Some Qualitative Properties for Geometric Flows and its Euler Implicit Discretization

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Abstract

We study the geometric flow parabolic equation and its implicit discretization which yields a family of nonlinear elliptic problems. We show that there are important differences in the study of those equations in which concerns the propagation of level sets of data. Our study is based on the previous study of radially symmetric solutions of the corresponding equation. Curiously, in radial coordinates both equations reduce to suitable singular Hamilton-Jacobi first order equations. After considering the case of monotone data we point out a new peculiar behavior for mon-monotone data with a profile of *Batman* type ($g = \min\{g_1, g_2\}$, $g_1(r)$ increasing, $g_2(r)$ decreasing and $g_1(r_d) = g_2(r_d)$ for some $r_d > 0$). In the parabolic regime, and when the velocity of the convexity part of the level sets is greater than the velocity of the concavity part, we show that the level set $\{u = g(r_d)\}$ develops a non-empty interior set for any $t > 0$. Nothing similar occurs in the stationary regime. We also present some numerical experiences.

Dedicated to an outstanding mathematician near his 70's: Juan Luis Vázquez

1 Introduction

This paper deals with several qualitative properties of geometric flows presented here, for simplicity in dimension 2, while all of our results admit generalizations to higher dimensions. A geometric flow can be defined as an operator $\mathrm{T}_t(f)$ which provides, for an original function f, a family of functions $\mathrm{T}_t(f)$ with $t \geq 0$. The geometric character of the flow is due to the assumption that the level set evolution of function f depends just on the geometry of the boundary of the level sets. We can formalize this geometric character using the named *morphological invariance assumption* which can be expressed as:

$$\mathrm{T}_t(f) \circ h = \mathrm{T}_t(f \circ h) \tag{1}$$

for any increasing function $h(\cdot)$. As it was proved in [4] (see also [35]) and [1], under some minimal architectural assumptions, all the geometric flows are generated by the partial differential equation:

$$\frac{\partial u}{\partial t} = \beta(\mathrm{curv}(u))|\mathrm{D}u|, \tag{2}$$

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where $\text{curv}(u)(x)$ is the curvature of the level line containing the point x, that is:

$$\text{curv}(u) = \text{div}\left(\frac{\text{D}u}{|\text{D}u|}\right), \tag{3}$$

and $\beta(\cdot)$ is a nondecreasing function given by:

$$\beta(s) = \begin{cases} b_+ s^q, & \text{if } s \geq 0, \\ -b_- (-s)^q, & \text{if } s < 0, \end{cases} \tag{4}$$

with $q, b_+, b_- \geq 0$ such that $b_+ + b_- > 0$. Therefore the geometric flow depends on three parameters, q, b_-, b_+. Among the different choices of these parameters we point out the cases $q = 1$, $b_- = b_+ = 1$, which corresponds to the *mean curvature operator* and $q = 1/3$, $b_- = b_+ = 1$, which corresponds to the *affine invariant equation* studied in [4] (in some works, as for instance [32, page 35], this last case is considered by assuming $b_- = 0$).

If $u(t,x)$ is the solution of equation (2), for the initial datum f then $u(t,x) = \text{T}_t(f)(x)$ and thus T_t represents the semigroup associated to the parabolic problem. For a level $l \in \mathbb{R}$, the associated level set of function f is defined as $L_f(l) = \overline{\{x : f(x) < l\}}$. The geometric character of the flow means that the evolution of a level set $L_f(l)$, given by $L_{\text{T}_t(f)}(l)$, depends just on the geometry of its boundary $\partial L_f(l)$. Due to its morphological invariance, geometric flows are commonly used in *Computer Vision* applications (see, *e.g.*, [37], [1] for an application to shape representation and [36], [2] for the use of affine invariant distances to a curve).

Our main interest in this paper will be centered in to get some new qualitative properties of solutions of the parabolic problem

$$\text{PP}(\mathbb{R}^2) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\text{div}\left(\dfrac{\text{D}u}{|\text{D}u|}\right)\right)|\text{D}u| = 0 & \text{in } (0,T) \times \mathbb{R}^2, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R}^2. \end{cases}$$

Notice that the above mentioned initial datum (the original *image function* f) is now denoted as u_0. We shall also pay attention to the case of an bounded regular domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions. A typical example is the case of homogeneous boundary conditions

$$\text{PP}(\Omega) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\text{div}\left(\dfrac{\text{D}u}{|\text{D}u|}\right)\right)|\text{D}u| = 0 & \text{in } (0,T) \times \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases}$$

but we shall also consider the case of non-homogeneous boundary data (see Corollary 1).

In some sense, this paper can be understood as continuation of several previous papers by the authors in which the qualitative properties of solutions of different parabolic problems are analyzed jointly with the suitable understanding of the qualitative properties of solutions of the stationary problems resulting from their the implicit Euler discretization. The main connection among the family of the diverse classes of considered problems comes from the fact that the involved *elliptic* operators generate a semigroup of contractions in some Banach space and thus, thanks to the abstract semigroup theory, the convergence of the *discretized solutions* is ensured in the corresponding Banach space. This point of view was applied in [5] to the study of the porous media equation and in [6] to a general doubly nonlinear equation involving the $p-$Laplace operator (see also the work [7] dealing with the *total variation flux* which formally can be understood as the limit case $p = 1$ of the $p-$Laplace operator: as a matter of fact, this is exactly the curv (u) operator mentioned before). Some of the qualitative properties can be connected in this way. In all those equations the correct Banach space was given by $\mathrm{L}^1(\Omega)$ (or $\mathrm{L}^1(\mathbb{R}^{\mathrm{N}})$ for the associated Cauchy problems). The case in which the Banach space was $\mathrm{L}^\infty(\Omega)$ (or $\mathrm{L}^\infty(\mathbb{R}^{\mathrm{N}})$) was considered in the series of papers [25, 26, 27] for the case of Monge-Ampere and k-Hessian operators.

Roughly speaking, the accretiveness of the formal operator $\mathcal{A}(u) = -\beta(\text{curv }(u))|\mathrm{D}u|$ in $BUC(\mathbb{R}^2)$, the set of bounded uniformly continuous functions on $\mathbb{R}^2$, was described in Theorem 2 of [4] (see also Theorem 3 of [35] and Section 9 of [22]) once that the differential operator is understood in the sense of the Crandall-Lions viscosity solutions framework (see *e.g.* the presentation made in [22]). Nevertheless, the usual notion of viscosity solutions requires some slight adaptation as to be applied to the operator $\mathcal{A}(u) = -\beta(\text{curv }(u))|\mathrm{D}u|$ such as it was shown in [34] (see also [33]), as we shall recall in Section 2. In any case, using the Euler implicit discretization we arrive to the family of stationary problems

$$-\beta\left(\text{div}\left(\frac{\mathrm{D}u_n}{|\mathrm{D}u_n|}\right)\right)|\mathrm{D}u_n| + \lambda u_n = \lambda u_{n-1} \qquad n \in \mathbb{N}, \tag{5}$$

where $\lambda > 0$ and thus u_n represents an approximation of $\mathrm{T}_{n/\lambda}(f)$. So, in this paper we shall study some qualitative properties of solutions of the stationary equation

$$-\beta\left(\text{div}\left(\frac{\mathrm{D}v}{|\mathrm{D}v|}\right)\right)|\mathrm{D}v| + \lambda v = g \tag{6}$$

and the stationary problems

$$\mathrm{SP}(\mathbb{R}^2) \qquad -\beta\left(\text{div}\left(\frac{\mathrm{D}v}{|\mathrm{D}v|}\right)\right)|\mathrm{D}v| + \lambda v = g \quad \text{in } \mathbb{R}^2, \tag{7}$$

as well as

$$\mathrm{SP}(\Omega) \qquad \begin{cases} -\beta\left(\text{div}\left(\frac{\mathrm{D}v}{|\mathrm{D}v|}\right)\right)|\mathrm{D}v| + \lambda v = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We shall consider too the case of non-homogeneous boundary data (see Corollary 2).

It is well-known that many qualitative properties of solutions for many different elliptic and parabolic equations are by-products of the previous study of radially symmetric solutions which later are used as barrier functions (see *e.g.* the monographs [28], [39]). This is the reason why we shall pay an special attention to the solvability of equations (2) and (5) in the class of radially symmetric solutions defined over balls and symmetric rings with different boundary conditions. A crucial aspect in our study is based on the fact that in the class of radially symmetric solutions $u(t, r)$ of (2) becomes the singular Hamilton-Jacobi first order equation

$$\frac{\partial u}{\partial t}(t,r) - \beta\left(\frac{\text{sign}\left(\frac{\partial u}{\partial r}(t,r)\right)}{r}\right)\left|\frac{\partial u}{\partial r}(t,r)\right| = 0, \tag{8}$$

(notice that sign $\left(\frac{\partial u}{\partial r}\right)\left|\frac{\partial u}{\partial r}\right| = 0$ if $\frac{\partial u}{\partial r} = 0$ and so the spatial operator is single valued). Similarly, radially symmetric solutions $v(r)$ of the equation (7) must solve the stationary singular Hamilton-Jacobi first order equation

$$-\beta\left(\frac{\operatorname{sign}\big(v'(r)\big)}{r}\right)|v'(r)| + \lambda v(r) = g(r). \tag{9}$$

We point out that equations (8) and (9) keep many resemblances with the hyperbolic and stationary *eikonal equations* (specially for solutions for which sign $(v'(r))$ remains constant for any r). Nevertheless the presence of the singular term $1/r$ is crucial. This singular term helps to justify the existence of solutions on balls but it is not enough as to justify the existence of solutions on rings if the boundary conditions are not well adapted to the transport flux given by the first derivative term. So, curiously enough we shall present here some non-existence of solutions results (see Corollaries 1 and 2) which seems to be unadvertised in the extensive literature available today on this type of equations (see *e.g.* the monographs [32], [17] and their references). In a separated work [3] we shall present a Lax–Oleinik type representation formula for radially symmetric solutions leading to many complementary properties.

In contrast to many other previous works dealing with level sets without interior area (see *e.g.* [10], [34] and their references) here we shall pay attention to the propagation of level sets which initially already have a non-empty interior. This methodology is what we adopted in the above mentioned papers by the authors on other equations. Nevertheless in the case of morphological invariant geometric flows the study of conditions ensuring that a given level set have a non-empty interior set or not is specially relevant. For instance, it was shown in [31] (see also [38]) that for a very special initial datum the solution of the *mean curvature geometric flow* ($q = 1$, $b_- = b_+ = 1$) may have some level sets with a non-empty interior set for $t > 0$ even if all the level sets of the initial datum have an empty interior set. The explanation given in [31] mentioned the lack of regularity of this initial datum. In contrast to that, here we shall show that something similar happens for problem PP($\mathbb{R}^2$) in the class of radially symmetric solutions once we assume $b_+ > b_-$ (roughly speaking the velocity of the level sets is privileging the convexity part of them over the concave one). To show this qualitative property (as well as many other properties) we shall consider the radially symmetric solutions of (2) and (9) corresponding to data $u_0(x) = g(|x|)$ which we shall call as *Batman* type profiles and are built as

$$\left\{\begin{array}{ll} g = \min\{g_1, g_2\}, \quad g_1(r) \text{ increasing}, \; g_2(r) \text{ decreasing} & \\ g_1(r_d) = g_2(r_d) & \text{for some } r_d > 0. \end{array}\right. \tag{10}$$

Notice that now the appearance of a non-empty interior level set is not due to any lack of regularity of the level set of the initial datum (since in our case all the level sets are circles) but to the condition $b_+ > b_-$ (which can be understood as corresponding to the velocity of the level sets is privileging the concavity part of them over the convex one) and the loss of monotonicity and of the initial datum.

The organization of the paper is as follows: in section 2 we shall recall several results on the existence and uniqueness of viscosity solutions of problems PP($\mathbb{R}^2$) and PP(Ω) and their adaptations to get the existence and uniqueness of viscosity solutions of the elliptic problems SP($\mathbb{R}^2$) and SP(Ω). Section 3 will be devoted to the study of several qualitative properties of solutions of PP($\mathbb{R}^2$) and PP(Ω), mainly the *finite speed of propagation*, the *finite extinction time property* and the *non instantaneous extinction* of level sets of solutions, independently of the value of the exponent $q > 0$

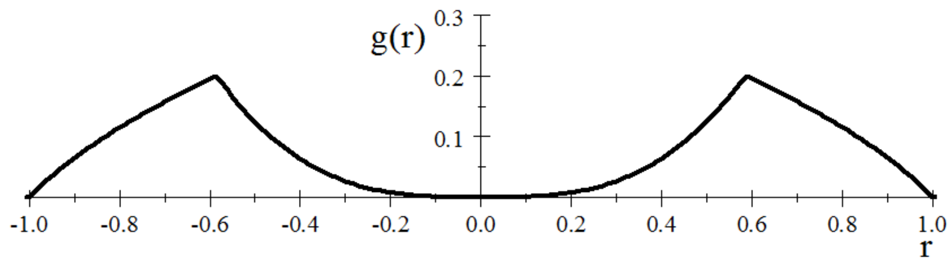

Figure 1: Non-monotone example of *Batman* profile type data.

involved at the operator (this contrast with which happens in the nonlinear diffusion or Monge-Ampere operators where only exponents strictly less than one leads to the behavior of solutions). We collected in Remark 5 many commentaries connecting and comparing those results with previous ones in the literature. As mentioned before, the crucial point will be the previous study of radially symmetric solutions. To this respect we shall present some necessary and sufficient conditions for the solvability of the parabolic equation on rings with different Dirichlet boundary data. The case of non-monotone initial data with a *Batman* type profile is considered in Subsection 3.2. We also present some numerical experiences in Subsection 3.3. The elliptic problem $\mathrm{SP}(\mathbb{R}^2)$ is considered in Section 4. We present a study of the problem following the same structure than in Section 3 but pointing out the many differences arising in it with respect the parabolic problem. For instance, there is no formation of interior free boundaries (of dead core type) neither the singular behavior for *Batman* type profiles independently of the values of b_+ and b_-. We also present some numerical experiences.

2 Some existence and uniqueness of solutions results

The differential operator under research can be written as $-\beta\,(\mathrm{curv}\ \ u)\,|\mathrm{D}u| := \mathrm{F}\,(\mathrm{D}u(x), \mathrm{D}^2u(x))$ with

$$\mathrm{F}\,(p, \mathcal{Z}) := -\beta\left(\frac{1}{|p|}\mathrm{trace}\left(\mathrm{I} - \frac{p\otimes p}{|p|^2}\right)\mathcal{Z}\right)|p|, \tag{11}$$

for $(p, \mathcal{Z}) \in \mathbb{R}^2 \times \mathcal{S}^2$, where $\mathcal{S}^2$ denotes the space of the 2×2 real and symmetric matrices (I is the identity matrix) and $\beta :\ \mathbb{R} \to \mathbb{R}$ is the nondecreasing continuous function given by (4) (this section of the paper remains valid for general nondecreasing continuous functions β such that $\beta(0) = 0$). Then $\mathrm{PP}(\mathbb{R}^2)$ and $\mathrm{SP}(\mathbb{R}^2)$ can be written as

$$\begin{cases} u_t + \mathrm{F}\,(\mathrm{D}u, \mathrm{D}^2u) = 0 & \text{in } (0,T)\times\mathbb{R}^2, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \mathbb{R}^2, \end{cases} \tag{12}$$

$$\mathrm{F}\left(\mathrm{D}u, \mathrm{D}^2u\right) + \lambda u = g \quad \text{in } \mathbb{R}^2. \tag{13}$$

Notice that $\mathrm{F}\,(p, \mathcal{Z})$ is continuous if $p \neq 0$ but it is not well defined for $p = 0$. It is easy to check (see *e.g.* [4], [35], [17]) that the operator $\mathrm{F}(p, \mathcal{Z})$ is *degenerate elliptic:* for any $p \in \mathbb{R}^2 \setminus \{0\}$,

$$\mathrm{F}(p, \mathcal{Z}_1) \geq \mathrm{F}(p, \mathcal{Z}_2) \quad \text{if } \mathcal{Z}_1 \leq \mathcal{Z}_2. \tag{14}$$

The operator $\mathrm{F}(p,\mathcal{Z})$ satisfies also the property of being *geometric* ($\mathrm{F}(\lambda p, \lambda\mathcal{Z}+\mu p\otimes p)=\lambda \mathrm{F}(p,\mathcal{Z})$ for all $\mathcal{Z}\in\mathcal{S}^2,\ p\in\mathbb{R}^2\setminus\{0\},\ \lambda>0$ and $\mu\in\mathbb{R}$).

