A gradient estimate to a degenerate parabolic equation with a singular absorption term: the global quenching phenomena

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Abstract. We prove global existence of nonnegative solutions to the one dimensional degenerate parabolic problems containing a singular term. We also show the global quenching phenomena for L^1 initial datums. Moreover, the free boundary problem is considered in this paper.

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1 Introduction

This paper deals with the one dimensional degenerate parabolic equation on a given open bounded interval $I = (L_1, L_2)$

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + \chi_{\{u>0\}}u^{-\beta} = 0 & \text{in } I \times (0,\infty), \\ u(L_1,t) = u(L_2,t) = 0 & t \in (0,\infty), \\ u(x,0) = u_0(x) & \text{in } I, \end{cases} \tag{1}$$

where $\beta \in (0,1)$, $p > 2$, $u_0 \geq 0$ and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points (x,t) where $u(x,t) > 0$. The absorption term $\chi_{\{u>0\}}u^{-\beta}$ becomes singular when u is near to 0 (but note that we are imposing $\chi_{\{u>0\}}u^{-\beta} = 0$ if $u = 0$). We shall also consider the associated Cauchy problem (formally equivalent (1) when $I = \mathbb{R}$).

Problem (1) can be considered as a limit model of a class of problems arising in Chemical Engineering corresponding to catalyst kinetics of Langmuir-Hinshelwood type(see, e.g. [24] p. 68). Here we assume that the diffusion coefficient, $D = |u_x|^{p-2}$, depends on the gradient of the concentration. From a mathematical point of view, the pioneering papers on this class of models were due to Phillips [22] and Bandle and Brauner [2], for the case $p = 2$ (even posed on an open bounded set Ω of $\mathbb{R}^N$). Besides, other authors also considered the semilinear case ($p = 2$); see, e.g. [20], [8], [25], [10], [7] and their references. The case of quasilinear diffusion operators was already considered in [17] (for a different diffusion term). We also mention here the case of the quasilinear problem of porous medium type studied in [18]. Recently, problem (1) was analyzed in the paper [14] (even under a more general formulation, see also the study of the associated stationary problem [16]) but the proof of the existence of a weak solution (as limit of solutions of approximate non-singular problems) is not completely well justified. One of the main goal of this paper is to get some sharper a priori estimates on the (spatial) gradient of the approximate solutions to pass to the limit in the approximation of the singular term of the equation.

Roughly speaking, the a priori gradient estimate that we shall prove is of the type

$$|\partial_x u(x,t)| \leq Cu^{1-\frac{1}{\gamma}}(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty), \tag{2}$$

for a suitable constant $C > 0$, and the exponent

$$\gamma = \frac{p}{p+\beta-1}. \tag{3}$$

Estimates of this type were already obtained (for the case of $p = 2$ and bounded initial data) in [22], [8] and [25]. The degeneracy of the diffusion operator when $p > 2$ leads, obviously, to a considerable amount of additional technical difficulties (see, e.g. the study of the unperturbed equation made in [15]). In addition, as in [7], we want to consider also the case of possibly unbounded initial data. Let us mention that the exponent γ given by (3) plays a fundamental role. It arises, in a natural way, when considering the associate stationary problem. It is not difficult to show that in that case the estimate (2) becomes an equality, for a suitable constant C. This is the reason why some authors call to this type of gradient estimates as "sharp gradient estimates" (see, e.g., [3] for a general exposition of this type of estimates).

As mentioned before, a very delicate point is to require a suitable integrability to the singular term of the equation. So, before stating our main results, let us define the notion of weak solution of equation (1) which we shall consider in this paper.

Definition 1 *Given $0 \leq u_0 \in L^1(I)$, a function u is called a weak solution of (1) if $u \in L^p_{loc}(0,\infty; W_0^{1,p}(I)) \cap L^\infty_{loc}(\overline{I} \times (0,\infty)) \cap \mathcal{C}([0,\infty); L^1(I))$, $u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,\infty))$, and u satisfies equation (1) in the sense of distributions $\mathcal{D}'(I \times (0,\infty))$, i.e:*

$$\int_0^\infty \int_I -u\phi_t + |u_x|^{p-2}u_x\phi_x + \chi_{\{u>0\}}u^{-\beta}\phi \, dxdt = 0, \quad \forall \phi \in \mathcal{C}_c^\infty(I \times (0,\infty)). \tag{4}$$

Our main existence result indicates also some additional regularity information on the weak solution:

Theorem 2 *Let $p > 2$, and $0 \leq u_0 \in L^1(I)$. Then, there exists a maximal weak solution $u \in L^p(0,T; W_0^{1,p}(I)) \cap \mathcal{C}([0,T]; L^1(I)$ of equation (1), i.e, for any weak solution v of equation (1) we have $v \leq u$ a.e in $I \times (0,\infty)$. Besides, u satisfies the additional regularity implied by the following estimates:*

(i) There is a positive constant $C = C(p, |I|)$ such that

$$\|u(.,t)\|_{L^\infty(I)} \leq C.t^{-\frac{1}{\lambda}}.\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}, \quad \textit{for } t \in (0,\infty),\ \lambda = 2(p-1). \tag{5}$$

(ii) For any $\tau > 0$, there exists a positive constant $C = C(\beta, p, |I|)$ such that

$$|\partial_x u(x,t)| \leq Cu^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + 1\right), \quad \textit{for a.e } (x,t) \in I \times (\tau,\infty), \tag{6}$$

(iii) For any $\tau > 0$ there is a positive constant $C = C(\beta, p, \tau, |I|, \|u_0\|_{L^1(I)})$ such that

$$|u(x,t) - u(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall x, y \in \overline{I}, \quad \forall t, s > \tau. \tag{7}$$

In fact, we shall derive previously estimates (6) and (7) for the case of bounded initial data. We also point out that conclusion (7) implies that u is continuous up to the boundary. This result answers an open question stated in the introduction of [25].

A second goal of this paper concerns the study of the quenching phenomenon of solutions. This property arises due to the presence of the singular term (even if $p = 2$): the absorption is stronger than the diffusion and thus there are internal regions of the $(x,t)-$space where the solutions vanishes. We shall prove here that this property remains valid also for $p > 2$. We start by proving that, even if there is a lack of uniqueness of solutions (see [25] for the case $p = 2$), any nonnegative weak solution of equation (1) vanishes in finite time even starting with a positive unbounded initial data:

Theorem 3 *Let $p > 2$, and $0 \leq u_0 \in L^1(I)$. Let v be any weak solution of equation (1). Then, there is a finite time $T_0 = T_0(\beta, p, |I|, \|u_0\|_{L^1(I)})$ such that*

$$v(x,t) = 0, \quad \textit{for a.e } x \in I, \ \textit{for } t \geq T_0.$$

We shall also prove that the quenching phenomenon takes place locally in space (previously to do that globally in spaces for a time large enough). In contrast to the energy method used, to this end, in the paper [10] we shall use here a suitable comparison argument showing the "uniform localization property" for solutions of the associated Cauchy problem. This also leads to a similar conclusion for the case of a bounded interval I, problem (1), once that I is large enough (depending on the support of u_0 and $\|u_0\|_{L^1(I)}$).

The paper is organized as follows: Section 2 is devoted to prove the a priori gradient estimate, which is the main key of proving the existence of solution. In section 3, we shall give the complete proof of Theorem 2. Section 4 is devoted to prove Theorem 3. Finally, Section 5 will concerns with the consideration of the associated Cauchy problem: after proving the existence of a maximal weak solution we study the free boundary defined as the boundary of the support of the solution, proving the "uniform localization property" and the extension of the global in time quenching phenomenon.

Several notations which will be used through this paper are the following: we denote by C a general positive constant, possibly varying from line to line. Furthermore, the constants which depend on parameters will be emphasized by using parentheses. For example, $C = C(p, \beta, \tau)$ means that C only depends on p, β, τ. We also denote by $B_r(x) = (x - r, x + r)$ to the open ball with center at x and radius $r > 0$.

2 Gradient estimates

In this section, we shall adapt to our framework the now classical Bernstein's technique to obtain an a priori estimate on $|u_x|$. As mentioned at the Introduction, our estimate of $|u_x|$ will involve a certain power of u. We recall that for the semilinear case, $p = 2$, it is well known that such type of gradient estimates plays a crucial role in proving the existence of solution (see, e.g. [22], [8], [25], and [18]). In the sequel, we shall denote simply as gradient estimate to such estimate on $|u_x|$.

To be similar to the case $p = 2$, we shall establish previously the gradient estimate for the solutions of a regularized family of problems. For any $\varepsilon > 0$, we define

$$g_\varepsilon(s) = \psi_\varepsilon(s)s^{-\beta}, \text{ with } \psi_\varepsilon(s) = \psi(\frac{s}{\varepsilon}),$$

and where $\psi \in \mathcal{C}^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$ is a non-decreasing function such that

$$\psi(s) = \begin{cases} 0, & \text{if } s \leq 1, \\ 1, & \text{if } s \geq 2. \end{cases}$$

Now, for a given initial data $0 \leq z_0 \in \mathcal{C}_c^\infty(I)$, $z_0 \neq 0$, we consider the regularizing problems

$$\begin{cases} \partial_t z - (a(z_x)z_x)_x + g_\varepsilon(z) = 0, & \text{in } I \times (0, \infty), \\ z(L_1, t) = z(L_2, t) = \eta, & t \in (0, \infty), \\ z(x, 0) = z_0(x) + \eta, & x \in I, \end{cases} \tag{8}$$

where $0 < \varepsilon < \|z_0\|_{L^\infty(I)}$, $0 < \eta < \varepsilon$, and

$$a(u) = b(u)^{\frac{p-2}{2}}, \quad b(u) = |u|^2 + \eta^\alpha, \text{ with } \alpha > 0 \text{ is chosen later.}$$

So, we replace the quasilinear coefficient $|z_x|^{p-2}$ by its regularization $a(z_x)$ and the singular term by its truncation-regularization $g_\varepsilon(z)$. Equation (8) can be understood as a regularization of equation (1).

In this framework, the gradient estimate can be presented as follows:

Lemma 4 *Let $\alpha > \frac{2(\gamma-1)}{\gamma}$, and z_0 be above. Then, there exists a unique classical solution $z_{\varepsilon,\eta}$ of equation (8). Moreover, there is a positive constant $C(\beta,p)$ such that*

$$|\partial_x z_{\varepsilon,\eta}(x,\tau)| \le C(\beta,p).z_{\varepsilon,\eta}^{1-\frac{1}{\gamma}}(x,\tau)\left(\tau^{-\frac{1}{p}}.\|z_0\|_{L^\infty(I)}^{\frac{1+\beta}{p}}+1\right), \quad \forall (x,\tau)\in I\times(0,\infty). \tag{9}$$

Remark 5 *Estimate (9) extends the similar ones for $p=2$, in [22], [8], and [25].*

Proof: Thanks to some classical results (see, e.g., [19], [26] and [27]), there exists a unique solution $z_{\varepsilon,\eta}\in\mathcal{C}^\infty(\overline{I}\times[0,\infty))$ of equation (8). For sake of brevity, let us drop dependence on ε,η in the notation and put $z=z_{\varepsilon,\eta}$.

