a) *The set  $M$  contains the interval  $[0, \mu^*)$ . That is, for every  $\mu \in [0, \mu^*)$ , the maximal solution of (2.3) with  $p = -1$  blows up in finite time.*
- (b) *The set  $M$  is open in  $\mathbb{R}_+$ , and  $\mu^*$  is not an element of  $M$ .*

*Proof.* Part (a) is clear by the definition of  $M$  and  $\mu^*$ . As to Part (b), recall from Lemma 2.4 that  $R_\mu < \infty$  if and only if the  $\phi$ -component of the corresponding maximal solution of (2.3) is eventually positive; since the solution depends continuously on  $\mu$ , this property is stable under small variations of  $\mu$ . Thus, the set  $M$  is open in  $\mathbb{R}_+$ , and it follows that  $\mu^*$  is not an element of  $M$ .  $\square$

**Remark 3.2** It is easy to see that the set  $M$  contains the interval  $[0, 1]$ ; that is, we have  $\mu^* > 1$ . Indeed, suppose that  $R_\mu = \infty$  for some  $\mu \in [0, 1]$  and let  $(w, \phi, \psi) \in C^1([0, \infty), \mathbb{R}^3)$  be the corresponding maximal solution of (2.3) with  $p = -1$ . Lemma 2.4 implies that  $\phi(r) \rightarrow 0$  as  $r \rightarrow \infty$ ; thus,  $\int_0^\infty \psi(s) ds = 1$ , due to (2.5). On the other hand, since  $\psi_\mu(r) := r^\mu \psi(r)$  is strictly increasing for  $r \geq 0$ , we have  $c := \psi_\mu(1) > 0$  and  $\psi(r) = \psi_\mu(r) r^{-\mu} \geq c r^{-\mu}$  for all  $r \geq 1$ ; thus,  $\int_0^\infty \psi(s) ds \geq c \int_1^\infty s^{-\mu} ds = \infty$ . The contradiction shows that  $R_\mu < \infty$  for every  $\mu \in [0, 1]$ .

Note that Remark 3.2 guarantees the existence of a second large radial solution of the system (1.3) at least for space dimensions 1 and 2. In the following, we will establish a-priori estimates to improve the lower bound on the number  $\mu^*$ .

In view of Lemma 2.4, the maximal solution  $(w, \phi, \psi)$  of the Cauchy problem (2.3) with  $p = -1$  blows up in finite time if and only if  $\phi$  crosses zero at some point  $r > 0$ . Due to the scaling property of the system (see Remark 2.1), this happens if and only if there exists a (necessarily negative and unique) initial value  $p$  such that the  $\phi$ -component of the corresponding maximal solution of (2.3) crosses zero at  $r = 1$ . In other words, the maximal

solution of (2.3) with  $p = -1$  blows up in finite time if and only if the boundary-value problem

$$\begin{cases} w' + \frac{\mu}{r} w = \phi, & w(0) = 0, \\ \phi' = \psi, & \phi(1) = 0, \\ \psi' + \frac{\mu}{r} \psi = w^2, & \psi(0) = 0, \end{cases} \quad (3.1)$$

has a (necessarily unique) nontrivial solution.

Clearly, (3.1) can be written as a parameter-dependent fixed-point equation of the form

$$u = T(\mu, u) \quad (3.2)$$

in  $X := C([0, 1], \mathbb{R}^3)$ , where  $T : \mathbb{R}_+ \times X \rightarrow X$  is a completely continuous operator, defined by

$$T(\mu, u)(r) := \left( \int_0^r \left(\frac{s}{r}\right)^\mu \phi(s) ds, -\int_r^1 \psi(s) ds, \int_0^r \left(\frac{s}{r}\right)^\mu w^2(s) ds \right),$$

for  $\mu \in \mathbb{R}_+$ ,  $u = (w, \phi, \psi) \in X$ , and  $r \in [0, 1]$ .

**Proposition 3.3** *For every  $\mu \in [0, \mu^*)$ , with  $\mu^*$  as in Lemma 3.1, Equation (3.2) has a unique nontrivial solution  $u_\mu \in X$ . The function  $\mu \mapsto u_\mu$  is continuous; its graph  $\mathcal{C} := \{(\mu, u_\mu) : \mu \in [0, \mu^*)\}$  is unbounded in  $\mathbb{R}_+ \times X$ , and the number  $\mu^*$  is greater than 9.*

*Proof.* The preceding discussion and Lemma 3.1(a) imply that for every  $\mu \in [0, \mu^*)$ , Equation (3.2) has a unique nontrivial solution  $u_\mu \in X$ . The continuity of the function  $\mu \mapsto u_\mu$  (as a mapping from  $[0, \mu^*)$  into  $X$ ) is easily verified, using the fact that the solution of the Cauchy problem (2.3) depends continuously on  $\mu$ , along with the scaling property. Hence, the graph  $\mathcal{C} := \{(\mu, u_\mu) : \mu \in [0, \mu^*)\}$  is a continuous curve in  $\mathbb{R}_+ \times X$ .

To prove the remaining assertions, we need to establish lower and upper a-priori bounds for the nontrivial solutions  $u_\mu = (w_\mu, \phi_\mu, \psi_\mu)$  of (3.2). Starting with the trivial estimate

$$\phi_\mu(r) \geq \phi_\mu(0) \quad \text{for } 0 \leq r \leq 1,$$

we obtain a lower bound for  $w_\mu(r) = \int_0^r \left(\frac{s}{r}\right)^\mu \phi_\mu(s) ds$ , then an upper bound for  $\psi_\mu(r) = \int_0^r \left(\frac{s}{r}\right)^\mu w_\mu^2(s) ds$ ; indeed,

$$w_\mu(r) \geq \frac{\phi_\mu(0)}{\mu + 1} r \quad \text{and} \quad \psi_\mu(r) \leq \frac{|\phi_\mu(0)|^2}{(\mu + 1)^2(\mu + 3)} r^3 \quad \text{for } 0 \leq r \leq 1.$$

Using the last estimate and the fact that  $|\phi_\mu(0)| = \int_0^1 \psi_\mu(s) ds$ , we infer that

$$|\phi_\mu(0)| \geq 4(\mu + 1)^2(\mu + 3) \geq 12 \quad \text{for all } \mu \geq 0. \quad (3.3)$$

Moreover, the upper bound for  $\psi_\mu(r)$  can be used to derive an upper bound for  $\phi_\mu(r) = \phi_\mu(0) + \int_0^r \psi_\mu(s) ds$ ; indeed,

$$\phi_\mu(r) \leq |\phi_\mu(0)| \left( \left( \frac{r}{r_\mu} \right)^4 - 1 \right) \quad \text{for } 0 \leq r \leq 1,$$

where  $r_\mu := (4(\mu + 1)^2(\mu + 3)/|\phi_\mu(0)|)^{1/4}$ ; due to (3.3), we have  $r_\mu \leq 1$ .