As mentioned in [34], the possible singularity which may arise, when $\mathrm{D}u=0$, can be treated satisfactorily by means of the theory of viscosity solutions introduced by M.G. Crandall and P.L. Lions (see, *e.g.*, [22]) when $q\in[0,1]$. The treatment of this case corresponds to some slight variation of the important theory developed for *mean curvature geometric flows* ($q=1$, $b_-=b_+=1$) presented in [31] or [20] (see [35] and [11]). Nevertheless, if $q>1$ the notion of viscosity solution needs to be suitably adapted as to allow possible singularity around the points where $\mathrm{D}u=0$ (it corresponds to a special case of a class of equations which are called as *very singular equations* in [32]: a different example of very singular equation is the level set Gaussian curvature flow equation). Such adaptation was carried out in [34] (see also [33] and [32]). Before recalling this adaptation it is convenient to observe that merely from (14) there exists a function $c\in C((0,+\infty))$ such that

$$-c(|p|)\leq \mathrm{F}(p,\mathrm{I})\leq \mathrm{F}(p,-\mathrm{I})\leq c(|p|) \text{ for } p\in\mathbb{R}^2\setminus\{0\}.$$

In our case we can take $c(s)=\max\{b_+,b_-\}s^{q+1}$ for $s>0$. Since we shall deal with possible discontinuous functions, it is convenient to use the notion of upper and lower semi–continuous envelopes of such discontinuous function. We recall that given a local bounded function $v:\ \mathcal{O}\to\mathbb{R}\cup\{\pm\infty\}$, $\mathcal{O}$ is an open set of $(0,T)\times\Omega$ (Ω being an open set of $\mathbb{R}^2$) its *upper semi–continuous envelope* is defined by

$$v^*(z):=\lim_{r\searrow 0}\sup\ \{v(\zeta):\ |z-\zeta|\leq r\},\quad z\in\mathcal{O}.$$

Analogously, its *lower semi–continuous envelope* is defined by

$$v_*(z):=\lim_{r\nearrow 0}\inf\ \{v(\zeta):\ |z-\widehat{r}|\leq r\},\quad z\in\mathcal{O}.$$

Notice that $v^*=-(-v)_*$ and that $v_*\leq v\leq v^*$.

The definition of viscosity solution of equation (12) in $\mathcal{O}$ will follows by replacing the differential expressions of the candidate to solution u by the values of similar differential expressions for an arbitrary test function φ in points $z\in\mathcal{O}$ where $u^*-\varphi$ have a local finite maximum and $u_*-\varphi$ have a local finite minimum. If $q>1$ we shall need to control $\mathrm{D}^2\varphi$ and φ_t in points $\widehat{z}=(\widehat{t},\widehat{x})\in\mathcal{O}$ where $\mathrm{D}\varphi(\widehat{z})=0$ (since a change of variable allows to assume that $\widehat{z}$ is the point where $u^*-\varphi$ or $u_*-\varphi$ have a local finite maximum or a local finite minimum). Let us call for the moment by $\mathcal{A}(\beta)$ to the set of admissible test functions which we shall define in a while.

Definition 1 *i) A function $u:\mathcal{O}\to\mathbb{R}\cup\{-\infty\}$ is a viscosity sub solution of* (12) *in $\mathcal{O}$ if $u^*<\infty$ and for all $\varphi\in\mathcal{A}(\beta)$ and all local finite maximum points z of $u^*-\varphi$*

$$\begin{cases}\varphi_t(z)+\mathrm{F}(\mathrm{D}\varphi(z),\mathrm{D}^2\varphi(z))\leq 0, & \text{if } \mathrm{D}\varphi(z)\neq 0,\\ \varphi_t(z)\leq 0, & \text{if } \mathrm{D}\varphi(z)=0.\end{cases}$$

ii) A function $u:\mathcal{O}\to\mathbb{R}\cup\{\infty\}$ is a viscosity super solution of (12) *in $\mathcal{O}$ if $u_*>-\infty$ and for all $\varphi\in\mathcal{A}(\beta)$ and all local finite minimum points z of $u_*-\varphi$*

$$\begin{cases}\varphi_t(z)+\mathrm{F}(\mathrm{D}\varphi(z),\mathrm{D}^2\varphi(z))\geq 0, & \text{if } \mathrm{D}\varphi(z)\neq 0,\\ \varphi_t(z)\geq 0, & \text{if } \mathrm{D}\varphi(z)=0.\end{cases}$$

iii) A function $u:\mathcal{O}\to\mathbb{R}$ is a viscosity solution of (12) *in $\mathcal{O}$ if u is both a viscosity sub solution and a viscosity super solution of* (12) *in $\mathcal{O}$.*

We define now the class of admissible test functions. Following [34] we have:

Definition 2 *Let $\mathcal{O}$ be an open subset of $(0,\mathrm{T})\times\mathbb{R}^2$. A function $\varphi\in\mathcal{C}^2(\mathcal{O})$ is admissible ($\varphi\in\mathcal{A}(\beta)$) if for any $\widehat{z}=(\widehat{t},\widehat{x})\in\mathcal{O}$ such that $\mathrm{D}\varphi(\widehat{z})=0$ there is a constant $\delta>0$,*

$$\begin{cases} \text{a function } f\in\mathcal{C}\big([0,\infty)\big) \text{ such that } f(0)=f'(0)=f''(0)=0 \text{ and } f''(r)>0 \text{ for } r>0 \text{ which satisfy} \\ \qquad \lim\limits_{p\to0}\dfrac{f'(|p|)}{|p|}\mathrm{F}(p,\mathrm{I})=\lim\limits_{p\to0}\dfrac{f'(|p|)}{|p|}\mathrm{F}(p,-\mathrm{I})=0, \\ \text{and a function } \omega\in\mathcal{C}\big([0,\infty)\big) \text{ satisfying } \lim\limits_{r\searrow0}\dfrac{\omega(r)}{r}=0 \end{cases}$$

such that for all $(t,x)\in B_\delta(\widehat{z})$

$$|\varphi(t,x)-\varphi(\widehat{t},\widehat{x})-\varphi_t(\widehat{z})(t-\widehat{t})|\le f(|x-\widehat{x}|)+\omega(|t-\widehat{t}|).$$

As mentioned in [34] the introduction of a class of admissible test functions does not lessen but rather strengthen the usual requirements for functions to be viscosity solutions. Moreover, it is immediate that if $u\in C^2(\mathcal{O})$ satisfies

$$\begin{cases} u_t(z)+\mathrm{F}(\mathrm{D}u(z),\mathrm{D}^2u(z))\le0, & \text{if } \mathrm{D}u(z)\neq0,\\ u_t(z)\le0, & \text{if } \mathrm{D}u(z)=0,\end{cases}$$

or

$$\begin{cases} u_t(z)+\mathrm{F}(\mathrm{D}u(z),\mathrm{D}^2u(z))\ge0, & \text{if } \mathrm{D}u(z)\neq0,\\ u_t(z)\ge0, & \text{if } \mathrm{D}u(z)=0,\end{cases}$$

then u is, respectively, a viscosity subsolution or a viscosity supersolution of (12) in $\mathcal{O}$. In which follows, by simplicity, we drop the term *viscosity* and hereafter simply refer to sub solutions, super solution and solutions.

Next, we collect some results of [34] which are relevant to our purposes:

Theorem 1 (Comparison: Theorem 1.7 of [34]) *Let $\mathcal{R}_T=[0,T)\times\overline{\Omega},\ \Omega\subset\mathbb{R}^2$. Let $u\in USC(\mathcal{R}_T)$ and $v\in LSC(\mathcal{R}_T)$ be a super solution and a sub solution of (12) in $\mathcal{R}_T$. Assume that*

$$\limsup_{r\searrow0}\{u^*(z)-v_*(\zeta):(z,\zeta)\in\partial_p\mathcal{Q}_T\times\mathcal{R}_T)\cup\mathcal{R}_T\times\partial_p\mathcal{Q}_T,\ |z-\zeta|\le r\}\le0$$

with $\partial_p\mathcal{Q}_T=\{0\}\times\Omega\cup[0,T)\times\partial\Omega$. Then $u^*\le v_*$ in $\mathcal{R}_T$. Moreover

$$\limsup_{r\searrow0}\{u^*(z)-v_*(\zeta):\ |z-\zeta|\le r\}\le0.$$

Theorem 2 (Existence and uniqueness: Theorem 2.2 of [34]) *Let $u_0\in UC(G_R)$ for each $M>0$ where $G_M=\{|u_0|<M\}$. Then there is a unique solution u of $\mathrm{PP}(\mathbb{R}^2)$ such that $u\in UC(U_M)$ for each $M>0$ with $U_M=\{|u|<M\}$.*

Concerning problem $\mathrm{PP}(\Omega)$ we point out that the comparison result remains valid but the existence results require some additional commentaries. This question was not considered in [34] (neither in [33]) but it is not difficult to adapt other results in the literature to get a similar version to the above existence theorem. For instance, as indicated in [11] in other context (see also [22] Section 7 and Subsection 2.3 of [32]), it is convenient to reformulate $\mathrm{PP}(\Omega)$ as a problem posed on the closed set $[0,T)\times\overline{\Omega}$ by replacing the above differential operator $\mathrm{F}(p,\mathcal{Z})$ by a new one $\mathrm{F}^{\#}(x,u,p,\mathcal{Z})$ given by

$$\mathrm{F}^{\#}(x,u,p,\mathcal{Z}) := \begin{cases} \mathrm{F}(p,\mathcal{Z}) & \text{if } x\in\Omega, \\ u & \text{if } x\in\partial\Omega, \end{cases} \tag{15}$$

with $\mathrm{F}(p,\mathcal{Z})$ defined in (11). In this way, problem $\mathrm{PP}(\Omega)$ can be read as

$$\mathrm{PP}^{\#}(\Omega) \qquad \begin{cases} \dfrac{\partial u}{\partial t} + \mathrm{F}^{\#}(x,u,\mathrm{D}u,\mathrm{D}^2u) = 0 & \text{in } [0,T)\times\overline{\Omega}, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega. \end{cases}$$

Now the viscosity solution will satisfy the boundary conditions in the sense that

$$\begin{cases} \max\left\{\dfrac{\partial u}{\partial t} + \mathrm{F}(\mathrm{D}u,\mathrm{D}^2u), u\right\} \geq 0 & \text{on } [0,T)\times\partial\Omega, \\ \min\left\{\dfrac{\partial u}{\partial t} + \mathrm{F}(\mathrm{D}u,\mathrm{D}^2u), u\right\} \leq 0 & \text{on } [0,T)\times\partial\Omega. \end{cases}$$

Perron's method (see [22], [32] and its many references) can be applied in the above mentioned context of viscosity solutions with test functions φ in the admissible set $\mathcal{A}(\beta)$. The following result is a trivial adaptation of Theorem 2.4.9 of [32] and Theorem 2.2 of [34] to our framework.

Theorem 3 *Let h_- and h_+ be lower and supersolutions of $\mathrm{PP}^{\#}(\Omega)$ with $h_+^* < +\infty$, $h_{-*} > -\infty$. If $h_- \leq h_+$ in $[0,T)\times\overline{\Omega}$ then there is a unique solution u of $\mathrm{PP}(\Omega)$ that satisfies $h_- \leq u \leq h_+$ in $[0,T)\times\overline{\Omega}$. In particular, if $u_0 \in UC(G_{M,\Omega})$ for each $M>0$ where $G_{M,\Omega} = \{x\in\overline{\Omega} : |u_0(x)| < M\}$ then there is a unique solution u of $\mathrm{PP}(\Omega)$ such that $u\in UC(U_{M,\mathcal{Q}_T})$ for each $M>0$ with $U_{M,\mathcal{Q}_T} = \{(x,t)\in\mathcal{Q}_T : |u(t,x)| < M\}$.* □

The techniques of the proof of the above results apply without difficulty to the case of $\mathrm{SP}(\mathbb{R}^2)$ and $\mathrm{SP}(\Omega)$ once we adapt the notion of viscosity solution on an open subset $\omega\subset\Omega$. The presence of the term $\lambda v(x)$ in the right hand side of the equation (6) does not add any new special difficulty (see *e.g.* Subsections 2.4.3 and 3.1 of [32]).

Definition 3 *A function $\varphi\in C^2(\omega)$ is admissible for the elliptic equation associated to the differential operator F (which we shall write as $\varphi\in\mathcal{A}_e(\beta)$) if for any $\widehat{x}\in\omega$ such that $\mathrm{D}\varphi(\widehat{x}) = 0$ there is a constant $\delta>0$ and*

$$\begin{cases} \text{a function } f\in\mathcal{C}\big([0,\infty)\big) \text{ such that } f(0)=f'(0)=f''(0)=0 \text{ and } f''(r)>0 \text{ for } r>0 \text{ which satisfy} \\ \qquad \lim\limits_{p\to 0}\dfrac{f'(|p|)}{|p|}\mathrm{F}(p,I) = \lim\limits_{p\to 0}\dfrac{f'(|p|)}{|p|}\mathrm{F}(p,-I) = 0, \end{cases}$$

such that for all $x\in B_\delta(\widehat{x})$

$$|\varphi(x)-\varphi(\widehat{x})| \leq f(|x-\widehat{x}|).$$

Definition 4 *i) A function $v : \omega \to \mathbb{R} \cup \{-\infty\}$ is a viscosity sub solution of (6) in ω if $v^* < \infty$ and for all $\varphi \in \mathcal{A}_e(\beta)$ and all local finite maximum points x of $v^* - \varphi$*

$$\mathrm{F}(\mathrm{D}\varphi(x), \mathrm{D}^2\varphi(x)) + \lambda v^*(x) - g_*(x) \leq 0, \quad \textit{if } \mathrm{D}\varphi(\widehat{x}) \neq 0.$$

ii) A function $v : \omega \to \mathbb{R} \cup \{\infty\}$ is a viscosity super solution of (6) in ω if $v_* > -\infty$ and for all $\varphi \in \mathcal{A}_e(\beta)$ and all local finite minimum points x of $v_* - \varphi$

$$\mathrm{F}(\mathrm{D}\varphi(x), \mathrm{D}^2\varphi(x)) + \lambda v_*(x) - g^*(x) \geq 0, \quad \textit{if } \mathrm{D}\varphi(\widehat{x}) \neq 0.$$

iii) A function $v : \omega \to \mathbb{R}$ is a viscosity solution of (6) in ω if v is both a viscosity sub solution and a viscosity super solution of (6) in ω.

A similar adaptation must be done for $\mathrm{SP}(\Omega)$ in the lines of the above discussion on the parabolic Dirichlet problem $\mathrm{PP}(\Omega)$. In particular we must introduce the problem

$$\mathrm{F}^{\#}\left(x, v, \mathrm{D}v, \mathrm{D}^2 v\right) + \lambda v(x) = g(x) \quad \text{in } \overline{\Omega},$$

where $\mathrm{F}^{\#}$ was defined in (15). The following result collects some existence and uniqueness results for the elliptic problems $\mathrm{SP}(\mathbb{R}^2)$ and $\mathrm{SP}(\Omega)$:

Theorem 4 *a) Let $\omega \subset \Omega \subset \mathbb{R}^2$. Let $u \in USC(\overline{\omega})$ and $v \in LSC(\overline{\omega})$ be a super solution and a sub solution of (6) in ω. Assume that*

$$\limsup_{r \searrow 0} \{u^*(x) - v_*(\zeta) : (x, \zeta) \in \partial\omega \times \overline{\omega}) \cup \overline{\omega} \times \partial\omega,\ |z - \zeta| \leq r\} \leq 0.$$

Then $u^* \leq v_*$ in $\overline{\omega}$. Moreover

$$\limsup_{r \searrow 0} \{u^*(x) - v_*(\zeta) :\ |x - \zeta| \leq r\} \leq 0.$$

b) If $g \in UC(G_M)$ for each $M > 0$ where $G_M = \{x \in \mathbb{R}^2 : |g(x)| < M\}$ then there is a unique solution v of $\mathrm{SP}(\mathbb{R}^2)$ such that $v \in UC(V_M)$ for each $M > 0$ with $V_M = \{x \in \mathbb{R}^2 : |v(x)| < M\}$.
c) Let h_- and h_+ be sub and supersolution of $\mathrm{SP}^{\#}(\Omega)$ with $h_+^* < +\infty$, $h_{-*} > -\infty$. If $h_- \leq h_+$ in $\overline{\Omega}$ then there is a unique solution v of $\mathrm{SP}(\Omega)$ that satisfies $h_- \leq v \leq h_+$ in $\overline{\Omega}$. In particular, if $g \in UC(G_{M,\Omega})$ for each $M > 0$ where $G_{M,\Omega} = \{x \in \overline{\Omega} : |g(x)| < M\}$ then there is a unique solution v of $\mathrm{SP}(\Omega)$ such that $v \in UC(V_{M,\Omega})$ for each $M > 0$ with $V_{M,\Omega} = \{x \in \overline{\Omega} : |v(x)| < M\}$. □

Remark 1 *It was not pointed out clearly enough in the previous literature that the above existence results on unbounded domains do not require any special behavior on the data. This must be compared with many previous results on quasilinear equations which attracted the attention of many specialists and use very different methods adapted to the quasilinear equations under consideration. Restricting ourselves only to the case of elliptic problems we can mention the works by H. Brezis [16], J.I. Díaz and O.A. Oleinik [29], G. Díaz [24] and L. Boccardo, Th. Gallouet and J.L. Vázquez [14] to mention only some few of them. For instance, in [22] it is assumed that the data grows at most linearly at the infinity. A more general framework was considered in Barles* et al. *[9].* □

Remark 2 *The Perron's method could be applied also to discontinuous initial data or discontinuous right hand side data in* $\mathrm{PP}(\mathbb{R}^2)$ *or* $\mathrm{SP}(\mathbb{R}^2)$. *Nevertheless, this is a very delicate question (see the connection with the loss of uniqueness of solutions of* $\mathrm{PP}(\mathbb{R}^2)$ *in [10]) which require some detailed analysis ([3]).* □

Remark 3 *The convergence of solutions of the discretized family of elliptic problems to the solution of the parabolic problem was shown in [11] (see also [12]).*

As mentioned in the Introduction the qualitative properties we shall present in the next sections will be consequence of the study of the class of radially symmetric solutions of the parabolic and elliptic equations when they are defined over balls and symmetric rings with different boundary conditions. The existence of this type of radially symmetric solutions requires some kind of compatibility conditions on the data. In some sense, the existence of some radially symmetric lower and supersolutions h_- and h_+ of the associated problems is not possible unless we assume suitable compatibility conditions on the data. We shall present such result in the following sections (see Corollaries 1 and 2).