It is clear that η (resp. $\|z_0\|_{L^\infty(I)}+\eta$) is a sub-solution (resp. super-solution) of equation (8). Then, the comparison principle yields

$$\eta \le z \le \|z_0\|_{L^\infty(I)}+\eta \le 2\|z_0\|_{L^\infty(I)}, \quad \text{in } I\times(0,\infty). \tag{10}$$

For any $0<\tau<T<\infty$, let us consider a test function $\xi(t)\in\mathcal{C}_c^\infty(0,\infty)$, $0\le\xi(t)\le 1$ such that

$$\xi(t)=\begin{cases} 1, & \text{on } [\tau,T], \\ 0, & \text{outside } (\frac{\tau}{2},T+\frac{\tau}{2}). \end{cases}, \quad \text{and } |\xi_t|\le\frac{c_0}{\tau},$$

and put

$$z=\varphi(v)=v^\gamma, \quad w(x,t)=\xi(t)v_x^2.$$

We briefly denote

$$a=a(z_x), a_x=(a(z_x))_x, a_{xx}=(a(z_x))_{xx}.$$

Then, we have

$$w_t-aw_{xx}=\xi_t.v_x^2+2\xi v_x(v_t-av_{xx})_x-2\xi a v_{xx}^2+2\xi a_x v_{xx}. \tag{11}$$

From the equation satisfied by z we get

$$v_t-av_{xx}=a_xv_x+av_x^2\frac{\varphi''}{\varphi'}-\frac{g_\varepsilon(\varphi)}{\varphi'},$$

where φ' (resp. φ'') is the first (resp. second) derivative of φ. By combining the last two equations, we have

$$w_t-aw_{xx}=\xi_tv_x^2+2\xi v_x\left(a_xv_x+av_x^2\frac{\varphi''}{\varphi'}-\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x-2\xi av_{xx}^2+2\xi a_xv_{xx}.$$

Now, we define

$$L=\max_{\overline{I}\times[0,\infty)}\{w(x,t)\}.$$

If $L=0$, then the conclusion (9) is trivial, and $|z_x(x,\tau)|=0$, in I. If $L>0$, then the function w must attain its maximum at a point $(x_0,t_0)\in I\times(\frac{\tau}{2},T+\frac{\tau}{2})$ since $w(x,t)=0$ on $\partial I\times(0,\infty)$ and $w(.,t)=0$ outside the interval $(\frac{\tau}{2},T+\frac{\tau}{2})$. This implies

$$\begin{cases} w_t(x_0,t_0)=w_x(x_0,t_0)=0, \\ \text{and} \\ 0\ge w_{xx}(x_0,t_0)=2\xi(t_0)v_{xx}^2(x_0,t_0)+2\xi(t_0)v_x.v_{xxx}(x_0,t_0), \end{cases}$$

so we obtain

$$v_x.v_{xxx}(x_0,t_0) \le 0. \tag{12}$$

Since $v_x(x_0,t_0) \neq 0$, we get

$$w_x(x_0,t_0) = 0 \text{ if and only if } v_{xx}(x_0,t_0) = 0. \tag{13}$$

At the point (x_0,t_0), (11) and (13) provide us

$$0 \le w_t - aw_{xx} = \xi_t v_x^2 + 2\xi v_x \left(a_{xx}v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + av_x^2 \left(\frac{\varphi''}{\varphi'}\right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x \right).$$

$$0 \le \frac{1}{2}\xi_t \xi^{-1} v_x^2 + v_x \left(a_{xx}v_x + a_x v_x^2 \frac{\varphi''}{\varphi'} + av_x^2 \left(\frac{\varphi''}{\varphi'}\right)_x - \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x \right).$$

Or

$$-av_x^3 \left(\frac{\varphi''}{\varphi'}\right)_x \le \frac{1}{2}\xi_t\xi^{-1}v_x^2 + a_{xx}v_x^2 + a_x v_x^3 \frac{\varphi''}{\varphi'} - v_x \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x. \tag{14}$$

By the fact $v_{xx}(x_0,t_0) = 0$ and computation, we have

$$\begin{cases} a_x(z_x)(x_0,t_0) = (p-2)b^{\frac{p-4}{2}}(z_x)\varphi'\varphi''v_x^3, \\ \left(\frac{\varphi''}{\varphi'}\right)_x = \left(\frac{\varphi'''\varphi' - \varphi''^2}{\varphi'^2}\right) v_x = -(\gamma-1)v^{-2}v_x, \end{cases} \tag{15}$$

and

$$a_{xx}(z_x)(x_0,t_0) = (p-2)(p-4)b^{\frac{p-6}{2}}(z_x)(\varphi'.\varphi'')^2 v_x^6 + (p-2)b^{\frac{p-4}{2}}(z_x)(\varphi''^2 + \varphi'\varphi''')v_x^4 + \\ (p-2)b^{\frac{p-4}{2}}(z_x)\varphi'^2 v_x v_{xxx}.$$

By (12), we obtain from the last equation

$$a_{xx}(z_x)(x_0,t_0) \le (p-2)(p-4)b^{\frac{p-6}{2}}(z_x)(\varphi'.\varphi'')^2 v_x^6 + (p-2)b^{\frac{p-4}{2}}(z_x)(\varphi''^2 + \varphi'\varphi''')v_x^4. \tag{16}$$

Next, we have

$$v_x \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x = (g'_\varepsilon - g_\varepsilon \frac{\varphi''}{\varphi'^2})v_x^2 = \left(\psi'_\varepsilon(\varphi).\varphi^{-\beta}(v) - (\beta + \frac{\gamma-1}{\gamma})\psi_\varepsilon(\varphi)v^{-(1+\beta)\gamma} \right) v_x^2.$$

Since $\psi'_\varepsilon(.) \ge 0$ and $0 \le \psi_\varepsilon \le 1$, we get

$$v_x \left(\frac{g_\varepsilon(\varphi)}{\varphi'}\right)_x \ge -(\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2. \tag{17}$$

Inserting (15), (16), and (17) into (14) yields

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + (p-2)(p-4)b^{\frac{p-6}{2}}(z_x)(\varphi'\varphi'')^2 v_x^8 + (p-2)b^{\frac{p-4}{2}}(z_x)(2\varphi''^2 + \varphi'\varphi''')v_x^6 + \\ (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 \ge (\gamma-1)v^{-2}a(z_x).v_x^4. \tag{18}$$

It is useful to introduce the notation

$$\mathcal{B} := (p-2)(p-4)b^{\frac{p-6}{2}}(z_x)(\varphi'\varphi'')^2 v_x^8 + (p-2)b^{\frac{p-4}{2}}(z_x)(2\varphi''^2 + \varphi'\varphi''')v_x^6.$$

Next, we rewrite $\mathcal{B}$ as follows

$$\begin{gathered}\mathcal{B} = (p-2)b^{\frac{p-6}{2}}(z_x)v_x^6\left((p-4).(\varphi'\varphi'')^2.v_x^2 + (2\varphi''^2+\varphi'\varphi''')b(z_x)\right) = \\ (p-2)\varphi'^2 b^{\frac{p-6}{2}}(z_x)v_x^8\left((p-2)\varphi''^2 + \varphi'\varphi'''\right) + \eta^\alpha(p-2)(2\varphi''^2+\varphi'\varphi''')b^{\frac{p-6}{2}}(z_x)v_x^6 = \\ \underbrace{(p-2)(p(\gamma-1)-\gamma)\gamma^2(\gamma-1)v^{2(\gamma-2)}\varphi'^2 b^{\frac{p-6}{2}}(z_x)v_x^8}_{\mathcal{B}_1} + \underbrace{\eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)v^{2(\gamma-2)}b^{\frac{p-6}{2}}(z_x)v_x^6}_{\mathcal{B}_2}\end{gathered}$$

The fact $p(\gamma-1)-\gamma<0$ implies $\mathcal{B}_1 \le 0$, thereby proves

$$\mathcal{B} \le \mathcal{B}_2. \tag{19}$$

From (18) and (19), we get

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \mathcal{B}_2 \ge (\gamma-1)v^{-2}a(z_x).v_x^4.$$

The fact that $b^{\frac{p-2}{2}}(.)$ is an increasing function since $p>2$ leads to

$$a(z_x) = b^{\frac{p-2}{2}}(z_x) \ge (v_x^2\varphi'^2)^{\frac{p-2}{2}} = |v_x|^{p-2}\gamma^{p-2}v^{(\gamma-1)(p-2)}.$$

A combination of the last two inequalities deduces

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \mathcal{B}_2 \ge (\gamma-1)\gamma^{p-2}v^{(\gamma-1)(p-2)-2}|v_x|^{p+2}.$$

By noting that $2-(\gamma-1)(p-2) = (1+\beta)\gamma$, we obtain

$$\frac{1}{2}\xi_t\xi^{-1}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v^{-(1+\beta)\gamma}v_x^2 + \mathcal{B}_2 \ge (\gamma-1)\gamma^{p-2}v^{-(1+\beta)\gamma}|v_x|^{p+2}. \tag{20}$$

By multiplying both sides of inequality (20) with $v^{(1+\beta)\gamma}$, and recalling the expression of $\mathcal{B}_2$, we conclude

$$\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v_x^2 + v^{(1+\beta)\gamma}\mathcal{B}_2 \ge (\gamma-1)\gamma^{p-2}|v_x|^{p+2}. \tag{21}$$

Now, we shall divide the study of inequality (21) in two different subcases:

(i) Case: $3\gamma-4 \le 0$.

