A problem on slender, nearly cylindrical shells suggested by Torroja's structures

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Abstract

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1 Introduction.

This paper is devoted to some generalizations and improvements of a previous paper by the authors ([12]) on Torroja's structures. Such kind of structures by the outstanding engineer Eduardo Torroja (Madrid, 1899-1961) are very peculiar kind of curved slender nearly cylindrical elastic shells enjoying rigidity properties inherited from the geometry which furnish remarkable properties of strength. This were used in various realizations of the real world such as the shell roofs of the Madrid Racecourse (1935). Another example of this type of structures is, for instance, the "pedestrian access shell in the southwestern side of the UNESCO building (Paris, 1953-58) due to Marcel Breuer and Bernard Zehrfuss with the collaboration of Antonio and Pier Luigi Nervi.

Torroja's structures are intermediate between shells and beams: the mathematical asymptotic structure is hybrid of shells and beams.

Let us consider the slender shelldepicted in Figure 1 (details will be given later), where ε denotes the small parameter with the thickness of the shell, whereas its lenght is l_1 (independent of ε) and its wideness is ηl_2, where $\eta = \eta(\varepsilon)$ is an asymptotic gauge function with

$$\lim_{\varepsilon \searrow 0} \eta(\varepsilon) = 0, \tag{1.1}$$

$$\lim_{\varepsilon \searrow 0} \varepsilon/\eta(\varepsilon) = 0. \tag{1.2}$$

It is ??? by $x_1 = 0$ and free elsewere (ARISTA??).

The "transversal curvature" in the direction of x_2 is independent of ε. We mainly consider "normal loadings" such that the structure works in flection.

The key point is: should such kind of elastic structures be considered as a shell or a beam?

Basically, shells correponds to $\eta = O(1)$. Asymptotics for $\varepsilon \searrow 0$ lead to the Love-Kirchhoff asymptotics, where normal segments to the midle surface behave as rigid, but, obviously, segments on the same section, $x_1 = constant$, are not rigidly linked to each other. The limit behavior is described by a PDE in x_1 and x_2. Both variables play analogous roles. Oppositely, beams corresponds to $\eta = O(\varepsilon)$. Asymptotics leads to the Bernoulli-Euler's assymptotic, where normal sections, $x_1 = constant$, behavies as rigid (at

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the leading order of asymptotics), i.e. the various normal segments are linked. Th elimit behaviour is then described by one ODE in x_1.

The basic result of the previous paper ([12]) consists in the description and rigurous asymptotics for

$$\eta = \varepsilon^{1/4}, \tag{1.3}$$

or even

$$\varepsilon^{1/3} \leq \eta \leq 1. \tag{1.4}$$

The corresponding asymptotics involves a PDE in x_1 and x_2 but both variables play very different roles: the order of differentiation is higher in x_2 than in x_1. The corresponding energy space is highly anisotropic, with "hipper rigidity properties" in the transversal direction x_2. Correspondingly, the "limit equations" are parabolic.

It should be pointed out that the Torroja's asymptotics described by (1.3), or even (1.4), does not cover the whole interval (1.1) and (1.2). Other cases of "slender shells" remain out of the scope of this study. Without pretension of exhaustively, let us mention Vlasov's slender shells (see, for instance, [2]) which corresponds to sections with curvature radius of η order (instead of $O(1)$ order) and $\eta = \varepsilon^{1/3}$.

¿simplification , pie de página?

The basic geometry of the mid surface in ([12]) was exactly cylindrical, i.e.,

$$x_3 = bx_2^2. \tag{1.5}$$

Nevertheless, Torroja's structures in the real world were slightly different, incorporating a very small curvature in the longitudinal x_1 direction. Specifically, this curvature was of opposite sign to the transversal one, so that the midsurface was hyperbolic. A generalization of ([12]) to that case was addressed in ([13]). This generalization was allowed by using properties of hyperbolic differential equations (dependence domains in particular) in order to obtain pertinent a priori estimates.

In the present paper we use a new more general method for obtaining a priori estimates which is independent of the type (hyperbolic , elliptic or parabolic) of the midsurface. It uses a decomposition of the space of functions of the transversal variable x_2; flection rigidily term is effective up to a kernel of finite dimenssion. As a matter of fact, "geometrical rigidility" issued from other geometrica properties is only used in a subspace of functions of the longitudinal vatiable with values in the above mentioned finite-dimenssional kernel and is then effective for same what general perturbation of the cylindrical shape.

Nevertheless, the decompostition of the space $L^2(0, l_2)$ of the transversal variable y_2 into $K \otimes K^\perp$ where K is the kernel formed by polynomials of order not greater than 3, does not conmute with products of functions of y_2. Moreover, products with functions of y_1 are no longer allowed for other technical reasons. In particular, the presence of rectangular term b_{12} in the ...¿? As a matter of fact, the new method only applies to cases with constant coefficients. Nevertheless, it is interesting, as it proves that rigidity properties are not linked to hyperbolicity. They are also present at the same order of magnitude in the elliptic case. This is a little surprising as elliptic shells with a part of the boundary free are "sensitive", which amounts to some kind of unstability (see, for instance, [34] and its bibliography).

The reason is the asympthotic rigidification furnished by the flection terms.

We also note that according to $\eta << 1$, the whole surface is "close to" any one of its tangent planes and in fact, shallow shell theory is analogous to clasical "Koiter-like" theory, allowing some technical simplifications.

We note that the "slight modification of the geometry" produced by a longitudinal curvature $O\left(\eta^2\right)$ gives discrepancies with the "exact cylinder" of order $O\left(\eta^2\right)$, which is the same as the discrepancies with the $(x_1 x_2)-$plane (see figure).

As a matter of fact, the above described perturbation of the cylinder corresponds (in the framework of shallow theory) to a surface of the form:

$$x_3 = bx_2^2 + \eta^2 a \qquad \text{with } a,\, b \text{ constants} \tag{1.6}$$

and this is a very restricted perturbation of the exact cylinder $x_3 = bx_2^2$. Indeed, the curvature lines (within shallow approximation) are $x_2 =$ constant or $x_1 =$ constant, according to $\partial_1\partial_2 x_3 = 0$ in (1.6). This does not allow a kind of perturbation which was handled by Torroja (bla bla bla) corresponding to a "kind of cylinder" with curvature depending on the longitudinal variable, i.e.:

$$x_3 = b(x_1)x_2^2 \qquad (b(x_1) \geq c > 0) \tag{1.7}$$

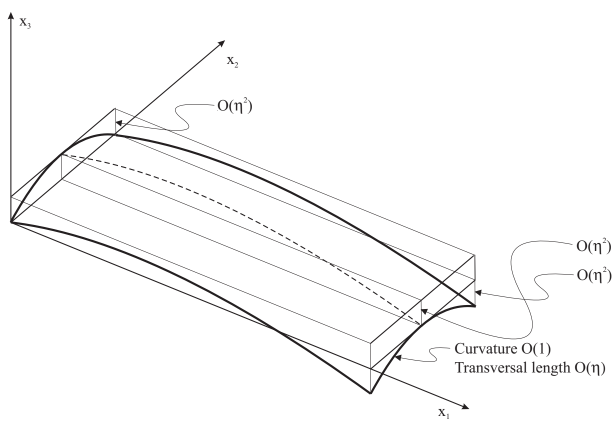
Muchas notas

In other words, the considered geometric perturbation does not destroy the relevant rigidity properties issued from the basic cylindrical shell scheme, but changes the specific coefficients (factors) describing them. In this sense, the structure is sensitive to the small perturbations of the geometry.

