

FINITE EXTINCTION TIME FOR A CLASS OF
NON-LINEAR PARABOLIC EQUATIONS

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Introduction. In this paper we study the behaviour, for large time, of the solution of the problem

$$\begin{aligned} (1) \quad & u_t(t,x) - \Delta\beta(u(t,x)) = 0 && 0 < t, \quad x \in \Omega \\ & \beta(u(t,x)) = 0 && 0 < t, \quad x \in \partial\Omega \\ & u(0,x) = u_0(x) && x \in \Omega \end{aligned}$$

where Ω is an open bounded set of \mathbb{R}^N and β is a continuous non-decreasing real function such that $\beta(0) = 0$ (or, more generally, a maximal monotone graph of \mathbb{R}^2 , such that $0 \in \beta(0)$, c.f. Brezis [6]). Equations of this sort arise in many applications: flow in a porous media, the Stefan problem, biological models, etc. It also arises in the study of phenomena of "electron heat conduction" in a plasma where the thermal conductivity is a function of the temperature (see [18] Chap. X and also [16]).

We are interested in characterizing the class of β for which the solution u of (1) has a finite extinction time, i.e. there exists a T_0 such that $u(t,x) = 0$ for $t \geq T_0$. Our main result establishes that the necessary and sufficient condition on β to have that extinction is that

$$(2) \quad \int_{-1}^1 \frac{ds}{\beta(s)} < +\infty$$

for a general class of data u_0 (e.g. $u_0 \in L^p(\Omega)$, $\max\{2N/N+2, 1\} < p \leq +\infty$).

Finite extinction time for particular formulations of (1) have been studied by Sabinina [15], [16], Evans [10], Berryman and Holland [5] and Benilan and Crandall [3]. Reference on this phenomena in other equations can be found in Díaz [9].

It is interesting to note that our main result does not apply if $\Omega = \mathbb{R}^N$, $N \geq 3$. If $\beta(r) = |r|^m \text{ sign } r$ the extinction only appears for $0 < m < \frac{N-2}{N}$ (see Proposition 10 of Benilan and Crandall [3]).

An essential tool in our study of (1) is the consideration of the "dual" problem

$$(3) \quad \begin{aligned} v_t(t,x) + \beta(-\Delta v(t,x)) &= 0 & t > 0, \quad x \in \Omega \\ v(t,x) &= 0 & t > 0, \quad x \in \partial\Omega \\ v(0,x) &= v_0(x) & x \in \Omega \end{aligned}$$

for whose solutions we establish the finite time of extinction. This will be obtained by means of comparison with suitable functions. Finally, duality arguments will be used to get the above main result.

The plan of this paper is as follows: In Section 1 results on existence and uniqueness of solutions of (1) and (3) are collected, specifying the sense of duality between both equations and giving some criteria of comparison. In Section 2, extinction for problem (3) is obtained. Finally, in Section 3, the main theorem is established. Among other remarks, the case of nonhomogeneous equation is also considered.

1.- Preliminaries.

Let Ω be a regular bounded open set of \mathbb{R}^N . Let us note by $\Lambda = -\Delta$ the canonical isomorphism from the Hilbert space $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ onto its dual $H^{-1}(\Omega)$. By abuse of notation we shall identify Λ^{-1} with $(-\Delta)^{-1}$. It will be useful to remind that by the L^p estimates one has that if $u \in L^p(\Omega) \cap H^{-1}(\Omega)$ $1 < p < +\infty$, then $(-\Delta)^{-1}u \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$. (See Agmon, Douglis; and Nirenberg [1]).

In the following β will be a continuous nondecreasing function from \mathbb{R} onto \mathbb{R} such that $0 = \beta(0)$, although the results that will be obtained remain valid in the general case in which β is a maximal monotone graph such that $R(\beta) = D(\beta) = \mathbb{R}$ and $0 \in \beta(0)$.

The abstract theory of evolution equations governed by accretive operators (see, e.g. [7] and [10]) allows to consider the problems (1) and (3) in the frame of several functional spaces.

Concretely, the problem (1) can be treated on $H^{-1}(\Omega)$ or on $L^1(\Omega)$. Indeed, the operators A and B given by

$$D(A) = \{u \in H^{-1}(\Omega) : \beta(u) \in H_0^1(\Omega)\}, \quad (\overline{D(A)} = H^{-1}(\Omega)),$$

$$A(u) = -\Delta\beta(u) \quad \text{if} \quad u \in D(A)$$

and

$$D(B) = \{u \in L^1(\Omega) : \beta(u) \in W_0^{1,1}(\Omega), \Delta\beta(u) \in L^1(\Omega)\},$$

$$Bu = -\Delta\beta(u) \quad (\text{in the weak-}L^1 \text{ sense}) \quad \text{if} \quad u \in D(B)$$

are m -accretives in $H^{-1}(\Omega)$ and $L^1(\Omega)$ respectively (see Brezis [6] and Benilan [2]).