3 On the parabolic problems

3.1 Level set propagation

Our main result on the propagation of level sets with non-empty interior is the following:

Theorem 5 *i) (Compact support type estimates and extinction in finite time). Let* $u_0 \in UC(G_M)$ *for each* $M > 0$, *where* $G_M := \{|u_0| < M\}$. *Assume that there is a level* s *such that*

$$\{u_0(x) = s\} \supset \mathbb{R}^2 - B_{R_s}(x_s) \text{ for some } x_s \in \mathbb{R}^2 \text{ and } R_s > 0, \tag{16}$$

(i.e. $\mathrm{supp}(u_0 - s) \subset \overline{B}_{R_s}(x_s)$*). Then if* u *is the solution of* $\mathrm{PP}(\mathbb{R}^2)$

$$\mathrm{supp}(u(t,.) - s) \subset \overline{B}_{R(t)}(x_s),\ R(t) = \left(R_s^{(q+1)} - \min(b_+, b_-)(q+1)t\right)^{1/(q+1)}. \tag{17}$$

In particular, $u(t,x) \equiv s$ *for any* $x \in \mathbb{R}^2$ *for any* $t \geq t_s := R_s^{(q+1)}/[\min(b_+, b_-)(q+1)]$ *and if* $s = 0$ *then* u *is solution of* $\mathrm{PP}(B_{R_0}(x_0))$.
ii) (Unbounded support and dead core type estimates). Assume that there is a level s *such that*

$$\mathrm{int}\{u_0(x) = s\} \neq \phi.$$

Assume that there exists a ball $B_{R_s}(x_s)$ *contained in* $\{u_0(x) = s\}$. *Then, for any*

$$t \leq \frac{R_s^{q+1}}{\max\{b_+, b_-\}(q+1)},$$

the level s *of the solution* u *satisfies*

$$\{x \in \mathbb{R}^2 : u(t,x) = s\} \supset \{x \in \mathbb{R}^2 : |x - x_s| \leq \left(R_s^{q+1} - \max\{b_+, b_-\}(q+1)t\right)^{\frac{1}{q+1}}\} \tag{18}$$

In particular, if the level set $\{u_0(x) = s\}$ *is unbounded then for any* $t > 0$ *the level set* s *of the solution,* $\{x \in \mathbb{R}^2 : u(t,x) = s\}$, *is also unbounded.*
iii) (No instantaneous level set extinction). Let $x_0 \in \mathrm{int}\big(\mathrm{supp}(u_0 - s)\big)$. *Then* $x_0 \in \mathrm{int}\big(\mathrm{supp}(u(t,.) - s)\big)$ *for any* $t > 0$ *small enough.*

The main tool of the proof is the following result on some special type of solutions which looks not too different to what some authors call as *self-similar solutions* for this type of equations when $q=1$ (see, *e.g.* [20, Lemma 6.1] and [32, page 53]). Nevertheless, our special solutions are not exactly the same than the indicated self-similar solutions. Moreover, in our framework $q>0$ is arbitrary.

Lemma 1 *i)* ***Exterior propagation for nonnegative data****: Assume $b_->0$. Let $g\in C^2([0,+\infty))$ be a nonincreasing function. Define*

$$\varsigma_-(t,r)=\left(r^{q+1}+b_-(q+1)t\right)^{\frac{1}{q+1}}.$$

Then the function

$$u(t,x)=g\big(\varsigma_-(t,|x|)\big) \tag{19}$$

is a $C^2((0,T]\times\mathbb{R}^2)$ solution of equation $\mathrm{PP}(\mathbb{R}^2)$ with $u_0(x)=g(|x|)$. In particular, if

$$g(r)\equiv s \quad \text{for } r\geq r_g\geq 0, \tag{20}$$

then

$$u(t,x)\equiv s \quad \text{for } |x|\geq \max\left\{\left(r_g^{q+1}-b_-(q+1)t\right)^{\frac{1}{q+1}},0\right\}.$$

ii) ***Interior propagation for nonnegative data****: Assume $b_+>0$. Let $g\in C^2([0,+\infty))$ be a nondecreasing function. Define*

$$\varsigma_+(t,r)=\left(r^{q+1}+b_+(q+1)t\right)^{\frac{1}{q+1}}.$$

Then the function

$$u(t,x)=g\left(\varsigma_+(t,|x|)\right) \tag{21}$$

is a radially symmetric $C^2((0,T]\times\mathbb{R}^2)$ solution of $\mathrm{PP}(\mathbb{R}^2)$ with $u_0(x)=g(|x|)$. In particular, if

$$g(r)\equiv s \quad \text{for } r\in[0,r_g], \tag{22}$$

for some $s\in\mathbb{R}$ then

$$u(t,x)\equiv s \quad \text{for } t\leq\frac{r_g^{q+1}}{b_+(q+1)} \text{ and } 0\leq|x|\leq\left(r_g^{q+1}-b_+(q+1)t\right)^{\frac{1}{q+1}}.$$

iii) Properties i) and ii) remain valid in the class of (viscosity) solutions if we replace the regularity $g\in C^2([0,+\infty))$ by $g\in C^0([0,+\infty))$.

Remark 4 *It seems remarkable that the above solutions does not develop any singularity at the origin, $x=0$, for $t>0$, even if the symmetric initial datum $g(|x|)$ have it (for instance when $g'(0)\neq 0$).* □

Proof of Lemma 1. First, we notice that for any radial function $u(t,r)$, in the points (t,r) where $\dfrac{\partial u}{\partial r}(t,r)\neq 0$ we have

$$\begin{cases} \mathrm{D}u(t,x)=\dfrac{\partial u}{\partial r}(t,|x|)\dfrac{x}{|x|}, \\[2ex] \mathrm{div}\left(\dfrac{\mathrm{D}u}{|\mathrm{D}u|}\right)=\dfrac{\mathrm{sign}\left(\dfrac{\partial u}{\partial r}(t,|x|)\right)}{|x|}. \end{cases}$$

Therefore, in radial coordinates equation (2) becomes equation (8)

$$\frac{\partial u}{\partial t}(t,r)-\beta\left(\frac{\operatorname{sign}\left(\dfrac{\partial u}{\partial r}(t,r)\right)}{r}\right)\left|\frac{\partial u}{\partial r}(t,r)\right|=0. \tag{23}$$

Let us start by the proof of ii). Let $u(t,r)=g(\varsigma_+)$, with $\varsigma_+(t,r)=(r^{q+1}+b_+(q+1)t)^{\frac{1}{q+1}}$ and $g\in C^1(0,+\infty))$ nondecreasing. Then straightforward computations lead to

$$\frac{\partial u}{\partial t}(t,r)-\frac{b_+}{r^q}\frac{\partial u}{\partial r}(t,r)=g'(\varsigma_+)\left(\frac{\partial\varsigma_+}{\partial t}-\frac{b_+}{r^q}\frac{\partial\varsigma_+}{\partial r}\right)=0.$$

Moreover, if $g\in C^2([0,+\infty))$ then $u\in C^2((0,T)\times\mathbb{R}^2)$ since from (8)

$$\frac{\partial u}{\partial r}(t,r)=\frac{r^q}{b_+}\frac{\partial u}{\partial t}(t,r)$$

and thus, the radial definition (21) does not generate any singularity at the origin for $t>0$.
The proof of i) is analogous since in this case

$$\beta(\operatorname{curv}(u))\,|\mathrm{D}u|=-\frac{b_-}{r^q}\left|\frac{\partial u}{\partial r}(t,r)\right|=\frac{b_-}{r^q}\frac{\partial u}{\partial r}(t,r).$$

Part iii) is consequence of Theorem 5.6 of [20] (which coincides with the *Grey-scale invariance* condition in [4]): if u is a solution then for any $\theta\in\mathcal{C}(\mathbb{R})$, θ nondecreasing then $\theta\circ u$ is also a viscosity solution of $\mathrm{PP}(\mathbb{R}^2)$. Then, if $g\in C\big([0,+\infty)\big)$ is nondecreasing, by starting with the special case of $g_0(r)=r^2$ then we arrive to the given $g\in C([0,+\infty))$ by taking $\theta(s)=g(\sqrt{s})$ for any $s\geq 0$. The case of $g\in C([0,+\infty))$ nonincreasing is similar. □

Proof of Theorem 5. The existence, comparison and uniqueness of solutions follows from Theorems 1 and 2. For the proof of i) we shall construct two global super and subsolutions leading to estimate (17). Let $x_s\in\operatorname{supp}(u_0-s)$ given in (16). It is clear that we can construct two functions $\overline{u}_0$ and $\underline{u}_0$, radially symmetric and of the form $\overline{g}(|x-x_s|)$ and $\underline{g}(|x-x_s|)$, with $\overline{g}$ nonincreasing and $\underline{g}$ nondecreasing, such that, $\underline{u}_0\leq u_0\leq\overline{u}_0$, $\overline{u}_0-s\geq 0$, $\underline{u}_0-s\leq 0$ and

$$\{\overline{u}_0(x)=s\}\supset\mathbb{R}^2-B_{R_s}(x_s)\text{ and }\{\underline{u}_0(x)=s\}\supset\mathbb{R}^2-B_{R_s}(x_s)$$

(use a regularization of the modulus of continuity of u_0, near the boundary of the support of (u_0-s)). Then, by Lemma 1 the solution $\overline{u}$ corresponding to $\overline{u}_0$ satisfies that

$$\overline{u}(t,x)=\overline{g}\left(\left(|x-x_s|^{q+1}+b_+(q+1)t\right)^{\frac{1}{q+1}}\right),\quad x\in\mathbb{R}^2,\ t>0.$$

Analogously the solution $\overline{u}$ corresponding to $\underline{u}_0$ is given by

$$\overline{u}(t,x)=\underline{g}\left(\left(|x-x_s|^{q+1}+b_-(q+1)t\right)^{\frac{1}{q+1}}\right).$$

From Theorem 1 we get that $\underline{u} \leq u \leq \overline{u}$ which leads to the conclusion. Notice that if $\min\{b_+, b_-\} = 0$ the conclusion holds trivially. So we can assume $\min(b_+, b_-) > 0$ and estimate (17) is always coherent. To prove ii) we shall use the method of local super and subsolutions (see,e.g., [28]). So, let $x_0 \in \{u_0(x) = s\} \subset G_M = \{|u_0| < M\}$ with $M > 0$. Then, since $u_0 \in UC(G_M)$ it is clear that that we can built two functions $\overline{u}_0, \underline{u}_0 \in C^2(\mathbb{R}^2)$ of the form $\overline{u}_0 = \overline{g}(|x - x_s|)$ with $\overline{g}$ nondecreasing, and $\underline{u}_0 = \underline{g}(|x - x_s|)$ with $\underline{g}$ nonincreasing, satisfying $\overline{u}_0(x) = \underline{u}_0(x) = s$ for any $x \in \mathbb{R}^2$ such that $|x - x_s| \leq R_s$ (use now a regularization of the modulus of continuity of u_0 on G_M). Define

$$t_s := \frac{R_s^{q+1}}{\max\{b_+, b_-\}(q+1)}.$$

Then by Lemma 1 and Theorem 1 we conclude that for any $x \in \mathbb{R}^2$ and $t \in [0, t_s]$

$$\underline{g}_i\left(\left(|x - x_s|^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) \leq u(t, x) \leq \overline{g}_i\left(\left(|x - x_s|^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right),$$

which proves the estimate (18).

In order to prove the no instantaneous level set disappearance property of part iii) we recall that, as indicated in the proof of Lemma 1, the function $\theta \circ u$ is also a solution of $\mathrm{PP}(\mathbb{R}^2)$ for any nondecreasing function $\theta : \mathbb{R} \to \mathbb{R}$. Noting that $\max\{s, u(t, x)\} = \theta(u(t, x))$ with

$$\theta(r) = \begin{cases} r & \text{if } r \geq s, \\ s & \text{if } r < s, \end{cases}$$

we conclude that $\max\{s, u(t, x)\}$ is the solution of $\mathrm{PP}(\mathbb{R}^2)$ corresponding to $\max\{s, u_0\}$ as initial datum. Now, take $x_0 \in \operatorname{int}\big(\operatorname{supp}(u_0 - s)\big)$. We can construct a function $\widehat{\underline{u}}_0(x) = \widehat{\underline{g}}(|x - x_0|)$ with $\widehat{\underline{g}}$ nonincreasing such that $\widehat{\underline{g}}(r) = s$ if $r \in [\frac{\epsilon}{2}, +\infty)$ and verifying

$$\widehat{\underline{u}}_0(x) \leq \max\{s, u_0(x)\}$$

for any $x \in \mathbb{R}$. Then, by Theorem 1 and Lemma 1

$$\widehat{\underline{g}}\left(\left(|x - x_0|^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) \leq u(t, x) \text{ on } B_{\frac{\epsilon}{2}}(x_0).$$

provided $t \in \left[0, \dfrac{\epsilon/2}{[b_-(q+1)]^{1/(q+1)}}\right]$. The analogous argument can be applied to $\min\{s, u\}$ and we get the result. □