Using this upper bound for  $\phi_\mu$  on the interval  $[0, r_\mu]$  and the trivial, yet better, estimate  $\phi_\mu \leq 0$  on  $(r_\mu, 1]$ , we obtain an upper bound for  $w_\mu$ , then a lower bound for  $\psi_\mu$ . Recalling, once again, that  $|\phi_\mu(0)| = \int_0^1 \psi_\mu(s) ds$ , we arrive at an inequality for  $|\phi_\mu(0)|$  of the form

$$a(\mu) |\phi_\mu(0)|^{(1-\mu)/2} - b(\mu) |\phi_\mu(0)|^{(1-\mu)/4} + c(\mu) \leq 0, \quad (3.4)$$

with coefficients  $a(\mu)$ ,  $b(\mu)$ ,  $c(\mu)$  defined for  $\mu \in \mathbb{R}_+ \setminus \{1\}$  and depending only on  $\mu$ ; moreover,  $a(\mu) > 0$ ,  $b(\mu) > 0$ , and  $b^2(\mu) - 4a(\mu)c(\mu) > 0$ , while  $c(\mu)$  has the same sign as  $3 + 29\mu + \mu^2 - \mu^3$  (we used a computer algebra system to compute the coefficients and verify the relevant properties). Hence, there exists a number  $\bar{\mu} \approx 5.955$  such that  $c(\mu) > 0$  for  $\mu \in [0, \bar{\mu}) \setminus \{1\}$  and  $c(\mu) < 0$  for  $\mu \in (\bar{\mu}, \infty)$ . It follows that the quadratic polynomial  $a(\mu)t^2 - b(\mu)t + c(\mu)$  has a pair of real roots  $t_1(\mu)$ ,  $t_2(\mu)$  with  $t_1(\mu) < t_2(\mu)$ ; while  $t_2(\mu)$  is always positive,  $t_1(\mu)$  is positive if and only if  $\mu < \bar{\mu}$ . Thus, the inequality (3.4) implies

$$|\phi_\mu(0)| \leq f(\mu) := \begin{cases} t_2(\mu)^{4/(1-\mu)} & \text{if } 0 \leq \mu < 1, \\ t_1(\mu)^{4/(1-\mu)} & \text{if } 1 < \mu < \bar{\mu}. \end{cases}$$

The function  $f$  can be extended to a continuous function  $\bar{f}$  on  $[0, \bar{\mu})$ , and by continuity,

$$|\phi_\mu(0)| \leq \bar{f}(\mu) \quad \text{for all } \mu \in [0, \bar{\mu}).$$

Note that  $\bar{f}(\mu) \rightarrow \infty$  as  $\mu \rightarrow \bar{\mu}$ . Of course, the inequality (3.4) yields also a lower bound for  $|\phi_\mu(0)|$ , which significantly improves (3.3), but this will not be needed. More importantly, with a refinement of the above argument and plenty of computer algebra, we are able to explicitly construct a continuous function  $\hat{f}$ , defined on an interval  $[0, \hat{\mu})$  with  $\hat{\mu} \approx 9.073$ , such that

$$|\phi_\mu(0)| \leq \hat{f}(\mu) \quad \text{for all } \mu \in [0, \hat{\mu}). \quad (3.5)$$

We omit the very technical details of this construction.

Now observe that

$$\|\phi_\mu\|_\infty = |\phi_\mu(0)|, \quad \|w_\mu\|_\infty \leq \frac{|\phi_\mu(0)|}{\mu + 1} \leq |\phi_\mu(0)|,$$

and

$$\|\psi_\mu\|_\infty \leq \frac{|\phi_\mu(0)|^2}{(\mu + 1)^2(\mu + 3)} \leq \frac{1}{3} |\phi_\mu(0)|^2,$$

from which we infer that  $\|(w_\mu, \phi_\mu, \psi_\mu)\|_\infty \leq |\phi_\mu(0)|^2$  (note that  $|\phi_\mu(0)| \geq 12$ , due to (3.3)). Consequently, we have

$$|\phi_\mu(0)| \leq \|u_\mu\|_\infty \leq |\phi_\mu(0)|^2,$$

and recalling (3.3) and (3.5), we conclude that

$$12 \leq 4(\mu + 1)^2(\mu + 3) \leq \|u_\mu\|_\infty \leq \hat{f}^2(\mu) \quad \text{for all } \mu \in [0, \hat{\mu}]. \quad (3.6)$$

Next we will compute the (Leray-Schauder) fixed-point index of  $T(0, \cdot)$  in  $u_0$ , the nontrivial solution of (3.2) for  $\mu = 0$ . (Note that the existence of  $u_0$  is already clear by Remark 3.2; but the following argument will prove it once again.) Inspired by similar reasoning in [1], we define a completely continuous operator  $S : \mathbb{R}_+ \times X \rightarrow X$  by

$$S(\lambda, u)(r) := \left( \int_0^r \phi(s) ds, -\int_r^1 \psi(s) ds, \int_0^r (w^2(s) + \lambda) ds \right),$$

for  $\lambda \in \mathbb{R}_+$ ,  $u = (w, \phi, \psi) \in X$ , and  $r \in [0, 1]$ , and consider the parameter-dependent fixed-point problem in  $X$ ,

$$u = S(\lambda, u). \quad (3.7)$$

Note that  $S(0, \cdot) = T(0, \cdot)$ ; that is, if  $\lambda = 0$ , then Equation (3.7) coincides with Equation (3.2) with  $\mu = 0$ .

