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MATHEMATICS

Homogenization of the p-Laplacian with Nonlinear Boundary Condition on Critical Size Particles: Identifying the Strange Term for the Some non Smooth and Multivalued Operators[1]

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Abstract—We extend previous papers in the literature concerning the homogenization of Robin type boundary conditions for quasilinear equations, in the case of microscopic obstacles of critical size: here we consider nonlinear boundary conditions involving some maximal monotone graphs which may correspond to discontinuous or non-Lipschitz functions arising in some catalysis problems.

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Papers [2–10] were devoted to the study of asymptotic behavior of the solution to the boundary value problem for the p-Laplacian ($p \in [2,n)$) in ε-periodically perforated domain with nonlinear Robin-type boundary condition that contains function $\sigma(x,u)$. It was supposed there that $\sigma(x,u)$ is a smooth function of it's arguments, monotone by variable u. In this paper we extend the method introduced in [3, 4, 7–10] to deal with the problems with more general conditions on the function $\sigma(x,u)$. As in all papers in which the holes are of critical size and the adsorption parameter has a critical power of ε (we will precise this later) we observe a change in the nature of the nonlinearity. Our aim is to present this change in the case $\sigma(u) = C|u|^{q-1}u$, $0 < q < 1$, which is not differentiable at 0, and in the case when σ is the maximal monotone operator for the Heaviside function, which is a multivalued operator, and $p \in [2,n)$. In a further paper [12] we extend the arguments to the case of general maximal monotone graphs σ and $p \in (1,n)$.

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Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary $\partial\Omega$ and let $Y = \left(-\frac{1}{2},\frac{1}{2}\right)^n$. Denote by G_0 the unit ball centered at the origin. For $\delta > 0$ and $0 < \varepsilon \ll 1$ we define sets $\delta B = \{x | \delta^{-1}x \in B\}$ and $\tilde{\Omega}_\varepsilon = \{x \in \Omega | \rho(x,\partial\Omega) > 2\varepsilon\}$. Let $a_\varepsilon = C_0\varepsilon^\alpha$, where $\alpha > 1$ and C_0 is positive number. Define

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n | (a_\varepsilon G_0 + \varepsilon j) \cap \bar{\tilde{\Omega}}_\varepsilon \neq \varnothing\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = \text{const} > 0$, Z^n is the set of vectors z with integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, where $j \in \Upsilon_\varepsilon$ and note that $\bar{G}_\varepsilon^j \subset Y_\varepsilon^j$ and center of the ball G_ε^j coincides with the center of the cubeY_ε^j. Define

$$\Omega_\varepsilon = \Omega \setminus \bar{G}_\varepsilon, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon.$$

Consider the problem

$$\begin{cases} -\Delta_p u_\varepsilon = f, \quad x \in \Omega_\varepsilon, \\ -\partial_{\nu_p} u_\varepsilon \in \varepsilon^{-\gamma}\sigma_q(u_\varepsilon), \quad x \in S_\varepsilon, \\ u_\varepsilon = 0, \quad x \in \partial\Omega, \end{cases} \tag{1}$$

where $\Delta_p u \equiv \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $p \in [2,n)$, $\partial_{\nu_p} u \equiv |\nabla u|^{p-2}(\nabla u, \nu)$, ν is the outward unit normal vector to S_ε, $\gamma = \alpha(p-1)$. We suppose that $f \in L^{p'}(\Omega)$, $p' = \dfrac{p}{p-1}$.

Function $\sigma_q(\lambda)$, $q \in [0,1]$, is the maximal monotone continuous mapping [11], that depends on parameter q

$$\sigma_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda^q, & \lambda \in (0,1), \\ 1, & \lambda > 1. \end{cases} \tag{2}$$

We note that $\sigma_0(\lambda)$ is the maximal monotone mapping for the Heaviside function, i.e. multivalued function

$$\sigma_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ [0,1], & \lambda = 0, \\ 1, & \lambda > 0. \end{cases} \tag{3}$$

Boundary conditions of this type correspond to the presence of so called chemical reaction of order q on the boundary of cavities [5, 6]. The motivation to truncate the powers comes from the chemical modeling, in which concentrations impose range in [0, 1], but it also corresponds to the case $f \in L^\infty(\Omega)$, for which the solution is bounded.