Remark 5 *Something similar to the propagation results presented in Theorem 5 could be also obtained with the techniques on the finite speed of propagation developed by Goto in [33]. Nevertheless, as we shall explain now our growth estimates on the location of the level sets are sharper than the possible ones which could be obtained by applying his technique. Indeed, if to fix ideas we consider the level $s = 0$ and we denote by Γ_0 the initial interface defined as the boundary of the support of u_0 (which in [33] it is assumed to be compact) then $\Gamma(t)$ represents the boundary of the support of $u(t) := u(t, \cdot)$. The partial differential equations can be understood in the sense that*

$$\frac{d}{dt}\Gamma(t) = \beta\big(\operatorname{curv}(u(t))\big)\overrightarrow{\mathbf{n}}(t) \tag{24}$$

with $\overrightarrow{\mathbf{n}}(t) := \overrightarrow{\mathbf{n}}(t,\cdot)$ *the unit exterior normal vector to the support of* $u(t)$*. So, the vector*

$$\overrightarrow{\mathbf{V}}(t,\cdot) := \beta\big(\mathrm{curv}(u(t,\cdot)\big)\,\overrightarrow{\mathbf{n}}(t,\cdot)$$

represents the speed of propagation *of the zero level set of the solution. The following notion was introduced in [33]: given a function* u *on* $\mathcal{Q}_T = [0,T)\times\Omega$*, given* $R>0$ *we say that* u *has an* upper speed bound $\nu(R)$ *for the level* $s=0$ *if* $\nu(R)\geq 0$ *is such that*

$$\sup_{(t,x)\in\Delta_\nu} u \leq c \qquad \text{with } \Delta_\nu = \{(t,x) : t\geq t_0, |x-x_0|\leq R-\nu(R)(t-t_0)\}, \tag{25}$$

whenever $c\in\mathbb{R}$, $(t_0,x_0)\in\mathcal{Q}_T$ *satisfy* $\sup_{|x-x_0|\leq R} u(t_0,x)\leq c$*. If* u *and* $-u$ *have upper speed bounds we say* u *has a finite speed. Particularizing Theorem 2.1 of [33] to our case (for any* $q, b_+, b_- \geq 0$ *with* $b_+ + b_- > 0$*) and if* Γ_0 *is compact then problem* $PP(\mathbb{R}^2)$ *have a unique solution with compact support for any* $t\in[0,T]$ *and with a finite speed. As consequence it could be possible to get some estimates on the location of the support of* $u(t)$ *through the application of the geometric estimate (25). Nevertheless our estimates are sharper in the sense that even for the special case of radially symmetric solutions the good estimate on the propagation of the level sets is not of* linear type *as (25) but have a nonlinear nature with explicit indication of the values of* q, b_+ *and* b_-*. For instance, in the case of a radially symmetric initial datum with* $g\in C\big([0,+\infty)\big)$ *nonincreasing and* $\Gamma_0 = \{|x| = r_g\}$ *the* (t,x) *domain where* u *vanishes is of the form*

$$|x|^{q+1} \geq \max\left\{r_g^{q+1} - b_-(q+1)t, 0\right\}.$$

Analogously, for $g\in C([0,+\infty))$ *nondecreasing with* $\Gamma_0 = \{|x| = r_g\}$ *the vanishing domain of* u *is*

$$|x|^{q+1} \leq r_g^{q+1} - b_+(q+1)t.$$

We recall that in [33] the superlinear case problem corresponding to $q>1$ *is reached as limit of a sequence of problems growing linearly and that the solutions of this family of problems already have (a uniformly bounded) finite speed. This explain why his definition is "too linear" in contrast to our direct approach. Finally we point out that in the special case of radially symmetric solutions the equation of the evolution of the level set (24) (which in fact resembles to the differential equation of the interface for free boundary problems like the porous media equation: see [39]) now becomes simply*

$$\frac{d}{dt}\Gamma(t) = b_- \frac{sign\ (g'(\varsigma_-))}{r^q}\overrightarrow{\mathbf{n}} \tag{26}$$

for the case g *nonincreasing (i.e. with sign* $\big(g'(\varsigma_-)\big) = 1$ *if* $g'<0$*) and*

$$\frac{d}{dt}\Gamma(t) = b_+ \frac{sign\ (g'(\varsigma_+))}{r^q}\overrightarrow{\mathbf{n}} \tag{27}$$

for the case g *nondecreasing (i.e. with sign*$\big(g'(\varsigma_+)\big) = 1$ *if* $g'>0$*). Since* $\overrightarrow{\mathbf{n}} = x/|x|$ *in the case of* g *nonincreasing satisfying (20) and* $\overrightarrow{\mathbf{n}} = -x/|x|$ *in the case of gnondecreasing satisfying (22) we get that curiously, in both cases* $\Gamma(t)$ *is moving monotonically in time contracting it towards the origin (it does not matter if* g *is either nonincreasing or nondecreasing). As far as we know, this kind of retracting free boundary (extended in i) of Theorem 5 to non-necessarily radially symmetric initial* *data) only appears, for the case of reaction-diffusion problems, in the presence of a strong absorption term. Moreover, Lemma 1 and part i) of Theorem 5 show that, in contrast with many other free boundary problems, there is no waiting time phenomenon (independently of the behavior of the initial data in a neighborhood of the boundary of its support). See e.g. [5], [6], [8] and [39]. Notice also that although estimate (??) is less precise that (17) it applies to the case in which* $supp(u_0 - s)$ *is unbounded (in contrast with part i) and the results of [33]). Several authors analyzed the singularity formed at the origin in the extinction time for the case of mean curvature geometric flows* ($q = 1$) *for many different initial hypersurfaces as level sets of suitable initial data: see e.g. the exposition made in the Introduction of [32]. Finally, we point out that in contrast with part iii) of Theorem 5, if* $int\big(supp(u_0 - s)\big)$ *is empty then the level set* $\{u - s\}$ *may extinct instantaneously (this result was initially proved in [31] and generalized later by different authors to different frameworks: see [32] Subsection 4.7).* □

Remark 6 *Radially symmetric geometric flows given by the Gauss curvature in* $\mathbb{R}^N$ *corresponds to* $q = N - 1$, *which is superlinear for* $N > 2$ *(see. e.g. [32] page 52). In this way, Lemma 1 is of interest in order to get qualitative properties of (non-necessarily radially symmetric) solutions of the equation*

$$\frac{\partial u}{\partial t} - |\mathrm{D}u| \det\left(\left(\mathrm{I} - \frac{\mathrm{D}u \otimes \mathrm{D}u}{|\mathrm{D}u|^2}\right)\frac{\mathrm{D}^2 u}{|\mathrm{D}u|}\left(\mathrm{I} - \frac{\mathrm{D}u \otimes \mathrm{D}u}{|\mathrm{D}u|^2}\right) + \frac{\mathrm{D}u \otimes \mathrm{D}u}{|\mathrm{D}u|^2}\right) = 0$$

(see [32] page 39 and [34]).This will be presented in a future work by the authors. □

Remark 7 *In the special case of* $q = 1/3$ *(corresponding to the affine curvature case) it is possible to replace the radially symmetric level sets of the above self-similar type solutions in Lemma 1 by solutions having all its level sets given by ellipses. Indeed, as it is shown in [4], for any* $a > 0$, *if* $u(t, x)$ *is a solution of* $PP(\mathbb{R}^2)$ *then* $u(t, x \cdot (a, 1/a)^T)$ *is also a solution of* $PP(\mathbb{R}^2)$. *In particular, if* $u(t, x)$ *is a radially symmetric solution then the level sets of* $u(t, x \cdot (a, 1/a)^T)$ *are ellipses for any* $t \geq 0$. *This could be used to get a variation of the estimates given in Theorem 5 for the special case* $q = 1/3$ *but we shall not enter into details here.* □

Remark 8 *Theorem 5 applies also to the solutions of problem* $PP(\Omega)$ *and in fact there is not any kind of peculiar behavior near the boundary similar to the Höpf maximum principle (ensuring that* $\mathrm{D}u \cdot \overrightarrow{\mathbf{n}} < 0$ *if* $u_0 \geq 0$*).* □

By arguing as in the proof of Lemma 1 it is possible to get some necessary and sufficient conditions for the existence of symmetric solutions on the equation of $\mathrm{PP}(\mathbb{R}^2)$ now on symmetric rings and balls. The following result collects this fact (we omit its proof since it is based on obvious adaptations).

Corollary 1 *Let* Ω *be the ring* $\{x \in \mathbb{R}^2\colon\, r_0 < |x| < r_1\}$ *for some* $0 < r_0 < r_1$, *or the ball* $B_{r_1}(0) = \{x \in \mathbb{R}^2\colon\, |x| < r_1\}$. *Given* $h_0(t)$, $h_1(t) \geq 0$ *we consider the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\mathrm{div}\,\left(\dfrac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right)|\mathrm{D}u| = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = h_0(t) & t \in (0,T),\ \ |x| = r_0, \\ u(t,x) = h_1(t) & t \in (0,T),\ \ |x| = r_1, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{28}$$

or simply

$$\begin{cases} \dfrac{\partial u}{\partial t} - \beta\left(\operatorname{div}\left(\dfrac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right)|\mathrm{D}u| = 0 & \text{in } (0,T)\times\Omega, \\ u(t,x) = h_1(t) & t\in(0,T),\ \ |x| = r_1, \\ u(0,\cdot) = u_0(\cdot) & \text{on } \Omega, \end{cases} \tag{29}$$

in the case of the ball $\Omega = B_{r_1}(0)$. *Let* $u_0(x) = g(|x|)$ *be a* $C^0[r_0, r_1]$ *radially symmetric function.*
i) Assume $b_+ > 0$ *and* $g(r)$ *nondecreasing. Then problem (28) have a nondecreasing solution if and only if*

$$h_0(t) = g\left(\left(r_0^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) \quad \text{for } t\in(0,T),$$

and

$$h_1(t) = g\left(\left(r_1^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) \quad \text{for } t\in(0,T). \tag{30}$$

In the case of problem (29) the necessary and sufficient condition is (30) $h_0(t)$*, the solution is given by*

$$u(t,x) = g\left(\left(|x|^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right), \tag{31}$$

and thus $u(t,0) = g\left((b_+(q+1)t)^{\frac{1}{q+1}}\right)$ *is nondecreasing in* t.
ii) Assume $b_- > 0$ *and let* $g(r)$ *be nonincreasing. Then problems (56) has a nonincreasing solution if and only if*

$$h_0(t) = g\left(\left(r_0^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right) \quad \text{for } t\in(0,T),$$

and

$$h_1(t) = g\left(\left(r_1^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right) \quad \text{for } t\in(0,T). \tag{32}$$

In the case of problem (29) the necessary and sufficient condition is (32), the solution is given by

$$u(t,x) = g\left(\left(|x|^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right), \tag{33}$$

and thus $u(t,0) = g\left((b_-(q+1)t)^{\frac{1}{q+1}}\right)$ *is nonincreasing in* t.

3.2 Non-monotone radially symmetric solutions

In Lemma 1 we have presented some special radially symmetric solutions of $\mathrm{PP}(\mathbb{R}^2)$. The crucial argument which allowed merely the continuity of the initial data was the monotonicity of the radial profile $g(r)$. In this section we shall construct several radially symmetric solutions with a non-monotone profile g. We shall see that the loss of monotonicity of $g(r)$ leads to new qualitative properties of the associated solutions of $\mathrm{PP}(\mathbb{R}^2)$.

Our main interest in this Subsection is on the case of an initial datum $u_0(x) = g(|x|)$ with g given by as we colloquially called as a Batman type profile (10) in the introduction. First, by using Lemma 1 we shall estimate a lower bound for the solution $u(t,r)$ of $\mathrm{PP}(\mathbb{R}^2)$ corresponding to $u_0(x) = g(|x|)$. We recall that the solutions $u_1(t,r)$, $u_2(t,r)$ associated to $g_1(r)$ and $g_2(r)$ are

$$\begin{cases} u_1(t,r) = g_1\left(\left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right), \\ u_2(t,r) = g_2\left(\left(r^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right). \end{cases} \tag{34}$$

We can make a short of puzzle with the above mentioned solutions by using a general principle:

Proposition 1 *For $i = 1, 2$, let u_i be a supersolution of* $\mathrm{PP}(\mathbb{R}^2)$ *corresponding to the initial datum* $u_{0,i}(x)$. *Then* $u = \min(u_1, u_2)$ *is a supersolution of* $\mathrm{PP}(\mathbb{R}^2)$ *corresponding to the initial datum* $u_0 = \min(u_{0,1}, u_{0,2})$.

PROOF. Since $u_{0,i} \geq \min(u_{0,1}, u_{0,2})$ then by Theorem 1 u_i is also a supersolution of $\mathrm{PP}(\mathbb{R}^2)$ corresponding to the initial datum $u_0 = \min\{u_{0,1}, u_{0,2}\}$. Moreover, since it is well-known that the minimum of two supersolutions is a supersolution (see [34]) we get the result. □

Remark 9 *The above simple result gives a parabolic viscosity version to the junction lemma of [13] applied in many contexts when the junction between two functions is not smooth.* □

Our main interest now is to show that in the case of initial data given by a radially symmetric *Batman* profile g satisfying (10) we can be more exact since as we shall prove $\min(u_1, u_2)$ coincides in fact (when $b_+ \leq b_-$) with the solution u corresponding to $u_0 = \min(u_{0,1}, u_{0,2})$ or (if $b_+ > b_-$) there is a level set $\{u = g(r_d)\}$ which have a non-empty interior for any $t > 0$ although this is not the case of the level set $\{u_0 = g(r_d)\}$. The crucial fact is the different velocities corresponding to the convex and concavity parts of the level sets.

Theorem 6 *Let g satisfying (10) with g_1 strictly increasing and g_2 strictly decreasing. Let u_1, u_2 be the corresponding solutions of* $\mathrm{PP}(\mathbb{R}^2)$ *given by (34). Let u be the unique continuous (viscosity) solution of* $\mathrm{PP}(\mathbb{R}^2)$ *of initial datum $u_0(x) = g(|x|)$. Then we have the following alternative:*
i) If $b_+ \leq b_-$ then

$$u(t, r) = \min\left\{u_1(t, r), u_2(t, r)\right\} \text{ for any } t > 0.$$

ii) If $b_+ > b_-$ then for any $t > 0$

$$u(t, r) = \begin{cases} g(r_d) & \text{if } r_+(t) < r < r_-(t) \\ \min\left\{u_1(t, r), u_2(t, r)\right\} & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} r_+(t) &= \max\left\{0, \left(r_d^{q+1} - b_+(q+1)t\right)^{\frac{1}{q+1}}\right\}, \\ r_-(t) &= \max\left\{0, \left(r_d^{q+1} - b_-(q+1)t\right)^{\frac{1}{q+1}}\right\}. \end{aligned} \tag{35}$$

Before to give the proof we anticipate now that several numerical experiences, giving idea of the above alternative, will be presented in the next subsection. We point out that $r_+(t)$, $r_-(t)$ can be understood as defined trough the characteristics of the Hamilton-Jacobi equation (8) and that the above results shows an anomalous behavior of solutions with respect to other hyperbolic first order equations once we assume $b_+ \neq b_-$. If $b_+ \leq b_-$ in the points where the characteristics meet there is no shocks of the solutions (but of its derivatives) and the solution is continuous. If $b_+ > b_-$ the characteristics do not fulfill all the (t, x) domain but instead to appear a *rarefaction wave* (see *e.g.* [30]) the not-covered region is occupied by a flat level set even if initially this level set was with an empty interior. As mentioned in the introduction, this rather simple example contrasts with the complexity of other examples on this phenomenon in the previous literature (see [31] and [38] for the case of *mean curvature geometric flows* $q = 1$, $b_- = b_+ = 1$).

In order to give the proof of Theorem 6 it is convenient to start by considering the easier case in which the non-monotone radially symmetric initial datum $u_0(x) = g(|x|)$ is given by a regularization of the characteristic function of an interval $[r_1, r_2]$. We will use the well-known family of mollifier functions $\varphi_\epsilon(r)$ given by

$$\varphi_\epsilon(r) = \begin{cases} \dfrac{K}{\epsilon} e^{-\frac{\epsilon^2}{(\epsilon^2 - r^2)}} & \text{if } r \leq \epsilon, \\ 0 & \text{if } r > \epsilon, \end{cases} \tag{36}$$

where K is chosen such that $\displaystyle\int_{\mathbb{R}} \varphi_\epsilon(r)dr = 1$. We consider $g(r) = g_{s,r_1,r_2,\epsilon}(r)$ given by

$$g_{s,r_1,r_2,\epsilon}(r) = s\chi_{[r_1,r_2]} * \varphi_\epsilon(r), \tag{37}$$

where $s > 0$, $\epsilon > 0$ and $\chi_{[r_1,r_2]}(r)$ denotes the characteristic function of the interval $[r_1, r_2]$.

Lemma 2 *The solution of* $\mathrm{PP}(\mathbb{R}^2)$ *for* $u_0(x) = g(|x|) = g_{s,r_1,r_2,\epsilon}(|x|)$ *is given by*

$$u_{s,r_1,r_2,\epsilon}(t, r) = \begin{cases} g_{s,r_1,r_2,\epsilon}\left(\left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) & \text{if } r < r_{\epsilon,1}(t), \\ s & \text{if } r_{\epsilon,1}(t) \leq r < r_{\epsilon,2}(t), \\ g_{s,r_1,r_2,\epsilon}\left(\left(r^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right) & \text{if } r \geq r_{\epsilon,2}(t), \end{cases} \tag{38}$$

where

$$r_{\epsilon,1}(t) = \max\left\{0, \left((r_1+\epsilon)^{q+1} - b_+(q+1)t\right)^{\frac{1}{q+1}}\right\},$$
$$r_{\epsilon,2}(t) = \max\left\{0, \left((r_2-\epsilon)^{q+1} - b_-(q+1)t\right)^{\frac{1}{q+1}}\right\},$$

for any t *such that*

$$r_{\epsilon,1}(t) \leq r_{\epsilon,2}(t).$$

Proof. We point out that the functions $g_1(r) = g_{s,r_1,\infty,\epsilon}(r)$ and $g_2(r) = g_{s,0,r_1,\epsilon}(r)$ are monotone nondecreasing and nonincreasing, respectively. Then, by applying Lemma 1, we have that the functions

$$u_1(t, r) = g_1(r^{q+1} + b_+(q+1)t),$$
$$u_2(t, r) = g_2(r^{q+1} + b_-(q+1)t),$$

are the corresponding solutions of $\mathrm{PP}(\mathbb{R}^2)$ and satisfy that

$$u_1(t, r) = s \quad \text{if} \quad r \geq r_{\epsilon,1}(t),$$
$$u_2(t, r) = s \quad \text{if} \quad r \leq r_{\epsilon,2}(t).$$

Therefore if

$$r_{\epsilon,1}(t) \leq r_{\epsilon,2}(t)$$

both solutions intersect in a smooth way on the level sets $\{u_1(t,.) \equiv s\}$ and $\{u_2(t,.) \equiv s\}$ and thus, by the uniqueness of solutions, we obtain that $u_{s,r_1,r_2,\epsilon}(t,r)$ given by (38) coincides with the solution of PP($\mathbb{R}^2$) corresponding to the initial datum $u_0(x) = g_{s,r_1,r_2,\epsilon}(|x|)$. □

In a next step we shall prove that in the case of Batman profile initial data $\min\{u_1, u_2\}$ is not always strictly greater that the solution u corresponding to $u_0 = \min\{u_{0,1}, u_{0,2}\}$.