Now let  $\lambda \in \mathbb{R}_+$  and suppose that  $u_\lambda = (w_\lambda, \phi_\lambda, \psi_\lambda) \in X$  is a solution of (3.7). Since  $\phi_\lambda'' = w_\lambda^2 + \lambda \geq 0$ , the function  $\phi_\lambda$  is convex; thus,  $\phi_\lambda(r) \leq \phi_\lambda(0)(1 - r)$  for all  $r \in [0, 1]$ . Arguing as in our earlier estimates, we derive an upper bound for  $w_\lambda$ , then a lower bound for  $\psi_\lambda$ . Again, we have  $|\phi_\lambda(0)| = \int_0^1 \psi_\lambda(s) ds$ ; this and the lower bound for  $\psi_\lambda$  yield a quadratic inequality for  $|\phi_\lambda(0)|$ , namely,  $|\phi_\lambda(0)|^2 - 24|\phi_\lambda(0)| + 12\lambda \leq 0$ . It follows that  $\lambda \leq 12$  and  $|\phi_\lambda(0)| \leq 24$ . This shows that (3.7) does not have any solutions if  $\lambda > 12$ ; moreover, the uniform bound on  $|\phi_\lambda(0)|$  implies a uniform bound on  $\|u_\lambda\|_\infty$ .

Choosing a sufficiently large constant  $\rho > 0$ , we infer that the number  $\deg(Id_X - S(\lambda, \cdot), B_\rho^X(0), 0)$  is well defined for  $\lambda \in \mathbb{R}_+$ , independent of  $\lambda$

(due to homotopy invariance), and in fact equal to zero (since (3.7) has no solutions for  $\lambda > 12$ ). It follows that

$$\deg(Id_X - T(0, \cdot), B_\rho^X(0), 0) = \deg(Id_X - S(0, \cdot), B_\rho^X(0), 0) = 0.$$

However, a routine homotopy argument shows that the index of  $T(0, \cdot)$  in the trivial fixed point 0 equals 1. This proves, once again, the existence of the nontrivial fixed point  $u_0$  and shows, more importantly, that the index of  $T(0, \cdot)$  in  $u_0$  is  $-1$ .

We are now in a position to complete the proof of the proposition by applying a Rabinowitz-type argument (see [16]). Suppose, by way of contradiction, that the solution branch  $\mathcal{C} := \{(\mu, u_\mu) : \mu \in [0, \mu^*)\}$  is bounded. Then  $\mu^* < \infty$ , and there exists a constant  $\rho > 0$  such that  $\|u_\mu\|_\infty < \rho$  for all  $\mu \in [0, \mu^*)$ . Also, due to (3.3),  $\|u_\mu\|_\infty \geq 12$  for all  $\mu \in [0, \mu^*)$ , and according to Lemma 3.1(b), the equation (3.2) has no nontrivial solution for  $\mu = \mu^*$ . It follows that  $\deg(Id_X - T(\mu, \cdot), B_\rho^X(0) \setminus \bar{B}_1^X(0), 0)$  is well defined for  $\mu \in [0, \mu^*]$ , independent of  $\mu$ , and in fact equal to zero. Clearly, this contradicts the fact that  $T(0, \cdot)$  has index  $-1$  in  $u_0$ . It follows that  $\mathcal{C}$  is unbounded in  $\mathbb{R}_+ \times X$ . Due to (3.6), the projection of  $\mathcal{C}$  onto  $\mathbb{R}_+ \times \{0\}$  contains the interval  $[0, \hat{\mu})$ , which implies  $\mu^* \geq \hat{\mu}$  and thus,  $\mu^* > 9$ .  $\square$

*Proof of Theorem 1.4.* In view of Remark 2.2, Lemma 3.1(a) implies that the elliptic system (1.3), for arbitrary  $R > 0$ , has exactly one large radial solution  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) < 0$ , provided that  $N - 1 < \mu^*$ ; since  $\mu^* > 9$ , this is the case if  $N \leq 10$ .

Lemma 2.4 shows that  $\phi$  is a strictly increasing function of the radial variable  $r$  and crosses zero at a point  $r_0 \in (0, R)$ . Hence, the function  $w_\mu(r) := r^\mu w(r)$ , with  $w = v'$ , is strictly decreasing for  $r < r_0$  and strictly increasing for  $r > r_0$ ; it crosses zero at a point  $r_1 \in (r_0, R)$ . Thus,  $v'$  is negative on  $(0, r_1)$ , positive on  $(r_1, R)$ , and consequently,  $v$  is strictly decreasing to a negative minimum at  $r_1$ , strictly increasing thereafter.  $\square$

**Remark 3.4** Numerical evidence suggests that  $\mu^* \approx 13.755$  and that the set  $M$  of Lemma 3.1 coincides with the interval  $[0, \mu^*)$ . In other words, the maximal solution of the Cauchy problem (2.3) with  $p = -1$  blows up in finite time if and only if  $\mu < \mu^* \approx 13.755$ . As a consequence, we conjecture that the large solution of Theorem 1.4 exists if and only if  $N \leq 14$ .

Figures 3–6 depict computed profiles of the large radial solutions  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) < 0$  of the system (1.3), with  $R = 1$ , for several values of the space dimension  $N$  (see Remark 4.4 for comments on the numerical method). Specifically, Figure 5 shows the solution for  $N = 10$ , the largest space dimension for which we *proved* its existence (Theorem 1.4); Figure 6 shows the solution for  $N = 14$ , the largest space dimension for which we found it numerically. Of course, we can compute the large solution  $(v, \phi)$  with  $v(0) = 0$  and  $\phi(0) < 0$  of the system (2.1) (the radial version of (1.3)) for every  $\mu \in [0, \mu^*)$ , where  $\mu^* \approx 13.755$ . It is noteworthy that as  $\mu \rightarrow \mu^*$ , the  $\phi$ -component of the solution appears to approach  $c_R \delta_R - c_0 \delta_0$ , for some positive constants  $c_0, c_R$ , where  $\delta_0, \delta_R$  denote the Dirac distributions centered at 0 and  $R$ , respectively.

