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# PARABOLIC MONGE-AMPERE EQUATIONS GIVING RISE TO A FREE BOUNDARY: THE WORN STONE MODEL

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Abstract. This paper deals with several qualitative properties of solutions of some parabolic equations associated to the Monge–Ampère operator arising in suitable formulations of the Gauss curvature flow and the worn stone problems.

1. **Introduction.** This paper deals with a free boundary parabolic problem in connection to the shape of worn stones model ([12], [4], [8], ...) governed by power Gauss curvature flow. Here we focus on a model following the one of Firey [12] for which we look for a convex function satisfying

$$\begin{cases} u_t = \left(\dfrac{\det \mathrm{D}^2 u}{g(|\mathrm{D}u|)}\right)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ u = \varphi, & \text{on } \partial\Omega \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0, & \text{in } \Omega, \end{cases} \tag{1}$$

where $\Omega$ is a convex open set of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} \geq 1$, the constant p is positive, $g$ is a continuos function on $\mathbb{R}_+ \cup \{0\}$ verifying $g(s) \geq 1$ and $u_0$ a non negative locally convex function in $\Omega$. In order to simplify the exposition, we assume that the boundary datum $\varphi$ is time independent.

We prove that problem (1) can be understood as a Cauchy problem on $\mathbb{X} = \mathcal{C}(\overline{\Omega})$

$$u_t + \mathcal{A}u = 0, \quad t > 0, \qquad u(0) = u_0, \tag{2}$$

for a suitable operator $\mathcal{A}$, (see (5)) whose mild solution $u$ is found by means of an implicit Euler scheme (see (7) below). Therefore, we must pay a special attention to the following stationary problem

$$\begin{cases} \det \mathrm{D}^2 u = \lambda g\big(|\mathrm{D}u|\big)\big(u - h\big)^{\frac{1}{\mathrm{p}}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Since the operator is only degenerate elliptic on the symmetric definite nonnegative matrices, a compatibility condition is assumed here

$$h \text{ is locally convex on } \overline{\Omega} \text{ and } h \leq \varphi \text{ on } \partial\Omega.$$

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**Remark 1.** When $u \geq h$ the ellipticity degeneracy occurs and classical solutions cannot be guaranteed. The generalized solutions or the viscosity solutions are suitable notions in order to negotiate the degeneracy of the operator. For convex domains $\Omega$ equivalence between both notions holds (see [11] and its bibliography therein). □

Since $h \leq u$ holds on $\overline{\Omega}$, the boundary, $\mathcal{F}$, between the sets $[u = h]$ and $[h < u]$, is a free boundary: the boundary of the set of points $x \in \Omega$ for which $\det \mathrm{D}^2 u(x) > 0$. Obviously, since the interior of the regions $[u = h]$ and $[\det \mathrm{D}^2 u = 0]$ coincide, if $h \in \mathcal{C}^2$ we must have that $\mathrm{D}^2 h = 0$, thus $h$ must be flat. Problem (3) and some properties of the free boundary $\mathcal{F}$ are studied in the previous work by the authors [11].

Here, the free boundary is studied whenever $h(x)$ has flat regions

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \mathrm{Flat}_\alpha(h), \quad \mathrm{Flat}_\alpha(h) = \{x : h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha\}, \tag{4}$$

where $\mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}}$ and $a_\alpha \in \mathbb{R}$. So that, in Section 3 we obtain some results essentially related to the shape of such worn stones. We shall prove that if $\mathrm{Np} \geq 1$ then the initial flat region persists for small times under some conditions on $u_0$ (the so called finite waiting time). By means of suitable self–similar solutions we shall show also that any flat region must disappear after a time large enough. Concerning the asymptotic behavior for large $t$, we shall prove that if a flat obstacle does not coincide with $u_0$ in a set with positive measure the same occurs in any $t > 0$ (the so called flattened retention). By the contrary, if $\mathrm{Np} < 1$ we shall prove that the solution becomes globally flat at a finite time.

In Section 2 existence of the solutions of the evolution problem is obtained by means of accretive operator theory.