Applying monotonicity tools (see, e.g., [11]) it is easy to see that problem (1) is equivalent to ask for $u_\varepsilon \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, satisfying the integral inequality

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla(\phi - u_\varepsilon) dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} (\psi_q(\phi) - \psi_q(u_\varepsilon)) ds \geq \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon) dx, \tag{4}$$

for any arbitrary function $\phi \in W^{1,p}(\Omega_\varepsilon, \partial\Omega)$, where $\psi_q(\lambda)$ for $q \in (0,1]$ is the primitive of σ_q. For $q \in (0,1]$ we have that

$$\psi_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \dfrac{\lambda^{q+1}}{q+1}, & \lambda \in (0,1), \\ \lambda - \dfrac{q}{q+1}, & \lambda > 1 \end{cases} \tag{5}$$

and if $q = 0$ then

$$\psi_0(\lambda) = \begin{cases} 0, & \lambda \leq 0, \\ \lambda, & \lambda > 0. \end{cases} \tag{6}$$

Space $W^{1,p}(\Omega_\varepsilon, \partial\Omega)$ is the closure in $W^{1,p}(\Omega_\varepsilon)$ of the set of infinitely differentiate functions in $\overline{\Omega}_\varepsilon$, that vanish near the boundary $\partial\Omega$.

It is well known that problem (1) has unique weak solution (see., e.g. [1, Theorem 8.5]). The following estimation is valid

$$\|u_\varepsilon\|^p_{L^p(\Omega_\varepsilon)} + \varepsilon^{-\gamma} \|u_\varepsilon\|_{L^1(S_\varepsilon \cap \{x | u_\varepsilon > 1\})} \leq K, \tag{7}$$

where constant K here and below is independent of ε.

Let $H_q(\lambda)$ be the function given through functional equation

$$B_0 |H_q|^{p-2} H_q \in \sigma_q(\lambda - H_q), \tag{8}$$

where $B_0 = \text{const} > 0$. Note that, for any prescribed λ equation (8) has unique solution. In the case if $q = 0$

$$H_0(\lambda) = \begin{cases} 0, & \lambda < 0, \\ \lambda, & 0 < \lambda < B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > B_0^{-\frac{1}{p-1}} \end{cases} \tag{9}$$

and if $q \in (0,1]$ then

$$H_q(\lambda) = \begin{cases} 0, & \lambda < 0, \\ (b_q)^{-1}(\lambda), & 0 < \lambda < 1 + B_0^{-\frac{1}{p-1}}, \\ B_0^{-\frac{1}{p-1}}, & \lambda > 1 + B_0^{-\frac{1}{p-1}}, \end{cases} \tag{10}$$

where $b_q(s)$ is the strictly monotone function given for $s \geq 0$ by

$$b_q(s) \equiv s + B_0^{\frac{1}{q}} s^{\frac{p-1}{q}} = \lambda. \tag{11}$$

Note that, in both cases, $H_q(\lambda)$ is bounded and Lipschitz continuous.

Denote by $\tilde{u}_\varepsilon$ a $W^{1,p}$-extension of function u_ε (see [10]). Using estimate (7) we get following inequality

$$\|\tilde{u}_\varepsilon\|_{W^{1,p}(\Omega)} \leq K. \tag{12}$$

Therefore, there exists a subsequence (denoted as the original sequence), such that $\varepsilon \to 0$:

$$\tilde{u}_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega). \tag{13}$$

The following theorem gives a description of the limit function u.

Theorem 1. *Let* $n \geq 3$, $p \in [2, n)$, $q \in [0,1]$, $\alpha = \dfrac{n}{n-p}$, $\gamma = \alpha(p-1)$ *and* u_ε *is the weak solution of the problem* (1). *Suppose that* $H_q(\lambda)$ *is the function given by equation* (8), *in which* $B_0 = \left(\dfrac{n-p}{p-1}\right)^{p-1} C_0^{1-p}$. *Then the limit function* u, *introduced in* (13), *is the weak solution of the problem*

$$\begin{cases} -\Delta_p u + A(n,p) |H_q(u)|^{p-2} H_q(u) = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{14}$$

which is understood as a function $u \in W_0^{1,p}(\Omega)$ *satisfying the variational inequality*

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v-u)dx +$$
$$+ A(n,p)\int_\Omega |H_q(v)|^{p-2}H_q(v)(v-u)dx \ge$$
$$\ge \int_\Omega f(v-u)dx,$$

for an arbitrary function $v \in W_0^{1,p}(\Omega)$. *Here*, $A(n,p) = \left(\frac{n-p}{p-1}\right)^{p-1} C_0^{n-p}\omega_n$, *with* ω_n *the surface area of the unit sphere in* $\mathbb{R}^n$.