2 The basic problem.

2.1 Setting of the basic problem

We consider a slender cylindrical shell as shown in Fig 1



According to standard notations in cylindrical shell theory (see, e.g. [25], [10], [30]) the "plane of parameters x_1, x_2" is merely the middle surface (cylinder) of the shell developed into a plane. We chose x_1 in the direction of the generators and x_2 normal to them, so that the principal curvatures are zero in the direction x_1 and $b = 1/R$ (we assume positive curvature see Remark 2.8 on the case $b = -1/R$) in the direction x_2, where R denotes the radius of the cross section of the cylinder. Accordingly, the second fundamental form of the surface has components $b_{11} = b_{12} = 0$ and $b_{22} = b$, which is considered as a free parameter for the time being. Moreover, the Christoffel symbols of the surface vanish identically, so that covariant and classical differentiation coincide. Since $b_{12}^2 - b_{11}b_{22} = 0$ the surface is parabolic, i.e. the directions of the principal curvatures coincide (see, e.g. [30]).

Remark 2.1 *As a matter of fact, the Torroja's structure mentioned at the introduction was not composed of cylindrical elements but by slightly hyperbolic ones. Nevertheless, the curvature in the longitudinal direction was much smaller (and even it vanished in early projects by Torroja: see [35] Chapter 1) than in the transversal direction, so that our model with zero longitudinal curvature may be considered as a* *first approximation. The case of elliptic or hyperbolic middle surfaces shells case will be analyzed in [15]. Another example is the "pedestrian access shell in the southwestern side of the UNESCO building (Paris, 1953-58) due to Marcel Breuer and Bernard Zehrfuss with the collaboration of Antonio and Pier Luigi Nervi ([**?**], [21]).*

Let ε be a small parameter, the relative thickness of the plate. Let $\eta = \eta(\varepsilon)$ be a new small parameter satisfying

$$\eta = O(\varepsilon^{\frac{1}{4}}) \tag{2.8}$$

(but the typical example will be $\eta = e\varepsilon^{\frac{1}{4}}$ for some constant $e > 0$, as announced in the introduction). Let us denote the shell domain by

$$\Omega_\varepsilon = (0, l_1) \times (0, \eta l_2), \tag{2.9}$$

with $\eta l_2 \leq 2R$. The corresponding tangential displacements are $\tilde{u}_1, \tilde{u}_2$, whereas $\tilde{u}_3$ is the displacement normal to the shell. Some times we shall use the notation $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^\varepsilon$ to indicate explicitly the ε-dependence.

We shall admit, in this section, that the shell is clamped by the "small curved boundary" $(\{0\} \times [0, \eta l_2]$ and free by the rest (see some comments on other cases in Remark **??**). This implies the kinematic boundary conditions:

$$0 = \tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = \tilde{\partial}_1 \tilde{u}_3 \ \text{ on } \{0\} \times [0, \eta l_2], \tag{2.10}$$

where

$$\tilde{\partial}_\alpha = \frac{\partial}{\partial x_\alpha}. \tag{2.11}$$

The space of configuration will be denoted by V_ε. It is the subspace of

$$H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$$

formed by the functions satisfying the kinematic boundary conditions (2.10).

Although it is possible to write the complete system of equations modeling the above elastic problem (the "strong formulation": see. e.g. [25]), here we shall follow a "variational or weak formulation" of the elasticity problem for this structure which takes the form

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \tag{2.12}$$

where the coefficients ε and ε^3 account for the fact that the membrane and flection rigidities are proportional to the thickness of the plate and to its third power, respectively. Moreover, the two bilinear forms $a(\mathbf{u}^\varepsilon, \mathbf{v})$ and $b(\mathbf{u}^\varepsilon, \mathbf{v})$ on the space $\mathbf{V}$ are defined thought the expressions (membrane strains in shell theory):

$$\left\{ \begin{array}{l} \tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \widetilde{\partial}_1 \widetilde{v}_1 + \eta^2 b_{11} \widetilde{v}_3 \\ \tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \tilde{\partial}_2 \tilde{v}_2 + b_{22} \tilde{v}_3 \\ \tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \frac{1}{2}(\tilde{\partial}_2 \tilde{v}_1 + \tilde{\partial}_1 \tilde{v}_2), \end{array} \right. \tag{2.13}$$

$$\left\{ \begin{array}{c} b_{11} \text{and } b_{22} \text{ are constants,} \\ b_{22} = b \text{ and} \\ b_{11} = \gamma b, \text{with } \gamma \gtreqless 0. \end{array} \right.$$

Note that $\gamma > 0$ corresponds to the so called *elliptic case* and that $\gamma < 0$ corresponds to the so called *hyperbolic case.* We also assume that

$$\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}) = \tilde{\partial}_{\alpha\beta} \tilde{v}_3 \tag{2.14}$$

for the triplets $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$.

Remark 2.2 *It should be noted that the very expression for* $\tilde{\rho}_{22}$ *in cylindrical shells is*

$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \tilde{\partial}_2^2\tilde{v}_3 + b\tilde{\partial}_2\tilde{v}_2$$

but, as we shall see in the sequel (2.25), (2.26), for instance, in the present framework the second term of the right hand side is always asymptotically small with respect to the first one. In order to avoid unnecessary cumbersome computations, we disregard it, according to (2.14).

The two bilinear forms on $\mathbf{V}$ are then defined by:

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}})\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}})d\mathbf{x} \tag{2.15}$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} B^{\alpha\beta\lambda\mu}\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}})\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}})d\mathbf{x}, \tag{2.16}$$

where the coefficients $A^{\alpha\beta\lambda\mu}$ and $B^{\alpha\beta\lambda\mu}$ satisfy the symmetry and positivity conditions

$$A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\lambda\mu\alpha\beta} \tag{2.17}$$

$$A^{\alpha\beta\lambda\mu}\theta_{\alpha\beta}\theta_{\lambda\mu} \geq c\theta_{\alpha\beta}\theta_{\alpha\beta} \qquad \text{for} \quad \theta_{\alpha\beta} = \theta_{\beta\alpha} \tag{2.18}$$

with some $c > 0$. Analogous hypotheses will be assumed for the coefficients B; for technical reasons, we shall assume that

$$B^{1222} = B^{2122} = B^{2212} = B^{2221} = 0. \tag{2.19}$$

(in some results we shall require some additional conditions: see (2.106)). In shell theory they are the membrane and flection rigidities (see, e.g. [30]); their specific values are classical in the isotropic case (satisfying in particular (2.19)), but this also covers many anisotropic cases. This also allows us to define the membrane stresses:

$$\tilde{T}^{\alpha\beta}(\tilde{\mathbf{u}}) = \tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}}) = A^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}). \tag{2.20}$$

It will prove useful to define the entries $C_{\alpha\beta\lambda\mu}$ of the inverse matrix of A; they are the "membrane compliances" (see, e.g. [30]) and (2.20) may equivalently be written:

$$\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}) = C_{\lambda\mu\alpha\beta}\tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}}) \tag{2.21}$$

As applied forces, we shall give a normal loading depending on ε by the factor ε^3 (see Remark **??** hereafter), specifically

$$\langle \mathbf{f}, \mathbf{v} \rangle = \varepsilon^3 \int_{\Omega_\varepsilon} F_3(x_1, x_2/\eta)\tilde{v}_3(x_1, x_2)d\mathbf{x}, \tag{2.22}$$

(for other loading see Remark 2.4). We note that the shape of the profile of the applied loading in x_2 is independent of ε but applied to the points x_2/η). Defining $y_2 = x_2/\eta$ (see also the scaling (2.25) hereafter), the function $F_3(x_1, y_2)$ is independent of ε. We shall admit in the sequel that

$$F_3 \in L^2(\Omega) \tag{2.23}$$

where

$$\Omega = (0, l_1) \times (0, l_2). \tag{2.24}$$

The specific definition of the problem in variational formulation is

Problem P_ε. *Find* $\tilde{\mathbf{u}}^\varepsilon \in \mathbf{V}_\varepsilon$ *satisfying* (2.12) *with* (2.15), (2.16) *and* (2.22) $\forall \tilde{\mathbf{v}} \in \mathbf{V}_\varepsilon$.

An easy application of the Lax-Milgram theorem allows to see that this problem has a unique solution depending on the parameter ε.

The objective of the rest of the section is to study its asymptotic behavior as $\varepsilon \downarrow 0$.