We observe from the expression of $\mathcal{B}_2$ that

$$\mathcal{B}_2 \le 0.$$

It follows then from (21) that

$$(\gamma-1)\gamma^{p-2}|v_x|^{p+2} \le \left(\frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma} + (\beta + \frac{\gamma-1}{\gamma})\right)v_x^2. \tag{22}$$

Remind that $z = \varphi(v) = v^\gamma$. We infer from (10) and (22) that there is a positive constant $C_1 = C_1(\beta, p)$ such that

$$|v_x(x_0,t_0)|^2 \le C_1 \left(|\xi_t(t_0)|\xi^{-1}(t_0).\|z_0\|^{1+\beta}_{L^\infty(I)} + 1 \right)^{\frac{2}{p}}. \tag{23}$$

Thus, from (23) we obtain

$$w(x_0,t_0) = \xi(t_0)|v_x(x_0,t_0)|^2 \le C_1\xi(t_0) \left(|\xi_t(t_0)|\xi^{-1}(t_0).\|z_0\|^{1+\beta}_{L^\infty(I)} + 1 \right)^{\frac{2}{p}}.$$

Using Young's inequality deduces

$$w(x_0,t_0) \le C_1\xi(t_0)^{1-\frac{2}{p}}|\xi_t(t_0)|^{\frac{2}{p}}.\|z_0\|^{\frac{2(1+\beta)}{p}}_{L^\infty(I)} + C_1\xi(t_0).$$

Since $0 \le \xi(t) \le 1$, $|\xi_t(t)| \le \frac{c_0}{\tau}$, and $w(x_0,t_0) = \max_{(x,t)\in I\times[0,\infty)} w(x,t)$, the last estimate yields

$$w(x,t) \le w(x_0,t_0) \le C_2.\tau^{-\frac{2}{p}}.\|z_0\|^{\frac{2(1+\beta)}{p}}_{L^\infty(I)} + C_2, \quad \forall (x,t) \in I \times (0,\infty),$$

with $C_2 = C_2(\beta,p) > 0$. Thus, at time $t = \tau$, we have

$$w(x,\tau) = |v_x(x,\tau)|^2 \le C_2.\tau^{-\frac{2}{p}}.\|z_0\|^{\frac{2(1+\beta)}{p}}_{L^\infty(I)} + C_2,$$

which implies

$$|z_x(x,\tau)| \le C_3.z^{1-\frac{1}{\gamma}} \left(\tau^{-\frac{1}{p}}.\|z_0\|^{\frac{(1+\beta)}{p}}_{L^\infty(I)} + 1 \right), \quad C_3 = C_3(\beta,p).$$

The last inequality holds for any $\tau > 0$, so we get conclusion (9).

(ii) Case: $3\gamma - 4 > 0 \Longleftrightarrow p < 4(1-\beta)$.
Now $b^{\frac{p-6}{2}}(.)$ is a decreasing function and we have

$$b^{\frac{p-6}{2}}(z_x) \le |z_x|^{p-6} = |v_x|^{p-6}\gamma^{p-6}v^{(\gamma-1)(p-6)}.$$

Thus, we obtain

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \le \eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}.v^{2(\gamma-2)+(1+\beta)\gamma+(\gamma-1)(p-6)}.|v_x|^p.$$

Note that $2(\gamma-2) + (1+\beta)\gamma + (\gamma-1)(p-6) = -2(\gamma-1)$, we get

$$v^{(1+\beta)\gamma}\mathcal{B}_2 \le \eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p.$$

Inserting this fact into (21) yields

$$(\gamma-1)\gamma^{p-2}|v_x|^{p+2} \le \frac{1}{2}\xi_t\xi^{-1}v^{(1+\beta)\gamma}v_x^2 + (\beta + \frac{\gamma-1}{\gamma})v_x^2 + \eta^\alpha(p-2)\gamma^2(\gamma-1)(3\gamma-4)\gamma^{p-6}v^{-2(\gamma-1)}|v_x|^p.$$

Therefore, there is a constant $C_4 = C_4(\beta,p) > 0$ such that

$$|v_x|^{p+2} \le C_4 \left(|\xi_t|\xi^{-1}v^{(1+\beta)\gamma} + 1 \right) v_x^2 + C_4.\eta^\alpha.v^{-2(\gamma-1)}|v_x|^p.$$

The fact $v = z^{\frac{1}{\gamma}} \geq \eta^{\frac{1}{\gamma}}$ implies

$$v^{-2(\gamma-1)} \leq \eta^{-\frac{2(\gamma-1)}{\gamma}}.$$

which leads to

$$|v_x|^{p+2} \leq C_4 \left(|\xi_t| \xi^{-1} v^{(1+\beta)\gamma} + 1 \right) v_x^2 + C_4 . \eta^{\alpha - \frac{2(\gamma-1)}{\gamma}} |v_x|^p. \tag{24}$$

At the moment, if $|v_x(x_0, t_0)| < 1$, then we have

$$\xi(t_0) |v_x(x_0, t_0)|^2 < 1,$$

likewise

$$w(x,t) \leq 1, \quad \text{in } I \times (0, \infty).$$

Thus, the conclusion (9) follows immediately.

If not $|v_x(x_0, t_0)| \geq 1$, then we have $|v_x|^p \leq |v_x|^{p+2}$. It follows from (24)

$$|v_x|^{p+2} \leq C_4 \left(|\xi_t| \xi^{-1} v^{(1+\beta)\gamma} + 1 \right) v_x^2 + C_4 \eta^{\alpha - \frac{2(\gamma-1)}{\gamma}} |v_x|^{p+2},$$

or

$$\left(1 - C_4 . \eta^{\alpha - \frac{2(\gamma-1)}{\gamma}} \right) |v_x|^{p+2} \leq C_4 \left(|\xi_t| \xi^{-1} v^{(1+\beta)\gamma} + 1 \right) v_x^2.$$

Since $\alpha > \frac{2(\gamma-1)}{\gamma}$ and $\eta > 0$ can be taken small enough, there exists a positive constant $C_5 = C_5(\beta, p) > 0$ such that

$$|v_x|^{p+2} \leq C_5 \left(|\xi_t| \xi^{-1} v^{(1+\beta)\gamma} + 1 \right) v_x^2. \tag{25}$$

Note that (25) is just a version of (22). By the same analysis as in (i), we also get (9). This puts an end to the proof of Lemma 4. □

Now we shall get the other a priori bound (7) for the regularizing problem. For any $\tau > 0$ we shall show that $z_{\varepsilon,\eta}$ is a Lipschitz function on $I \times (\tau, \infty)$ with a Lipschitz constant C being independent of ε, η.

Proposition 6 *Let $z_{\varepsilon,\eta}$ be the solution of equation* (8) *above. Then, for any $\tau > 0$ there is a positive constant $C = C(\beta, p, \tau, |I|, \|z_0\|_{L^\infty(I)})$ such that*

$$|z_{\varepsilon,\eta}(x,t) - z_{\varepsilon,\eta}(y,s)| \leq C \left(|x-y| + |t-s|^{\frac{1}{3}} \right), \quad \forall x, y \in \overline{I}, \quad \forall t, s > \tau. \tag{26}$$

Proof: We first extend $z_{\varepsilon,\eta}$ by η outside I, still denoted as $z_{\varepsilon,\eta}$. Assume without loss of generality that $t > s$. To simplify the notation, we denote $z = z_{\varepsilon,\eta}$ as above. For any $\tau > 0$ and for $t > s \geq \tau$, after multiplying equation (8) by $\partial_t z$, and using integration by parts we get

$$\int_s^t \int_I |\partial_t z|^2 + a(z_x) z_x \partial_t z_x + g_\varepsilon(z) \partial_t z \; dx d\sigma = 0. \tag{27}$$

We observe that

$$a(z_x) z_x \partial_t z_x = \left(|z_x|^2 + \eta^\alpha \right)^{\frac{p-2}{2}} . \frac{1}{2} \partial_t (|z_x|^2) = \frac{1}{p} \partial_t (|z_x|^2 + \eta^\alpha)^{\frac{p}{2}}.$$

Inserting this fact into equation (27) we deduce

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq \int_I \frac{1}{p} \left(|z_x(x,s)|^2 + \eta^\alpha\right)^{\frac{p}{2}} dx + \int_I G_\varepsilon(z(x,s))dx,$$

with

$$G_\varepsilon(r) = \int_0^r g_\varepsilon(s)ds \leq \int_0^r s^{-\beta} ds = \frac{r^{1-\beta}}{1-\beta}.$$

Then, we get

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq \frac{1}{p} \int_I \left(|z_x(x,s)|^2 + \eta^\alpha\right)^{\frac{p}{2}} dx + \frac{1}{1-\beta} \int_I z(x,s)^{1-\beta} dx.$$

Or

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq \frac{1}{p} \int_I \left(\|z_x(s)\|^2_{L^\infty(I)} + \eta^\alpha\right)^{\frac{p}{2}} dx + \frac{1}{1-\beta} \int_I \left(\|z_0\|_{L^\infty(I)} + \eta\right)^{1-\beta} dx. \quad (28)$$

By applying Young's inequality in (28), we obtain

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq C_6 \left(\|z_x(s)\|^p_{L^\infty(I)} + \|z(s)\|^{1-\beta}_{L^\infty(I)}\right) + O(\eta), \quad (29)$$

with $C_6 = C_6(\beta, p, |I|)$, and $\lim_{\eta \to 0} O(\eta) = 0$.
By combining (9) and (29), we deduce that there is a constant $C_7 = C_7(\beta, p, \tau, |I|, \|z_0\|_{L^\infty(I)}) > 0$ such that

$$\int_s^t \int_I |\partial_t z|^2 dx d\sigma \leq C_7, \quad \forall t > s \geq \tau. \quad (30)$$

Thus $\|\partial_t z_{\varepsilon,\eta}\|_{L^2(I\times(s,t))}$ is bounded by a constant which is independent of ε and η.

Next, for any $x, y \in I$, we set

$$r = |x-y| + |t-s|^{\frac{1}{3}}.$$

According to the Mean Value Theorem, there is a real number $\bar{x} \in B_r(y)$ such that

$$|\partial_t z(\bar{x},\sigma)|^2 = \frac{1}{|B_r(y)|} \int_{B_r(y)} |\partial_t z(l,\sigma)|^2 dl = \frac{1}{2r} \int_{B_r(y)\cap I} |\partial_t z(l,\sigma)|^2 dl \leq \frac{1}{2r} \int_I |\partial_t z(l,\sigma)|^2 dl \quad (31)$$

(Note that $\partial_t z(.,t) = 0$ outside I).
Next, we have from Holder's inequality

$$|z(\bar{x},t) - z(\bar{x},s)|^2 \leq (t-s) \int_s^t |\partial_t z(\bar{x},\sigma)|^2 d\sigma \overset{(31)}{\leq} \frac{(t-s)}{2r} \int_s^t \int_I |\partial_t z(l,\sigma)|^2 dl d\sigma,$$

or

$$|z(\bar{x},t) - z(\bar{x},s)|^2 \overset{(30)}{\leq} \frac{1}{2} C_7.(t-s)^{\frac{2}{3}}.$$

Then, we obtain

$$|z(\bar{x},t) - z(\bar{x},s)| \leq C_8.(t-s)^{\frac{1}{3}}, \quad \forall t > s \geq \tau, \quad (32)$$

with $C_8 = \sqrt{\frac{1}{2}C_7}$. Now, it is sufficient to show (26). Indeed, we have the triangular inequality

$$\begin{aligned}|z(x,t)-z(y,s)| &\leq |z(x,t)-z(y,t)| + |z(y,t)-z(y,s)| \\ &\leq |z(x,t)-z(y,t)| + |z(y,t)-z(\bar{x},t)| + |z(\bar{x},t)-z(\bar{x},s)| + +|z(\bar{x},s)-z(y,s)|,\end{aligned}$$

where $\bar{x} \in I_r(y)$ is above. Then, the conclusion (26) just follows from (32), gradient estimate (9), and the Mean Value Theorem. This puts an end to the proof of Proposition 6. ☐

Next, we will pass to the limit as $\eta \to 0$ in order to get gradient estimate (9) for the "least regularized problem"

$$\begin{cases} \partial_t z_\varepsilon - \left(|\partial_x z_\varepsilon|^{p-2}\partial_x z_\varepsilon\right)_x + g_\varepsilon(z_\varepsilon) = 0 & \text{in } I \times (0,\infty), \\ z_\varepsilon(L_1,t) = z_\varepsilon(L_2,t) = 0 & t \in (0,\infty), \\ z_\varepsilon(x,0) = z_0(x) & \text{on } I. \end{cases} \tag{33}$$

Theorem 7 *Let $p > 2$, and $0 \leq z_0 \in L^\infty(I)$. Then, there exists a unique weak solution z_ε of equation (33). Furthermore, z_ε fulfills the gradient estimate (9)*

$$|\partial_x z_\varepsilon(x,t)| \leq C(\beta,p).z_\varepsilon^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}.\|z_0\|_{L^\infty(I)}^{\frac{1+\beta}{p}} + 1\right), \quad \textit{for a.e } (x,t) \in I \times (0,\infty), \tag{34}$$

Moreover, z_ε also satisfies (26), i.e., z_ε is a Lipschitz function.