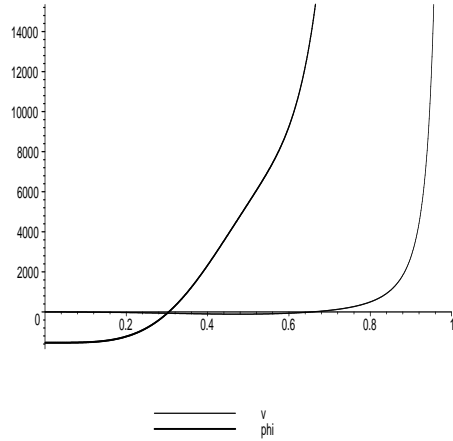


Fig. 3:  $N=1$

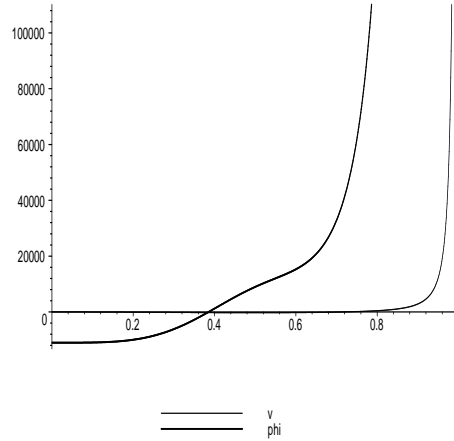


Fig. 4:  $N=3$

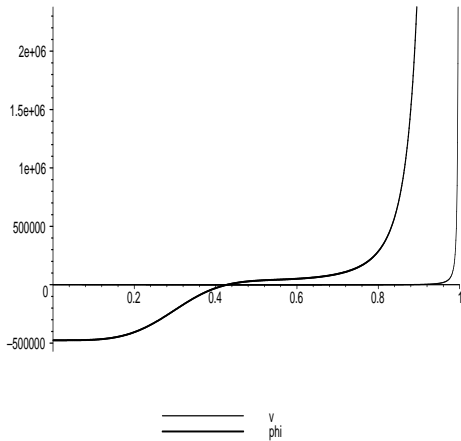


Fig. 5:  $N=10$

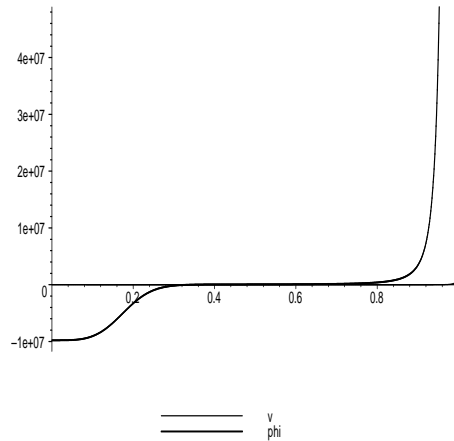


Fig. 6:  $N=14$



We conclude this section with a comment on the Dirichlet problem for the elliptic system (1.2) on a ball in  $\mathbb{R}^N$ ,

$$\begin{cases} -\Delta v = \theta & \text{in } B_R^N(0), \\ -\Delta \theta = |\nabla v|^2 & \text{in } B_R^N(0), \\ v = \theta = 0 & \text{on } \partial B_R^N(0). \end{cases} \quad (3.8)$$

Assuming that  $R = 1$  (due to the scaling property, this entails no loss of generality), there is a one-to-one correspondence between the radial solutions of (3.8) and the solutions of the boundary-value problem (3.1) or the equivalent fixed-point equation (3.2), with  $\mu = N - 1$ ,  $\theta = -\phi$ , and  $v(r) = -\int_r^1 w(s) ds$  for  $r \in [0, 1]$ . Hence, Proposition 3.3 implies that (3.8) has a unique nontrivial radial solution  $(v, \theta)$  at least for  $N \leq 10$ ; it is easily checked that both components of this solution are strictly decreasing functions of the radial variable  $r$ .

**Corollary 3.5** *For every  $N \in \mathbb{N}$  with  $N \leq 10$  and every  $R > 0$ , the Dirichlet problem (3.8) has a unique nontrivial radially symmetric solution  $(v, \theta)$ . Both components of this solution are decreasing functions of the radial variable  $r$ .*

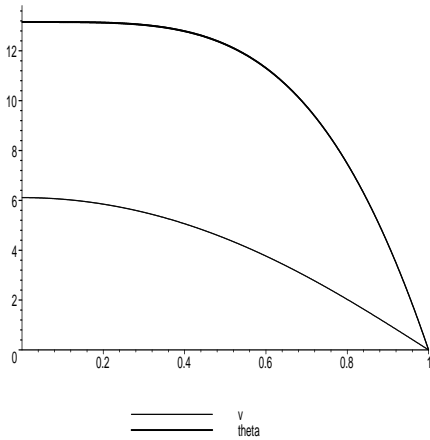


Fig. 7:  $N=1$

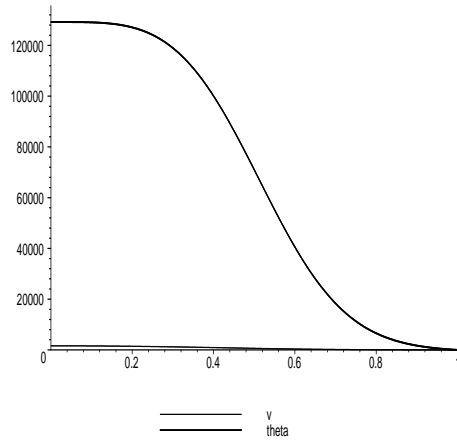


Fig. 8:  $N=14$

The numerical evidence cited in Remark 3.4 suggests that the (necessarily unique) nontrivial radial solution of (3.8) exists, in fact, if and only if  $N \leq 14$ . Figures 7 and 8 show computed profiles of this solution for  $R = 1$  and the two extreme values of the space dimension,  $N = 1$  and  $N = 14$ .

Of course, we can compute the nontrivial solution  $(v, \theta)$  of the radial version of (3.8) for any  $\mu \in [0, \mu^*)$  in place of the integer  $N - 1$ . As  $\mu \rightarrow \mu^* \approx 13.755$ , the  $\theta$ -component of the solution appears to approach a multiple of  $\delta_0$  (the Dirac distribution centered at 0).