2. **The evolution model: A Cauchy problem.** From now on $f \in \mathcal{C}(\mathbb{R})$ denotes an odd increasing function such that $f(0) = 0$. Then, we say $u \in \mathrm{D}(\mathcal{A})$ if $u \in \mathcal{C}(\overline{\Omega})$ is a locally convex function satisfying on $\partial\Omega$ the data $\varphi \in \mathcal{C}(\partial\Omega)$ and there exists a nonpositive continuos function $v$ in $\Omega$ such that $u$ is a generalized solution of

$$\begin{cases} \det \mathrm{D}^2 u = -g\big(|\mathrm{D}u|\big) f(v) & \text{in } \Omega, \\ u = \varphi & \text{in } \partial\Omega. \end{cases} \tag{5}$$

Then we denote by $\mathcal{A}u$ the set of all such $v \in \mathcal{C}(\overline{\Omega})$. As a mere application of [11, Proposition 1] we obtain

**Theorem 2.1.** *The operator $\mathcal{A}$ is T-accretive on the Banach space $\mathbb{X} = \mathcal{C}(\Omega)$ equipped with the supreme norm.* □

Certainly, one has

$$\mathrm{D}(\mathcal{A}) \subset \widehat{\mathbb{X}}_\varphi \doteq \{w \in \mathcal{C}(\overline{\Omega}) : \ w \text{ locally convex on } \overline{\Omega} \text{ and } w = \varphi \text{ on } \partial\Omega\}.$$

In fact, we have

**Corollary 1.** *The operator $\mathcal{A}$ satisfies $\overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi$ as well as the range condition*

$$R(\mathbf{I} + \varepsilon\mathcal{A}) \supset \overline{\mathrm{D}(\mathcal{A})}, \quad \varepsilon > 0.$$

*Proof.* It is well-known that, any $w$ locally convex function can be approximate uniformly by a sequence of smooth locally convex functions $w_n \in \mathcal{C}(\overline{\Omega})$ such that $w_n = \varphi$ on $\partial\Omega$. Then we can assume that $\det \mathrm{D}^2 w_n \in \mathcal{C}(\overline{\Omega})$, and so $w_n \in \mathrm{D}(\mathcal{A})$ (note that we merely have that $\det \mathrm{D}^2 w_n \rightharpoonup \det \mathrm{D}^2 w$ weakly in the sense of measures). Therefore $\overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi$. On the other hand, for each $h \in \overline{\mathrm{D}(\mathcal{A})}$ and $\varepsilon > 0$ by a simple adaptation of the proof of [11, Theorem 2.5] one proves the existence of a unique solution of

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f\left(\dfrac{u-h}{\varepsilon}\right) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

thus $\big(\mathbf{I} + \varepsilon\mathcal{A}\big)u = h$. □

Crandall–Liggett generation theorem and Corollary 1 enables us to show that $\mathcal{A}$ generates a nonlinear semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathbb{X}$

$$S(t)u_0 = \lim_{\substack{n\to\infty \\ \varepsilon n \to t}} (\mathbf{I} + \varepsilon\mathcal{A})^{-1} u_0 \quad \text{for any } u_0 \in \overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi,$$

uniformly for $t$ in bounded subsets of $]0,\infty[$. Furthermore, the mapping $t \mapsto S(t)u_0$ is continuous from $[0,\infty[$ into $\mathbb{X}$. In general the semigroup generated by such accretive operators $\mathcal{A}$ can be regarded as the so called "mild solution" of the Cauchy problem (2). A different characterization is possible.

**Proposition 1.** *Assume $u_0 \in \overline{\mathrm{D}(\mathcal{A})} \subset \mathbb{X}$. Then*

$$u(x,t) = S(t)u_0(x), \quad x \in \Omega,\ 0 < t < \infty, \tag{6}$$

*satisfies*

$$\begin{cases} u_t = f\left(\dfrac{\det \mathrm{D}^2 u}{g(|\mathrm{D}u|)}\right) & \textit{in } \Omega \times \mathbb{R}_+, \\ u = \varphi, & \textit{on } \partial\Omega \times \mathbb{R}_+, \\ u(\cdot, 0) = u_0, & \textit{in } \Omega, \end{cases}$$

*in the viscosity sense.*

*Proof.* Given $\mathrm{T} > 0$ arbitrary, assume $\Phi \in \mathcal{C}^2(\Omega\times]0,\mathrm{T}[)$ such that $\Phi(\cdot,t)$ is convex on $\Omega$ for all $t \in [0,\mathrm{T}]$ and $u - \Phi$ attains a strict local maximum at $(x_0,t_0) \in \Omega\times]0,\mathrm{T}[$. For each $\varepsilon > 0$, consider the step function $u_\varepsilon(t) \in \mathrm{D}(\mathcal{A})$ solving

$$\begin{cases} \det \mathrm{D}_x^2 u_\varepsilon(t+\varepsilon) = g\big(|\mathrm{D}_x u_\varepsilon(t+\varepsilon)|\big) f\left(\dfrac{u_\varepsilon(t+\varepsilon) - u_\varepsilon(t)}{\varepsilon}\right) & \text{in } \Omega, \quad t \in ]0,\mathrm{T}[, \\ u_\varepsilon(t) = u_0 \quad \text{if } 0 \leq t \leq \varepsilon. \end{cases} \tag{7}$$