Proof: Equation (33) is just the limit of equation (8) as $\eta \to 0$, see [27], or [26]. Note that one can regularize initial data z_0 if necessary. Thus, estimate (34) follows from (9). ☐

3 Proof of Theorem 2

The proof of Theorem 2 is divided into three parts. In the first part, we show the existence and uniqueness of solution u_ε of equation (33) with initial data $u_0 \in L^1(I)$. Moreover, we also prove a gradient estimate for $|\partial_x u_\varepsilon|$ involving the terms of u_ε and $\|u_0\|_{L^1(I)}$ (see Theorem 8 below). After that, passing $\varepsilon \to 0$ yields equation (1). Finally, the conclusion that u is a maximal solution will be proven in Proposition 11 below.

We first have the following result.

Theorem 8 *Let $p > 2$, and $u_0 \in L^1(I)$, $u_0 \geq 0$. Then, there exists a unique weak solution u_ε of equation (33). Moreover, u_ε satisfies the following additional estimates:*
(i) There is a constant $C(p,|I|) > 0$ such that

$$\|u_\varepsilon(.,t)\|_{L^\infty(I)} \leq C(p,|I|).t^{-\frac{1}{\lambda}}.\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}}, \quad \textit{for } t \in (0,\infty). \tag{35}$$

Recall here $\lambda = 2(p-1)$.
(ii) For any $\tau > 0$, there is a constant $C(\beta,p,|I|) > 0$ such that

$$|\partial_x u_\varepsilon(x,t)| \leq C(\beta,p,|I|).u_\varepsilon^{1-\frac{1}{\gamma}}(x,t).\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}} + 1\right), \quad \textit{for a.e } (x,t) \in (\tau,\infty). \tag{36}$$

(iii) There exists a constant $C = C(\beta,p,\tau,|I|,\|u_0\|_{L^1(I)}) > 0$ such that

$$|u_\varepsilon(x,t) - u_\varepsilon(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall x,y \in I, \quad \forall t,s > \tau. \tag{37}$$

Proof: (I) Uniqueness. The uniqueness result follows from the lemma below.

Lemma 9 *Let v_1 (resp. v_2) be a weak sub-solution (resp. super-solution) of equation* (33). *Then, we have*

$$v_1 \le v_2, \quad \text{in } I \times (0,\infty).$$

We skip the proof of Lemma 9 and give its proof in the Appendix.

(II) Existence. We make a regularization to initial data u_0 by considering a sequence $\{u_{0,n}\}_{n\ge1} \subset \mathcal{C}_c^{\infty}(I)$ such that

$$u_{0,n} \overset{n\to\infty}{\longrightarrow} u_0, \quad \text{in } L^1(I); \quad \text{and } \|u_{0,n}\|_{L^1(I)} \le \|u_0\|_{L^1(I)}.$$

Let $u_{\varepsilon,n}$ be a unique (weak) solution of the equation (see details in [27], or [26])

$$\begin{cases} \partial_t u_{\varepsilon,n} - \left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n}\right)_x + g_\varepsilon(u_{\varepsilon,n}) = 0 & \text{in } I \times (0,\infty), \\ u_{\varepsilon,n}(L_1,t) = u_{\varepsilon,n}(L_2,t) = 0 & t \in (0,\infty), \\ u_{\varepsilon,n}(x,0) = u_{0,n}(x) & \text{on } I \end{cases} \tag{38}$$

We will show that $u_{\varepsilon,n}$ converges to u_ε, which is a solution of equation (33). The proof contains some steps.

Step 1: A priori estimates.

First of all, we observe that $u_{\varepsilon,n}$ is a sub-solution of the following equation

$$\begin{cases} \partial_t v_n - \left(|\partial_x v_n|^{p-2}\partial_x v_n\right)_x = 0 & \text{in } I \times (0,\infty), \\ v_n(L_1,t) = v_n(L_2,t) = 0 & \forall t \in (0,\infty), \\ v_n(x,0) = u_{0,n}(x) & \text{in } I \end{cases} \tag{39}$$

Therefore, the comparison principle yields

$$u_{\varepsilon,n} \le v_n, \quad \text{in } I \times (0,\infty). \tag{40}$$

Using smoothing effect $L^1 - L^\infty$ deduces (see, e.g., Theorem 4.3, [12])

$$\|v_n(.,t)\|_{L^\infty(I)} \le C(p,|I|).t^{-\frac{1}{\lambda}}.\|v_n(0)\|^{\frac{p}{\lambda}}_{L^1(I)} \le C(p,|I|).t^{-\frac{1}{\lambda}}.\|u_0\|^{\frac{p}{\lambda}}_{L^1(I)}, \quad \forall t > 0, \tag{41}$$

By (40) and (41), we obtain

$$\|u_{\varepsilon,n}(.,t)\|_{L^\infty(I)} \le C(p,|I|).t^{-\frac{1}{\lambda}}.\|u_0\|^{\frac{p}{\lambda}}_{L^1(I)}, \quad \forall t > 0. \tag{42}$$

Now, for any $\tau > 0$, we apply Theorem 7 to $u_{\varepsilon,n}$ by considering $u_{\varepsilon,n}(\frac{\tau}{2})$ as the initial data instead of $u_{\varepsilon,n}(0)$ in order to get

$$|\partial_x u_{\varepsilon,n}(x,t)| \le C(\beta,p).u_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left((t-\frac{\tau}{2})^{-\frac{1}{p}}\|u_{\varepsilon,n}(\frac{\tau}{2})\|^{\frac{1+\beta}{p}}_{L^\infty(I)} + 1\right), \quad \text{for a.e } (x,t) \in I \times (\frac{\tau}{2},\infty),$$

which implies

$$|\partial_x u_{\varepsilon,n}(x,t)| \le C(\beta,p)u_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left(\left(\frac{\tau}{2}\right)^{-\frac{1}{p}}\|u_{\varepsilon,n}(\frac{\tau}{2})\|^{\frac{1+\beta}{p}}_{L^\infty(I)} + 1\right), \tag{43}$$

for a.e $(x,t) \in I \times (\tau,\infty)$. It follows from (42) and (43) that there exists a positive constant $C(\beta,p,|I|)$ such that

$$|\partial_x u_{\varepsilon,n}(x,t)| \leq C(\beta,p,|I|).u_{\varepsilon,n}^{1-\frac{1}{\gamma}}(x,t)\left(\tau^{-\frac{\lambda+\beta+1}{\lambda p}}\|u_0\|_{L^1(I)}^{\frac{1+\beta}{\lambda}}+1\right), \quad \text{for a.e } (x,t) \in (\tau,\infty). \quad (44)$$

In view of (42) and (44), $u_{\varepsilon,n}(t)$ and $|\partial_x u_{\varepsilon,n}(t)|$ are bounded on $I \times (\tau,\infty)$ by the positive constants which are independent of ε and η.
Thanks to Proposition 6, there is a positive constant $C = C(\beta,p,\tau,|I|,\|u_0\|_{L^1(I)})$ such that

$$|u_{\varepsilon,n}(x,t) - u_{\varepsilon,n}(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall x,y \in \overline{I},\ \forall t,s > \tau, \quad (45)$$

Step 2: Passing to the limit as $n \to \infty$. To avoid relabeling after any passage to the limit, we want to keep the same label. Now, we observe that (45) allows us to apply the Ascoli-Arzela Theorem to $u_{\varepsilon,n}$, so there is a subsequence of $\{u_{\varepsilon,n}\}_{n\geq 1}$ such that

$$u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} u_\varepsilon, \quad \text{uniformly on every compact of } \overline{I} \times (\tau,\infty).$$

Furthermore, the diagonal argument asserts that there is a subsequence of $\{u_{\varepsilon,n}\}_{n\geq 1}$ such that

$$u_{\varepsilon,n}(x,t) \overset{n\to\infty}{\longrightarrow} u_\varepsilon(x,t), \quad \text{pointwise in } \overline{I} \times (0,\infty). \quad (46)$$

Thus, u_ε also satisfies (42) and (45).
Next, we claim that for any $0 < \tau < T < \infty$

$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{in } L^1(I \times (\tau,T)). \quad (47)$$

To prove (47), we borrow an idea of L. Boccardo and F. Murat [5] (the so called almost everywhere convergence of the gradients, see also in [4]). Let us put

$$w_{n,m} = u_{\varepsilon,n} - u_{\varepsilon,m}, \quad \text{for } n,m \in \mathbb{N},$$

and

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k.sign(s), & \text{if } |s| > k, \end{cases}$$

and

$$S_k(u) = \int_0^u T_k(s)ds = \frac{1}{2}|u|^2\chi_{\{|u|<k\}} + k(|u| - \frac{1}{2}k)\chi_{\{|u|\geq k\}}.$$

Then, we have

$$\partial_t w_{n,m} - \left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right)_x + g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m}) = 0.$$

Multiplying both sides of the last equation with $T_\delta(w_{n,m})$ and using the integration by part yield

$$\int_I S_\delta(w_{n,m}(x,T))dx + \int_\tau^T \int_I \left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right).\partial_x T_\delta(w_{n,m})(x,s)dxds +$$

$$\int_\tau^T \int_I \left(g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})\right) T_\delta(w_{n,m})dxds = \int_I S_\delta(w_{n,m}(x,\tau))dx. \quad (48)$$