## 4 Asymptotic Behavior

Let  $(w, \phi, \psi) \in C^1([0, R], \mathbb{R}^3)$  be the maximal solution of the Cauchy problem (2.3) for a given  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$ . By Lemma 2.4, we know that  $R$  is finite if and only if  $\phi(r)$  is eventually positive and that in this case,  $w(r), \phi(r), \psi(r) \rightarrow \infty$  as  $r \rightarrow R^-$ . In view of the existing literature on boundary blow-up in elliptic equations (see, for example, [3, 18]), it is natural to expect asymptotic behavior of the form  $Q/(R - r)^q$ , with positive constants  $Q$  and  $q$ . In fact, we will prove the following result.

**Proposition 4.1** *Let  $(w, \phi, \psi) \in C^1([0, R], \mathbb{R}^3)$  denote the maximal solution of (2.3), for a given  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$ , and suppose that  $R$  is finite. Then, as  $r \rightarrow R^-$ ,*

$$w(r) \sim \frac{60}{(R - r)^3}, \quad \phi(r) \sim \frac{180}{(R - r)^4}, \quad \psi(r) \sim \frac{720}{(R - r)^5}.$$

Let us note that if all three of the functions  $w, \phi, \psi$  exhibit asymptotic behavior of the form  $Q/(R - r)^q$ , it is easy to see that the constants  $Q$  and  $q$  are necessarily as above.

We will prove Proposition 4.1 under the assumption that  $R = 1$ ; thanks to the scaling property, this entails no loss of generality. The proof will be achieved by analyzing a system of equations derived from (2.3) by a suitable change of variables.

Given any solution  $(w, \phi, \psi)$  of (2.3), define functions  $\alpha, \beta, \gamma$  by

$$\alpha(r) := \frac{(1 - r)^3}{60} w(r), \quad \beta(r) := \frac{(1 - r)^4}{180} \phi(r), \quad \gamma(r) := \frac{(1 - r)^5}{720} \psi(r);$$

then  $(\alpha, \beta, \gamma)$  is a solution of

$$\begin{cases} (1 - r) \left( \alpha' + \frac{\mu}{r} \alpha \right) = 3(\beta - \alpha), & \alpha(0) = 0, \\ (1 - r) \beta' = 4(\gamma - \beta), & \beta(0) = p/180, \\ (1 - r) \left( \gamma' + \frac{\mu}{r} \gamma \right) = 5(\alpha^2 - \gamma), & \gamma(0) = 0. \end{cases} \quad (4.1)$$

Just like (2.3), the system (4.1) is singular at  $r = 0$ , but this does not affect the well-posedness of the initial value problem; in addition, (4.1) is singular at  $r = 1$ .

**Remark 4.2** Suppose that  $(w, \phi, \psi)$  is the maximal solution of (2.3), for a given  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$ , with interval of existence  $[0, R_p)$ ; let  $(\alpha, \beta, \gamma)$  be the corresponding solution of (4.1), as defined above. Clearly, if  $R_p < 1$ , then  $(\alpha, \beta, \gamma)$  ceases to exist before reaching the singularity at  $r = 1$ ; in fact,  $\alpha(r), \beta(r), \gamma(r) \rightarrow \infty$  as  $r \rightarrow R_p^-$ . Also, if  $R_p > 1$ , then  $(\alpha, \beta, \gamma)$  can be continued beyond the singularity at  $r = 1$ , and  $\alpha(r), \beta(r), \gamma(r) \rightarrow 0$  as  $r \rightarrow 1^-$ . Finally, if  $R_p = 1$ , then  $(\alpha, \beta, \gamma)$  exists up to the singularity at  $r = 1$ , but the behavior near the singularity is not obvious. The assertion of Proposition 4.1 (with  $R = 1$ ) is that, in this case,  $\alpha(r), \beta(r), \gamma(r) \rightarrow 1$  as  $r \rightarrow 1^-$ .

**Remark 4.3** Recall that for every  $\mu \in \mathbb{R}_+$ , there is exactly one initial value  $p_\mu^+ > 0$  and at most one initial value  $p_\mu^- < 0$  such that the maximal solution of (2.3) with  $p = p_\mu^\pm$  blows up at  $r = 1$ . Let  $p_\mu^*$  denote one such value. Due to the scaling property of the system (2.3), all solutions with  $\text{sign}(p) = \text{sign}(p_\mu^*)$  blow up in finite time. Moreover, there is a one-to-one correspondence between the initial value  $p$  and the exit time  $R_p$  of the solution; in fact,  $R_p^4 = p_\mu^*/p$  (see Remark 2.1). It follows that if  $p > p_\mu^* > 0$  or  $p < p_\mu^* < 0$ , then  $R_p < 1$ , and consequently,  $\alpha(r), \beta(r), \gamma(r) \rightarrow \infty$  as  $r \rightarrow R_p^-$ . Also, if  $0 < p < p_\mu^*$  or  $p_\mu^* < p < 0$ , then  $R_p > 1$ , and consequently,  $\alpha(r), \beta(r), \gamma(r) \rightarrow 0$  as  $r \rightarrow 1^-$ .

**Remark 4.4** The observations in the preceding remark allow us to use a shooting method to numerically approximate  $p_\mu^+$  and, if it exists,  $p_\mu^-$ , that is, the critical initial values  $p$  for which the maximal solution of (2.3) blows up at  $r = 1$ . Solving the Cauchy problem (4.1) with  $p = p_\mu^\pm$ , we can then construct the solutions of (2.3) that blow up at  $r = 1$ , and thereby, the large radial solutions of the elliptic system (1.3) on the unit ball. All the graphs in the preceding sections were generated in this way (with a suitable rescaling in the case of Figures 7 and 8).

Our experiments suggest that  $p_\mu^-$  exists if and only if  $\mu$  is smaller than a certain number, approximately equal to 13.755, which, due to the scaling property, must coincide with the number  $\mu^*$  in Lemma 3.1. In fact, we find that  $p_\mu^-$  is a strictly decreasing function of  $\mu$  that approaches  $-\infty$  as  $\mu$  approaches the above number.