We may assume $t_0 \neq k\varepsilon$ by appropriate choice of $\varepsilon$. Since $u_\varepsilon(t) \to S(t)u_0$ uniformely on $[0,\mathrm{T}]$ in $\mathbb{X}$ as $\varepsilon \to 0$, $u_\varepsilon(x,t+\varepsilon) - \Phi(x,t)$ has a local maximum at some point $(x_\varepsilon,t_\varepsilon)$, such that $(x_\varepsilon,t_\varepsilon) \in \Omega\times]0,\mathrm{T}[$, $x_\varepsilon \to x_0$, $t_\varepsilon \to t_0$, as $\varepsilon \to 0$. Hence,

$$\det \mathrm{D}_x^2 u_\varepsilon(t_\varepsilon+\varepsilon) - \det \mathrm{D}_x^2 \Phi \leq 0 \quad \text{at } x_\varepsilon,$$

according to the definition of $\mathcal{A}$ (note $u_\varepsilon(\cdot+\varepsilon) \in \mathrm{D}(\mathcal{A})$). Moreover, we have

$$\frac{u_\varepsilon(x_\varepsilon,t_\varepsilon+\varepsilon) - u_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon} \geq \frac{\Phi(x_\varepsilon,t_\varepsilon) - \Phi(x_\varepsilon,t_\varepsilon-\varepsilon)}{\varepsilon},$$

provided $\varepsilon$ small enough, whence

$$\det \mathrm{D}^2\Phi(x_\varepsilon,t_\varepsilon) \geq g\big(|\mathrm{D}_x\Phi(x_\varepsilon,t_\varepsilon-\varepsilon)|\big) f\left(\frac{\Phi(x_\varepsilon,t_\varepsilon-\varepsilon) - \Phi(x_\varepsilon,t_\varepsilon)}{\varepsilon}\right).$$

If we let $\varepsilon \to 0$, then, since $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$, we obtain

$$-\det \mathrm{D}_x^2\Phi(x_0,t_0) + g\big(|\mathrm{D}_x\Phi(x_0,t_0)|\big) f\big(\Phi_t(x_0,t_0)\big) \leq 0.$$

The opposite inequality follows when $u - \Phi$ attains a local minimum at $(x_0,t_0)$. □

**Remark 2.** Since $u(t) \in \overline{\mathrm{D}(\mathcal{A})}$ the property $0 \leq \det \mathrm{D}_x^2 u(t)$ holds in the generalized sense *a.e.* $t > 0$. Note that $S(t)\big(\overline{\mathrm{D}(\mathcal{A})}\big) \subset \overline{\mathrm{D}(\mathcal{A})}$ and so the time derivative $u_t$ must be understood in a large sense. Nevertheless, it is possible to apply different regularity results according $f$ (see, for instance, [8] and its references). In any case, at least $u_t$ is a nonnegative measure and $u(\cdot,t)$ is a locally convex function. □

So that, from the viscosity notion point of view we may rewrite (2) in the usual form (see (1) above).

3. **On the free boundary.** The first goal on the free boundary consists in studying how a possible region of flatness of the initial datum $u_0$ shrinks when $t$ increases. We use the notation of (4), for $u_0 \in \overline{\Omega}$ we denote

$$\mathrm{Flat}(u_0) = \bigcup_{\alpha} \mathrm{Flat}_{\alpha}(u_0).$$

Here by $u$ one denotes the viscosity solution of (1).

**Theorem 3.1** (Interior initial shrinking)**.** *Let* $\mathrm{Np} > 1$ *and*

$$\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_{\alpha}(u_0) \tag{8}$$

*for some* $\mathrm{R} > 0$*. Then there exists* $t^* = t^*(u_0) > 0$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_{\alpha}, x_0 \rangle + a_{\alpha}, \quad 0 \leq t < t^*.$$

*Proof.* We need a local supersolution $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$ where $\eta(t)$ is given by

$$\eta'(t) = \delta\big(\eta(t)\big)^{\mathrm{Np}}, \quad t > 0, \quad \text{for some } \delta > 0, \tag{9}$$

whose solution is

$$\eta(t) = \left[\frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}} - \frac{\delta}{\mathrm{Np}-1} t\right]^{-\frac{1}{\mathrm{Np}-1}}. \tag{10}$$