Since $S_k(.) \geq 0$, and $S_k(s) \leq k|s|$, we get

$$\int_\tau^T \int_I \left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right).\partial_x T_\delta(w_{n,m})(x,s)dxds$$

$$\leq \delta \int_\tau^T \int_I \left(g_\varepsilon(u_{\varepsilon,n}) + g_\varepsilon(u_{\varepsilon,m})\right) dxds + \delta \int_I |w_{n,m}(\tau)|dx. \tag{49}$$

Next, for any $t > 0$, we have L^1-estimate

$$\int_I u_{\varepsilon,n}(t)dx + \int_0^t \int_I g_\varepsilon(u_{\varepsilon,n})dxds \leq \int_I u_{\varepsilon,n}(0)dx \leq \|u_0\|_{L^1(I)}, \quad \forall n \geq 1. \tag{50}$$

Combining (49) and (50) yields

$$\int_\tau^T \int_I \left(|\partial_x u_{\varepsilon,n}|^{p-2}\partial_x u_{\varepsilon,n} - |\partial_x u_{\varepsilon,m}|^{p-2}\partial_x u_{\varepsilon,m}\right)\partial_x T_\delta(w_{n,m})(x,s)dxds \leq 4\delta\|u_0\|_{L^1(I)}. \tag{51}$$

Thus, it follows from the strong monotonicity of $p-$Laplace operator (see Lemma 22) that there is a positive constant c such that

$$c\int_{\{w_{n,m}<\delta\}\cap I\times(\tau,T)} |\partial_x w_{n,m}(x,s)|^p dxds \leq 4\delta\|u_0\|_{L^1(I)}. \tag{52}$$

By Holder's inequality, we obtain

$$\int_{\{w_{n,m}<\delta\}\cap I\times(\tau,T)} |\partial_x w_{n,m}(x,s)|dxds \leq C(|I|,T)\left(\int_{\{w_{n,m}(x,t)<\delta\}} |\partial_x w_{n,m}(x,s)|^p dxds\right)^{\frac{1}{p}}. \tag{53}$$

From (52) and (53), we deduce

$$\int_{\{w_{n,m}<\delta\}\cap I\times(\tau,T)} |\partial_x w_{n,m}(x,s)|dxds \leq C\delta^{\frac{1}{p}}, \tag{54}$$

with $C = C(|I|,T,c,\|u_0\|_{L^1(I)})$.
On the other hand, we have

$$\int_{\{w_{n,m}(x,t)\geq\delta\}\cap I\times(\tau,T)} |\partial_x w_{n,m}(x,s)|dxds \leq \|\partial_x w_{n,m}\|_{L^\infty(I\times(\tau,T))}.mes\left(\{w_{n,m}(x,t)\geq\delta\}\cap I\times(\tau,T)\right).$$

Insert gradient estimate (44) into the last inequality to get

$$\int_{\{w_{n,m}(x,t)\geq\delta\}\cap I\times(\tau,T)} |\partial_x w_{n,m}(x,s)|dxds \leq C_1.mes\left(\{w_{n,m}(x,t)\geq\delta\}\cap I\times(\tau,T)\right), \tag{55}$$

where the constant C_1 only depends on $\beta, p, |I|, \tau, \|u_0\|_{L^1(I)}$.
A combination of (54) and (55) provides us

$$\int_\tau^T \int_I |\partial_x w_{n,m}(x,s)|dxds \leq C_2.\left(mes\left(\{w_{n,m}(x,t)\geq\delta\}\cap I\times(\tau,T)\right) + \delta^{\frac{1}{p}}\right).$$

Let $n, m \to \infty$ in the above inequality. Note that (46) implies

$$\lim_{n,m\to\infty} mes\left(\{w_{n,m}(x,t) \geq \delta\} \cap I \times (\tau, T)\right) = 0,$$

thereby proves

$$\lim_{n,m\to\infty} \int_\tau^T \int_I |\partial_x w_{n,m}(x,s)| dxds \leq C\delta^{\frac{1}{p}}.$$

The last estimate holds for any $\delta > 0$, so we get claim (47) after passing $\delta \to 0$.
According to (47) and (44), we obtain

$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{in } L^q(I \times (\tau, T)),\ \forall q \in [1, \infty), \tag{56}$$

and there is a subsequence of $\{\partial_x u_{\varepsilon,n}\}$ such that

$$\partial_x u_{\varepsilon,n} \overset{n\to\infty}{\longrightarrow} \partial_x u_\varepsilon, \quad \text{for a.e } (x,t) \in I \times (0, \infty). \tag{57}$$

Thus, the conclusion (36) follows from (57) and (44).
Next, we claim that

$$u_\varepsilon \in \mathcal{C}([0,\infty); L^1(I)). \tag{58}$$

It suffices to demonstrate that

$$u_\varepsilon \in \mathcal{C}([0,T]; L^1(I)), \quad \text{for any } T \in (0, \infty). \tag{59}$$

Indeed, we first observe that for any $\varepsilon > 0$ fixed, $g_\varepsilon(u_{\varepsilon,n})$ is bounded by $\varepsilon^{-\beta}$. Moreover, (46) deduces

$$g_\varepsilon(u_{\varepsilon,n}) \overset{n\to\infty}{\longrightarrow} g_\varepsilon(u_\varepsilon), \quad \text{pointwise in } I \times (0, \infty).$$

Therefore, the Dominated Convergence Theorem yields

$$g_\varepsilon(u_{\varepsilon,n}) \overset{n\to\infty}{\longrightarrow} g_\varepsilon(u_\varepsilon), \quad \text{in } L^1(I \times (0, T)). \tag{60}$$

As a consequence of (60) and (50), we get

$$\int_0^\infty \int_I g_\varepsilon(u_\varepsilon(x,s)) dxds \leq \|u_0\|_{L^1(I)}. \tag{61}$$

Next, we take $\delta = 1$ in equation (48) to obtain

$$\int_I S(w_{n,m}(t))dx \leq \int_\tau^t \int_I |g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})| dxds + \int_I |w_{n,m}(\tau)| dxds, \quad \text{for } t \in (\tau, T).$$

Passing $\tau \to 0$ in the above inequality provides us

$$\int_I S(w_{n,m}(x,t))dx \leq \int_0^t \int_I |g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})| dxds + \int_I |w_{n,m}(0)| dxds, \quad \text{for } 0 < t < T.$$

By (60), we derive

$$\int_I S(w_{n,m}(x,t))dx \leq \int_0^T \int_I |g_\varepsilon(u_{\varepsilon,n}) - g_\varepsilon(u_{\varepsilon,m})| dxds + \int_I |u_{0,n} - u_{0,m}| dxds = o(n,m) \tag{62}$$

where $o(n,m) \overset{n,m\to\infty}{\longrightarrow} 0$.
Moreover, we have a relation between $w_{n,m}$ and $S(w_{n,m})$ as follows (see also in [7])

$$\int_I w_{n,m}(x,t)dx \le \sqrt{2|I| \int_I S(w_{n,m}(t))dx} + 2\int_I S(w_{n,m}(x,t))dx, \quad \forall t > 0. \tag{63}$$

Combining (62) and (63) yields

$$\int_I w_{n,m}(x,t)dx \le C(|I|)\left(\sqrt{o(n,m)} + o(n,m)\right), \quad \forall t > 0. \tag{64}$$

Then

$$\lim_{n,m\to\infty} \sup_{t\in[0,\infty)} \int_I w_{n,m}(x,t)dx = 0, \quad \text{uniformly on } [0,T].$$

This implies claim (58).

Now, it is enough to show that u_ε is a weak solution of equation (33). In fact, we observe that (56) and (60) allows us to pass to the limit as $n \to \infty$ in the equation satisfied by $u_{\varepsilon,n}$ to obtain

$$\partial_t u_\varepsilon - \left(|\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\right)_x + g_\varepsilon(u_\varepsilon) = 0, \quad \text{in } \mathcal{D}'(I \times (0,\infty)).$$

Or, we get the proof of Theorem 8. □

To complete the proof of Theorem 2, it remains to pass to the limit as $\varepsilon \to 0$. We first show that $\{u_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence, thus we have $u_\varepsilon(x,t) \downarrow u(x,t)$. We note that the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$ will be intensively used in what follows.
In fact, for any $\varepsilon > \varepsilon' > 0$, it is clear that

$$g_{\varepsilon'}(v) = \psi(\frac{v}{\varepsilon'})v^{-\beta} \ge \psi(\frac{v}{\varepsilon})v^{-\beta} = g_\varepsilon(v), \quad \text{for } v \in \mathbb{R}.$$

Thus

$$\partial_t u_\varepsilon - \left(|\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\right)_x + g_{\varepsilon'}(u_\varepsilon) \ge \partial_t u_\varepsilon - \left(|\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon\right)_x + g_\varepsilon(u_\varepsilon) = 0,$$

which implies that u_ε is a super-solution of equation satisfied by $u_{\varepsilon'}$, so Lemma 9 yields

$$u_\varepsilon(x,t) \ge u_{\varepsilon'}(x,t), \quad \text{in } I \times (0,\infty), \tag{65}$$

or we get the result.

It is obvious that the estimates in the proof of Theorem 8 are independent of ε. Thus, a similar analysis as in the proof of Theorem 8 implies that there exists a function u such that

$$\begin{cases} \partial_x u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \partial_x u, & \text{for a.e } (x,t) \in I \times (0,\infty), \\ \partial_x u_\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} \partial_x u, & \text{in } L^q(I \times (\tau,T)), \quad \text{for } 0 < \tau < T < \infty,\ \forall q \ge 1, \end{cases} \tag{66}$$

so u satisfies the estimates (5), (6) and (7) of Theorem 2.
Next, we shall show that there is a subsequence of $\{g_\varepsilon(u_\varepsilon)\}_{\varepsilon>0}$ such that

$$g_\varepsilon(u_\varepsilon) \overset{\varepsilon\to 0}{\longrightarrow} u^{-\beta}\chi_{\{u>0\}}, \quad \text{in } L^1(I \times (0,\infty)). \tag{67}$$

Let us emphasize that (67) implies the conclusion

$$u \in \mathcal{C}([0,\infty); L^1(I)) \tag{68}$$

by following the proof of (58).
By (61) and Fatou's lemma, there is a function $\Phi \in L^1(I \times (0,\infty))$ such that

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon) = \Phi, \quad \text{in } L^1(I \times (0,\infty)). \tag{69}$$

By the monotonicity of $\{u_\varepsilon\}_{\varepsilon>0}$, we have

$$g_\varepsilon(u_\varepsilon)(x,t) \geq g_\varepsilon(u_\varepsilon)\chi_{\{u>0\}}(x,t),$$

which implies

$$\liminf_{\varepsilon \to 0} g_\varepsilon(u_\varepsilon)(x,t) \geq u^{-\beta}\chi_{\{u>0\}}(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty). \tag{70}$$