Proof. (a) Consider the case $q = 0$. Denote $B_1 = B_0^{-\frac{1}{p-1}}$. Note that the integral inequality in the case when $q = 0$ has the form

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+ - u_\varepsilon^+)ds \ge \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx, \tag{15}$$

where ϕ^+ is the positive part of function ϕ, $\phi = \phi^+ - \phi^-$. From (15) we conclude

$$\varepsilon^{-\gamma}\left\|u_\varepsilon^+\right\|_{L^1(S_\varepsilon)} \le K. \tag{16}$$

Using the monotonicity of function $|\lambda|^{p-2}\lambda$ for $p > 1$ we derive inequality for u_ε

$$\int_{\Omega_\varepsilon} |\nabla \phi|^{p-2}\nabla\phi\nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+ - u_\varepsilon^+)ds \ge \int_{\Omega_\varepsilon} f(\phi - u_\varepsilon)dx, \tag{17}$$

that is valid for an arbitrary function $\varphi \in W_0^{1,p}(\Omega)$.

Let us take a test function in inequality (17)

$$\phi(x) = v(x) - W_\varepsilon(x)H_0(v(x)),$$

where $v \in C_0^\infty(\Omega)$, $H_0(\lambda)$ is given by the formula (9), function $W_\varepsilon(x)$ is defined as follows

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j, & x \in T_\varepsilon^j \setminus \overline{G_\varepsilon^j}, \quad j \in \Upsilon_\varepsilon, \\ 1, & x \in G_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j\in\Upsilon_\varepsilon} T_\varepsilon^j. \end{cases} \tag{18}$$

Here, T_ε^j is the ball of radius $\varepsilon/4$ center of which coincides with the center P_ε^j of G_ε^j,

$$w_\varepsilon^j(x) = \frac{|x - P_\varepsilon^j|^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}{a_\varepsilon^{(p-n)/(p-1)} - (\varepsilon/4)^{(p-n)/(p-1)}}.$$

Note that

$$W_\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad W^{1,p}(\Omega) \tag{19}$$

as $\varepsilon \to 0$. Using that

$$v = H_0(v) \Leftrightarrow v \in [0, B_1],$$
$$v < H_0(v) \Leftrightarrow v < 0,$$
$$v > H_0(v) \Leftrightarrow v > B_1$$

and $\phi^+|_{S_\varepsilon} = (v - H_0(v))^+ = v - H_0(v)$ if $v > H_0(v)$ we get

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(\phi^+ - u_\varepsilon^+)ds \le \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v - B_1 - u_\varepsilon^+)ds - \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ ds. \tag{20}$$

Substituting introduced test function in inequality (17) and using (19) and (20), we get that the limit as $\varepsilon \to 0$ of the left-hand side of the inequality (17) doesn't exceed the limit of

$$\int_{\Omega_\varepsilon}|\nabla v|^{p-2}\nabla v\nabla(v - u_\varepsilon)dx + \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v - B_1 - u_\varepsilon^+)ds$$
$$- \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ ds - \sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2} \tag{21}$$
$$\times H_0(v - H_0(v) - u_\varepsilon)ds$$
$$- \sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2}H_0(v - u_\varepsilon)ds.$$

The limit of the right-hand side of inequality (17) is equal to

$$\int_\Omega f(v-u)dx. \tag{22}$$

Consider the integrals over S_ε, included in the expression (21):

$$-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial G_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0|^{p-2} H_0(v - H_0(v) - u_\varepsilon)ds$$
$$+ \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v - B_1 - u_\varepsilon^+)ds - \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ ds$$
$$= \varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0|^{p-2}H_0(v - H_0(v) - u_\varepsilon)ds + \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v - B_1 - u_\varepsilon^+)ds$$
$$- \varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}} u_\varepsilon^+ ds$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-\varepsilon^{-\gamma}B_0\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds \quad (23)$$

$$-\frac{B_0\varepsilon^{-\gamma}\alpha_\varepsilon}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+ds+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-B_0\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}B_0^{-1}(v-B_1-u_\varepsilon^+)ds$$

$$+\varepsilon^{-\gamma}B_0\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}v^{p-1}u_\varepsilon^+ds$$

$$-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>B_1\}}(v-B_1-u_\varepsilon^+)ds-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+ds$$

$$= -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$

$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+(1-B_0v^{p-1}(x))ds$$

$$-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds$$

$$= J^\varepsilon-\frac{\alpha_\varepsilon B_0\varepsilon^{-\gamma}}{1-\alpha_\varepsilon}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)(v-H_0(v)-u_\varepsilon^+)ds,$$

where

$$J^\varepsilon \equiv -\varepsilon^{-\gamma}\frac{B_0}{(1-\alpha_\varepsilon)}\int_{S_\varepsilon}|H_0(v)|^{p-2}H_0(v)u_\varepsilon^- ds$$
$$-\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,B_1)\}}u_\varepsilon^+(1-B_0v^{p-1})ds\le 0, \quad (24)$$

and $\alpha_\varepsilon\to 0$ if $\varepsilon\to 0$.