Note that $\eta(t)$ blows up at

$$t^*\big(\delta, \eta(0)\big) \doteq \frac{\mathrm{Np}-1}{\delta} \frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}}.$$

The spatial dependence is given by the function

$$\mathrm{U}(r) = \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}} r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0,$$

where the positive constant $\mathrm{C}_{\mathrm{p,N}}$ will be chosen below. From the regularity $\mathrm{D}u \in \mathrm{L}^{\infty}\big(0, \infty : \mathrm{L}^{\infty}(\Omega)\big)$, the convexity of solution enables us to choose $\eta(0)$, $\delta$ and R such that

$$\max_{\mathbf{B}_{\mathrm{R}}(x_0) \times \overline{\mathbb{R}}_+} u \leq \eta(0) \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}} \mathrm{R}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}. \tag{11}$$

We consider now the function

$$\mathrm{V}(x,t) = u(x,t) - \big(\langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha}\big)$$

for which

$$\mathrm{V}(x, 0) = 0, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),$$

(see (8)) and

$$\mathrm{V}(x,t) \leq \overline{\mathrm{U}}(x,t), \quad (x,t) \in \partial \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[$$

hold (see (11)). On the other hand, we have that

$$\frac{1}{\delta} \geq g\big(|\mathrm{D}_x \mathrm{V}(x,t)|\big) \geq 1, \quad (x,t) \in \mathbf{B}_{\mathrm{R}}(x_0) \times \big]0, t^*\big(\delta, \eta(0)\big)\big[$$

for a suitable choice of $\delta$ for which

$$-\frac{\big(\det \mathrm{D}_x^2 \mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x \mathrm{V}|\big)} \leq -\delta\big(\det \mathrm{D}_x^2 \mathrm{V}\big)^{\mathrm{p}} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \times \big]0, t^*\big(\delta, \eta(0)\big)\big[.$$

Moreover, following [11, Theorem 3.3] we may choose $\mathrm{C}_{\mathrm{p,N}}$ in order to

$$\mathrm{V}_t - \frac{\big(\det \mathrm{D}_x^2 \mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x \mathrm{V}(x,t)|\big)} = 0 \leq \overline{\mathrm{U}}_t - \frac{\big(\det \mathrm{D}_x^2 \overline{\mathrm{U}}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x \overline{\mathrm{U}}(x,t)|\big)} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \times \big]0, t^*\big(\delta, \eta(0)\big)\big[$$

holds. Then the proof concludes by the comparison

$$0 \le \mathrm{V}(x,t) \le \overline{\mathrm{U}}(x,t), \quad (x,t) \in \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[.$$

□

**Remark 3.** From the above proof it follows the estimate

$$\limsup_{t\to 0} \operatorname{dist}\big(\mathcal{F}_\alpha(t), \mathcal{F}_\alpha(0)\big) t^{-\frac{1}{\mathrm{Np}-1}} \in \mathbb{R}_+,$$

on the shrinking of the free boundary $\mathcal{F}_\alpha(t) = \partial\{(x,t): \; u(x,t) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha\}$. □

We prove that under a big flatness the corner remains during a finite time.

**Theorem 3.2** (Finite waiting time)**.** *Let* $\mathrm{Np} > 1$ *and let* $x_0 \in \Omega$ *be such that*

$$u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{12}$$

*for some suitable positive constants* K *and* R*. Then, there exists* $\widetilde{t} = \widetilde{t}(x_0)$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha \quad \textit{if } 0 \le t < \widetilde{t}.$$

*Proof.* Now the function $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$ is a local supersolution, where $\eta(t)$ as in (9) and

$$\mathrm{U}(r) = \mathrm{C}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \ge 0,$$

for $\mathrm{C} > 0$. Then

$$-\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\Big[\big(\mathrm{U}'(r)\big)^{\mathrm{N}}\Big] + \delta\big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{p}}} = \delta\left[1 - \frac{\lambda_{\mathrm{p}}}{\delta}\mathrm{C}^{\frac{\mathrm{Np}-1}{\mathrm{p}}}\right]\big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{p}}} > 0$$

if $0 < \mathrm{C} < \left(\dfrac{\delta}{\lambda_{\mathrm{p}}}\right)^{\frac{\mathrm{p}}{\mathrm{Np}-1}}$. Then the reasonings are similar to those of the proof of Theorem 3.1 because now (12) provides the inequality

$$\mathrm{V}(x,0) \le \overline{\mathrm{U}}(x,0).$$

□

**Remark 4.** A similar waiting time result was obtained in [4] for the special case $\mathrm{p} = 1$ and $\mathrm{N} = 2$. Essentially, their assumption is $u_0 \in \mathcal{C}^4(\Omega)$. Note that for this special case $\mathrm{p} = 1$ and $\mathrm{N} = 2$ the condition (12) becomes

$$u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^4, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0). \tag{13}$$