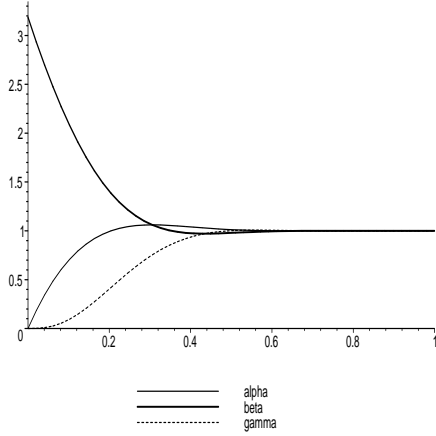


Fig. 9:  $\mu=0$ ,  $p/180=3.185739467$

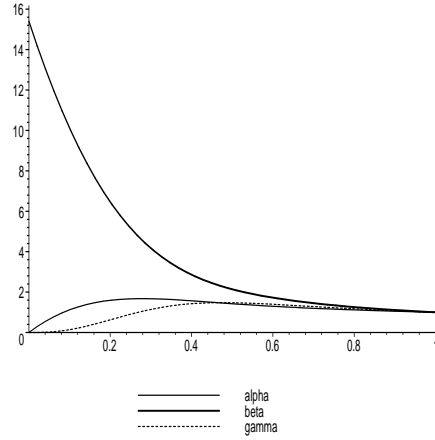


Fig. 10:  $\mu=2$ ,  $p/180=15.39874038$

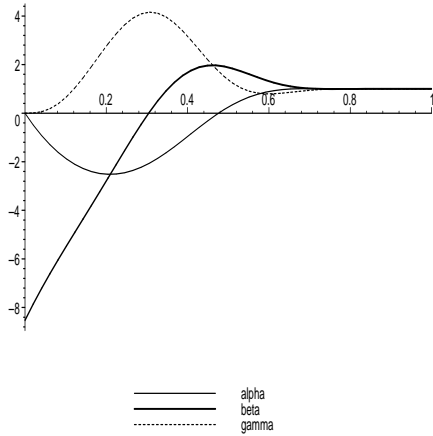


Fig. 11:  $\mu=0$ ,  $p/180=-8.525812707$

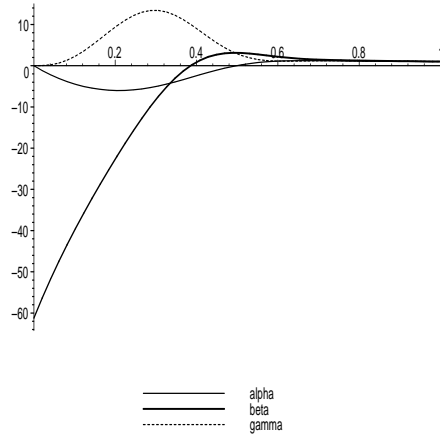


Fig. 12:  $\mu=2$ ,  $p/180=-61.20559852$

In order to prove Proposition 4.1 for  $R = 1$ , we must show that all three components of the maximal solution  $(\alpha, \beta, \gamma)$  of the Cauchy problem (4.1) with  $\mu \in \mathbb{R}_+$  and  $p = p_\mu^\pm$  converge to 1 as  $r \rightarrow 1^-$  (recall Remarks 4.2 and 4.3). While our numerical experiments leave no doubt about this (see Figures 9–12 for examples of computed solutions), the proof requires a small detour in dynamical systems; we refer to [17] for terminology and basic properties.

It is very convenient to perform a change of variables in the system (4.1), letting  $r = 1 - e^{-t}$  and  $a(t) = \alpha(r)$ ,  $b(t) = \beta(r)$ ,  $c(t) = \gamma(r)$ . With this rescaling of the independent variable, (4.1) is equivalent to

$$\begin{cases} a' + \frac{\mu}{e^t - 1} a = 3(b - a), & a(0) = 0, \\ b' = 4(c - b), & b(0) = p/180, \\ c' + \frac{\mu}{e^t - 1} c = 5(a^2 - c), & c(0) = 0. \end{cases} \quad (4.2)$$

Note that the singularity of (4.1) at  $r = 1$  has been moved to  $t = \infty$ ; moreover, the system (4.2) is *autonomous* for  $\mu = 0$  and *asymptotically autonomous* for  $\mu > 0$ . For notational convenience, we write the system of differential equations in (4.2) as

$$x' + \frac{\mu}{e^t - 1} E(x) = F(x), \quad (4.3)$$

where  $x = (a, b, c)$ ,  $E : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the linear mapping defined by  $E(a, b, c) := (a, 0, c)$ , and  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is the vector field defined by

$$F(a, b, c) := (3(b - a), 4(c - b), 5(a^2 - c)).$$

**Remark 4.5** For  $t > 0$  and  $a \geq 0$ , the system (4.3), with arbitrary  $\mu \in \mathbb{R}_+$ , satisfies the Kamke condition and thus, a comparison principle (see, for example, [19]).

In fact, let  $t_0, t_1 \in [0, \infty]$  with  $t_0 < t_1$  and suppose that  $x_1, x_2 \in C([t_0, t_1], \mathbb{R}^3) \cap C^1((t_0, t_1), \mathbb{R}^3)$ . If  $x_1$  is a subsolution of (4.3) with  $a_1 \geq 0$ ,  $x_2$  is a supersolution of (4.3), and  $x_1(t_0) \leq x_2(t_0)$  ( $x_1(t_0) < x_2(t_0)$ ), then  $x_1(t) \leq x_2(t)$  ( $x_1(t) < x_2(t)$ ) for all  $t \in [t_0, t_1)$ .

By a subsolution (supersolution) of (4.3) we mean, of course, a function  $x = (a, b, c)$  satisfying the differential inequality obtained from (4.3) by replacing “=” with “ $\leq$ ” (“ $\geq$ ”). Also, given vectors  $x_1, x_2 \in \mathbb{R}^3$ , we write  $x_1 \leq x_2$  or  $x_2 \geq x_1$  ( $x_1 < x_2$  or  $x_2 > x_1$ ), if the respective inequality holds componentwise, and we call a vector  $x \in \mathbb{R}^3$  nonnegative (positive) if  $x \geq \bar{0}$  ( $x > \bar{0}$ ), where  $\bar{0} := (0, 0, 0)$ .