2.2 Scaling and a priori estimates in the basic problem.

Let us perform the change of variables :

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2), \\ y_1 = x_1, \quad y_2 = \eta^{-1} x_2 \end{cases} \tag{2.25}$$

so, the domain Ω_ε is transformed into Ω and

$$\partial_1 = \tilde{\partial}_1, \quad \partial_2 = \eta \tilde{\partial}_2; \quad \partial_\alpha = \frac{\partial}{\partial y_\alpha}. \tag{2.26}$$

Moreover, we shall perform the change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^6 u_1(\mathbf{y}), \\ \tilde{u}_2(\mathbf{x}) = \eta^5 u_2(\mathbf{y}), \\ \tilde{u}_3(\mathbf{x}) = \eta^4 b_{22}^{-1} u_3(\mathbf{y}), \end{cases} \tag{2.27}$$

(as before, some times we shall use the notation $\mathbf{u} = \mathbf{u}^\varepsilon$ to indicate explicitly the ε-dependence). The specific values of the exponents of η and $b(\varepsilon)$ will be found later (see (**??**) and (**??**)). Let us explain a little the meaning of (2.27). As θ is not defined, the total level of the scaling is not specified, only the mutual ratios of dilatation of the three components are fixed. They are chosen in analogy with layers in parabolic shells. Specifically, the ratio between the components 1 and 2 is fixed in order that the new form of the shear membrane strain $\tilde{e}_{12}$ be formed by two terms of the same order (which, on the other hand, are asymptotically large, forming a constraint for the limit problem). The ratio between the components 2 and 3 is also fixed in such a way that the new form of the membrane strain $\tilde{e}_{22}$ be formed by two terms of the same order.

We then perform the previous change for $\tilde{\mathbf{u}}^\varepsilon$ as well as for $\tilde{\mathbf{v}}$ in P_ε and we have

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^6(\partial_1 v_1 + \gamma v_3) \tag{2.28}$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^5 \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2), \tag{2.29}$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^4(\partial_2 v_2 + v_3), \tag{2.30}$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^4 b^{-1} \partial_1^2 v_3, \tag{2.31}$$

$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^3 b_{22}^{-1} \partial_1 \partial_2 v_3, \tag{2.32}$$

$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^2 b_{22}^{-1} \partial_2^2 v_3. \tag{2.33}$$

It will prove useful to define

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \partial_1 v_1 + \gamma v_3 \tag{2.34}$$

$$\gamma_{12}^\varepsilon(\mathbf{v}) = \gamma_{21}^\varepsilon(\mathbf{v}) = \eta^{-1} \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2), \tag{2.35}$$

$$\gamma_{22}^\varepsilon(\mathbf{v}) = \eta^{-2}(\partial_2 v_2 + v_3); \tag{2.36}$$

$$\rho_{11}^\varepsilon(\mathbf{v}) = \eta^2 \partial_1^2 v_3, \tag{2.37}$$

$$\rho_{12}^\varepsilon(\mathbf{v}) = \rho_{21}(\mathbf{v}) = \eta \partial_1 \partial_2 v_3, \tag{2.38}$$

$$\rho_{22}^\varepsilon(\tilde{\mathbf{v}}) = \partial_2^2 v_3. \tag{2.39}$$

so that:

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^6 \gamma_{11}^\varepsilon(v)$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^6 \gamma_{12}^\varepsilon(\mathbf{v})$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^6 \gamma_{22}^\varepsilon(\mathbf{v})$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^2 b^{-1} \rho^\varepsilon_{11}(\mathbf{v})$$
$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^5 b^{-1} \rho^\varepsilon_{12}(\mathbf{v})$$
$$\tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^2 b^{-1} \rho^\varepsilon_{2}(\mathbf{v}).$$

We recall that the spatial domain is now $\Omega = (0, l_1) \times (0, l_2)$. The space of configuration, after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$$

formed by the functions satisfying the kinematic boundary conditions

$$0 = u_1 = u_2 = u_3 = \partial_1 u_3 \text{ on } \{0\} \times [0, l_2]. \tag{2.40}$$

Once we normalize

$$b_{22} = \frac{\varepsilon}{\eta^4}$$

the expression (2.12) then becomes:

$$\int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} = \int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y}. \tag{2.41}$$

Summing up, the problem P_ε becomes after scaling:
Problem Π_ε. *Find* $\mathbf{u}^\varepsilon \in \mathbf{V}$ *satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \textstyle\int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y} \tag{2.42}$$

$\forall v \in \mathbf{V}$, *where*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) \stackrel{def}{=} \int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y}.$$

It should be emphasized that, by virtue of the definitions (2.34) to (2.39), the coefficients in (2.42) involve various powers of η, running from -4 to $+4$. The terms in η^{-4} to η^{-1} are "penalty terms", whereas those in η^1 to η^4 are "singular perturbation terms". Only the terms of order 1 will remain in the limit expression.

Let us proceed to the a priori estimates. We first estate a series of estimates in order to prove that the functional in the right hand side of the (2.42) remains bounded with respect to the energy norm of the left hand side. From the expression of $a^\varepsilon(\mathbf{v}, \mathbf{v})$ with $\mathbf{u}^\varepsilon = \mathbf{v}$, written under the form (2.42) and using the positivity of the coefficients $A^{\alpha\beta\lambda\mu}$ we see that each term in the left hand side is majorized by the right hand side. Specifically, using (2.17) - (2.18), we have:

Lemma 2.1 *The estimates:*

$$\|\partial_1 v_1 + \gamma v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.43}$$

$$\|\eta^{-1} \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2)\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.44}$$

$$\|\eta^{-2}(\partial_2 v_2 + v_3)\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.45}$$

$$\|\partial_2^2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.46}$$

$$\|\eta \partial_1 \partial_2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.47}$$

$$\|\eta^2 \partial_1^2 v_3\|^2_{L^2(\Omega)} \le c a^\varepsilon(\mathbf{v}, \mathbf{v}) \tag{2.48}$$

hold true for a certain $c > 0$ *independent of* ε *and* $\mathbf{v} \in \mathbf{V}$.

Now, in order to prove that the functional in the right hand side is bounded independently of ε, we need an estimate on u_3 itself.

Lemma 2.2 *The estimate:*

$$\|v_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.49}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Proof. Discarding the factors in η in (2.44) and (2.45) and differentiating we have:

$$\|\partial_2^2 v_1 + \partial_2\partial_1 v_2)\|^2_{L^2((0,l_1);H^{-1}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.50}$$

$$\|\partial_1\partial_2 v_2 + \partial_1 v_3)\|^2_{H^{-1}((0,l_1);L^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}). \tag{2.51}$$

Differenciating with respect ∂_2^2 in (2.43) we get

$$\left\|\partial_1\partial_2^2 v_1 + \gamma\partial_2^2 v_3\right\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.52}$$

using (2.46)

$$\left\|\partial_1\partial_2^2 v_1\right\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.53}$$

On the other hand, from (2.43), using the fact that v_1 vanishes on $\{0\} \times [0, l_2]$, by using the *generalized Poincaré inequality* (see Section 9 of [8]) we obtain:

$$\left\|\partial_2^2 v_1\right\|^2_{H^1((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.54}$$

From the last estimate and (2.50), taking the weaker norm, it follows that

$$\|\partial_2\partial_1 v_2\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}) \tag{2.55}$$

and using (2.51)

$$\|\partial_1 v_3\|^2_{H^{-1}((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}),$$

or even by applying the *generalized Poincaré inequality* of Section 9 of [8]) on account of the vanishing of the trace on $\{0\} \times [0, l_2]$):

$$\|v_3\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v},\mathbf{v}). \tag{2.56}$$

We then use (2.46). Concerning the space $H^2(0, l_2)$, its norm up to affine functions is merely the norm in $L^2(0, l_2)$ of the second derivative; the kernel of affine functions is of finite dimension, so that in it the norms L^2 and H^{-2} are equivalent. The conclusion follows. □

Then, provided that $F_3 \in L^2(\Omega)$ (this hypothesis is not optimal), we have

Lemma 2.3 *The estimate*

$$|\int_\Omega F_3 v_3 d\mathbf{y}| \leq ca^\varepsilon(\mathbf{v},\mathbf{v})^{1/2} \tag{2.57}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Now, taking $\mathbf{v} = \mathbf{u}^\varepsilon$ in (2.42) and using (2.55) we get the energy estimate:

Lemma 2.4 *Let $\mathbf{u}^\varepsilon$ be the solution of problem Π_ε. The energy remains bounded independently of ε, i. e. the estimate*

$$a^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) \leq C \tag{2.58}$$

holds true for a certain $C > 0$ independent of ε.