In particular, any $\mathrm{C}^4$ partially flat function satisfies (13) on the boundary of $\mathcal{F}_\alpha(0)$. Our result can be regarded as a local and generalized version of the result of [4]. □

Our next goal is the study of cases in which $\mathcal{F}_\alpha(t)$ is shrinking.

**Theorem 3.3.** *Let* $\mathrm{Np} > 1$ *and assume* $x_0 \in \mathrm{Flat}_\alpha(u_0)$*. Then there exists* $\widehat{t} > 0$ *such that*

$$u(x_0, t) > \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha, \quad \textit{for any } t > \widehat{t}. \tag{14}$$

As a matter of fact, it is enough to show that

$$u(x_0, \widehat{t}) > \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha, \tag{15}$$

because since $u_t \ge 0$ we get (14) for any $t > \widehat{t}$. The reasonings are near to the ones those used in [3], [8] or [14] for other problems. In order to prove Theorem 3.3 we shall use other suitable supersolution based on the self–similar solution of the Cauchy problem associated to $g(s) \equiv \delta$, for suitable $\delta \ge 1$. We start with the adimensionalization of the equation

**Lemma 3.4.** *Let $u(x,t)$ be a viscosity solution of*

$$u_t = \big(\det \mathrm{D}_x^2 u\big)^p \quad \textit{in } \mathbb{R}^{\mathrm{N}} \times \mathbb{R}_+. \tag{16}$$

*Then the change of scales $x' = \mathrm{L}x,\ t' = \mathrm{T}t$ allows to define*

$$u'(x',t') = \mathrm{L}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \mathrm{T}^{-\frac{1}{\mathrm{Np}-1}} u(x,t)$$

*which is also a viscosity solution of (16).* □

A deeper conclusion on the self–similar solution of (16) is the following

**Theorem 3.5.** *Assume* $\mathrm{Np} > 1$. *Then, there exists a family of convex compactly supported similarity solutions of (16) given by*

$$u(r,t;\sigma,\beta) \doteq t^{-\sigma}\Lambda(\eta), \quad \eta \doteq \frac{r}{t^{\beta}} \quad \textit{for some suitable } \sigma,\ \beta > 0.$$

The proof of Theorem 3.5 requires the study of the corresponding phase–plane system

$$\begin{cases} \dfrac{dq}{d\eta} = -\left[\dfrac{\mathrm{N}^{\mathrm{p}}\beta}{\eta}\big[\dfrac{\sigma}{\beta}\Lambda(\eta) + \eta q^{\frac{1}{\mathrm{N}}}\big]\right]^{\frac{1}{\mathrm{p}}}, \\ \dfrac{d\Lambda}{d\eta} = q^{\frac{1}{\mathrm{N}}}, \end{cases}$$

where $q = (\Lambda')^{\mathrm{N}}\big(\mathrm{sign}\ \Lambda'\big)$. We indicate that the proof is a non–difficult variation of some results in the literature (see, for instance, [1] and [14]). See also [7] for the case of the "focusing problem" associated to (16).

*Proof of Theorem 3.3.* Assume again that the solution $u$ has a bounded gradient $\mathrm{D}u \in \mathrm{L}^{\infty}\big(\Omega\times]0,\widehat{t}[\big)$ for any given $\widehat{t}$. So $\mathrm{V}(x,t) = u(x,t) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ verifies

$$\mathrm{V}_t(x,t) - \delta\big(\det \mathrm{D}_x^2 \mathrm{V}(x,t)\big)^{\mathrm{p}} \geq 0,$$

for a suitable $\delta > 1$. By means of simple scaling we may consider the above equation with $\delta = 1$. Let $x_0 \in \Omega \cap \mathrm{Flat}_\alpha(u_0)$. Then we consider $\sigma$ and $\beta$ for which

$$u\big(|x - x_0|, t; \sigma, \beta\big) \leq \mathrm{V}(x,t) \quad \text{for any } t \geq 0 \text{ and } x \in \mathbf{B}_{\mathrm{R}}(x_0)$$

for some $\mathrm{R} > 0$ (see Theorem 3.5). Since $u\big(r_0, t; \sigma, \beta\big) > 0$ for any $r_0 > 0$ once that $t$ is large enough, we conclude the result. □