Lemma 3 *Let $g = \min\{g_1, g_2\}$ satisfying (10). Then the solution $u(t,x)$ of the parabolic problem corresponding to $u_0(x) = g(|x|)$ with $g = \min\{g_1, g_2\}$ satisfies that*

$$u(t,r) \geq \min\{u_1(t,r), u_2(t,r)\}$$

for any $t, r \geq 0$ such that

$$s = \min\{u_1(t,r), u_2(t,r)\} \leq g(r_d).$$

PROOF. Assume first that

$$s = \min\{u_1(t,r), u_2(t,r)\} = u_1(t,r).$$

We consider r_1^s, r_2^s given by

$$[r_1^s, r_2^s] = \{r : \min\{g_1, g_2\} \leq s\}.$$

We observe that as $s \leq g(r_d)$, then $r_2^s \geq r_1^s$ and as g_2 is decreasing we have that

$$r_1^s = \left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}},$$
$$r_2^s \geq \left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}},$$

and

$$\min\{g_1, g_2\}(r) \geq g_{s,r_1^s+\epsilon,r_2^s-\epsilon,\epsilon}(r) \quad \text{for} \quad r \geq 0.$$

Then by applying Lemma 2 (using the variable t', r' to avoid misleading with respect to the variables t, r used here) and by comparison we have that

$$u(t', r') \geq s$$

for any ϵ small enough and t', r' given in Lemma 2. Then, passing to the limit when $\epsilon \to 0^+$ we obtain

$$u(t', r') \geq s \quad \text{if} \quad \max\left\{0, \left((r_1^s)^{q+1} - b_+(q+1)t'\right)^{\frac{1}{q+1}}\right\} \leq r' \leq \max\left\{0, \left((r_2^s)^{q+1} - b_-(q+1)t'\right)^{\frac{1}{q+1}}\right\}$$

for any t' such that

$$\max\left\{0, \left((r_1^s)^{q+1} - b_+(q+1)t'\right)^{\frac{1}{q+1}}\right\} \leq \max\left\{0, \left((r_2^s)^{q+1} - b_-(q+1)t'\right)^{\frac{1}{q+1}}\right\}.$$

By applying the above inequality we have that $u(t', r') \geq g_1\left(\left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right)$ if

$$\max\left\{0, \left(r^{q+1} + b_+(q+1)t - b_+(q+1)t'\right)^{\frac{1}{q+1}}\right\} \leq r' \leq \left(\max\{0, (r_2^s)^{q+1} - b_-(q+1)t'\}\right)^{\frac{1}{q+1}}$$

for any t' such that

$$\max\left\{0, \left(r^{q+1} + b_+(q+1)t - b_+(q+1)t'\right)^{\frac{1}{q+1}}\right\} \leq \max\left\{0, \left((r_2^s)^{q+1} - b_-(q+1)t'\right)^{\frac{1}{q+1}}\right\}.$$

Now we observe that the values $r' = r$ and $t' = t$ satisfy the above inequalities and then we obtain that

$$u(t,r) \geq u_1(t,r).$$

The case $\min\{u_1(t,r), u_2(t,r)\} = u_2(t,r)$ is analogous. □

Proof of Theorem 6. First we observe that the function $\bar{u}(t,r) = \min\{u_1(t,r), u_2(t,r), g(r_d)\}$ is a supersolution because is the minima of 3 solutions and $g(r) \leq \min\{g_1(r), g_2(r), g(r_d)\}$ (see Proposition 1). On the other hand

$$\min\{u_1(t,r), u_2(t,r)\} > g(r_d)$$

if and only if

$$\left(r^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}} < r_d < \left(r^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}},$$

which is only possible in the case $b_+ > b_-$. Operating in this inequality we obtain

$$\max\left\{0, \left(r_d^{q+1} - b_+(q+1)t\right)^{\frac{1}{q+1}}\right\} < r < \max\left\{0, \left(r_d^{q+1} - b_-(q+1)t\right)^{\frac{1}{q+1}}\right\}, \quad (39)$$

therefore, by Lemma 3 if (t,r) does not satisfy the above inequality then

$$u(t,r) \geq \min\{u_1(t,r), u_2(t,r)\},$$

which concludes $i)$ and part of the statement of $ii)$. In the case (t,r) satisfying (39), to show that $u(t,r) = g(r_d)$ we use a comparison strategy. Let $\delta > 0$ and $s_\delta = g(r_d) - \delta$. We consider r_1^s, r_2^s given by

$$[r_1^{s_\delta}, r_2^{s_\delta}] = \{r : \min\{g_1, g_2\} \geq s\},$$

for $\epsilon > 0$ small enough. Consider the function $g_{s_\delta, r_1^{s_\delta}-\epsilon, r_2^{s_\delta}+\epsilon, \epsilon}(r)$. Then

$$g_{s_\delta, r_1^{s_\delta}-\epsilon, r_2^{s_\delta}+\epsilon, \epsilon}(r) \leq \min\{g_1, g_2\}(r),$$

and therefore, by comparison and passing to the limit when $\epsilon \to 0^+$ we obtain

$$u(t,r) \geq s \quad \text{if} \quad \max\left\{\left((r_1^{s_\delta})^{q+1} - b_+(q+1)t\right)^{\frac{1}{q+1}}\right\} \leq r \leq \left(\max\{0, (r_2^{s_\delta})^{q+1} - b_-(q+1)t'\}\right)^{\frac{1}{q+1}}.$$

Next, passing to the limit when $\delta \to 0^+$ we obtain

$$u(t,r) \geq g_1(r_d) \quad \text{if} \quad \max\left\{0, \left(r_d^{q+1} - b_+(q+1)t\right)^{\frac{1}{q+1}}\right\} \leq r \leq \max\left\{0, \left(r_d^{q+1} - b_-(q+1)t'\right)^{\frac{1}{q+1}}\right\},$$

which concludes the proof of $ii)$ taking into account that $g(r_d)$ is an upper bound of $u(t,r)$. □

Remark 10 *We point out that from Theorem 6 we obtain that in the case $b_+ \leq b_-$, the solution $u(t, r)$ can be expressed as*

$$u(t, r) = \begin{cases} g_1\left((r^{q+1} + b_+(q+1)t)^{\frac{1}{q+1}}\right) & if \quad r \leq r(t) \\ g_2\left((r^{q+1} + b_-(q+1)t)^{\frac{1}{q+1}}\right) & if \quad r > r(t), \end{cases}$$

where $r(t)$ represents the interface separating the regions where $u(t, r)$ is defined according to g_1 or g_2. The curve $r(t)$ is implicitly defined by the relation

$$g_1\left(\left(r(t)^{q+1} + b_+(q+1)t\right)^{\frac{1}{q+1}}\right) = g_2\left(\left(r(t)^{q+1} + b_-(q+1)t\right)^{\frac{1}{q+1}}\right). \tag{40}$$

It is easy to see that $r(t)$ is at least a Lipschitz continuous function once that $g_1, g_2 \in C^1(0, +\infty)$. In the next subsection we shall study this interface $r(t)$ which is not any characteristic curve of the evolution problem. □

3.3 On the interface of discontinuity of derivatives if $b_+ \leq b_-$

The following result allows to identify the interface $r(t)$ mentioned in Theorem 6 when $b_+ \leq b_-$.

Proposition 2 *Let u as in Theorem 6 with $g_1, g_2 \in C^1(0, +\infty)$ and assume $b_+ \leq b_-$. Let $r = r(t)$ with $r(0) = r_d$ be the curve C separating two regions of the (t, x) space where $u = u_1$ to the left of the curve C from where $u = u_2$, to the right under the assumption that u_1 and u_2 are radially symmetric solutions of the equation of* PP($\mathbb{R}^2$). *Define*

$$w_l(t) = \frac{\partial u}{\partial r}(t, r(t)-) \text{ and } w_r(t) = \frac{\partial u}{\partial r}(t, r(t)+).$$

Then u is a solution of the equation of PP($\mathbb{R}^2$) *if and only if*

$$r'(t) = -\frac{b_+ r(t)^{-p} w_l(t) - b_- r(t)^{-p} w_r(t)}{(w_l(t) - w_r(t))} \quad a.e. \ t > 0. \tag{41}$$

Proof. Let us calculate the jump of the derivatives across the interface $r(t)$. In the class of radially symmetric solutions, since $\frac{\partial u}{\partial r}(0, t) = 0$, by using the odd prolongation ($f(-r) = f(r)$) problem PP($\mathbb{R}^2$) becomes the Hamilton-Jacobi type problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + H\left(r, \dfrac{\partial u}{\partial r}\right) = 0 & t \in (0, T), \text{ in } \mathbb{R}, \\ u(0, r) = u_0(r) & \text{on } \mathbb{R}, \end{cases} \tag{42}$$

with $H(r, q)$ the (convex) Hamiltonian

$$H(r, q) = \begin{cases} -\dfrac{b_+}{|r|^p} q & \text{if } q > 0 \\ -\dfrac{b_-}{|r|^p} q & \text{if } q < 0. \end{cases}$$

As in [19], by making $w = \dfrac{\partial u}{\partial r}$, differentiating in (42) we see that the viscosity solutions are transformed in Kruzkov solutions of the conservation laws problem

$$\begin{cases} \dfrac{\partial w}{\partial t} + \dfrac{\partial}{\partial r} H(r,w) = 0 & t \in (0,T), \text{ in } \mathbb{R}, \\ w(0,r) = w_0(r) & \text{on } \mathbb{R}, \end{cases} \tag{43}$$

and reciprocally any Kruzkhov solution of (43) generate a (viscosity) solution of (42). We know that w is smooth on either side of the smooth curve C given by $r = r(t)$ with $r(0) = r_d$. Let V be an open region $V \subset (0,T) \times \mathbb{R}$, let V_l be the part of V on the left of the curve and V_r that part on the right. By taking a smooth test function $\psi(t,r)$ with support on V_l (respectively V_r) we know that

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial r} H(r,w) = 0 \text{ in } V_l \cup V_r \,. \tag{44}$$

Selecting now a smooth test function $\psi(t,r)$ with compact support in V we get

$$0 = \int_0^T \int_0^{+\infty} (w \frac{\partial \psi}{\partial t} + H(r,w) \frac{\partial \psi}{\partial r}) dr dt = \int\int_{V_l} [\cdots] + \int\int_{V_r} [\cdots].$$

But

$$\int\int_{V_l} \left(w \frac{\partial \psi}{\partial t} + H(r,w) \frac{\partial \psi}{\partial r} \right) dr dt = -\int\int_{V_l} \left(\frac{\partial w}{\partial t} + \frac{\partial}{\partial r}(H(r,w)) \right) \psi dr dt + \int_C \big(w_l \nu^t + H(r,w_l) \nu^r \big) \psi dl$$

and analogously

$$\int\int_{V_r} \left(w \frac{\partial \psi}{\partial t} + H(r,w) \frac{\partial \psi}{\partial r} \right) dr dt = -\int\int_{V_r} \left(\frac{\partial w}{\partial t} + \frac{\partial}{\partial r} H(r,w) \right) \psi dr dt - \int_C \big(w_r \nu^t + H(r,w_r) \nu^r \big) \psi dl$$

where $\overrightarrow{\nu} = (\nu^r, \nu^t)$ is the unit normal to the curve C, pointing from V_l to V_r (the subscripts "l", "r" mean the limit from the left and the limit from the right respectively). Then, adding both identities and recalling (44) we get

$$\int_C \big[(w_l - w_r) \nu^t + \big(H(r,w_l) - H(r,w_r) \big) \nu^r \big] \psi dl = 0,$$

and it must holds for any test function ψ as above. We conclude that

$$(w_l - w_r) \nu^t + \big(H(r,w_l) - H(r,w_r) \big) \nu^r = 0 \quad \text{ along } C.$$

Since the curve C is represented by $\{(t,r) \mid r = r(t)\}$ for some smooth $r : [0,+\infty) \to [0,+\infty)$ we can take

$$\overrightarrow{\nu} = (\nu^r, \nu^t) = \frac{1}{\sqrt{1 + r'(t)}} (1, r'(t)).$$

In consequence

$$H\big(r(t), w_l(t)\big) - H\big(r(t), w_r(t)\big) = r'(t)(w_l(t) - w_r(t))$$

which is (41). $\square$

3.4 Numerical experiments for the parabolic problem

In this subsection we present some numerical experiments to illustrate the theoretical results obtained for the problem $PP(\mathbb{R}^2)$. In Remark 11 below we present some details on the used implementation. We point out that the numerical experiments have just illustration purposes and we do not intend to perform any rigorous numerical analysis of the equation (see to this respect, e.g. [21] and [18]). In Figure 2 we illustrate the solution of equation (8) in the case of $g(r) = \chi_{[1,2]} * \varphi_{0.1}(r)$. In Figures 3 and 4 we illustrate the solution of (8) in the case of the *Batman* profile presented in Figure 1 (given by $g(r) = \min\{|r|^3, \max\{0, 0.5(\sqrt{1.1-|r|} - \sqrt{0.1})\}\}$). In Figure 5 we illustrate the solution of (8) in the case of $g(r) = \min\{r, 2-r\}$. Figure 5 also presents the interfaces given by (35) and (40). An straightforward computation yields the following expressions for the interfaces of Figure 5

$$r(t) = \sqrt{\max\{0, 1-2t\}} \qquad \text{if} \quad b_+ = b_- = 1$$

$$r(t) = \sqrt{\max\left\{0, \frac{t^2 - 40t + 16}{16}\right\}} \qquad \text{if} \quad b_+ = 0.5 \text{ and } b_- = 1$$

$$\begin{cases} r_+(t) = \sqrt{\max\{0, 1-2t\}} \\ r_+(t) = \sqrt{\max\{0, 1-t\}} \end{cases} \qquad \text{if} \quad b_+ = 1 \text{ and } b_- = 0.5.$$

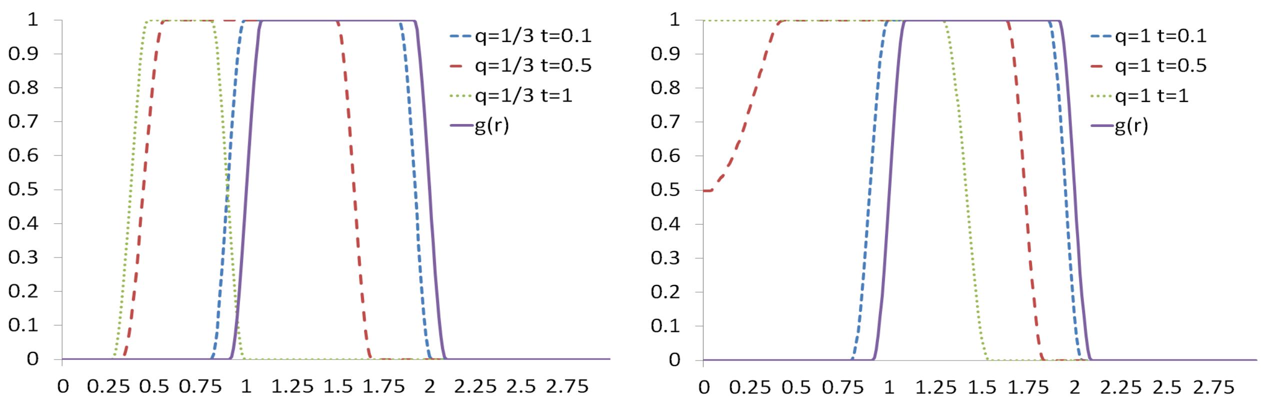

Figure 2: Shape of the solutions of (8) with $b_+ = b_- = 1$ for $g(r) = \chi_{[1,2]} * \varphi_{0.1}(r)$ and different values of t. On the left, shape of $u(t, r)$ for $q = 1/3$ and, on the right, shape of $u(t, r)$ for $q = 1$.