Using that $\varepsilon^{-\gamma}\|u_\varepsilon^+\|_{L^1(S_\varepsilon)}\le K$, we get that the limit of the expression (21) doesn't exceed the limit of the following expression

$$\int_\Omega|\nabla v|^{p-2}\nabla v\nabla(v-u)dx$$
$$-\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_{\nu_p}w_\varepsilon^j|H_0(v)|^{p-2}H_0(v)(v-u_\varepsilon)ds. \quad (25)$$

Using an equality proved in [3] we get

$$\lim_{\varepsilon\to 0}\sum_{j\in\Upsilon_\varepsilon}\int_{\partial T_\varepsilon^j}|\nabla w_\varepsilon^j|^{p-2}\partial_\nu w_\varepsilon^j|H_0(v)|^{p-2}H_0(v-u_\varepsilon)ds$$
$$= -A(n,p)\int_\Omega|H_0(v)|^{p-2}H_0(v)(v-u)dx. \quad (26)$$

It follows from (20)–(26) that u satisfies inequality, for any $v\in W_0^{1,p}(\Omega)$,

$$\int_\Omega|\nabla v|^{p-2}\nabla v\nabla(v-u)dx+A(n,p)$$
$$\times\int_\Omega|H_0(v)|^{p-2}H_0(v)(v-u)dx\ge\int_\Omega f(v-u)dx. \quad (27)$$

Taking $v=u+\lambda w$, with $w\in W_0^{1,p}(\Omega)$ arbitrary, and making $\lambda\to 0$, since $H_0$is Lipschits continuous and bounded, we get that u is a weak solution of problem (14) for $q=0$ in the usual sense.

(b) Now we consider the case $q\in(0,1]$. In this case we make similar reasoning. We set $\phi=v-W_\varepsilon H_q(v)$ in inequality (17) as a test function, where $H_q(\lambda)$ is defined by (10). Further, we only need to explain the method of the comparison of the integrals over S_ε, included in the obtained variational inequality. Note that in this case variational inequality has the form

$$\int_{\Omega_\varepsilon}|\nabla\phi|^{p-2}\nabla\phi(\nabla\phi-\nabla u_\varepsilon)dx+\varepsilon^{-\gamma}$$
$$\times\int_{S_\varepsilon}(\psi_q(\phi)-\psi_q(u_\varepsilon))ds\ge\int_{\Omega_\varepsilon}f(\phi-u_\varepsilon)dx, \quad (28)$$

where

$$\varepsilon^{-\gamma}\int_{S_\varepsilon}(\psi_q(\varphi)-\psi_q(u_\varepsilon))ds$$
$$\le\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\in(0,1)\}}\left(\frac{(v-H_q(v))^{q+1}}{q+1}-\frac{u_\varepsilon^{q+1}}{q+1}\right)ds$$
$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon\le 0\}}\frac{(v-H_q(v))^{q+1}}{q+1}ds$$
$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v\in(0,1+B_1)\}\cap\{u_\varepsilon>1\}}\left(\frac{(v-H_q)^{q+1}}{q+1}-u_\varepsilon+\frac{q}{q+1}\right)ds \quad (29)$$
$$+\varepsilon^{-\gamma}\int_{S_\varepsilon\cap\{v>1+B_1\}\cap\{u_\varepsilon\in(0,1)\}}\left(v-H_q-\frac{q}{q+1}-\frac{u_\varepsilon^{q+1}}{q+1}\right)ds$$

$$+ \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \le 0\}} \left(v - H_q - \frac{q}{q+1} \right) ds$$

$$+ \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon > 1\}} \left(v - H_q - \frac{q}{q+1} - u_\varepsilon + \frac{q}{q+1} \right) ds.$$

We substitute the test function in (17) and we consider the remaining integrals over S_ε in the left-hand side of variational inequality (17):

$$\begin{aligned} - \sum_{j \in \Upsilon_\varepsilon} \int\limits_{\partial G_\varepsilon^j} |\nabla w_\varepsilon^j|^{p-2} \partial_\nu w_\varepsilon^j |H_q|^{p-2} H_q (v - H_q(v) - u_\varepsilon) ds \\ = - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} B_0 H_q^{p-1}(v)(v - H_q(v) - u_\varepsilon^+) ds \\ - \frac{B_0 \varepsilon^{-\gamma} \alpha_\varepsilon}{1 - \alpha_\varepsilon} \int\limits_{S_\varepsilon} H_q^{p-1}(v)(v - H_q(v) - u_\varepsilon^+) ds \\ - \varepsilon^{-\gamma} \frac{B_0}{1 - \alpha_\varepsilon} \int\limits_{S_\varepsilon} H_q^{\,p-1} u_\varepsilon^- ds. \end{aligned} \tag{30}$$