**Remark 5.** Other properties on the solution can be obtained from the family of self-similar solutions given in Theorem 3.5. □

Our last goal is the study a kind of asymptotic behavior of $u$ (see [2] for other qualitative properties obtained via perimeter symmetrization techniques). We start by proving that if $\mathrm{Np} \geq 1$ then the stabilization to a stationary solution requires an infinite time. From now on, any locally convex function on $\overline{\Omega}$ such that $\det \mathrm{D}^2 h = 0$, *a.e.* in $\Omega$, will be called a *flat convex function*. By simplicity we assume $g\big(|\mathbf{p}|\big) \equiv 1$.

**Theorem 3.6.** *Assume* $\mathrm{Np} \geq 1$. *Let $\overline{h}$ be a flat convex function on $\overline{\Omega}$ such that $\varphi \leq \overline{h}$ on $\partial\Omega$. Then for each $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ such that $u_0 \leq \overline{h}$ on $\overline{\Omega}$ and $u_0 < \overline{h}$ in some set $\Omega' \subset \Omega$ with positive measure, there exists a positive constant $\mathrm{C}_{\Omega'}$ such that*

$$\begin{cases} \liminf\limits_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big)t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} > 1, \\ \liminf\limits_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big)e^{t} \geq \mathrm{C}_{\Omega'} & \textit{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega', \tag{17}$$

*where $u$ is the viscosity solution of (1). Analogously, let $\underline{h}$ be a flat convex function on $\overline{\Omega}$ such that $\varphi \geq \underline{h}$ on $\partial\Omega$ verifying $u_0 \geq \underline{h}$ on $\overline{\Omega}$ and $u_0 > \overline{h}$ in some set $\Omega' \subset \Omega$ with positive measure. Then there exists a positive constant $\mathrm{C}_{\Omega'}$ such that*

$$\begin{cases} \liminf\limits_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big) t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} > 1, \\ \liminf\limits_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big) e^{t} \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega'. \tag{18}$$

*Proof.* Let $\overline{h}$ be a flat function such that $\varphi \leq \overline{h}$ on $\partial\Omega$. Then, one has

$$\overline{h}_t = \big(\det \mathrm{D}_x^2 \overline{h}\big)^{\mathrm{p}}$$

whence $\underline{u}(x,t) = u(x,t) - \overline{h}(x)$ verifies

$$\begin{cases} \underline{u}_t = \big(\det \mathrm{D}_x^2 \underline{u}\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \quad \text{(in the viscosity sense)}, \\ \underline{u} \leq 0,\ \underline{u} \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \\ \underline{u}(x,t) \leq 0, & (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+. \end{cases} \tag{19}$$

Here the key idea is to consider the auxiliar problem

$$\begin{cases} \phi'(t) + \dfrac{2}{\mathrm{m}} \big(k\phi(t)\big)^{\mathrm{Np}} = 0, \quad t \geq 0, \\ \phi(0) = 1, \quad \phi(\infty) = 0, \end{cases} \tag{20}$$

whose solution is

$$\phi(t) = \begin{cases} \left[1 + \dfrac{2}{\mathrm{m}} \dfrac{k^{\mathrm{Np}}}{\mathrm{Np}-1} t\right]^{-\frac{1}{\mathrm{Np}-1}}, & \text{if } \mathrm{Np} > 1, \\ e^{-\frac{2k}{\mathrm{m}} t}, & \text{if } \mathrm{Np} = 1, \end{cases}$$

where $k$ is a positive constant to be choosen and m is a positive constant such that $\overline{h} - u_0 \geq \mathrm{m}$ in some ball $\mathbf{B}_{2\mathrm{R}} \subset \Omega$. Let $\psi(x_1) \in \mathcal{C}^2$ be such that

$$\begin{cases} \psi(x_1) = 0, \quad x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ -\mathrm{m} < \psi(x_1) < -\dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, \quad x \in \mathbf{B}_{\mathrm{R}}, \\ -\dfrac{\mathrm{m}}{2} < \psi(x_1) < 0 \quad \text{and} \quad \psi''(x_1) \leq 0, \quad x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}. \end{cases}$$