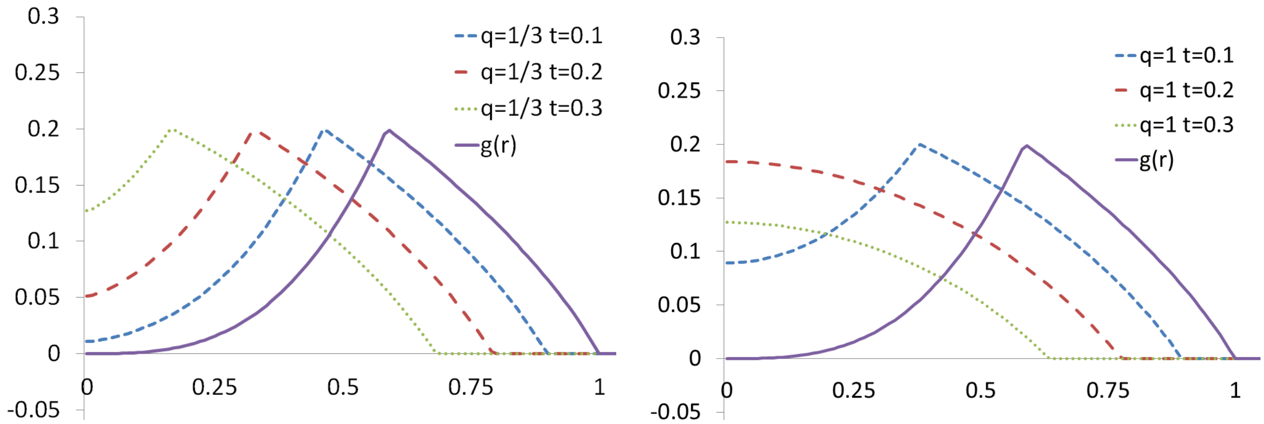

Figure 3: Shape of the solution of (8) for the *Batman* profile appearing in Figure 1 when $b_+ = b_- = 1$, for different values of t. On the left, shape $u(t, r)$ for $q = 1/3$ and, on the right, shape of $u(t, r)$ for $q = 1$.

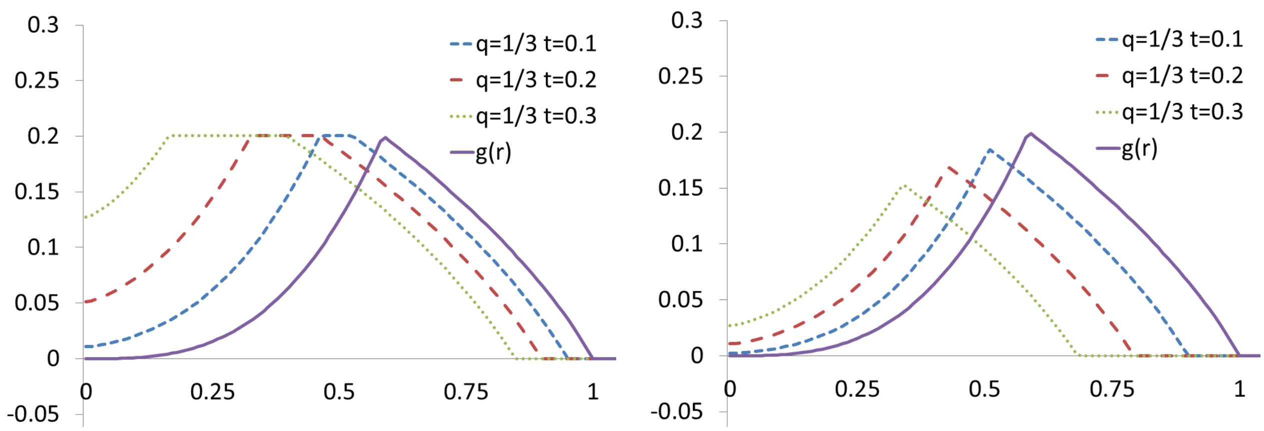

Figure 4: Shape of the solution of (8) for the *Batman* profile appearing in Figure 1 for $q = 1/3$ using different values for b_+ and b_-. On the left, $b_+ = 1$ and $b_- = 0.5$., and, on the right, $b_+ = 0.5$ and $b_- = 1$.

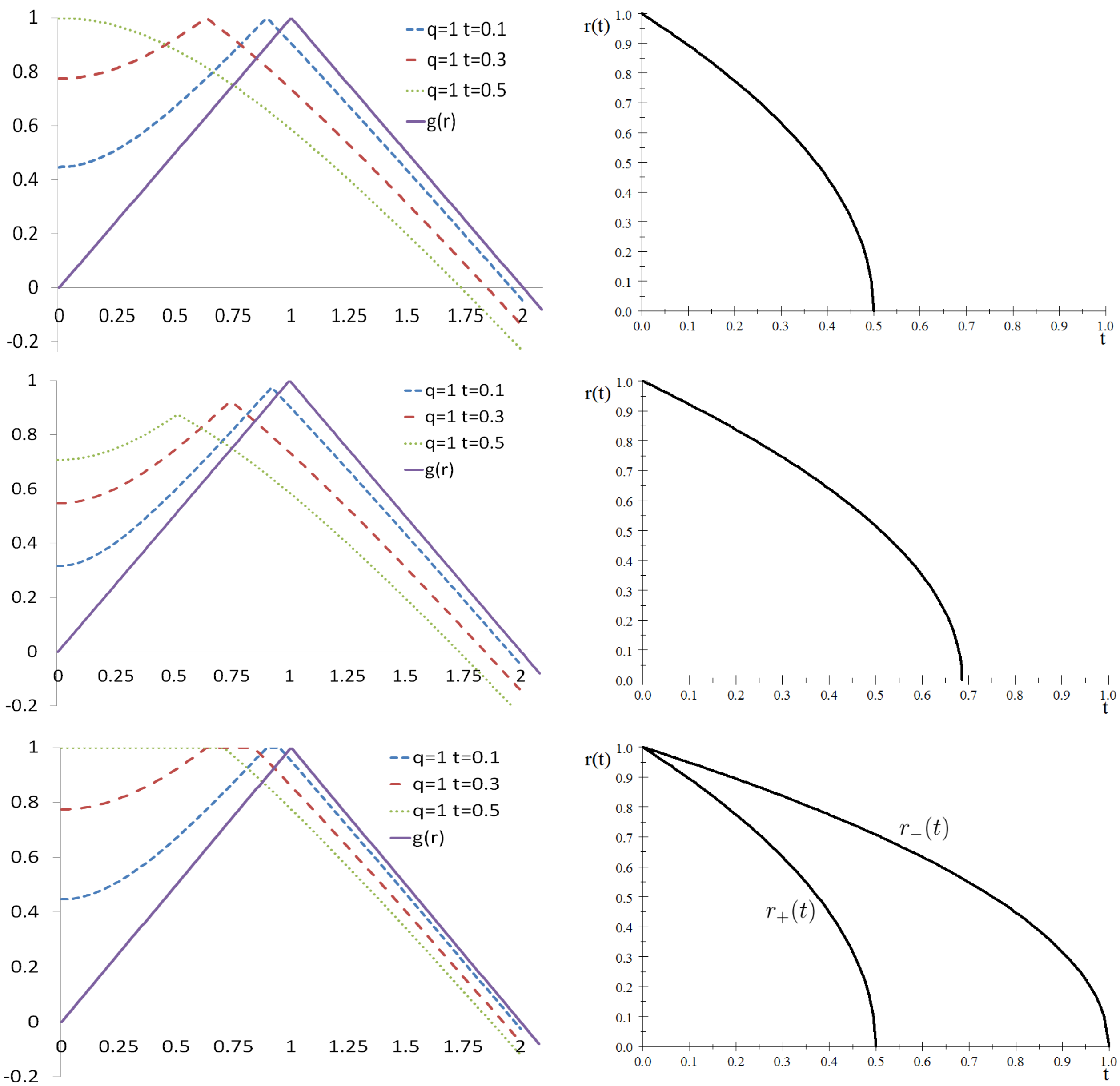

Figure 5: Shape of the solution $u(t, r)$ of (8) with $q = 1$ and the interfaces given by (35) and (40) for $g(r) = \min\{r, 2-r\}$ and different values of t. In the first row we present the case $b_+ = b_- = 1$, in the second row $b_+ = 0.5$ and $b_- = 1$, and in the third row $b_+ = 1$ and $b_- = 0.5$.

Remark 11 *In the case of monotone radially symmetric solutions of the parabolic problem the performed numerical computation is straightforward: for a given* $t > 0$, *we discretize the spatial variable* r *using* $r_n = n \cdot \delta r$ *with* $n \in \mathbb{N}$ *and we just evaluate* $\mathrm{u}(\mathrm{t}, \mathrm{r}_n) = \mathrm{g}\left((r_n^{p+1} + \beta_1(p+1)t)^{\frac{1}{p+1}}\right)$. *When* g *is not monotone we follows the arguments of the proof of Theorem 6. Slightly more involved arguments are needed for the stationary case (see Remark 17 below).* □

4 On the elliptic problems

4.1 Level sets propagation for the stationary problem

Now we consider the stationary problem $\mathrm{SP}(\mathbb{R}^2)$. Our main goal is to study when it is possible that the level $\{v(x) = s\}$ have a non-empty interior and then to study some properties of the interfaces $\partial\{v(x) = s\}$. As we shall see, some of the propagations properties collected in Theorem 5 admits a stationary version but, which is quite curious, some other properties do not have any similar correspondent statement in the time-discretized problem.

The global finite speed of propagation, (property i) of Theorem 5, remains valid thanks to the construction of global super and subsolutions in the same style than the 1974 pioneering paper by H. Brezis [15].

Theorem 7 *Let* $g \in UC(G_M)$ *for each* $M > 0$, *where* $G_M := \{|g| < M\}$. *Let* $v \in UC(V_M)$ *for each* $M > 0$, *with* $V_M := \{|v| < M\}$, *be the unique solution of* $\mathrm{SP}(\mathbb{R}^2)$. *Assume that there is a level* s *such that*

$$\{g(x) = \lambda s\} \supset \mathbb{R}^2 - B_{R_s}(x_s) \text{ for some } x_s \in \mathbb{R}^2 \text{ and } R_s > 0. \tag{45}$$

Then, for any $q > 0$, *the* $s-$*level set of* v *satisfies also that*

$$\{v(x) = s\} \supset \mathbb{R}^2 - B_{R_s}(x_s). \tag{46}$$

Remark 12 . *Notice that, in fact, the above result implies that* $v(x) = s$ *for any* $x \in \partial\{g(x) = \lambda s\} \cap \partial B_{R_s}, (x_s)$. *So, there is no dilatation of the part of* $\partial\{g(x) = \lambda s\}$ *which coincides with* $\partial B_{R_s}(x_s)$, *independently of the growth of the continuous function* g. *This contrasts with what occurs with solutions of most of the quasilinear second order stationary problems where such non-dilatation properties of the support of the right hand side data* g *only appears for sufficiently flat growing decay of* g *near the boundary of its support (see [28], [5] and [6]). We do not know if this non-dilation phenomenon occurs on the parts of* $\partial\{g(x) = \lambda s\}$ *which are different of a part of* $\partial B_{R_s}(x_s)$. □

The following result shows the limitations of the application of the method of local and supersolutions to the stationary problems $\mathrm{SP}(\mathbb{R}^2)$ and $\mathrm{SP}(\Omega)$ (for a general presentation of this method see, *e.g.*, [28]). Although a stationary version of property iii) of Theorem 5 remains valid now this is not the case of property ii) dealing with *dead core type estimates*. So, in contrast with the evolution problem, if the level set $\{g(x) = \lambda s\}$ have a non-empty interior and it is compact then the corresponding $s-$level set of the solution $\{v(x) = s\}$ not only loses the property of having a non-empty interior but it can even disappear.

Theorem 8 *Let $g \in UC(G_M)$ for each $M > 0$, where $G_M := \{|g| < M\}$. Let $v \in UC(V_M)$ for each $M > 0$, where $V_M := \{|v| < M\}$, be the unique solution of* $\mathrm{SP}(\mathbb{R}^2)$*. Let $s \in \mathbb{R}$ be an arbitrary level.*
i) If $g(x_0) > \lambda s$ for some $x_0 \in \mathbb{R}^2$ then $v(x_0) > s$.
ii) If $g(x) > \lambda s$ in the boundary of a ball, $\partial B_{r_0}(x_0)$, for some $x_0 \in \mathbb{R}^2$ and $r_0 > 0$, then $v(x) > s$ in the whole ball $B_{r_0}(x_0)$.

iii) In the special case of $g(x) = \lambda \chi_{[R_0,\infty)} * \varphi_\epsilon(|x|)$ (i.e. with $g(x) = 0$ on $B_{R_0-\epsilon}(0)$ and $g(x) = 1$ on $\mathbb{R}^2 - B_{R_0+\epsilon}(0)$) we have $v(x) > 0$ for any $x \in \mathbb{R}^2$, and if $g(r) = \lambda \chi_{[0,R_0]} * \varphi_\epsilon(r)$ (i.e. with $g(x) = 1$ on $B_{R_0-\epsilon}(0)$ and $g(x) = 0$ on $\mathbb{R}^2 - B_{R_0+\epsilon}(0)$) we have $v(x) < 1$ for any $x \in \mathbb{R}^2$.

Remark 13 *Parts ii) and iii) show that the behaviors of the parabolic and elliptic problem are completely different in which concerns level set propagation: as shown in Theorem 5, the interior area of a level set evolves in a smooth way with respect to t, however, in the elliptic problem, some level sets of g can completely disappear for v, for any $\lambda > 0$.* □

4.2 Monotone radially symmetric solutions

As in the parabolic case, the key stone of the proofs of the above level set propagation results is the study of radially symmetric solutions of $\mathrm{SP}(\mathbb{R}^2)$. For convenience, in which follows, in some occasions we shall use the notation

$$g_\lambda(x) = \frac{g(x)}{\lambda}.$$

As for the parabolic case, in radial coordinates the equation of $\mathrm{SP}(\mathbb{R}^2)$ becomes simpler. In this case it is reduced to a first order ordinary differential equation which keeps many resemblances with the *eikonal equation* for solutions in which sign $(v'(r))$ remains constant for any r. It is the equation (9) quoted in the introduction and that we can also write as

$$-\beta\left(\frac{\mathrm{sign}\big(v'(r)\big)}{r}\right)|v'(r)| + \lambda v(r) = \lambda g_\lambda(x). \tag{47}$$

The following result analyzes the shape of some radially symmetric monotone solutions of equation (47) on generic intervals $[r_0, r_1]$.

Lemma 4 *i) Let $b_+ > 0$, $r_1 > r_0 \geq 0$, and $g \in C[r_0, r_1)$. Let $v_{r_0} \in \mathbb{R}$ be such that*

$$v_{r_0} > \lambda g(r) e^{-\frac{\lambda}{b_+(q+1)}(r^{q+1}-r_0^{q+1})} + \frac{e^{\frac{\lambda}{b_+(q+1)}r_0^{q+1}}}{b_+}\int_{r_0}^{r} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}}d\sigma \quad \forall r \in [r_0, r_1). \tag{48}$$

Then the function $v(r)$ given by

$$v(r) = v_{r_0} e^{\frac{\lambda}{b_+(q+1)}(r^{q+1}-r_0^{q+1})} - \frac{e^{\frac{\lambda}{b_+(q+1)}r^{q+1}}}{b_+}\int_{r_0}^{r} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}}d\sigma \tag{49}$$

is a $C^1[r_0, r_1)$ radially nondecreasing solution of (47) in $[r_0, r_1)$ and satisfies $v(r_0) = v_{r_0}$.

ii) Let $b_- > 0$, $r_1 > r_0 \geq 0$ and $g \in C[r_0, r_1)$. Let $v_{r_0} \in \mathbb{R}$ be such that

$$v_{r_0} < \lambda g(r) e^{-\frac{\lambda}{b_-(q+1)}(r^{q+1} - r_0^{q+1})} + \frac{e^{\frac{\lambda}{b_+(q+1)} r_0^{q+1}}}{b_-} \int_{r_0}^{r} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma \quad \forall r \in [r_0, r_1). \tag{50}$$

Then the function $v(r)$ given by

$$v(r) = v_{r_0} e^{\frac{\lambda}{b_-(q+1)}(r^{q+1} - r_0^{q+1})} - \frac{e^{\frac{\lambda}{b_-(q+1)} r^{q+1}}}{b_-} \int_{r_0}^{r} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma \tag{51}$$

is a $C^1[r_0, r_1)$ radially nonincreasing solution of (47) in $[r_0, r_1)$ and satisfies $v(r_0) = v_{r_0}$.
iii) In the above cases, if $r_0 = 0$ then $v'(0) = 0$.
iv) Statements i) and ii) are also true for $r_1 = \infty$ or when $r_1 < r_0$ (in this last case we must replace the interval $[r_0, r_1)$ in the above statements by $(r_1, r_0]$).