From (69) and (70), we deduce

$$u^{-\beta}\chi_{\{u>0\}} \leq \Phi, \quad \text{and } u^{-\beta}\chi_{\{u>0\}} \in L^1(I \times (0,\infty)). \tag{71}$$

Now, for any $\eta > 0$ fixed, we use the test function $\psi_\eta(u_\varepsilon)\phi$, $\phi \in \mathcal{C}_c^\infty(I \times (0,T))$ in the equation satisfied by u_ε. Then, the integration by parts yields

$$\int_{Supp(\phi)} -\Psi_\eta(u_\varepsilon)\phi_t + \frac{1}{\eta}|\partial_x u_\varepsilon|^p \psi'(\frac{u_\varepsilon}{\eta})\phi + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon.\partial_x\phi.\psi_\eta(u_\varepsilon) + g_\varepsilon(u_\varepsilon)\psi_\eta(u_\varepsilon)\phi \, dxds = 0,$$

where

$$\Psi_\eta(u) = \int_0^u \psi_\eta(s)ds.$$

Thanks to the Dominated Convergence Theorem and (66), going to the limit as $\varepsilon \to 0$ in the indicated equation yields

$$\int_{Supp(\phi)} -\Psi_\eta(u)\phi_t + \frac{1}{\eta}|\partial_x u|^p \psi'(\frac{u}{\eta})\phi + |\partial_x u|^{p-2}\partial_x u \partial_x \phi \psi_\eta(u) + u^{-\beta}\psi_\eta(u)\phi \, dxds = 0. \tag{72}$$

After that, we pass to the limit as $\eta \to 0$ in equation (72). It is not difficult to verify that

$$\begin{cases} \lim_{\eta \to 0} \int_{Supp(\phi)} \Psi_\eta(u)\phi_t dxds = \int_{Supp(\phi)} u.\phi_t dxds, \\ \lim_{\eta \to 0} \int_{Supp(\phi)} |\partial_x u|^{p-2}\partial_x u.\partial_x\phi.\psi_\eta(u) dxds = \int_{Supp(\phi)} |\partial_x u|^{p-2}\partial_x u.\partial_x \phi dxds, \\ \lim_{\eta \to 0} \int_{Supp(\phi)} u^{-\beta}\psi_\eta(u)\phi dxds = \int_{Supp(\phi)} u^{-\beta}\chi_{\{u>0\}}\phi dxds. \end{cases} \tag{73}$$

While

$$\lim_{\eta \to 0} \int_{Supp(\phi)} \frac{1}{\eta}|\partial_x u|^p \psi'(\frac{u}{\eta})\phi dxds = 0. \tag{74}$$

Indeed, the fact that u satisfies gradient estimate (6) leads to

$$\frac{1}{\eta}\int_{Supp(\phi)} |\partial_x u|^p |\psi'(\frac{u}{\eta}).\phi| dxds \leq C\frac{1}{\eta}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{1-\beta} dxds$$
$$\leq 2C\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta} dxds,$$

where the constant $C > 0$ is independent of η.
Thanks to (71), and the Dominated Convergence Theorem, we obtain

$$\lim_{\eta\to 0}\int_{Supp(\phi)\cap\{\eta<u<2\eta\}} u^{-\beta} dxds = 0,$$

which implies the conclusion (74). Combining (72), (73) and (74) it yields

$$\int_{Supp(\phi)} \left(-u\phi_t + |\partial_x u|^{p-2}\partial_x u \partial_x \phi + u^{-\beta}\chi_{\{u>0\}}\phi\right) dxds = 0. \tag{75}$$

Therefore, u satisfies equation (1) in $\mathcal{D}'(I \times (0,\infty))$.
Next, the fact that u_ε is a weak solution of (33) gives us

$$\int_{Supp(\phi)} \left(-u_\varepsilon\phi_t + |\partial_x u_\varepsilon|^{p-2}\partial_x u_\varepsilon \partial_x \phi + g_\varepsilon(u_\varepsilon)\phi\right) dxds = 0.$$

Letting $\varepsilon \to 0$ deduces

$$\int_{Supp(\phi)} \left(-u\phi_t + |\partial_x u|^{p-2}\partial_x u \partial_x \phi\right) dxds + \lim_{\varepsilon\to 0}\int_{Supp(\phi)} g_\varepsilon(u_\varepsilon)\phi dxds = 0. \tag{76}$$

A comparison between (75) and (76) leads to

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_I g_\varepsilon(u_\varepsilon)\phi dxds = \int_0^\infty\int_I u^{-\beta}\chi_{\{u>0\}}\phi dxds. \tag{77}$$

According to (69) and (77), we obtain

$$\int_0^\infty\int_I u^{-\beta}\chi_{\{u>0\}}\phi dxds \geq \int_0^\infty\int_I \Phi\phi dxds, \quad \forall\phi \in \mathcal{C}_c^\infty(I \times (0,\infty)), \phi \geq 0.$$

The last inequality and (71) imply

$$u^{-\beta}\chi_{\{u>0\}} = \Phi, \quad \text{in } I \times (0,\infty).$$

Thereby, we get (67). Thanks to (66), (68) and (75), u is a weak solution of equation (1).

Remark 10 *The reader should note that* (75) *is not sufficient to conclude that u is a weak solution by following Definition* 1. *Thus, it is necessary to prove* (67) *in order to get* (68).

We end this Section by proving that u is the maximal solution of equation (1).

Proposition 11 *Let v be any weak solution of equation* (1). *Then, we have*

$$v(x,t) \leq u(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty).$$

Proof: For any $\varepsilon > 0$, we observe that

$$g_\varepsilon(v) \le v^{-\beta}\chi_{\{v>0\}}.$$

Then

$$\partial_t v - \left(|\partial_x v|^{p-2}\partial_x v\right)_x + g_\varepsilon(v) \le \partial_t v - \left(|\partial_x v|^{p-2}\partial_x v\right)_x + v^{-\beta}\chi_{\{v>0\}} = 0,$$

which implies that v is a sub-solution of equation satisfied by u_ε.
Thanks to Lemma 9, we get

$$v(x,t) \le u_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in I \times (0,\infty).$$

Letting $\varepsilon \to 0$ yields the result. This puts an end to the proof of Theorem 2. □

Remark 12 *If $u_0 \in L^\infty(I)$, then u also satisfies estimate* (34).

4 Global quenching phenomenon in a finite time

In this section, we will show that any weak solution of equation (1) must quench (Theorem 3). According to Proposition 11, it is enough to prove that the maximal solution u vanishes identically after a finite time. Then, we have the following result

Theorem 13 *Let $u_0 \in L^1(I)$, $u_0 \ge 0$. Then, there exists a finite time T_0 such that*

$$u(x,t) = 0, \quad \forall x \in \overline{I},\ \forall t > T_0.$$

Furthermore, T_0 can be estimated by a constant depending on $\beta, p, |I|, \|u_0\|_{L^1(I)}$.

Proof: For any $\tau > 0$, we put

$$m(\tau, u_0) = C(p,|I|).\tau^{-\frac{1}{\lambda}}.\|u_0\|_{L^1(I)}^{\frac{p}{\lambda}},$$

the a priori bound of $u(x,t)$ on $[\tau,\infty)$, see (42) or (5).

Let $\Gamma_\varepsilon(t)$ be a flat solution of equation (33), i.e,

$$\begin{cases} \partial_t \Gamma_\varepsilon(t) + g_\varepsilon(\Gamma_\varepsilon) = 0, \quad t > 0, \\ \Gamma_\varepsilon(0) = m(\tau, u_0). \end{cases} \tag{78}$$

Then, the strong comparison deduces

$$u_\varepsilon(x, s+\tau) \le \Gamma_\varepsilon(s), \quad \forall (x,s) \in I \times (0,\infty).$$

It is straightforward to show that

$$\Gamma_\varepsilon(t) \xrightarrow{\varepsilon\to 0} \Gamma(t) = \left(m(\tau,u_0)^{1+\beta} - (1+\beta)t\right)_+^{\frac{1}{1+\beta}}, \quad \text{for } t > 0.$$

Then, we obtain

$$u(x, s+\tau) \le \Gamma(s), \quad \text{for } (x,s) \in I \times (0,\infty),$$

which implies

$$u(x,t) = 0, \quad \text{for any } t \geq \tau + \frac{m^{1+\beta}(\tau, u_0)}{1+\beta}, \text{ and for } x \in I. \tag{79}$$

Now, we try to estimate the value of the quenching time T_0. By (79), we can choose T_0 as follows

$$T_0 = \min_{\tau>0}\{\tau + \frac{m^{1+\beta}(\tau, u_0)}{1+\beta}\} = C(\beta, p, |I|).\|u_0\|_{L^1(I)}^{\frac{(1+\beta)p}{1+\beta+\lambda}}.$$

This completes the proof of Theorem 13, thereby proves Theorem 3. □

Remark 14 *If $u_0 \in L^\infty(I)$, then we can take $T_0 = \dfrac{\|u_0\|_{L^\infty(I)}^{1+\beta}}{1+\beta}$.*

Remark 15 *In the previous works, (see e.g, [14], [8] and references therein) the estimate of quenching time T_0 depends on $\|u_0\|_{L^\infty(I)}$, which obviously requires $u_0 \in L^\infty(I)$. Thus, our result is sharp because we merely assume $u_0 \in L^1(I)$.*

Next, we will point out an upper bound and a lower bound of any solution of equation (1) at the quenching time.

4.1 Upper bound at the quenching time

Assume that $T_{\min}$ is the minimal extinction time. It is clear that $T_{\min} \leq T_0$. Then, it follows from Proposition 6 that

$$|u(x,t) - u(x, T_{\min})| \leq C.|T_{\min} - t|^{\frac{1}{3}}, \quad \text{for } (x,t) \in I \times (\frac{T_{\min}}{2}, T_{\min}),$$

with constant $C = C(\beta, p, |I|, T_{\min}, \|u_0\|_{L^1(I)}) > 0$. Therefore, we get

$$u(x,t) \leq C.(T_{\min} - t)_+^{\frac{1}{3}}, \quad \text{for } (x,t) \in I \times (\frac{T_{\min}}{2}, T_{\min}),$$

which implies

$$\limsup_{t \to T_{\min}^-} \Big((T_{\min} - t)^{-\frac{1}{3}}.\|u(t)\|_{L^\infty(I)} \Big) \leq C.$$

This conclusion also holds for any solution of equation (1), since u is the maximal solution.