Note that

$$\begin{aligned} -\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} B_0 H_q^{p-1}(v)(v - H_q(v) - u_\varepsilon^+) ds \\ = -\varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \in (0,1)\}} (v - H_q(v))^q \\ \times (v - H_q(v) - u_\varepsilon) ds \\ - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon > 1\}} (v - H_q(v))^q (v - H_q(v) - u_\varepsilon) ds \\ - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \le 0\}} (v - H_q(v))^{q+1} ds \\ - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \le 0\}} (v - B_1) ds \\ - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon \in (0,1)\}} (v - B_1 - u_\varepsilon) ds \\ - \varepsilon^{-\gamma} \int\limits_{S_\varepsilon \cap \{v > 1 + B_1\} \cap \{u_\varepsilon > 1\}} (v - B_1 - u_\varepsilon) ds. \end{aligned} \tag{31}$$

Next we compare integrals over the same subsets of S_ε in the left-hand side of inequality (17). We have

$$\begin{aligned} I_\varepsilon \equiv \varepsilon^{-\gamma} \int\limits_{M_\varepsilon} \Bigg\{ \frac{(v - H_q(v))^{q+1}}{q+1} - \frac{u_\varepsilon^{q+1}}{q+1} \\ - (v - H_q(v))^{q+1} + (v - H_q(v))^q u_\varepsilon \Bigg\} ds \\ = -\varepsilon^{-\gamma} \int\limits_{M_\varepsilon} \Bigg(\frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1} \\ - u_\varepsilon (v - H_q(v))^q \Bigg) ds, \end{aligned} \tag{32}$$

where $M_\varepsilon = S_\varepsilon \cap \{v \in (0, 1 + B_1)\} \cap \{u_\varepsilon \in (0,1)\}$. Using Young's inequality we get

$$u_\varepsilon (v - H_q(v))^q \le \frac{q(v - H_q(v))^{q+1}}{q+1} + \frac{u_\varepsilon^{q+1}}{q+1}. \tag{33}$$

Therefore,

$$I_\varepsilon \le 0. \tag{34}$$

The remaining integrals over subsets of S_ε are considered in the similar way and we establish that the sum of all integrals over the corresponding subsets of S_ε is non-positive. Therefore, the limit function u satisfy variational inequality

$$\int\limits_\Omega |\nabla v|^{p-2} \nabla v \nabla (v - u) dx + A(n, p) \times \int\limits_\Omega H_q^{\,p-1}(v)(v - u) dx \ge \int\limits_\Omega f(v - u),$$

for an arbitrary function $v \in W_0^{1,p}(\Omega)$. Again taking $v = u + \lambda w$, with $w \in W_0^{1,p}(\Omega)$arbitrary, and making $\lambda \to 0$ we obtain that u is a weak solution the usual sense.

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REFERENCES

1. J.-L. Lions, *Quelques methodes de resolution des problemes aux limites non lineires* (Dunod, Paris, 1969; Mir, Moscow, 1972).
2. M. Goncharenko, GAKUTO Int. Ser. Math. Sci. Appl. **9**, 203–213 (1997).
3. M. N. Zubova and T. A. Shaposhnikova, Differ. Equations **47** (1), 78–90 (2011).
4. W. Jäger, M. Neuss-Radu, and T. A. Shaposhnikova, Nonlin. Anal. Real World Appl. **15**, 367–380 (2014).
5. C. Conca, J. I. Diaz, A. Linan, and C. Timofte, Electron. J. Differ. Equations **40**, 1–22 (2004).
6. J. I. Diaz, D. Gómez-Castro, and C. Timofte, *Proceedings of the 24th Congress on Differential Equations and Application, 14th Congress on Applied Mathematics* (Cadiz, 2015), pp. 571–576.
7. D. Gómez, M. Lobo, M. E. Pérez, and T. A. Shaposhnikova, Appl. Anal. **92** (2), 218–237 (2013).
8. D. Gómez, M. E. Pérez, and T. A. Shaposhnikova, Asymptot. Anal. **80**, 289–322 (2012).
9. A. V. Podol'skii, Dokl. Math. **82** (3), 942–945 (2010).
10. A. V. Podol'skii, Dokl. Math. **91** (1) 30–34 (2015).
11. H. Brezis, J. Math. Pures Appl. **51**, 1–168 (1972).