From this, Lemma 2.2 gives the main estimates of the solutions

Lemma 2.5 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The estimates*

$$\|\gamma^\varepsilon_{\alpha\beta}(u^\varepsilon)\| \leq C \qquad \alpha,\beta = 1,2 \tag{2.59}$$

$$\left\|\partial_1 v_1^2 + \gamma u_3^\varepsilon\right\|_{L^2(\Omega)} \leq c \tag{2.60}$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon)\|^2_{L^2(\Omega)} \leq C \tag{2.61}$$

$$\|\eta^{-2}(\partial_2 u_2^\varepsilon + u_3^\varepsilon)\|^2_{L^2(\Omega)} \leq C \tag{2.62}$$

$$\|\partial_2^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.63}$$

$$\|\eta\partial_1\partial_2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.64}$$

$$\|\eta^2\partial_1^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \leq C \tag{2.65}$$

hold true for a certain $C > 0$ *independent of* ε.

We note that (2.59) is merely a new form of (2.60) - (2.65). We shall need an estimate on u_2^ε itself. We shall obtain it by differentiating with respect to y_2 and integrating in y_1.

Lemma 2.6 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The estimates*

$$\|u_1^\varepsilon\|_{H^1((0,\mathrm{l}_1);L^2(0,\mathrm{l}_2))} \leq C \tag{2.66}$$

$$\|u_2^\varepsilon\|_{\widetilde{H}^1_0((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2))} \leq C \tag{2.67}$$

$$\|u_3\|^2_{L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2))} \leq C, \tag{2.68}$$

holds true for a certain $C > 0$ *independent of* ε, *where*

$$\widetilde{H}^1_0((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) = \{w \in H^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) \text{ such that } w(0,\cdot) = 0\}. \tag{2.69}$$

Proof. From (2.60) and (2.49) we get that

$$\|\partial_1 v_1^\varepsilon\|_{L^2(\Omega)} \leq c.$$

Using the Poincaré inequality on account of the fact that the trace of u_1^ε vanishes on $\{0\}\times[0,l_2]$ we see that u_1^ε remains bounded in $H^1((0,l_1);L^2(0,l_2))$ (which proves (2.66)) and then $\partial_2 u_1^\varepsilon$ remains bounded in $H^1((0,l_1);H^{-1}(0,l_2))$. Using (2.61) we then see that $\partial_1 u_2^\varepsilon$ remains bounded in $L^2((0,l_1);H^{-1}(0,l_2))$. As the trace of u_2^ε vanishes on $\{0\}\times[0,l_2]$, integrating in y_1 we get the conclusion by applying the Poincaré inequality. Finally, (2.68) follows from (2.49) and (2.57). □

A first result of convergence is

Lemma 2.7 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* Π_ε. *The following convergences (as* $\varepsilon \to 0$*) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \qquad \text{weakly in } \widetilde{H}^1_0((0,\mathrm{l}_1);L^2(0,\mathrm{l}_2)) \tag{2.70}$$

$$u_2^\varepsilon \to u_2^* \qquad \text{weakly in } \widetilde{H}^1_0((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) \tag{2.71}$$

$$u_3^\varepsilon \to u_3^* \qquad \text{weakly in } L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2)) \tag{2.72}$$

where $\mathbf{u}^* = (u_1^*, u_2^*, u_3^*)$ *are distributions on* Ω, *belonging to the spaces specified in (2.70) - (2.72). Moreover, they satisfy:*

$$\partial_2 u_1^* + \partial_1 u_2^* = 0$$
$$\partial_2 u_2^* + u_3^* = 0.$$

Finally,

$$\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \to \gamma^*_{\alpha\beta} \qquad \text{weakly in } L^2(\Omega), \ \alpha,\beta = 1,2, \tag{2.73}$$

for some $\gamma^*_{\alpha\beta} \in L^2(\Omega)$.

Proof. By weak compactness, the conclusions are obvious consequences of the estimates in lemmas 2.5 and 2.6. □

2.3 Limit and convergence in the basic problem.

Let us define the space $\mathbf{G}$ for the definition of the limit problem:

$$\begin{aligned}&\mathbf{G} = \{\mathbf{v} = (v_1, v_2, v_3) \in \widetilde{H}_0^1((0, l_1); L^2(0, l_2)) \times \widetilde{H}_0^1((0, l_1); H^{-1}(0, l_2)) \times L^2((0, l_1); H^2(0, l_2)),\\ &\partial_2 v_1 + \partial_1 v_2 = 0, \quad \partial_2 v_2 + v_3 = 0\},\end{aligned} \tag{2.74}$$

where we observe that v_1 defines completely v_2 and then v_3. Clearly, $\mathbf{G}$ is a Hilbert space with the norm

$$\begin{cases} \|\mathbf{v}\|^2_{\mathbf{G}} = \|v_1\|^2_{\widetilde{H}_0^1((0,l_1);L^2(0,l_2))} + \|\partial_2^2 v_3\|^2_{L^2(\Omega)} \\ \simeq \|\partial_1 v_1\|^2_{L^2(\Omega)} + \|\partial_2^3 v_2\|^2_{L^2(\Omega)} \end{cases}$$

Remark 2.3 *A straightforward comparison with the space* $\mathbf{V}$ *shows that the space* $\mathbf{G}$ *for the limit problem incorporates the two constraints corresponding to the "penalty terms" in* Π_ε *(2.42), whereas the boundary conditions for* u_3*, which are concerned with the "singular perturbation terms" in* Π_ε *(2.42) are lost.* □

It is worthwhile to state an equivalent definition of the space $\mathbf{G}$ where the functions are defined in terms of a scalar "potential ψ":

Lemma 2.8 *The space* $\mathbf{G}$ *may equivalently be defined as the space of the triplets* $\mathbf{v} = (v_1, v_2, v_3)$ *such that:*

$$v_1 = \partial_1 \psi, \qquad v_2 = -\partial_2 \psi, \qquad v_3 = \partial_2^2 \psi. \tag{2.75}$$

where ψ *is an element of*

$$\widetilde{G} = \tilde{H}_0^2((0, \mathrm{l}_1); L^2(0, \mathrm{l}_2)) \cap L^2((0, \mathrm{l}_1); H^4(0, \mathrm{l}_2)) \tag{2.76}$$

where

$$\tilde{H}_0^2((0, \mathrm{l}_1); L^2(0, \mathrm{l}_2)) = \{\psi \in H^2((0, \mathrm{l}_1); L^2(0, \mathrm{l}_2)); \psi(0, y_2) = \partial_1 \psi(0, y_2) = 0\}. \tag{2.77}$$

Proof. Let $\mathbf{v} \in \mathbf{G}$. Because of the first constraint indicated in (2.74), there exist a distribution ψ, defined up to an additive constant, such that v_1 and v_2 are given by the two first relations in (2.75). The second constraint then shows that v_3 is then given by the last relation in (2.75). As the traces of v_1 and v_2 vanish for $y_1 = 0$, we see that

$$\psi(0, y_2) = C_1 \qquad \partial_1 \psi(0, y_2) = 0$$

with C_1 a constant. We then fix the arbitrary constant of ψ to have $C_1 = 0$. Using the Poincaré inequality, it follows from $\partial_1 v_1 \in L^2(\Omega)$ and the above boundary conditions for ψ that $\psi \in \tilde{H}_0^2((0, l_1); L^2(0, l_2))$, so that ψ is in the first one of the two spaces on the right hand side of (2.76). Belonging to the second space follows easily from the last relations in (2.74) and (2.75). Conversely, it is straightforward that $\psi \in \widetilde{G}$ implies $v \in \mathbf{G}$. □

It should prove useful to prove a lemma on density in $\mathbf{G}$.