Proof. Since we are searching nondecreasing solutions $v(r)$ of (47) then necessarily we must have

$$-v'(r) + \frac{\lambda}{b_+} r^q v(r) = \frac{\lambda}{b_+} r^q g_\lambda(r). \tag{52}$$

From (52) we obtain that in fact $v(r)$ is nondecreasing iff $v(r) < g_\lambda(r)$ for all $r \geq r_0$. By operating, we obtain that (52) can be equivalently written as

$$\left(v e^{-\frac{\lambda}{b_+(q+1)} r^{q+1}}\right)' = -\frac{\lambda}{b_+} r^q e^{-\frac{\lambda}{b_+(q+1)} r^{q+1}} g_\lambda(r).$$

Then by integrating and assuming that $v(r_0) = v_{r_0}$ we get

$$v(r) = v_{r_0} e^{\frac{\lambda}{b_+(q+1)}(r^{q+1} - r_0^{q+1})} - \frac{\lambda}{b_+} e^{\frac{\lambda}{b_+(q+1)} r^{q+1}} \int_{r_0}^{r} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma. \tag{53}$$

Thus, we conclude that $v'(r) \geq 0$ if and only if (48) holds. The proof of ii) is analogous. The proof of iii) and iv) are obvious from (52) and its similar version in case ii). □

As an application of Lemma 4, we can construct now some explicit solutions of (47) corresponding to monotone data g (what could be consider as a stationary alternative to the explicit solutions built in Lemma 1 for the parabolic case). In fact, by an approximation argument, we can drop the continuity assumption on g.

Lemma 5 *i) Assume $g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} \in L^1(r_0, +\infty)$. Let*

$$v_{r_0} = \frac{e^{\frac{\lambda}{b_+}\frac{r_0^{q+1}}{q+1}}}{b_+} \int_{r_0}^{\infty} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma,$$

and assume that $g(r)$ is nondecreasing for $r \geq r_0$. Then the function $v(r)$ given by

$$v(r) = \frac{e^{\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}}}{b_+} \int_{r}^{\infty} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma$$

is a radially nondecreasing solution of (9) satisfying $g(r) \leq \lambda v(r)$ for $r \geq r_0$
ii) Assume $g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} \in L^1(r_0, +\infty)$. Let

$$v_{r_0} = \frac{e^{\frac{\lambda}{b_-}\frac{r_0^{q+1}}{q+1}}}{b_-} \int_{r_0}^{\infty} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma,$$

and assume that $g(r)$ is nonincreasing for $r \geq r_0$. Then the function $v(r)$ given by

$$v(r) = \frac{e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}}{b_-} \int_{r}^{\infty} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma$$

is a radially nonincreasing solution of (9) satisfying $g(r) \geq \lambda v(r)$ for $r \geq r_0$.

Proof. By an approximation argument (as in [34]), it is enough to check condition (48) of Lemma 4 when g is continuous. We have that

$$g_\lambda(r)e^{-\frac{\lambda}{b_+(q+1)}(r^{q+1}-r_0^{q+1})} + \frac{\lambda}{b_+}e^{\frac{\lambda}{b_+(q+1)}r_0^{q+1}} \int_{r_0}^{r} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma$$
$$= g_\lambda(r)\frac{\lambda}{b_+}e^{\frac{\lambda}{b_-}\frac{r_0^{q+1}}{q+1}} \int_{r}^{\infty} \sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma + \frac{\lambda}{b_+}e^{\frac{\lambda}{b_+(q+1)}r_0^{q+1}} \int_{r_0}^{r} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma := I.$$

Moreover, as $g_\lambda(r)$ is nondecreasing for $r \geq r_0$

$$I < \frac{\lambda}{b_+}e^{\frac{\lambda}{b_-}\frac{r_0^{q+1}}{q+1}} \int_{r}^{\infty} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma + \frac{\lambda}{b_+}e^{\frac{\lambda}{b_+(q+1)}r_0^{q+1}} \int_{r_0}^{r} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma$$
$$= e^{\frac{\lambda}{b_+(q+1)}r_0^{q+1}} \int_{r_0}^{\infty} \frac{\lambda}{b_+} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma = v_{r_0}.$$

Then by applying Lemma 4 we obtain that the function $v(r)$ given by

$$v(r) = e^{\frac{\lambda}{b_+(q+1)}r^{q+1}} \int_{r_0}^{\infty} \frac{\lambda}{b_-} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma - \frac{\lambda}{b_+}e^{\frac{\lambda}{b_+(q+1)}r^{q+1}} \int_{r_0}^{r} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma$$
$$= e^{\frac{\lambda}{b_+(q+1)}r^{q+1}} \int_{r}^{\infty} \frac{\lambda}{b_-} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma$$

is a radially nondecreasing solution of (47). Moreover, as $g_\lambda(r)$ is nondecreasing for $r \geq r_0$ we have

$$v(r) = e^{\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}} \int_{r}^{\infty} \frac{\lambda}{b_+} g_\lambda(\sigma)\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma \geq g_\lambda(r)e^{\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}} \int_{r}^{\infty} \frac{\lambda}{b_+}\sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma = g_\lambda(r).$$

The proof of $ii)$ is analogous. □

Remark 14 *Motivated by the study made in [22] for data growing linearly at the infinity, it is illustrative to mention that function*

$$v(r)=e^{\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}}\left(\left(\frac{b_+(q+1)}{\lambda}\right)^{\frac{1}{q+1}}\Gamma\left(\frac{q+2}{q+1},\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}\right)\right)$$

is the radially nonincreasing solution of (9) corresponding to $g_\lambda(r)=r$, *where the function* Γ *([23]) is given by*

$$\Gamma(a,z)=\int_z^\infty s^{a-1}e^{-s}ds.$$

$\square$

An interesting application of Lemma 5 corresponds to the choice of $g_\lambda(r)$ given by $g_{s,0,r_1,\epsilon}(r)=s\chi_{[0,r_1]}*\varphi_\epsilon(r)$ (with $\chi_{[0,r_1]}$ and $\varphi_\epsilon(r)$ defined by (37) and (36) respectively). In such case, a solution $v_{s,0,r_1,\epsilon}(r)$ of (47) is given by

$$v_{s,0,r_1,\epsilon}(r)=e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}\int_r^\infty\frac{\lambda}{b_-}s\chi_{[0,r_1]}*\varphi_\epsilon(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}}d\sigma. \tag{54}$$

We observe that $v_{s,0,r_1,\epsilon}(r)$ is nonnegative, nonincreasing and satisfies $v_{s,0,r_1,\epsilon}(r)=0$ if $r\geq r_1+\epsilon$ and $v_{s,0,r_1,\epsilon}(r)=s$ if $r\leq r_1-\epsilon$. In fact, as the following result shows (54) is the only nonnegative and bounded solution of (47) for $g_\lambda(r)=g_{s,0,r_1,\epsilon}(r)$. Let us consider, more in general, the case of nonincreasing nonnegative data g with compact support:

Lemma 6 *Let* $g(r)\geq 0$ *be nonincreasing for* $r>0$ *and with a compact support given by the interval* $[0,R_0]$. *Then:*
i) the function

$$v(r)=\frac{e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}}{b_-}\int_r^{R_0}g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}}d\sigma \tag{55}$$

is a solution of (47) for $r\geq 0$ *and satisfies* $v(r)=0$ *for* $r\geq R_0$ *and* $0\leq v(r)\leq\lambda g(0)$.
ii) if $v(r)$ *is any other solution of (47) for* $r\geq R_0$ *such that* $v(R_0)>0$ *then* $\lim_{r\to\infty}v(r)=\infty$. *In particular, the function* $v(|x|)$ *with* v *given by (55) is the only solution of* $\mathrm{SP}(\mathbb{R}^2)$ *corresponding to* $g(|x|)$.

Proof. *i)* $v(r)$ is the solution provided by Lemma 5 corresponding to $r_0=0$ and as $g(r)=0$ for $r\geq R_0$ the integral in the expression of $v(r)$ is equal to 0 and then $v(r)=0$ for $r\geq R_0$. On the other hand, as $0\leq g(r)\leq g(0)$ we have

$$0\leq v(r)\leq g_\lambda(0)e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}\int_r^\infty\frac{\lambda}{b_-}\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}}d\sigma=g_\lambda(0).$$

To show *ii)* we point out that if $v(R_0)>0$, $v(r)$ is solution of (47) and $g_\lambda(r)=0$ for $r\geq R_0$. Then for $r\geq R_0$ we have

$$v'(R_0)=\frac{\lambda}{b_+}r^q v(R_0)>0.$$

Therefore, for any $r \geq R_0$ the function $v(r)$ can not decreases and it satisfies

$$v'(r) = \frac{\lambda}{b_+} r^q v(r) \geq \frac{\lambda}{b_+} r^q v(R_0) > 0.$$

Then $\lim_{r\to\infty} v(r) = \infty$. Therefore, if $v(r)$ is a nonnegative bounded solution of (47) then $v(R_0) = 0$ and from the uniqueness of solutions for $\mathrm{SP}(\mathbb{R}^2)$ it has to be the one provided in i) (since such solution must be nonnegative and bounded). □

Remark 15 *Theorems 7 and 8 apply also to solutions of the Dirichlet problem $SP(\Omega)$ and in fact there is not any kind of peculiar behavior near the boundary of Ω (in contrast with Höpf maximum principle ensuring that $\mathrm{D}v \cdot \overrightarrow{\mathbf{n}} < 0$ if $v \geq 0$).*

By arguing as in the previous lemmas it is possible to get some necessary and sufficient conditions for the existence of symmetric solutions on the stationary equation of $\mathrm{SP}(\mathbb{R}^2)$ but now on symmetric rings and balls. The following result collects this fact. We drop its proof since it based on obvious adaptations. To simplify the statement we shall assume $g \in C^1$ and thus since the radial solutions are C^2 they are automatically (viscosity) solutions of the second order equation.

Corollary 2 *Let Ω be the ring $\{x \in \mathbb{R}^2\colon\ r_0 < |x| < r_1\}$ for some $0 < r_0 < r_1$ or the ball $B_{r_1}(0) = \{x \in \mathbb{R}^2\colon\ |x| < r_1\}$ if $r_0 = 0$. Given $h_0, h_1 \geq 0$ we consider the problem*

$$\begin{cases} -\beta \left(\operatorname{div} \left(\dfrac{\mathrm{D}v}{|\mathrm{D}v|} \right) \right) |\mathrm{D}v| + \lambda v = g(x) & \text{in } \Omega, \\ v = h_0 & |x| = r_0, \\ v = h_1 & |x| = r_1, \end{cases} \tag{56}$$

or simply

$$\begin{cases} -\beta \left(\operatorname{div} \left(\dfrac{\mathrm{D}v}{|\mathrm{D}v|} \right) \right) |\mathrm{D}v| + \lambda v = g(x) & \text{in } \Omega, \\ v = h_1 & \text{if } |x| = r_1, \end{cases} \tag{57}$$

in the case of the ball $\Omega = B_{r_1}(0)$. Let $g(x) = g(|x|)$ be a $C^1[r_0, r_1]$ radially symmetric function.
i) Assume $b_+ > 0$ and $g(r)$ nondecreasing. Then problems (56) or (57) have a C^2 nondecreasing solution if and only if

$$h_1 e^{-\frac{\lambda r_1^{(q+1)}}{b_+(q+1)}} - h_0 e^{-\frac{\lambda r_0^{(q+1)}}{b_+(q+1)}} = -\frac{\lambda}{b_+} \int_{r_0}^{r_1} g(\sigma) \sigma^q e^{-\frac{\lambda}{b_+(q+1)} \sigma^{q+1}} d\sigma. \tag{58}$$

In particular, in the case of a ball ($r_0 = 0$) and $g \equiv 0$, for any $h_1 > 0$ the solution is given by

$$v(x) = h_1 e^{-\frac{\lambda}{b_+(q+1)} (r_1^{(q+1)} - |x|^{(q+1)})}, \tag{59}$$

and thus $v(0) = h_1 e^{-\frac{\lambda r_1^{(q+1)}}{b_+(q+1)}} > 0$. Moreover, if $r_0 > 0$ and $h_0 = 0$ then problem (56) has a nontrivial C^2 positive solution if and only if

$$\int_{r_0}^{r_1} g(\sigma) \sigma^q e^{-\frac{\lambda}{b_+(q+1)} \sigma^{q+1}} d\sigma < 0.$$

ii) Assume $b_- > 0$ and let $g(r)$ be nonincreasing. Then problems (56) or (57) have a C^2 nonincreasing solution if and only if

$$h_1 e^{-\frac{\lambda r_1^{(q+1)}}{b_-(q+1)}} - h_0 e^{-\frac{\lambda r_0^{(q+1)}}{b_-(q+1)}} = -\frac{\lambda}{b_-}\int_{r_0}^{r_1} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma. \tag{60}$$

In particular, in the case of a ball ($r_0 = 0$) and $h_1 \geq 0$ there exists a nontrivial nonincreasing C^2 positive solution (and then $v(0) > 0$) if and only if (60) and

$$h_1 = -\frac{e^{\frac{\lambda r_1^{(q+1)}}{b_-(q+1)}}}{b_-}\int_{r_0}^{r_1} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma$$

hold. Moreover, if $r_0 > 0$, $h_0 > 0$ and $h_1 = 0$ then problem (56) has a nontrivial positive solution if and only if

$$h_0 = \frac{e^{\frac{\lambda r_0^{(q+1)}}{b_-(q+1)}}}{b_-}\int_{r_0}^{r_1} g(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma.$$

4.3 Non monotone radially symmetric solutions

As in the parabolic case, the loss of monotonicity of the datum $g(r)$ leads to new qualitative properties of the associated solutions of $\mathrm{SP}(\mathbb{R}^2)$. Again, as in subsection 3.2, we shall concentrate our attention on the case of *Batman* profiles type $g(|x|)$ satisfying (10). As in the parabolic case, we can produce a puzzle of monotone solutions by means of a general principle which now can be stated as follows (the proof is similar to the one of the parabolic case).

Proposition 3 *i) Let v_i be a supersolution of $\mathrm{SP}(\mathbb{R}^2)$ corresponding to the datum $g_i(x)$, $i = 1, 2$. Then $v = \min(v_1, v_2)$ is a supersolution of $\mathrm{SP}(\mathbb{R}^2)$ corresponding to the datum $g = \min(g_1, g_2)$.*
ii) Let g_1, g_2 be $C^1[r_0, r_1]$ radially symmetric functions with g_1 nondecreasing and g_2 nonincreasing and let $h_0^i, h_1^i \geq 0$ with $i = 1, 2$. Assume that the corresponding conditions (58) (60) hold with $g = g_i$ and $h_0 = h_0^i$ and $h_1 = h_1^i$ respectively. Let v_1, v_2 be the corresponding C^2 solutions of (56) or (57). Then the function $v = \min\{v_1, v_2\}$ (respectively $v = \max\{v_1, v_2\}$) is a (viscosity) supersolution (respectively subsolution) of (56) or (57) with $h_0 = \min\{h_0^1, h_0^2\}$ and $h_1 = \min\{h_1^1, h_1^2\}$ (respectively $h_0 = \max\{h_0^1, h_0^2\}$ and $h_1 = \max\{h_1^1, h_1^2\}$).

The following result proves, that in contrast with what happens for the parabolic problem (see Theorem 6) now it is not so relevant the possible difference among velocities corresponding to the convex and concavity parts of the level sets of solutions v. In fact, we shall consider a slightly more general case than the *Batman* profiles of subsection 3.2.