**Remark 4.6** For every  $\lambda \in [0, \infty]$ , let  $\bar{\lambda}$  denote the vector  $(\lambda, \lambda, \lambda)$ . For arbitrary  $\mu \in \mathbb{R}_+$ ,  $\bar{0}$  is a solution of (4.3), and  $\bar{1}$  is a supersolution (a solution if  $\mu = 0$ ). More generally, for every  $\lambda \in [0, 1]$ ,  $\bar{\lambda}$  is a supersolution. Furthermore, for every  $\lambda \in (1, \infty)$ , there exists a number  $\tau \in [0, \infty)$ , depending only on  $\lambda$  and  $\mu$ , such that  $(\frac{\lambda+1}{2}, \lambda, \lambda)$  is a subsolution on the interval  $(\tau, \infty)$  (where  $\tau = 0$  if  $\mu = 0$ ).

**Proposition 4.7** *For a given  $\mu \in \mathbb{R}_+$ , let  $x$  be a nonnegative maximal forward solution of (4.3).*

- (a) *If  $x$  is unbounded, then  $x$  blows up in finite time and approaches  $\overline{\infty}$ .*
- (b) *If  $x$  is bounded, then  $x$  converges to either  $\bar{0}$  or  $\bar{1}$ .*

*Proof.* Fix  $\mu \in \mathbb{R}_+$  and let  $x = (a, b, c)$  be a nonnegative maximal forward solution of (4.3).

(a) Suppose that  $x = (a, b, c)$  is unbounded. First we will show that  $b$  is unbounded. By way of contradiction, suppose that  $b \leq b_0$  for some positive constant  $b_0$ . Then we have

$$a' \leq a' + \frac{\mu}{e^t - 1} a = 3(b - a) \leq 3(b_0 - a),$$

which implies that  $a$  is bounded. A similar argument then shows that  $c$  is bounded as well, and this contradicts the unboundedness of  $x = (a, b, c)$ . Thus,  $b$  is unbounded.

Now fix a number  $p_0 > 0$  such that the solution  $(w_0, \phi_0, \psi_0)$  of the initial-value problem (2.3) with  $p = p_0$  blows up at a point  $R_0 < 1$ . The corresponding solution  $x_0 = (a_0, b_0, c_0)$  of the initial-value problem (4.2) then blows up at  $-\ln(1 - R_0)$ . Recall that the  $b$ -component of the trajectory  $x = (a, b, c)$  is unbounded and choose a point  $\tau$  in the interval of existence of  $x$  such that  $b(\tau) \geq p_0/180$ ; define  $\tilde{x} := x(\tau + \cdot)$ . Then we have

$$\tilde{x}' + \frac{\mu}{e^t - 1} E(\tilde{x}) \geq \tilde{x}' + \frac{\mu}{e^{\tau+t} - 1} E(\tilde{x}) = F(\tilde{x})$$

and

$$\tilde{x}(0) = x(\tau) = (a(\tau), b(\tau), c(\tau)) \geq (0, p_0/180, 0) = x_0(0).$$

Thanks to the comparison principle in Remark 4.5, it follows that  $\tilde{x} \geq x_0$ . In particular,  $\tilde{x}$  blows up in finite time, and then, so does  $x$ . Reasoning as in the proof of Lemma 2.4(b), it is now easy to verify that all three components of  $x$  approach infinity.

(b) Suppose that  $x$  is bounded. First, consider the autonomous case,  $\mu = 0$ . The vector field  $F$  is cooperative in the half-space  $a \geq 0$  and, in particular, in the nonnegative cone  $\mathbb{R}_+^3$ ; moreover,  $\operatorname{div}(F) \equiv -12$ , and  $F$  has exactly two zeros, at  $\bar{0}$  and  $\bar{1}$ . Thus,  $F$  generates a monotone, volume-contracting semiflow  $\Phi$  in  $\mathbb{R}_+^3$ , with exactly two equilibria, at  $\bar{0}$  and  $\bar{1}$ . The equilibrium at  $\bar{0}$  is a stable node, the one at  $\bar{1}$  is unstable, with a two-dimensional stable manifold (the eigenvalues are 1 and  $-13/2 \pm i\sqrt{71}/2$ ).

Moreover, the system can be embedded into a cooperative system in all of  $\mathbb{R}^3$  by replacing the nonlinear term  $a^2$  in the third component of the

vector field  $F$  with  $a|a|$ ; the extended system still has negative divergence and hyperbolic equilibria, at  $\bar{0}$  and  $\pm\bar{1}$ . Morris Hirsch proved (see [9], Theorem 1) that every compact  $\alpha$ - or  $\omega$ -limit set of a cooperative or competitive system in  $\mathbb{R}^3$  is either a cycle or contains an equilibrium. Another result of Hirsch (see [8], Theorem 7) guarantees that a cooperative system in  $\mathbb{R}^3$  with negative divergence cannot have any cycles. Moreover, if a limit set contains a *hyperbolic* equilibrium, then the limit set is a singleton. Combining these results, we infer that in a cooperative system in  $\mathbb{R}^3$  with negative divergence and hyperbolic equilibria, every bounded (forward or backward) trajectory converges. Since by assumption, the trajectory  $x$  is nonnegative, it follows that  $x$  converges to either  $\bar{0}$  or  $\bar{1}$ .

Now consider the nonautonomous case,  $\mu > 0$ . As we observed before, the system (4.3) is asymptotically autonomous. An old result of Markus [13] implies that the  $\omega$ -limit set  $K$  of the trajectory  $x$  is a nonempty compact and connected subset of  $\mathbb{R}_+^3$ ; moreover,  $\text{dist}(x(t), K) \rightarrow 0$  as  $t \rightarrow \infty$ , and  $K$  is invariant under the semiflow  $\Phi$  of the autonomous limit system, that is, (4.3) with  $\mu = 0$ . A more recent result by Mischaikow, Smith and Thieme (see [14], Theorem 1.8) implies that  $K$  is also *chain-recurrent* under  $\Phi$ . Roughly speaking, this means that the  $\Phi$ -trajectory starting at any given point  $z \in K$  will return to the vicinity of  $z$ , time and again, as  $t \rightarrow \infty$ .