Lemma 2.9 *The subspace of* $\mathbf{G}$ *formed by the elements* $\mathbf{v} = (v_1, v_2, v_3)$ *which are smooth, vanish in a neighborhood of* $\{0\} \times [0, \mathrm{l}_2]$ *and derive from a "potential"* ψ *according to (2.75) is dense in* $\mathbf{G}$*. In other words, the set of functions of* $\mathbf{G} \cap \mathbf{V}$ *which are smooth and vanish in a neighborhood of* $\{0\} \times [0, \mathrm{l}_2]$ *is dense in* $\mathbf{G}$*.*

Proof. As in Lemma 2.9 of ([12]), thanks to the equivalence of the spaces $\mathbf{G}$ and $\widetilde{G}$ given by lemma 2.8, the proof (in the unconstrained space $\widetilde{G}$) is almost classical (see, for instance, lemma 5.2 of [31] or lemma 8.1 of [7]). □

Obviously, the norm of $\widetilde{G}$ is

$$\|\varphi\|_G^2 = \int_\Omega \left(|\partial_1\varphi|^2 + \left|\partial_2^4\varphi\right|^2\right) dy. \tag{2.78}$$

Lemma nuevo 1 - *The expression*

$$e(\varphi) = \int_\Omega \left(\left|\partial_1\varphi + \gamma\partial_2^2\varphi\right|^2 + \left|\partial_2^4\varphi\right|^2\right) dy \tag{2.79}$$

is the square of a norm on $\widetilde{G}$, which is equivalent to the natural norm (2.78).

Obviously the norm form (2.79) is continuous on the norm (2.78), so that the Lemma (nuevo 1) is equivalent to the coerciveness of $e(\varphi)$ on $\|\varphi\|_G^2$. So, it follows from

Lemma nuevo 2 - There exists a constant c such that for any $\varphi \in \widetilde{G}$;

$$\left\|\partial_2^2\varphi\right\|_{L^2(\Omega)}^2 \leq ce(\varphi) \tag{2.80}$$

Proof of Lemma2. Let us decompose $L^2(0,l_2)$ as the product of the subspace of functions which are polynomials of order ≤ 3, (that we shall denote K, as it is the kernel of ∂_2^4) and its orthogonal, denoted by $K_{K^\perp}$. The respective dimensions are ... and .... Accordingly, any function φ with values in $L^2(0,l_2)$ will be decomposed in the form:

$$\varphi = \varphi_K + \varphi_{K^\perp} \tag{2.81}$$

by taking the orthogonal projection on K and $K^\perp$ of the values of the function. The projectors obviously conmute with defferentiation ∂_1 and traces on $y_1 =$ constant. Obviously, φ_K takes the form:

$$\varphi_K = A(y_1) + B(y_2)y_2 + C(y_1)\frac{y_2^2}{2} + D(y_2)\frac{y_2^3}{6} \tag{2.82}$$

with A, B, C, D functions of y_1 depending on φ. From the nulling of traces on $y_1 = 0$, we have

$$\left\{ \begin{array}{c} A(0) = B(0) = C(0) = D(0) = 0 \\ A'(0) = B'(0) = C'(0) = D'(0) = 0 \end{array} \right. \tag{2.83}$$

As $\partial_2^4\varphi = \partial_2^4\varphi_{K^\perp}$, it follows from (2.79) taking a weaker norm that

$$\|\varphi_{K^\perp}\|_{L^2(0,l_1);H^2(0,l_2)}^2 \leq ce(\varphi) \tag{2.84}$$

Now, in order to get an analogous estimate for $\varphi_{K^\perp}$, we use the first term in the right hand side of (2.79). We have

$$\left\|\partial_1^2\varphi + \gamma\partial_2^2\varphi\right\|_{L^2(\Omega)}^2 = \left\|\left(\partial_1^2 + \gamma\partial_2^1\right)\left(\varphi_K + \varphi_{K^\perp}\right\|_{L^2(\Omega)}^2 \leq ce(\varphi) \tag{2.85}$$

From (2.84), taking a weaker norm;

$$\left\|(\partial_1^2 + \gamma\partial_2^2)\varphi_{K^\perp}\right\|_{H^{-2}(0,l_1);H^{-2}(0,l_2)}^2 \leq ce(\varphi) \tag{2.86}$$

so that, from (2.85) (always with a weaker norm);

$$\left\|(\partial_1^2 + \gamma\partial_2^2)\varphi_K\right\|_{H^{-2}(0,l_1);H^{-2}(0,l_2)}^2 \leq ce(\varphi) \tag{2.87}$$

But:

$$(\partial_1^2 + \gamma\partial_2^2)\varphi_K = (A'' + \gamma C) + (B'' + \gamma D)y_2 + C''\frac{y_2^2}{2} + D''\frac{y_2^3}{6} \tag{2.88}$$

We now note that $\left\{1, y_2, \frac{y_2^2}{2}, \frac{y_2^3}{6}\right\}$ form a basis in the 4-dimensional space K, so that the norm in $H^{-2}(0,l_2)$ is equivalent to any other norm, for instance the sum of the modulli of the components in that basis, so that

$$\begin{aligned}
\|A'' + \gamma C\|_{H^{-2}(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}} \\
\|B'' + \gamma C\|_{H^{-2}(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}} \\
\|C''\|_{H^{-2}(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}} \\
\|D''\|_{H^{-2}(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}}
\end{aligned}$$

From this, using the initial conditions (2.83) and the generalized Poincaré inequality, we have:

$$\begin{aligned}
\|C\|_{L^2(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}} \\
\|D\|_{L^2(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}}
\end{aligned}$$

and then

$$\begin{aligned}
\|A''\|_{L^2(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}} \\
\|B''\|_{L^2(0,l_1)} &\leq ce(\varphi)^{\frac{1}{2}}
\end{aligned}$$

so that

$$\|\varphi_K\|_{L^2(\Omega)} \leq ce(\varphi)^{\frac{1}{2}} \tag{2.89}$$

and with (2.84):

$$\|\varphi\|_{L^2(\Omega)} \leq ce(\varphi)^{\frac{1}{2}} \tag{2.90}$$

From this, using the last term in (2.79):

$$\|\varphi\|_{L^2(0,l_1);H^4(0,l)} \leq ce(\varphi)^{\frac{1}{2}} \tag{2.91}$$

and (2.80) is proven.
6666666666666666666666666

We are now defining the limit problem. It involves the numerical coefficients $1/C_{1111}$, and B^{2222} where $C_{\alpha\beta\lambda\mu}$ is the matrix inverse of $A^{\alpha\beta\lambda\mu}$, i. e. the matrix of membrane compliances, and $\mathbf{B}$ is the matrix of flection rigidities. They are both strictly positive.

Problem Π_0. *Find* $\mathbf{u} \in \mathbf{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}}(\partial_1 u_1 + \gamma u_3)(\partial_1 v_1 + \gamma v_3) d\mathbf{y} + \int_\Omega B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}. \tag{2.92}$$

$\forall \mathbf{v} \in \mathbf{G}$, *or equivalently, in terms of the potential, find* $\varphi \in \widetilde{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}}(\partial_1^2 + \gamma\partial_2^2)\varphi(\partial_1^2 + \gamma\partial_2^2)\psi d\mathbf{y} + \int_\Omega B^{2222} \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y} = -\int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}, \tag{2.93}$$

$\forall \psi \in \widetilde{G}$.