Then the function

$$W(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} W(x,t) < 0, & (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \overline{\mathbb{R}}_+, \\ W(x,t) = 0, & (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \\ W(x,0) = \phi(0)\psi(x_1) \geq -\mathrm{m} > \big(u_0 - h\big)(x), & x \in \mathbf{B}_{2\mathrm{R}}, \\ W(x,0) = \phi(0)\psi(x_1) = 0 \geq \big(u_0 - h\big)(x), & x \in \overline{\Omega} \setminus \overline{\mathbf{B}}_{2\mathrm{R}}. \end{cases}$$

because $\phi(0) = 1$. Moreover, from (20) we get

$$W_t(x,t) + \big(-\det \mathrm{D}_x^2 W(x,t)\big)^{\mathrm{p}} \doteq r(x,t)$$

where

$$r(x,t) \geq \begin{cases} \big(k\phi(t)\big)^{\mathrm{Np}} + \Big(-\big(\phi(t)\big)^{\mathrm{Np}}\Big) \big(\psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \mathbf{B}_{\mathrm{R}}, \\ \phi'(t)\psi(x_1) + \big(\phi(t)\big)^{\mathrm{Np}} \big(-\psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \Omega \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

for $t > 0$, provided large $k$. Then, from (19), comparison results lead to

$$u(x,t) - \overline{h}(x) \leq W(x,t) \leq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$\overline{h}(x) - u(x,t) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+.$$

We may repeat the reasoning with a flat function $\underline{h}$ such that $\varphi \geq \underline{h}$ on $\partial\Omega$. So, the function $\overline{u}(x,t) = u(x,t) - \underline{h}(x)$ verifies

$$\begin{cases} \overline{u}_t = \left(\det \mathrm{D}_x^2 \overline{u}\right)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ \overline{u} \geq 0, \ \overline{u} \not\equiv 0 & \text{on } \left(\partial\Omega \times \mathbb{R}_+\right) \cup \left(\overline{\Omega} \times \{0\}\right), \end{cases}$$

in the viscosity sense and

$$\overline{u}(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Now, we consider a non negative function $\psi(x_1) \in \mathcal{C}^2$ such that

$$\begin{cases} \psi(x_1) = 0, \quad x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ \dfrac{\mathrm{m}}{2} < \psi(x_1) < \mathrm{m} \quad \text{and} \quad \psi''(x_1) \leq 0, \quad x \in \mathbf{B}_{\mathrm{R}}, \\ 0 < \psi(x_1) < \dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, \quad x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

where m is a positive constant such that $u_0 - \underline{h} \geq \mathrm{m}$ in some $\mathbf{B}_{2\mathrm{R}} \subset \Omega$. Arguing as above one proves that the function

$$w(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} w_t(x,t) + \left(-\det \mathrm{D}_x^2 w(x,t)\right)^{\mathrm{p}} \leq 0 & \text{in } \Omega \times \mathbb{R}_+, \\ w \leq \overline{u} & \text{on } \left(\partial\Omega \times \mathbb{R}_+\right) \cup \left(\overline{\Omega} \times \{0\}\right), \end{cases}$$

provided $k$ is large enough. Then, we obtain

$$u(x,t) - \underline{h}(x) \geq w(x,t) \geq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$u(x,t) - \underline{h}(x) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+.$$

□

Note that Theorem 3.6 implies a kind of non flattened global retention property:

$$\begin{cases} u_0(x) < \overline{h}(x), \ x \in \Omega' \subset \Omega \quad \Rightarrow \quad u(x,t) < \overline{h}(x), \ x \in \Omega' \text{ for all } t \geq 0 \\ \underline{h}(x) < u_0(x), \ x \in \Omega' \subset \Omega \quad \Rightarrow \quad \underline{h}(x) < u(x,t), \ x \in \Omega' \text{ for all } t \geq 0. \end{cases} \tag{21}$$

Clearly, the second retention property also follows from $u_t \geq 0$. Theorem 3.6 is according to the results of [5] and [4] for parametric compact surfaces.

Our final result in this paper shows that when $\mathrm{Np} < 1$ the asymptotic behavior is very fast. It is the property of "finite global flattened time" which we prove by means of some ideas used by first time in [9]. Again, we assume $g\left(|\mathbf{p}|\right) \equiv 1$.