4.2 Lower bound at the quenching time

For any $\tau > 0$, let Γ_ε be a solution of equation (78) with initial data $\|u(\tau)\|_{L^\infty(I)}$. By the same argument with the proof of Theorem 13, we obtain

$$\Gamma_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} \Gamma(t) = \Big(\|u(\tau)\|_{L^\infty(I)}^{1+\beta} - (1+\beta)t \Big)_+^{\frac{1}{1+\beta}}, \quad \text{for } t > 0.$$

This leads to

$$T_{\min} \leq T_0 \leq \tau + \frac{\|u(\tau)\|_{L^\infty(I)}^{1+\beta}}{1+\beta}.$$

Thus, we obtain

$$\liminf_{\tau \to T_{\min}^-} \Big((T_{\min} - \tau)^{-\frac{1}{1+\beta}}.\|u(\tau)\|_{L^\infty(I)} \Big) \geq (1+\beta)^{\frac{1}{1+\beta}}.$$

5 On the associated Cauchy problem

In this section, we extend the result of the existence of weak solutions of equation (1) to the Cauchy problem:

$$\begin{cases} \partial_t U - (|U_x|^{p-2}U_x)_x + \chi_{\{U>0\}}U^{-\beta} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ U(x,0) = U_0(x), & \text{in } \mathbb{R}, \end{cases} \tag{80}$$

Besides, we also study the quenching phenomenon and the free boundary of solutions of equation (80), which arise due to the singular absorption term.

5.1 The existence of a weak solution

We have a existence result of problem (80).

Theorem 16 *Let $p > 2$, and $\beta \in (0,1)$, and $0 \leq U_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, there exists a bounded solution, $U \in \mathcal{C}([0,\infty); L^1(\mathbb{R})) \cap L^p(0,T; W^{1,p}(\mathbb{R}))$ satisfying equation* (80) *in $\mathcal{D}'(\mathbb{R}\times(0,\infty))$. Besides, there is a positive constant $C = C(\beta,p)$ such that*

$$|\partial_x U(x,t)| \leq C.U^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}.\|U_0\|^{\frac{1+\beta}{p}}_{L^\infty(\mathbb{R})} + 1\right), \quad \textit{for a.e } (x,t) \in \mathbb{R}\times(0,\infty). \tag{81}$$

As a consequence of (81) *and Proposition* 6*, U is a locally Lipschitz function, i.e., for any $\tau > 0$ and for $r > 0$, there is a positive constant $C = C(\beta,p,r,\tau,\|U_0\|_{L^\infty(\mathbb{R})})$ such that*

$$|U(x,t) - U(y,s)| \leq C\left(|x-y| + |t-s|^{\frac{1}{3}}\right), \quad \forall t,s > \tau,\ \forall x,y \in B_r. \tag{82}$$

Proof: The proof of this theorem is most likely to the one of Theorem 2 at many points, so we just point out the main different ideas. For any $\varepsilon > 0$ and for $r > 0$, let $u_{r,\varepsilon}$ be the unique solution of the problem

$$\begin{cases} \partial_t u - (|u_x|^{p-2}u_x)_x + g_\varepsilon(u) = 0 & \text{in } B_r\times(0,\infty), \\ u(-r,t) = u(r,t) = 0, & \forall t \in (0,\infty), \\ u(x,0) = U_0(x), & \text{in } B_r, \end{cases} \tag{83}$$

see Theorem 8. Thanks to the comparison principle, we have

$$\|u_{r,\varepsilon}(.,t)\|_{L^\infty(B_r)} \leq \|U_0\|_{L^\infty(\mathbb{R})}, \quad \text{for any } t \in (0,\infty). \tag{84}$$

And L^1-estimate yields

$$\|u_{r,\varepsilon}(.,t)\|_{L^1(B_r)} \leq \|U_0\|_{L^1(\mathbb{R})}, \quad \text{for any } t \in (0,\infty). \tag{85}$$

We infer from (34) and (84) that there is a constant $C(\beta,p) > 0$ such that

$$|\partial_x u_{r,\varepsilon}(x,t)| \leq C(\beta,p).u_{r,\varepsilon}^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}.\|U_0\|^{\frac{1+\beta}{p}}_{L^\infty(\mathbb{R})} + 1\right), \quad \text{for a.e } (x,t) \in B_r\times(0,\infty). \tag{86}$$

Next, we will pass to the limit when $r \to \infty$, and $\varepsilon \to 0$. Let us start by passing firstly to the limit as $r \to \infty$. For any $\varepsilon > 0$ fixed, we observe that $\{u_{r,\varepsilon}\}_{r>0}$ is a non-decreasing sequence. Then, there exists a nonnegative function U_ε such that

$$u_{r,\varepsilon}(x,t) \uparrow U_\varepsilon(x,t), \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty), \tag{87}$$

so, we have from (84), (85), (87), and the Monotone Convergence Theorem

$$\begin{cases} \|U_\varepsilon(.,t)\|_{L^\infty(\mathbb{R})} \le \|U_0\|_{L^\infty(\mathbb{R})}, & \text{for any } t \in (0,\infty), \\ u_{r,\varepsilon}(.,t) \to U_\varepsilon(t), \quad \text{in } L^1(\mathbb{R}), & \text{for any } t \in (0,\infty), \\ \|U_\varepsilon(.,t)\|_{L^1(\mathbb{R})} \le \|U_0\|_{L^1(\mathbb{R})}, & \text{for any } t \in (0,\infty). \end{cases} \tag{88}$$

By the same analysis as in the proof of (47), we also have

$$\partial_x u_{r,\varepsilon}(x,t) \overset{r\to\infty}{\longrightarrow} \partial_x U_\varepsilon(x,t), \quad \text{for a.e } (x,t) \in \mathbb{R} \times (0,\infty),$$

up to a subsequence. Thus, it follows from (86)

$$|\partial_x U_\varepsilon(x,t)| \le C(\beta,p).U_\varepsilon^{1-\frac{1}{\gamma}}(x,t)\left(t^{-\frac{1}{p}}.\|U_0\|_{L^\infty(\mathbb{R})}^{\frac{1+\beta}{p}} + 1\right), \quad \text{for a.e } (x,t) \in \mathbb{R} \times (0,\infty), \tag{89}$$

and

$$\partial_x u_{r,\varepsilon} \longrightarrow \partial_x U_\varepsilon, \quad \text{in } L^q_{loc}(\mathbb{R} \times (0,\infty)), \quad \forall q \in [1,\infty). \tag{90}$$

Thanks to (87), (88) and (90), passing to the limit as $r \to \infty$ in the equation satisfied by $u_{r,\varepsilon}$ yields

$$\partial_t U_\varepsilon - \left(|\partial_x U_\varepsilon|^{p-2}\partial_x U_\varepsilon\right)_x + g_\varepsilon(U_\varepsilon) = 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,\infty)). \tag{91}$$

Now, we shall pass to the limit when $\varepsilon \to 0$. We first claim that $\{U_\varepsilon\}_{\varepsilon>0}$ is a non-decreasing sequence. Indeed, we mimic the proof of (65) to get for any $r > 0$,

$$u_{r,\varepsilon} \ge u_{r,\varepsilon'}, \quad \text{in } B_r \times (0,\infty), \quad \forall \varepsilon > \varepsilon' > 0,$$

so the above claim follows when $r \to \infty$. Then, there exists a function U such that

$$U_\varepsilon(x,t) \downarrow U(x,t), \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty). \tag{92}$$

In similar, we also get

$$\partial_x U_\varepsilon \to \partial_x U, \quad \text{for a.e } (x,t) \in \mathbb{R} \times (0,\infty).$$

Therefore, the conclusions (81) follows from (89) when $\varepsilon \to 0$.
In addition, by repeating the argument of (67), there is a subsequence of $\{g_\varepsilon(U_\varepsilon)\}_{\varepsilon>0}$ such that

$$g_\varepsilon(U_\varepsilon) \to U^{-\beta}\chi_{\{U>0\}}, \quad \text{in } L^1(\mathbb{R} \times (0,\infty)). \tag{93}$$

The above results allows us to mimic the proof $(72) - (75)$ in order to pass to the limit as $\varepsilon \to 0$ in equation (91) to get

$$\partial_t U - \left(|U_x|^{p-2}U_x\right)_x + U^{-\beta}\chi_{\{U>0\}} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0,\infty)), \tag{94}$$

Next, using the local argument as in the proof of (58) yields

$$U \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R})).$$

Now, to prove $u \in \mathcal{C}([0,\infty); L^1(\mathbb{R}))$, it suffices to show that $u(t)$ is continuous at $t = 0$ in $L^1(\mathbb{R})$, i.e

$$\lim_{t\to 0} \int_{\mathbb{R}} |U(x,t) - U_0(x)|dx = 0,$$

and the conclusion for $t > 0$ is proved in the same way. In fact, we have for any $m \geq 1$

$$\int_{\mathbb{R}} |U(x,t) - U_0(x)|dx \leq \int_{I_m} |U(x,t) - U_0(x)|dx + \int_{\mathbb{R}\setminus I_m} |U(x,t) - U_0(x)|dx$$

$$\leq \int_{I_m} |U(x,t) - U_0(x)|dx + \int_{\mathbb{R}\setminus I_m} U(x,t)dx + \int_{\mathbb{R}\setminus I_m} U_0(x)dx =$$

$$\int_{I_m} |U(x,t) - U_0(x)|dx - \left(\int_{I_m} (U(x,t) - U_0(x))dx\right) + \int_{\mathbb{R}} U(x,t)dx - \int_{I_m} U_0(x)dx + \int_{\mathbb{R}\setminus I_m} U_0(x)dx.$$

By (88) and (92), we have

$$\int_{\mathbb{R}} U(x,t)dx \leq \int_{\mathbb{R}} U_0(x)dx,$$

which implies

$$\int_{\mathbb{R}} |U(x,t) - U_0(x)|dx \leq 2\int_{I_m} |U(x,t) - U_0(x)|dx + \int_{\mathbb{R}} U_0(x)dx - \int_{I_m} U_0(x)dx + \int_{\mathbb{R}\setminus I_m} U_0(x)dx =$$

$$2\int_{I_m} |U(x,t) - U_0(x)|dx + 2\int_{\mathbb{R}\setminus I_m} U_0(x)dx.$$

Taking $\limsup\limits_{t\to 0}$ both sides of the indicated inequality deduces

$$\limsup_{t\to 0} \int_{\mathbb{R}} |U(x,t) - U_0(x)|dx \leq 2\limsup_{t\to 0} \int_{I_m} |U(x,t) - U_0(x)|dx + 2\int_{\mathbb{R}\setminus I_m} U_0(x)dx.$$

By $U \in \mathcal{C}([0,\infty); L^1_{loc}(\mathbb{R}))$, we obtain from the last inequality

$$\limsup_{t\to 0} \int_{\mathbb{R}} |U(x,t) - U_0(x)|dx \leq 2\int_{\mathbb{R}\setminus I_m} U_0(x)dx.$$

Then the result follows as $m \to \infty$.
Finally, the conclusion $U \in L^p(0,T; W^{1,p}(\mathbb{R}))$ is a classical result for the initial data $U_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, we leave the detail for the reader. In summary, we complete the proof of the above theorem. ☐

Remark 17 *By the boundedness of* U*, it is clear that* $U \in \mathcal{C}([0,\infty); L^q(\mathbb{R}))$*, for* $q \in [1,\infty)$*.*

From the construction of U above, we have an observation as follows

Corollary 18 *Assume that* I *is a bounded interval in* $\mathbb{R}$*. Let* U *be the solution of equation* (80)*, and* u *be the maximal solution of equation* (1) *in* $I \times (0,\infty)$*. Then, we have*

$$u(x,t) \leq U(x,t), \quad \forall (x,t) \in I \times (0,\infty). \tag{95}$$

Proof: In fact, we have for any r large enough such that $I \subset B_r$

$$u_{I,\varepsilon} \le u_{r,\varepsilon}, \quad \text{in } I \times (0,\infty). \tag{96}$$

Passing $r \to \infty$ and $\varepsilon \to 0$ in (96) yields conclusion (95). □

Next, we will show that any weak solution W of equation (80) quenches after a finite time.