Obviously, this problem is in the Lax - Milgram framework, as the right hand side of (2.92) is a continuous functional on $\mathbf{G}$. We then have

Theorem 2.1 *Under the assumption $F_3 \in L^2(\Omega)$, Problem Π_0 has a unique solution.*

Remark 2.4 *Clearly, the case considered here, $F_1 = F_2 = 0$ and $F_3 \in L^2(\Omega)$ is not the more general case we can deal with the above arguments. So, for instance, taking into account the previous a priori estimate we can consider other loadings $\mathbf{F}$ satisfying*

$$|\int_\Omega F_i v_i dy| \leq c a^\varepsilon(v,v)^{1/2},\ i = 1,2,3,$$

as it is the special case of $F_2 = 0$ and $F_1, F_3 \in L^2(\Omega)$. This is interesting for the special case in which $\mathbf{F}$ is the gravity and the middle surface $x_3 = 0$ makes an angle with respect to the horizontal, as it is the case of the Torroja's structure mentioned at the Introduction. Other possible choices are the concentrated loadings of the type $F_1 = F_2 = 0$ and F_3 given in terms of the Dirac delta and its derivatives as in [7].

Our main convergence result is:

Theorem 2.2 *Let $\mathbf{u}^\varepsilon$ and $\mathbf{u}$ be the solutions of Π_ε and Π_0 respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

in the topologies indicated in (2.70) - (2.72) (lemma 2.7). In other words, the limit $\mathbf{u}^*$ in lemma 2.7 is the solution of the limit problem (2.92).

Before proving this theorem, let us define certain limits which will be useful in the sequel. We know by (2.73) that the $\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)$ have limits $\gamma^*_{\alpha\beta}$. Correspondingly, we define:

$$T^{\alpha\beta\varepsilon}(\mathbf{u}^\varepsilon) = A^{\alpha\beta\lambda\mu}\gamma^\varepsilon_{\lambda\mu}(\mathbf{u}^\varepsilon)$$

and

$$T^{\alpha\beta *} = A^{\alpha\beta\lambda\mu}\gamma^*_{\lambda\mu}$$

so that

$$T^{\alpha\beta\varepsilon}(\mathbf{u}^\varepsilon) \to T^{\alpha\beta *} \qquad \text{weakly in } L^2(\Omega),\ \ \alpha,\beta = 1,2.$$

Remark 2.5 *It seems important to point out that the a priori estimates (2.61) and (2.62) does not allow to conclude the identification $\gamma^*_{12} = 0$ and $\gamma^*_{22} = 0$ in spite to know that $\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)$ weakly converge to $\gamma^*_{\alpha\beta}$ and that necessarily $\partial_2 u^*_1 + \partial_1 u^*_2 = 0$ and $\partial_2 u^*_2 + u^*_3 = 0$. The reason is due to the presence of the terms η^{-1} (respectively η^{-2}) in the definition of γ^ε_{12} (respectively γ^ε_{22}). Notice that, in fact, in most of the cases we must have that $\gamma^*_{12} \neq 0$ or $\gamma^*_{22} \neq 0$, since otherwise we could get that $T^{11\varepsilon}(\mathbf{u}^\varepsilon) = A^{1111}\gamma^\varepsilon_{11}(\mathbf{u}^\varepsilon) + 2A^{1112}\gamma^\varepsilon_{12}(\mathbf{u}^\varepsilon) + A^{1122}\gamma^\varepsilon_{22}(\mathbf{u}^\varepsilon)$ converges (weakly in $L^2(\Omega)$) to $T^{11*}(\mathbf{u}^*) = A^{1111}\gamma_{11}(\mathbf{u}^*) = A^{1111}\partial_1 u^*_1$ and this would imply (thanks to Theorem 2.2) that necessarily*

$$A^{1111} = \frac{1}{C_{1111}}, \tag{2.94}$$

which is not necessarily true since it depends of the constitutive assumptions made on the elastic medium.

Proof of Theorem 2.2. That the limit $\mathbf{u}^*$ in lemma 2.7 belongs to $\mathbf{G}$ follows from the definition of this space. Let us now prove that $\mathbf{u}^*$ is the solution of (2.92). Let us take in (2.42) $\mathbf{v}$ in the dense set of $\mathbf{G}$ indicated in Lemma 2.9. From the definition of $\mathbf{G}$ and (2.34) - (2.36) we see that the only non vanishing $\gamma^\varepsilon_{\alpha\beta}$ is

$$\gamma^\varepsilon_{11}(\mathbf{v}) = \partial_1 v_1 + \gamma v_3$$

and we have

$$\int_\Omega T^{11\varepsilon}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{11}(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}.$$

The term in the first integral obviously pass to the limit (note that $\gamma^\varepsilon_{11}(v)$ does not depend on ε: see (2.34)). Concerning the terms in ρ, we have, as we know, an estimate in the spaces involved in the definition of $\mathbf{G}$, so that the term

$$B^{2222}\partial_2^2 u_3^\varepsilon \partial_2^2 v_3$$

also pass to the limit. For the same reason, with the estimates (2.63) - (2.65) all the other terms tend to zero, with the exception of

$$\eta\int_\Omega B^{1222}\partial_1\partial_2 u_3^\varepsilon \partial_2^2 v_3 d\mathbf{y}$$

and

$$\eta^2\int_\Omega B^{1122}\partial_1^2 u_3^\varepsilon \partial_2^2 v_3 d\mathbf{y}$$

which are not evident. The first one vanishes as, according to our hypotheses, B^{1222} is taken to be zero (see (2.19)). As for the second one, according to distribution theory (integration by parts) it is equal to

$$\eta^2\int_\Omega B^{1122} u_3^\varepsilon \partial_1^2\partial_2^2 v_3 d\mathbf{y}.$$

We note that u_3^ε remains bounded in $L^2(\Omega)$. Then, because of the factor η^2, the expression tends to 0. As a result, the limit is

$$\int_\Omega T^{11*}(\partial_1 v_1 + \gamma v_3)d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3^*\partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}. \tag{2.95}$$

We are now transforming the term in T^{11*} in the previous equation. To this end, let us take

$$w_1 = w_2 = 0, \qquad w_3 \in C_0^\infty(\Omega) \tag{2.96}$$

and let us take in (2.42) the test function

$$\mathbf{v} = \eta^2\mathbf{w} \tag{2.97}$$

so that from (2.34) - (2.36) we see that the only non vanishing $\gamma^\varepsilon_{\alpha\beta}$ are

$$\gamma^\varepsilon_{11}(v) = \eta^2\gamma w_3$$
$$\gamma^\varepsilon_{22}(v) = -w_3$$

and passing to the limit in (2.42) we have

$$\int_\Omega T^{22*}(-w_3)d\mathbf{y} = 0 \tag{2.98}$$

so that

$$T^{22*} = 0. \tag{2.99}$$

Let us now take

$$w_1 \in C_0^\infty((0,l_1);C^\infty(0,l_2)), \qquad w_2 = w_3 = 0, \tag{2.100}$$

and let us take in (2.42) the test function

$$\mathbf{v} = \eta \mathbf{w} \tag{2.101}$$

so that from (2.34) - (2.36) we see that the only non vanishing $\gamma^\varepsilon_{\alpha\beta}$ are

$$\gamma^\varepsilon_{11}(\mathbf{v}) = \eta\partial_1 w_1, \qquad \gamma^\varepsilon_{12}(\mathbf{v}) = \frac{1}{2}\partial_2 w_1 \tag{2.102}$$

and passing to the limit in (2.42) we have

$$\int_\Omega T^{12*}(\frac{1}{2}\partial_2 w_1)d\mathbf{y} = 0 \tag{2.103}$$

so that, as $\partial_2 w_1$ is "arbitrary",

$$T^{12*} = 0. \tag{2.104}$$

As the only non zero $T^{\alpha\beta*}$ is T^{11*}, the $\gamma^*_{\alpha\beta}$ are given by the expressions

$$\gamma^*_{\alpha\beta} = C_{\alpha\beta 11}T^{11*}$$

and in particular

$$\gamma^*_{11} = C_{1111}T^{11*}.$$

Moreover, it follows from (2.70) and (2.73) that

$$\gamma^\varepsilon_{11}(\mathbf{u}^\varepsilon) = \partial_1 u^\varepsilon_1 + \gamma u^\varepsilon_3 \to \partial_1 u^*_1 + \gamma u^\varepsilon_3 \text{ weakly in } L^2(\Omega).$$

so that

$$T^{11*} = \frac{1}{C_{1111}}(\partial_1 u^*_1 + \gamma u_3) \tag{2.105}$$

and replacing it in (2.95) we obtain (2.92). This expression holds true for $\mathbf{v}$ in a dense set in $\mathbf{G}$ (see lemma 2.9) so that $\mathbf{u}^*$ is the unique solution of (2.92). The proof is finished. □

Our next result improves the convergence under some additional condition on the coefficients.