We claim that  $K \subset \{\bar{0}, \bar{1}\}$ . By way of contradiction, suppose there is a point  $z \in K \setminus \{\bar{0}, \bar{1}\}$ . In light of what we proved for the autonomous case, since  $K$  is compact and  $\Phi$ -invariant, the trajectory  $\Phi(\cdot, z)$  must converge, both forward and backward in time. Backward in time,  $\Phi(\cdot, z)$  can only converge to  $\bar{1}$  (since  $\bar{0}$  is stable). Thus,  $z$  belongs to the unstable manifold of  $\bar{1}$ , and it follows that, forward in time,  $\Phi(\cdot, z)$  can only converge to  $\bar{0}$  (see the remarks on the dynamics of  $\Phi$  at the end of this section). Hence,  $K$  consists of the two equilibria,  $\bar{0}$  and  $\bar{1}$ , and a heteroclinic orbit connecting the two; such a set is obviously not chain-recurrent. The contradiction proves that  $K \subset \{\bar{0}, \bar{1}\}$ . In fact, since  $K$  is nonempty and connected, we have either  $K = \{\bar{0}\}$  or  $K = \{\bar{1}\}$ ; that is,  $x$  converges to either  $\bar{0}$  or  $\bar{1}$ .  $\square$

**Corollary 4.8** *Let  $\mu \in \mathbb{R}_+$  and  $p \in \mathbb{R}$  be such that the maximal solution  $(w, \phi, \psi)$  of the Cauchy problem (2.3) blows up at  $r = 1$ . Then the maximal solution  $(a, b, c)$  of (4.2) converges to  $\bar{1}$ .*

*Proof.* Under the assumptions of the corollary,  $(w, \phi, \psi)$  approaches  $\infty$  at  $r = 1$ ; thus, the corresponding solution  $(\alpha, \beta, \gamma)$  of (4.1) is eventually positive and exists on  $[0, 1)$  (see Remark 4.2). This means that  $x = (a, b, c)$  is eventually positive and exists on  $[0, \infty)$ . By Part (a) of Proposition 4.7,

it follows that  $x$  is bounded, and then, Part (b) of the same proposition implies that  $x$  converges to either  $\bar{0}$  or  $\bar{1}$ . Now, suppose that  $x(t) \rightarrow \bar{0}$  as  $t \rightarrow \infty$ , and choose  $t_0 \in (0, \infty)$  such that  $\bar{0} < x(t_0) < \bar{1}$ . Since the solution of (4.2) depends continuously on  $p$ , we can find a value  $\tilde{p}$ , close to  $p$ , with  $|\tilde{p}| > |p|$  such that the corresponding maximal solution  $\tilde{x}$  of (4.2) exists at  $t = t_0$  and satisfies  $\bar{0} < \tilde{x}(t_0) < \bar{1}$ . Since  $\bar{0}$  is a solution and  $\bar{1}$  is a supersolution of (4.3), the comparison principle in Remark 4.5 implies that  $\bar{0} < \tilde{x}(t) < \bar{1}$  for all  $t \geq t_0$ , as long as  $\tilde{x}$  exists. From Remark 4.3, however, we know that  $\tilde{x}$  goes to  $\infty$  (in finite time). This contradiction proves that  $x$  does not converge to  $\bar{0}$ , and so, it must converge to  $\bar{1}$ .  $\square$

*Proof of Proposition 4.1.* Due to the scaling property of the system (2.3) (see Remark 2.1), it suffices to prove the assertion of the proposition for  $R = 1$ ; in this case, it is an immediate consequence of Corollary 4.8 (see Remark 4.2).  $\square$

*Proof of Theorem 1.5.* Since every large radial solution  $(v, \phi)$  of the system (1.3) on  $B_R^N(0)$  corresponds to a solution  $(w, \phi, \psi)$  of (2.3) that blows up at  $R$ , the asymptotic behavior of  $\phi$  is clear from Proposition 4.1. Moreover, the asymptotic behavior of  $v$ , given by  $v(r) = \int_0^r w(s) ds$  for  $0 \leq r < R$ , follows readily from that of  $w$ .  $\square$

In closing, we note that Hirsch's results on cooperative systems in  $\mathbb{R}^3$  (see [8, 9]) allow us to completely describe the dynamics of the monotone, volume-contracting semiflow  $\Phi$  in  $\mathbb{R}_+^3$ , induced by the vector field  $F$ . First, it can be easily verified that  $\Phi$  is, in fact, *strongly* monotone (even though  $F$  is irreducible only in the open half-space  $a > 0$ ). As shown in the first part of the proof of Proposition 4.7(b), Hirsch's results imply that every forward trajectory of  $\Phi$  either converges to  $\bar{0}$  (a stable node) or to  $\bar{1}$  (a saddle point), or it approaches  $\infty$ , necessarily in finite time. Clearly, both  $\bar{0}$  and  $\infty$  are stable attractors. In fact, using the sub and supersolutions constructed in Remark 4.6, we see that the open order intervals  $(\bar{0}, \bar{1})$  and  $(\bar{1}, \infty)$  are positively invariant and contained in the basins of attraction of  $\bar{0}$  and  $\infty$ , respectively. The two basins of attraction are separated by the (two-dimensional) stable manifold  $W_s(\bar{1})$  of the saddle point. The (one-dimensional) unstable manifold  $W_u(\bar{1})$  has a *positive* tangent vector at  $\bar{1}$ , which implies that  $W_u(\bar{1}) \setminus \{\bar{1}\}$  is contained in the union of the order intervals  $(\bar{0}, \bar{1})$  and  $(\bar{1}, \infty)$ . Thus, every forward trajectory on  $W_u(\bar{1}) \setminus \{\bar{1}\}$  either converges to  $\bar{0}$  or approaches  $\infty$ . It follows that  $W_u(\bar{1}) \setminus \{\bar{1}\}$  consists of two heteroclinic orbits, connecting  $\bar{1}$  to  $\bar{0}$  and  $\infty$ , respectively.



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