Theorem 19 *Let $p > 2$, and $\beta \in (0,1)$, and $U_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, there exists a finite time T_0 so that W satisfies*

$$W(t) = 0, \quad \forall t \ge T_0, \quad \textit{with } T_0 = \frac{\|U_0\|^{1+\beta}_{L^\infty(\mathbb{R})}}{1+\beta}.$$

Proof: Recall here Γ_ε is the solution of the equation

$$\begin{cases} \partial_t \Gamma_\varepsilon(t) + g_\varepsilon(\Gamma_\varepsilon) = 0, & t > 0, \\ \Gamma_\varepsilon(0) = \|U_0\|_{L^\infty(\mathbb{R})}. \end{cases}$$

It is straightforward to show that

$$\Gamma_\varepsilon(t) \to \Gamma(t) = \Big(m_0^{1+\beta} - (1+\beta)t\Big)_+^{\frac{1}{1+\beta}}, \quad \text{for } t > 0.$$

We observe that W is a sub-solution of equation (33) in $\mathbb{R} \times (0,\infty)$. By the strong comparison theorem, we obtain

$$W(x,t) \le \Gamma_\varepsilon(t), \quad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$

which implies the result as $\varepsilon \to 0$. □

5.2 The uniform localization property and the global quenching in a finite fime

Here, we study the uniform localization property of solutions of Cauchy problem (80). This implies the finite speed of propagation of solutions, that any solution with compact support initially has compact support at all later times $t > 0$. In fact, we shall show that $Supp(W(t))$ is uniformly bounded for any $t > 0$ (the uniform localization property), if $Supp(U_0) \subset\subset \mathbb{R}$, where W is a weak solution of equation (80).

Let us first make a simple argument to show the finite speed of propagation property. Indeed, let V be the unique solution of the unperturbed equation

$$\begin{cases} \partial_t V - (|V_x|^{p-2} V_x)_x = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ V(x,0) = U_0(x), & \text{in } \mathbb{R}. \end{cases} \tag{97}$$

Thanks to the strong comparison theorem, we have

$$W(x,t) \le V(x,t), \quad \text{for any } (x,t) \in \mathbb{R} \times (0,\infty).$$

Moreover, it is well known that for any $t > 0$, $Supp(V(t))$ is bounded by a function of t (see [11]). This implies the result.

Besides, we have (see [12])

$$Supp(V(t_1)) \subseteq Supp(V(t_2)), \quad \forall t_2 > t_1 > 0. \tag{98}$$

However, property (98) is not true for W, see Theorem 19 above. Nevertheless, we will show that $Supp(W(t))$ can be contained in a ball with its radius independent of t.

Theorem 20 *Let $p > 2$, and $\beta \in (0, 1)$, and $0 \leq U_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Assume $Supp(U_0) \subset B(0, r_0)$, for some $r_0 > 0$. Then, any weak solution W of equation (80) satisfies*

$$Supp\,(W(t)) \subset B(0, r_0 + \frac{m_0^{\frac{1}{\gamma}}}{l_0}), \quad \textit{for any } t > 0,$$

with $l_0 = (\frac{1}{\gamma^{p-1}(\gamma-1)(p-1)})^{\frac{1}{p}}$, and $m_0 = \|U_0\|_{L^\infty(\mathbb{R})}$.

Proof: For any $\varepsilon > 0$, let w_ε be a non-negative solution of the following equation

$$\begin{cases} -(|w'_\varepsilon|^{p-2} w'_\varepsilon)' + g_\varepsilon(w_\varepsilon) = 0, & \text{in } \mathbb{R}^+, \\ w_\varepsilon(0) = m_0, \\ \lim_{x \to \infty} w_\varepsilon(x) = 0. \end{cases} \tag{99}$$

It is not difficult to show that

$$w_\varepsilon(x) \to w(x) = \left(m_0^{\frac{1}{\gamma}} - l_0 x \right)_+^\gamma, \quad \text{for } x > 0,$$

To obtain the conclusion, it is sufficient to show that

$$W(x, t) \leq w(x - r_0), \quad \text{for } x > r_0, \ t > 0, \tag{100}$$

then $v(x, t) = 0$, for any $x \geq m_0$, and for $t > 0$. The same argument for the case $x < -R_0$ implies $v(x, t) = 0$, for any $x \leq -m_0$, and for $t > 0$, thereby proves the above Lemma.

Now, we prove (100). It is clear that W is a sub-solution of equation (33) in $(R_0, \infty) \times (0, \infty)$. Moreover, we have

$$\begin{cases} W(x, t) \mid_{x=R_0} \leq \|u_0\|_{L^\infty} = w_\varepsilon(x - R_0) \mid_{x=R_0}, \\ \\ W(x, 0) = 0 \leq w_\varepsilon(x - R_0), \quad \text{for } x > R_0. \end{cases}$$

By the comparison principle, we obtain

$$W(x, t) \leq w_\varepsilon(x), \quad \text{for } (x, t) \in (R_0, \infty) \times (0, \infty).$$

Letting $\varepsilon \to 0$ yields conclusion (100). This puts an end to the proof of Theorem 20. $\square$

As a consequence of Theorem 20, we have the following corollary

Corollary 21 *Let $u_0 \in L^\infty(I)$. Assume $Supp(u_0) \subset B(0, r_0)$, for some $r_0 > 0$. Assume more that*

$$B(0, r_0 + \frac{m_0^{\frac{1}{\gamma}}}{l_0}) \subset I \tag{101}$$

with l_0 and m_0 above. Then, the Cauchy solution U of (80) *coincides with the maximal solution u of* (1) *in $I \times (0, \infty)$.*

Proof: Thanks to the condition (101) and Theorem 20, we observe that the restriction to I of U is a weak solution of the homogeneous zero Dirichlet boundary condition of problem (1) in $I \times (0, \infty)$. This implies that

$$U(t) \leq u(t), \quad \text{in } I \times (0, \infty), \tag{102}$$

because u is the maximal solution of equation (1). Thus, the conclusion follows from (102) and Corollary 18. □

6 Appendix

We first have a well-known result because of the strong monotonicity of the diffusion operator.

Lemma 22 *For any $v_1, v_2 \in W_0^{1,p}(I)$, there is a constant $c > 0$ such that*

$$\int_I \left(|\partial_x v_1|^{p-2}.\partial_x v_1 - |\partial_x v_2|^{p-2}.\partial_x v_2\right)(\partial_x v_1 - \partial_x v_2)\, dx \geq c.\|\partial_x v_1 - \partial_x v_2\|^p_{L^p(I)}.$$

(see, e.g., [9] or [23]). Before giving the proof of Lemma 9, let us define a weak sub-solution (resp. super-solution) of equation (33).

Definition 23 *v is called a weak sub-solution (resp. super-solution) of equation* (33) *if $v \in \mathcal{C}([0,\infty); L^1(I)) \cap L^\infty_{loc}(\overline{I} \times (0,\infty)) \cap L^p_{loc}(0, \infty; W_0^{1,p}(I))$ satisfies*

$$\partial_t v_1 - (|\partial_x v_1|^{p-2}\partial_x v_1)_x + g_\varepsilon(v_1) \leq 0, \quad \text{in } \mathcal{D}'(I \times (0,\infty)) \quad (\text{resp. } \geq 0).$$

The proof of Lemma 9: We recall the function $T_k(s)$ and $S_k(s)$ as in the proof of Theorem 8 (see 13-pages). Then, a subtraction between two equations satisfied by v_1 and v_2 gives us

$$\partial_t(v_1 - v_2) - \partial_x\left(|\partial_x v_1|^{p-2}\partial_x v_1 - |\partial_x v_2|^{p-2}\partial_x v_2\right) + g_\varepsilon(v_1) - g_\varepsilon(v_2) \leq 0.$$

Multiplying both sides of the above equation with the test function $T_1(w)$, $w = (v_1 - v_2)_+$; and using integration by part yield

$$\int_I S_1(w(x,t))dx + \int_\tau^t \int_I \left(|\partial_x v_1|^{p-2}\partial_x v_1 - |\partial_x v_2|^{p-2}\partial_x v_2\right)(\partial_x T_1(w))\, dxds +$$
$$\int_\tau^t \int_I \left(g_\varepsilon(v_1) - g_\varepsilon(v_2)\right).T_1(w) dxds \leq \int_I S_1(w(x,\tau))dx, \quad \text{for } t > \tau > 0.$$

It follows from Lemma 22, and the fact that g_ε is a global Lipschitz function

$$\int_I S_1(w(x,t))dx \leq C(\varepsilon)\int_\tau^t \int_I |v_1 - v_2| T_1(w) dxds + \int_I S_1(w(x,\tau))dx,$$

where $C(\varepsilon) > 0$ is the Lipschitz constant of g_ε. Letting $\tau \to 0$ in the above inequality deduces

$$\int_I S_1(w(x,t))dx \le C(\varepsilon) \int_0^t \int_I |v_1 - v_2| T_1(w) dx ds.$$

In addition, we have

$$|v_1 - v_2| T_1(w)(x,t) \le 2 S_1(w(x,t)).$$

Inserting this fact into the indicated inequality yields

$$\int_I S_1(w(x,t))dx \le 2C(\varepsilon) \int_0^t \int_I S_1(w(x,t)) dx ds.$$

Then, we arrive to the following ordinary differential equation

$$\begin{cases} \frac{d}{dt} y(t) \le 2C(\varepsilon) y(t), & t > 0, \\ y(0) = 0. \end{cases}$$

with

$$y(t) = \int_I S_1(w(x,t))dx.$$

It follows from Gronwall's lemma that

$$y(t) = 0, \quad \forall t > 0,$$

which implies

$$w(t) = 0, \quad \forall t > 0.$$

In other words, we get the above lemma.

Remark 24 *The result of Lemma 9 also holds for any sub-solution v_1 and super-solution v_2 of equation* (33) *satisfying $v_2 \ge v_1$ on the boundary.*

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