Theorem 2.3 *Assume that*

$$A^{11\lambda\mu} = 0 \textit{ if } \lambda > 1 \textit{ or } \mu > 1. \tag{2.106}$$

Let $\mathbf{u}^\varepsilon$ *and* $\mathbf{u}$ *be the solutions of* Π_ε *and* Π_0 *respectively. Then*

$$u^\varepsilon_1 \to u^*_1 \qquad \textit{strongly in } \widetilde{H}^1_0((0,\mathrm{l}_1); L^2(0,\mathrm{l}_2)) \tag{2.107}$$

$$u^\varepsilon_2 \to u^*_2 \qquad \textit{strongly in } \widetilde{H}^1_0((0,\mathrm{l}_1); H^{-1}(0,\mathrm{l}_2)) \tag{2.108}$$

$$u^\varepsilon_3 \to u^*_3 \qquad \textit{strongly in } L^2((0,\mathrm{l}_1); H^2(0,\mathrm{l}_2)) \tag{2.109}$$

for $\varepsilon \downarrow 0$.

Proof. We follows closely our proof of Theorem 2.3 in [12] (an argument inspired in some ideas by J. L. Lions : see, e.g. Theorem 10.1 Chapter I of [22]). We reformulate the bilinear form as

$$\begin{aligned} a^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}) &= \int_\Omega A^{\alpha\beta\lambda\mu}\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} \qquad (2.110) \\ &= \mathfrak{a}_0(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon,\mathbf{v}), \end{aligned}$$

for the (positive) symmetric bilinear forms $\mathfrak{a}_{1/2}$, $\mathfrak{a}_1$, $\mathfrak{a}_{-1/4}$, $\mathfrak{a}_{-1/2}$ given by

$$\mathfrak{a}_{1/2}(\mathbf{u},\mathbf{v}) = \int_\Omega \partial_1\partial_2 u_3 \partial_1\partial_2 v_3 d\mathbf{y},$$

$$\mathfrak{a}_1(\mathbf{u},\mathbf{v}) = \int_\Omega \partial_1^2 u_3 \partial_1^2 v_3 d\mathbf{y},$$
$$\mathfrak{a}_{-1/2}(\mathbf{u},\mathbf{v}) = \int_\Omega (\partial_2 u_1 + \partial_1 u_2)(\partial_2 v_1 + \partial_1 v_2) d\mathbf{y},$$
$$\mathfrak{a}_{-1/4}(\mathbf{u},\mathbf{v}) = \frac{1}{4}\int_\Omega (\partial_2 u_2 + u_3)(\partial_2 v_2 + v_3) d\mathbf{y},$$

(for the sake of simplicity, we assumed here that different coefficients are identically equal to 1 but the general case can be treated in the same way since which is relevant is the order of ε in the above expansion) and where, due to the assumption (2.106),

$$\mathfrak{a}_0(\mathbf{u},\mathbf{v}) = \int_\Omega A^{1111}(\partial_1 u_1 + \gamma u_3)(\partial_1 v_1 + \gamma v_3) d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y}. \tag{2.111}$$

We have that

$$\begin{aligned} &\mathfrak{a}_0(\mathbf{u}^\varepsilon - \mathbf{u}^*, \mathbf{u}^\varepsilon - \mathbf{u}^*) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) \\ = &\int_\Omega F_3(y_1,y_2) u_3^\varepsilon(y_1,y_2) d\mathbf{y} - 2\mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^\varepsilon) + \mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^*) \to \\ \to &\int_\Omega F_3(y_1,y_2) u_3^*(y_1,y_2) d\mathbf{y} - \mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^*) = 0. \end{aligned}$$

Then, by the above theorem (and since (2.106) implies (2.94))

$$\int_\Omega \frac{1}{C_{1111}}((\partial_1 u_1^* + \gamma u_3^*) - (\partial_1 u_1^\varepsilon + \gamma u_3^\varepsilon))^2 d\mathbf{y} + \int_\Omega B^{2222}(\partial_2^2 u_3^* - \partial_2^2 u_3^\varepsilon) d\mathbf{y} \to 0,$$

which, by using the a priori estimates, leads to the result. □

We emphasize that the limit problem (in terms of φ) is given by the variational formulation (2.93). The corresponding higher order partial differential equation for φ is obviously

$$(\frac{1}{C_{1111}}\partial_1^4 + 2\gamma\partial_1^2\partial_2^2 + \gamma^2\partial_2^4)\varphi + B^{2222}\partial_2^8\varphi = -\partial_2^2 F_3. \tag{2.112}$$

which may be a little misstating when considered without the corresponding boundary conditions (on $\Gamma_l := [0,l_1]\times\{0\}\cup[0,l_1]\times\{l_2\}$). Indeed, looking at (2.112) one may think that the data (and then the solution) vanishes when F_3 is affine with respect to y_2 (as in that case the right hand side of (2.112) vanishes). In fact, this is not the case as the natural boundary conditions are not homogeneous in general. This is a consequence of the very peculiar form of the right hand side of the variational formulation (2.93), which involves $\partial_2^2\psi$ instead of the test function ψ itself.

Let us write down the natural boundary conditions assuming, as usual, that F_3 and the solution are sufficiently smooth (e.g. $F_3, \partial_2^2 F_3 \in L^2(\Omega)$) then

$$\begin{aligned} \textstyle\int_\Omega F_3\partial_2^2\psi d\mathbf{y} &= \textstyle\int_\Omega \partial_2 F_3(\partial_2\psi) d\mathbf{y} - \int_\Omega (\partial_2 F_3)\partial_2\psi d\mathbf{y} \\ &= \textstyle\int_0^{l_1} dy_1\ (F_3\partial_2\psi)|_0^{l_2} - \int_\Omega \partial_2[(\partial_2 F_3)\psi] d\mathbf{y} + \int_\Omega (\partial_2^2 F_3)\psi d\mathbf{y} \\ &= \textstyle\int_0^{l_1} dy_1\ (F_3\partial_2\psi)|_0^{l_2} - \int_0^{l_1} dy_1\ [(\partial_2 F_3)\psi]|_0^{l_2} + \int_\Omega (\partial_2^2 F_3)\psi d\mathbf{y}. \end{aligned} \tag{2.113}$$

Analogously, if we assume that $\varphi, \partial_2^8\varphi \in L^2(\Omega)$ then

$$\begin{aligned} \textstyle\int_\Omega \partial_2^4\varphi\partial_2^4\psi d\mathbf{y} &= \textstyle\int_0^{l_1} dy_1\ (\partial_2^4\varphi\partial_2^3\psi)\big|_0^{l_2} \\ &- \textstyle\int_0^{l_1} dy_1\ ((\partial_2^5\varphi\partial_2^2\psi)\big|_0^{l_2} + \int_0^{l_1} dy_1\ ((\partial_2^6\varphi\partial_2\psi)\big|_0^{l_2} \\ &- \textstyle\int_0^{l_1} dy_1\ ((\partial_2^7\varphi\psi)\big|_0^{l_2} + \int_\Omega (\partial_2^8\varphi)\psi d\mathbf{y}. \end{aligned} \tag{2.114}$$

Then, from (2.113) and (2.114), and as the *test* functions ψ, $\partial_2\psi$, $\partial_2^2\psi$ and $\partial_2^3\psi$ are arbitrary, we deduce that the natural boundary conditions on $\Gamma_l := [0, l_1] \times \{0\} \cup [0, l_1] \times \{l_2\}$ are

$$\begin{cases} B^{2222}\partial_2^7\varphi = -\partial_2 F_3, \quad B^{2222}\partial_2^6\varphi = -F_3 & \text{on } \Gamma_l, \\ \partial_2^5\varphi = \partial_2^4\varphi = 0 & \text{on } \Gamma_l, \end{cases} \tag{2.115}$$

and so, the two first boundary conditions depend on the right hand side of the partial differential equation.