Theorem 9 *Let g_1,$g_2 \in C[0, \infty)$ be such that $g_1(r)$ is a nondecreasing function, $g_2(r)$ is a nonincreasing function, $\{r : g_1(r) = g_2(r)\} \neq \emptyset$ and $r_d = \min\{r : g_1(r) = g_2(r)\}$. Let $v_\infty(r)$ be given by*

$$v_\infty(r) = \frac{e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}}{b_-}\int_r^\infty g_2(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma.$$

Consider the function

$$v_d(r) = v_\infty(r_d) e^{\frac{\lambda}{b_- (q+1)}(r^{q+1} - r_d^{q+1})} - \frac{e^{\frac{\lambda}{b_-(q+1)} r^{q+1}}}{b_-} \int_{r_d}^{r} g_1(\sigma) \sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}} d\sigma.$$

Let $r_m = \min\{r \geq 0 : \lambda v_d(r) \leq g_1(r)\}$ *and define the function*

$$v_m(r) = v_d(r_m) e^{\frac{\lambda}{b_+ (q+1)}(r^{q+1} - r_m^{q+1})} - \frac{e^{\frac{\lambda}{b_+(q+1)} r^{q+1}}}{b_+} \int_{r_m}^{r} g_1(\sigma) \sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma.$$

Then, the function

$$v(r) = \begin{cases} v_m(r) & for \quad r \in [0, r_m] \\ v_d(r) & for \quad r \in [r_m, r_d] \\ v_\infty(r) & for \quad r \geq r_d \end{cases}$$

is a $C^1[0,\infty)$ *solution of equation (47) for* $g(r) = \min\{g_1(r), g_2(r)\}$*.*

Proof. First we point out that by Lemma 5, $v_\infty(r)$ is a solution of (47) for $r \geq r_d$ satisfying $v_\infty(r) \leq g_{\lambda,2}(r)$. On the other hand as $v_d(r_d) = v_\infty(r_d) \leq g_{\lambda,2}(r_d) = g_{\lambda,1}(r_d)$ then $r_m \leq r_d$ is well defined. We point out that since $v_d(r) \leq g_{\lambda,1}(r)$ in $[r_m, r_d]$ then $v_d'(r) = \lambda(v_d(r) - g_{\lambda,1}(r)) \leq 0$ in $[r_m, r_d]$ and then $v_d(r)$ is a solution of (47) in $[r_m, r_d]$. Finally, in the interval $[0, r_m]$, as $g_{\lambda,1}(r)$ is nondecreasing and $v_m(r_m) = v_d(r_m) = g_{\lambda,1}(r_m)$ we obtain that for $r < r_m$, $g_{\lambda,1}(r) \leq g_{\lambda,1}(r_m) = v_d(r_m)$ and then

$$\begin{aligned} v_m(r) &= v_d(r_m) e^{\frac{\lambda}{b_+(q+1)}(r^{q+1} - r_m^{q+1})} + \frac{\lambda}{b_+} e^{\frac{\lambda}{b_+(q+1)} r^{q+1}} \int_r^{r_m} g_{\lambda,1}(\sigma) \sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma \geq \\ &\geq v_d(r_m) e^{\frac{\lambda}{b_+(q+1)}(r^{q+1} - r_m^{q+1})} + \frac{\lambda}{b_+} e^{\frac{\lambda}{b_+(q+1)} r^{q+1}} g_{\lambda,1}(r) \int_r^{r_m} \sigma^q e^{-\frac{\lambda}{b_+(q+1)}\sigma^{q+1}} d\sigma = \\ &= (v_d(r_m) - g_{\lambda,1}(r)) e^{\frac{\lambda}{b_+(q+1)}(r^{q+1} - r_m^{q+1})} + g_{\lambda,1}(r) \geq g_{\lambda,1}(r). \end{aligned}$$

Therefore $v_m'(r) = \lambda(v_m(r) - g_{\lambda,1}(r)) \geq 0$ in $[0, r_m]$ and then v_m is a solution of equation (47). Concerning the regularity of $v(r)$ notice that $v_d'(r_d) = v_\infty'(r_d)$ because $g_{\lambda,1}(r)$, $g_{\lambda,2}(r)$ are continuous and $g_{\lambda,1}(r_d) = g_{\lambda,2}(r_d)$. Finally since $v_m(r_m) = v_d(r_m) = g_{\lambda,1}(r_m)$ then $v_m'(r_m) = v_d'(r_m) = \lambda(v_m(r_m) - g_{\lambda,1}(r_m)) = 0$. □

Remark 16 *We point out that if* $g_1(r)$, $g_2(r)$ *are nonnegative and* $g(r_0) = \min\left\{g_1(r_0), g_2(r_0)\right\} > 0$ *then the solution provided by the previous theorem satisfies that* $v(r) > 0$ *for any* $r < r_0$*. Indeed, we observe that the functions* $v_m(r)$*,* $v_d(r)$ *and* $v_\infty(r)$ *are nonnegative and if* $g(r) > 0$ *in a region, then this positive value is propagated towards the left. We also observe that in this stationary case the assumption* $b_+ \neq b_-$ *does not introduce any kind of singularity in the derivatives of the solution.* □

4.4 Proof of the general level propagation results

Proof of Theorem 7. The existence, comparison and uniqueness of solutions follows from Theorem 4. For the proof of i) we shall construct two global super and subsolutions leading to estimate (46). Let $x_s \in \mathrm{supp}(g - s)$ given in (45). It is clear that we can construct two radially symmetric functions of the form $\overline{g}(|x-x_s|)$ and $\underline{g}(|x-x_s|)$, with $\overline{g}$ nonincreasing and $\underline{g}$ nondecreasing, such that, $\underline{g} \le g \le \overline{g}$, $\overline{g}-s \ge 0$, $\underline{g}-s \le 0$ and

$$\{\overline{g}=s\} \supset \mathbb{R}^2 - B_{R_s}(x_s) \text{ and } \{\underline{g}=s\} \supset \mathbb{R}^2 - B_{R_s}(x_s),$$

(use a regularization of the modulus of continuity of g near the boundary of the support of $(g-s)$). Then, by Lemma 6 the solution $\overline{v}$ corresponding to $\overline{g}$ is given by

$$\overline{v}(r) = \frac{e^{\frac{\lambda}{b_-}\frac{r^{q+1}}{q+1}}}{b_-}\int_r^{Rs} \overline{g}(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}}d\sigma, \tag{61}$$

and satisfies $\overline{v}(r)=0$ for $r \ge Rs$ and $0 \le \overline{v}(r) \le \lambda\overline{g}(0)$. Analogously, the solution $\underline{v}$ corresponding to $\underline{g}$ is given by

$$\underline{v}(r) = \frac{e^{\frac{\lambda}{b_+}\frac{r^{q+1}}{q+1}}}{b_+}\int_r^{Rs} \underline{g}(\sigma)\sigma^q e^{-\frac{\lambda}{b_-(q+1)}\sigma^{q+1}}d\sigma, \tag{62}$$

and satisfies $\underline{v}\ (r)=0$ for $r \ge Rs$ and $0 \ge \underline{v}(r) \ge \lambda\overline{g}(0)$. From Theorem 4 we get that $\underline{v} \le v \le \overline{v}$ which leads to the conclusion. Notice that if $\min\{b_+,b_-\}=0$ the conclusion holds trivially. So we can assume $\min(b_+,b_-)>0$ and estimate (46) is coherent in both cases. □

Proof of Theorem 8. i) If $g(x_0)>0$ and g is continuous then there exists a ball $B_{r_0}(x_0)$ such that $\lambda g(x) \ge s > 0$ for any $x \in B_{r_0}(x_0)$. Next we consider the solution $v_{s,0,r_1-\epsilon,\epsilon}(|x-x_0|)$ given in (54). Then, for $\epsilon>0$ small enough by comparison we obtain that

$$v(x_0) \ge v_{s,0,r_1-\epsilon,\epsilon}(0) = s > 0.$$

ii) As $\partial B_{r_0}(x_0)$ is a compact set and g is continuous there exist $s,\delta>0$ such that

$$\lambda g(x) \ge s \qquad \text{for } x \in B_{r_0}(x_0)\backslash B_{r_0-\delta}(x_0).$$

Therefore for $\epsilon>0$ small enough

$$\lambda g(x) \ge s\chi_{[r_0-\epsilon,,r_0-\delta+\epsilon]} * \varphi_\epsilon(|x-x_0|) \qquad \text{for } x \in \mathbb{R}^2.$$

Then, by comparison and applying Theorem 9 and Remark 16 we obtain that $v(x)>0$ in $B_{r_0}(x_0)$. The proof of iii) is consequence of formula (54). □

4.5 Numerical experiments for the elliptic problem $\mathrm{SP}(\mathbb{R}^2)$

In this section we present some numerical experiments to illustrate the theoretical results obtained for problem $\mathrm{SP}(\mathbb{R}^2)$. In Figure 6 we illustrate the solution of equation (9) provided by Lemma 5 in the case of $g_\lambda(r)=\chi_{[0,2]}*\varphi_{0.1}(r)$ and $g_\lambda(r)=\chi_{[1,\infty)}*\varphi_{0.1}(r)$. In Figure 7 we illustrate the solution of equation (9) in the case of $g_\lambda(r)=\chi_{[1,2]}*\varphi_{0.1}(r)=\min\{\chi_{[0,2]}*\varphi_{0.1}(r),\chi_{[1,\infty)}*\varphi_{0.1}(r)\}$. We illustrate in this figure that for the elliptic problem (9) the minimum of two solutions is not a solution of the equation (in contrast to the result for the parabolic problem quoted in Theorem 6 when $b_+ \le b_-$). In Figure 8 we show the solution of equation (9) in the case of $g(r)=\min\{|r|^3,\max\{0,0.5(\sqrt{1.1-|r|}-\sqrt{0.1})\}\}$. We point out, again, that for the elliptic problem (9) no singularity appears in the case $b_+ \ne b_-$.

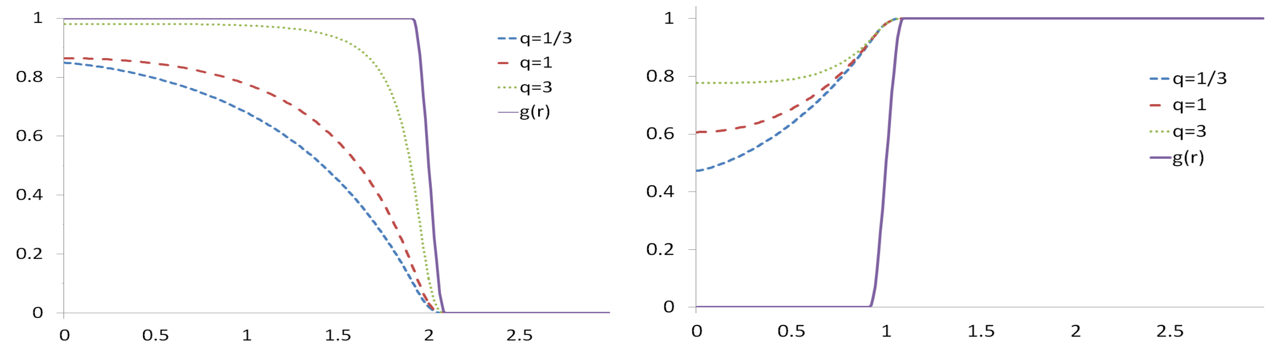

Figure 6: Shape of the solution of for (9) for $\lambda = 1$, $b_+ = b_- = 1$ and different values of q. On the left we present the shape of $v(r)$ for $g_\lambda(r) = \chi_{[0,2]} * \varphi_{0.1}(r)$ and on the right for $g_\lambda(r) = \chi_{[1,\infty)} * \varphi_{0.1}(r)$.

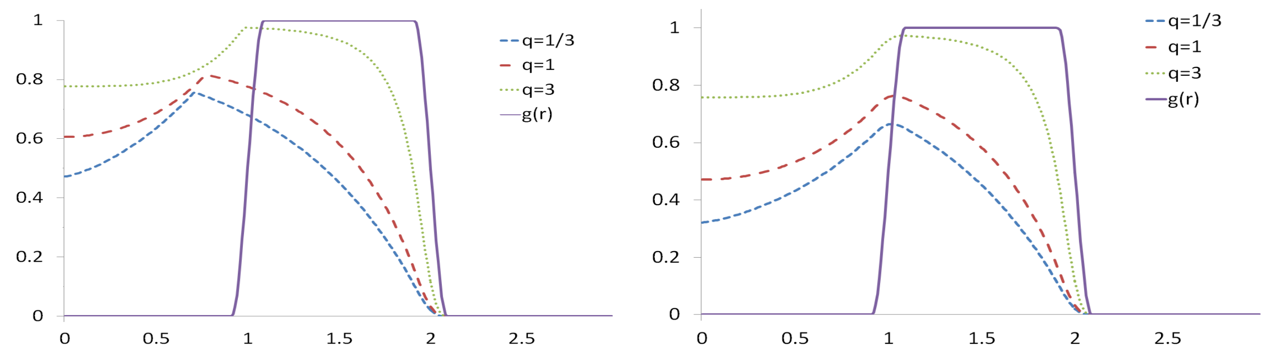

Figure 7: On the left we present the function $v(r) = min\{v_1(r), v2(r)\}$, where $v_1(r)$, $v_2(r)$ are the solutions of (9) for $g_{\lambda,1}(r) = \chi_{[0,2]} * \varphi_{0.1}(r)$ and $g_{\lambda.2}(r) = \chi_{[1,\infty)} * \varphi_{0.1}(r)$. On the right we show the solution $v(r)$ of (9) for $g_\lambda(r) = \min\{g_{\lambda,1}(r), g_{\lambda,2}(r)\}$.

Remark 17 *Such as it was proved before, a nonincreasing radially symmetric solution of the elliptic problem* $\mathrm{SP}(\mathbb{R}^2)$ *is given by*

$$v(r) = \frac{e^{\frac{\lambda}{b_-}\frac{r^{p+1}}{p+1}}}{b_-}\int_r^\infty g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}}d\sigma.$$

As in the parabolic problem, we discretize the spatial variable r *using* $r_n = n\cdot\delta r$ *and we approximate* $v(r_n)$ *by using the expression*

$$v(r_n) = \lim_{A\to\infty}\frac{e^{\frac{\lambda}{b_-}\frac{r^{p+1}}{p+1}}}{b_-}\int_{r_n}^A g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}}d\sigma. \tag{63}$$

To compute numerically the integral in the above expression we use the following scheme based on the basic Simpson rule

$$\int_a^b f(\sigma)d\sigma \approx \sum_{k=0}^{M-1}\frac{f\left(a+k\frac{b-a}{M}\right)+4f\left(a+(k+0.5)\frac{b-a}{M}\right)+f\left(a+(k+1)\frac{b-a}{M}\right)}{6}\frac{b-a}{M},$$

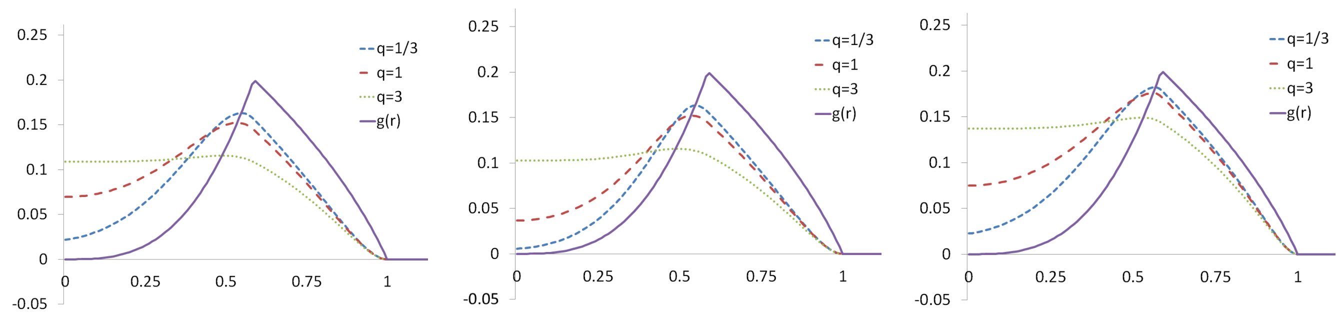

Figure 8: Shape of the solution of (9) for the Batman profile appearing in Figure 1, $\lambda = 10$ and different values of q. From left to right we present the results for $b_+ = b_- = 1$, $b_+ = 0.5$, $b_- = 1$ and $b_+ = 1$, $b_- = 0.5$.

where $M \in \mathbb{N}$ is big enough. In practice, in this paper we fix M as $\mathrm{M} = \left\lfloor \frac{b-a}{10^{-4}} \right\rfloor + 1$, *where $\lfloor s \rfloor$ is the largest integer not greater than s. Finally, to compute numerically the limit in expression (63) we use the following iterative algorithm: we fix $A_0 \gg r_n$ and we define $A_m = A_0 + m$ for $m \in \mathbb{N}$. We initially define*

$$v(r_n) = \frac{e^{\frac{\lambda}{b_-}\frac{r^{p+1}}{p+1}}}{b_-} \int_{r_n}^{A_0} g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}} d\sigma.$$

Then we iteratively update the $v(r_n)$ value in the following way

$$v(r_n) = v(r_n) + \frac{e^{\frac{\lambda}{b_-}\frac{r_n^{p+1}}{p+1}}}{b_-} \int_{A_m}^{A_{m+1}} g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}} d\sigma \quad \text{for } m \in \mathbb{N}.$$

In practice, we stop iterations when

$$\left| e^{\frac{\lambda}{b_-}\frac{r_n^{p+1}}{p+1}} \int_{A_m}^{A_{m+1}} g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}} d\sigma \right| < 10^{-10} \left| v(r_n) \right|.$$

We recall that if

$$\int_0^\infty g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}} d\sigma < \infty$$

then

$$\lim_{m\to\infty} \int_{A_m}^{A_{m+1}} g(\sigma)\sigma^p e^{-\frac{\lambda}{\beta_{-1}(p+1)}\sigma^{p+1}} d\sigma = 0.$$

In the case of using as function $g(.)$ given by the convolution with a mollifier function, that is $g(r) = f * \varphi_\epsilon(r)$ where $supp\{\varphi_\epsilon\} = [-\epsilon, \epsilon]$, then to compute numerically $f * \varphi_\epsilon(r)$ we discretize the convolution operator in the following way:

$$f * \varphi_\epsilon(r) = \sum_{k=-M}^{k=M} f\left(r + k\frac{\epsilon}{M}\right) \varphi_\epsilon\left(k\frac{\epsilon}{M}\right) \frac{\epsilon}{M},$$

where $M \in \mathbb{N}$ is big enough. In practice, our numerical experiments assumed $M = 10^4$.

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