**Theorem 3.7.** *Let* $h(x) = \langle \mathbf{p}, x\rangle + a$ *on* $\overline{\Omega}$ *and suppose* $\varphi = h$ *in the definition of the operator* $\mathcal{A}$. *Assume* $\mathrm{Np} < 1$. *Then for each* $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ *such that* $u_0 \leq h$ *on* $\overline{\Omega}$ *there exists a time* $\mathrm{T}_0$, *depending on* $h - u_0$, *such that*

$$u(x,t) = \langle \mathbf{p}, x\rangle + a, \quad x \in \overline{\Omega}, \quad t \geq \mathrm{T}_0.$$

*Proof.* Let us denote $u_h(x,t) = u(x,t) - h(x)$. One verifies (19), thus

$$\begin{cases} (u_h)_t = \left(\det \mathrm{D}_x^2 u_h\right)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ u_h \leq 0, \ u_h \not\equiv 0 & \text{on } \left(\partial\Omega \times \mathbb{R}_+\right) \cup \left(\overline{\Omega} \times \{0\}\right), \end{cases}$$

in the viscosity sense, whence

$$u_h(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

In fact, if $u_0 = h$ one derives the coincidence

$$u_h(x,t) = 0 \quad \text{for any } (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Suppose $u_0 \leq h,\ u_0 \not\equiv h$. It is clear that the "finite flattened time property" is strongly based on the initial value problem

$$\mathrm{m}\Theta'(t) = \big(2\Theta(t)\big)^{\mathrm{Np}}, \quad t \geq 0, \qquad \Theta(0) = 0$$

whose solution is

$$\Theta(t) = \left(\frac{2^{\mathrm{Np}}(1-\mathrm{Np})}{\mathrm{m}}\right)^{\frac{1}{1-\mathrm{Np}}} t^{\frac{1}{1-\mathrm{Np}}},$$

provided $\mathrm{Np} < 1$ and m is a positive constant. Then, for each $\mathrm{T}_0 > 0$ the function

$$\mathcal{T}(t) = \Theta\big(\mathrm{T}_0 - t\big)_+ \quad \text{if } 0 < t \leq \mathrm{T}_0,$$

satisfies

$$\mathcal{T}'(t)\mathrm{m} + \big(2\mathcal{T}(t)\big)^{\mathrm{Np}} = 0, \quad 0 < t \neq \mathrm{T}_0. \tag{22}$$

On the other hand, for $\mathrm{R} > 0$ large, we consider the function

$$\zeta(x) = 2^{\mathrm{N}-1}\big(x_1^2 - \mathrm{R}^2\big) \leq 0, \quad x \in \overline{\Omega}$$

which verifies

$$\begin{cases} -m < \zeta(x) < -\mathrm{M} < 0, \quad \mathrm{x} \in \overline{\Omega}, \quad -\mathrm{m} \doteq \min\limits_{\mathrm{x}\in\overline{\Omega}} \zeta, \ -\mathrm{M} \doteq \max\limits_{\mathrm{x}\in\overline{\Omega}} \zeta \\ \det \mathrm{D}^2\zeta(x) \equiv 2^{\mathrm{N}}, \quad x \in \overline{\Omega}. \end{cases}$$

It enables us to define

$$V(x,t) = \mathcal{T}(t)\zeta(x), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

for which $v(x,t) \leq 0,\ (x,t) \in \partial\Omega \times \mathbb{R}_+$ and $V(x,0) \leq -\Theta\big(\mathrm{T}_0\big)\mathrm{M},\ \mathrm{x} \in \overline{\Omega}$, whence

$$v(x,0) \leq \big(u_0 - h\big)(x), \quad x \in \overline{\Omega},$$

provided $\mathrm{T}_0 = \Theta^{-1}\big(\|h - u_0\|_\infty \mathrm{M}^{-1}\big)$. Moreover, for each $(x,t) \in \Omega \times \mathbb{R}_+$ one has

$$v_t(x,t) + \big(-\det \mathrm{D}_x^2 v(x,t)\big)^{\mathrm{p}} \leq -\mathcal{T}'(t)\mathrm{m} + f_{\mathrm{p}}^{-1}\left(-2\big(\mathcal{T}(t)\big)^{\mathrm{N}}\right) = 0$$

(see (22)). Thus

$$v_t(x,t) - \big(\det \mathrm{D}_x^2 v(x,t)\big)^{\mathrm{p}} \leq 0, \quad (x,t) \in \Omega \times [0,\mathrm{T}].$$

This function $V$ can be considered as an eventual test function for the viscosity solution $u_h$ (see (19)). Then we deduce

$$v(x,t) \leq u_h(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times [0,\mathrm{T}[,$$

whence the finite global flattened time property holds. □

**Remark 6.** In fact, the condition $\mathrm{Np} < 1$ is also necessary for the property of "finite global flattened time" as we have shown in Theorem 3.6. □

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