In fact, the previous Theorem 2.1 can be applied when, merely, $F_3 \in L^2(\Omega)$. Then, although $\widetilde{G} \subset \tilde{H}^2((0, l_1); L^2(0, l_2))$ the variational formulation is not enough as to formulate separately the partial differential equation (2.112) from the boundary conditions (2.115). For instance, let us consider the function $F_3(y_1, y_2) = (l_2 - y_2)^\alpha$ with $\alpha \in (-\frac{1}{2}, 0)$. Then, since $F_3 \in L^2(\Omega)$, the variational formulation makes sense whereas boundary conditions (2.115) do not, as the traces of F_3 and $\partial_2 F_3$ on Γ_l do not exist. It should be noticed that in the Lions-Magenes ([23]) theory (which is nevertheless only concerned with elliptic problems, and so out of our framework) singular right hand side terms are only allowed when their singularities are located at the interior of Ω and not when they are in a vicinity of the boundary. This is associated with the fact that the allowed right hand side should belongs to the space $\Xi^s(\Omega)$, $s < 0$, which are analogous to the space $H^s(\Omega)$, $s < 0$, inside of Ω but not near the boundary $\partial\Omega$ where $\Xi^s(\Omega)$ only contains smother functions (see [23], Section 6.3, Chapter 2).

Remark 2.6 *REVISAR The problem (2.112) is parabolic according the theory of linear partial differential equations (see, e.g. [37]). Indeed, the characteristics are find as normal curves to the vectors (ξ_1, ξ_2) satisfying that $P^m(\xi_1, \xi_2) = 0$, where P^m is the "principal symbol" of the differential operator. In our case, $P^m(\xi_1, \xi_2) = B^{2222}\xi_2^8$, and so, $\xi_2 = 0$ is a multiple characteristic (of 8th-order). Thus, the passing to the limit arguments show that the parabolicity of the middle surface leads to a limit equation as (2.112) of parabolic type, with characteristic of multiplicity 8.*

Remark 2.7 *Obviously, to the implicit boundary condition on Γ_l given in (2.115) we must add the rest of boundary conditions. So, for instance, the fact that the boundary $\{\mathrm{l}_1\} \times [0, \mathrm{l}_2]$ is free leads to*

$$\begin{cases} T^{11} = 0 & \text{on } \{\mathrm{l}_1\} \times [0, \mathrm{l}_2], \\ \partial_1 T^{11} = 0 & \text{on } \{\mathrm{l}_1\} \times [0, \mathrm{l}_2]. \end{cases} \tag{2.116}$$

Remark 2.8 *The case $b = -1/R$ can be also considered with obvious modifications. For instance, in the rescaling change (2.27) we must assume now that $\tilde{u}_3(\mathbf{x}) = \eta^{\theta-2}|b|^{-1} u_3(\mathbf{y})$. We point out that corresponding sign changes at the different equations may justify the different behavior of solutions with respect the case of $b = 1/R$. Easy comparison experiences can be made by using a flexible steel retractable meter tape measure in its normal and reverse positions.*

3 Generalizations and Remarks

It is worth while noticing that all the results hold true when the fixation conditions

$$u_1(0, y_2) = u_2(0, y_2)$$

are prescribed *only on a part* (with positive measure) of the boundary $y_1 = 0$. Indeed, this is sufficient to get

$$A(0) = B(0) = C(0) = D(0) = A'(0) = B'(0) = C'(0) = D'(0) = 0$$

and the whole proof holds true.

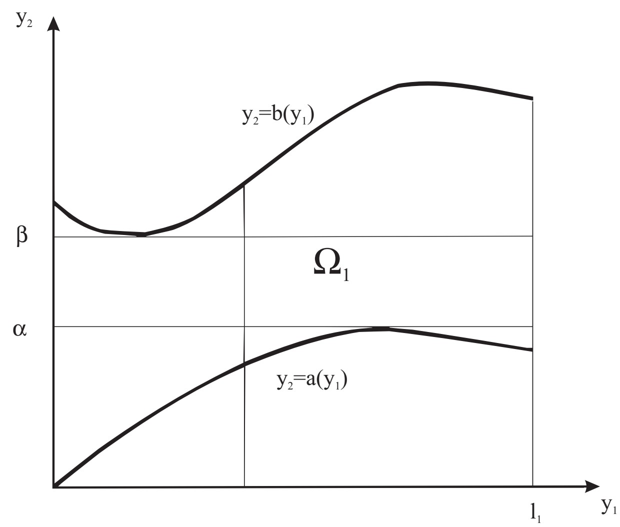
Generally speaking, the former expression of the limit problem may be obtained by a formal expression in powers of η (after scaling), without proving rigorously the convergence. Nevertheless in that case, it is important to prove that the limit problem admits loadings $F_1 \in L^2(\Omega)$ for instance. This needs to prove that the corresponding functional

$$\int_\Omega F_3 \partial_2^2 \psi dy$$

is continous on $\widetilde{G}$, or, equivalently, that

$$\|\varphi\|_G^2 \geq c \int_\Omega \left|\partial_2^2 \varphi\right|^2 dy.$$

It is not hard to prove such kind of estimates in more general situations. For instance in cases when the shape of the shell is not exactly rectangular. Let us consider for example the case described (after scaling to the varables y_1, y_2) in the Figure 2.



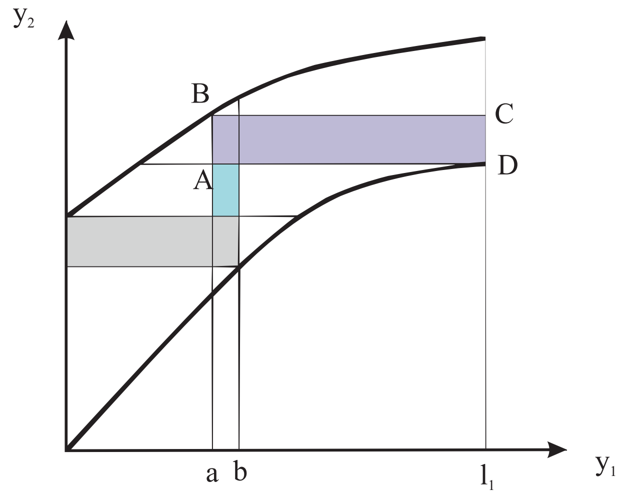
Indeed, the inequalities are obtained as before on the rectangle Ω_1. Then, we pass to the whole Ω noting that on each section $y_1 =$ constant,

$$\left(\int_{a(y_1)}^{b(y_1)} \left|\partial_2^4 \varphi\right|^2 dy_2 + \int_\alpha^\beta |\varphi|^2 dy_2 \right)^{\frac{1}{2}}$$

is an *equivalent* norm to the standard one

$$\left(\sum_{j=0}^{3} \int_{a(y_1)}^{b(y_1)} \left|\partial_2^j \varphi\right|^2 dy_2 \right)^{\frac{1}{2}}.$$

The case of Figure 3 may also be considered but the proof needs a slight modification:



Indeed, the previous method allows us to prove the inequalities in the region of Ω on the left of b. In order to go on, we should consider the rectangle $ABCD$. To this end, we have not "boundary conditions" on $y_1 = a$; they are replaced by the fact that the inequality holds true in the portion $a < y_1 < b$. As a matter of fact, instead of the "generalized Poincaré inequality", we can use the fact that the norms

$$\|\varphi\|_{L^2(a,l_1)} \quad \text{and} \quad \left(\left\| \partial_1^2 \varphi \right\|^2_{H^{-2}(a,l_1)} + \|u\|^2_{L^2(a,b)} \right)^{\frac{1}{2}}$$

are equivalent (as the term $\|u\|^2_{L^2(a,b)}$ controls the finite-dimensional kernel of ∂_1^2).

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