On the other hand, problem (3) can be studied on $H_0^1(\Omega)$ or on $L^\infty(\Omega)$. As a matter of fact, the m -accretivity in $H_0^1(\Omega)$ of the operator given by

$$D(C) = \{v \in H_0^1(\Omega) : -\Delta v \in L^1(\Omega) \text{ and } \beta(-\Delta v) \in H_0^1(\Omega)\},$$

$$Cv = \beta(-\Delta v) \quad \text{if} \quad v \in D(C)$$

is an unpublished result by Brezis. It is also known that the operator E , defined by

$$D(E) = \{v \in L^\infty(\Omega) \cap H_0^1(\Omega) \cap H^2(\Omega) : \beta(-\Delta v) \in L^\infty(\Omega)\}$$

$$Ev = \beta(-\Delta v) \quad \text{if} \quad v \in D(E)$$

is m -accretive in $L^\infty(\Omega)$. (See Benilan and Ha [4] and also Konishi [13]).

A fact that we will exploit is the existing duality between problem (1) and (3) being governed by the operators A and C respectively. Concretely, given $u_0 \in H^{-1}(\Omega)$ such that $(-\Delta)^{-1} u_0 \in \overline{D(C)}$, if $u \in C([0, \infty) : H^{-1}(\Omega))$ is the solution of (1), then the function $v(t) = (-\Delta)^{-1} u(t)$, $v \in C([0, \infty) : H_0^1(\Omega))$ is the unique solution of (3) corresponding to $v_0 = (-\Delta)^{-1} u_0$. Indeed, let be $\lambda > 0$ and define $a_0 = u_0$, and $a_{n+1} \in H^{-1}(\Omega)$ be means of

$$\frac{a_{n+1} - a_n}{\lambda} - \Delta \beta(a_{n+1}) = 0$$

or, more precisely, $a_{n+1} = (I + \lambda A)^{-1} a_n$. By defining $b_n = (-\Delta)^{-1} a_n$, the linearity of $(-\Delta)^{-1}$ determines that $b_{n+1} = (I + \lambda C)^{-1} b_n$. Considering now the function $u_\lambda(t) = a_n$ (resp. $v_\lambda(t) = b_n$) for $n\lambda \leq t \leq (n+1)\lambda$, then the theory of semigroups implies that $u_\lambda \rightarrow u$ (resp. $v_\lambda \rightarrow v$), solution of (1) (resp. (3)) in $H^{-1}(\Omega)$ (resp. $H_0^1(\Omega)$) uniformly on compact subsets of $[0, \infty)$. Finally, by the continuity of $(-\Delta)^{-1}$, $v(t) = (-\Delta)^{-1} u(t)$.

In what follows it will also be of great importance the fact that the problems (1) and (3) have properties of comparison, being governed by B and E . For example, in [2] it is shown

that B is T -accretive and hence if $u_{0,i} \in L^1(\Omega)$ and u_i is the corresponding solution of (1) ($i=1,2$), then

$$(4) \quad u_{0,1} \leq u_{0,2} \text{ in } \Omega \Rightarrow u_1(t,x) \leq u_2(t,x) \text{ for all } t \in [0, \infty), \\ \text{a.e. } x \in \Omega.$$

When (3) is governed by E one also has similar results. This can be obtained from the following lemma, which we will use in Section 2:

Lemma 1. Let $v_0 \in \overline{D(E)}^{L^\infty}$ and v be the solution of (3).
Let be $v_i \in W_{loc}^{1,1}(0, \infty; L^\infty(\Omega))$ with $\Delta v_i \in L_{loc}^1(0, \infty; L^\infty(\Omega))$ and
such that

$$\frac{dv_i}{dt}(t, \cdot) + \beta(-\Delta v_i(t, \cdot)) = f_i(t, \cdot) \text{ a.e. } t \in (0, \infty), \text{ in } L^\infty$$

$$v_i(t, x) = g_i(t, x) \text{ a.e. } (t, x) \in (0, \infty) \times \partial\Omega$$

$$v_i(0, x) = v_{0,i}(x) \text{ a.e. } x \in \Omega.$$

$i = 1, 2$. Let us also assume that $f_1 \leq 0 \leq f_2$, $g_1 \leq 0 \leq g_2$
and $v_{0,1} \leq v_0 \leq v_{0,2}$. Then $v_1(t, x) \leq v(t, x) \leq v_2(t, x)$ for
all $t \in [0, \infty)$ and a.e. $x \in \Omega$.

Proof. Due to the continuity in $L^\infty(\Omega)$ of the semigroup generated by E , it will suffice to establish the result for $v_0 \in D(E)$. The Theorem 2 of [4] then ensures that the solution in the sense of semigroups is such that $v(t, \cdot) \in D(E)$ for a.e. t and also $v(t, \cdot)$ is derivable in the $\sigma(L^\infty(\Omega), L^1(\Omega))$ topology. We will show that $v \leq v_2$. In a similar way one can

prove the other inequality. Let us note by convenience

$v(t, \cdot) = v(t)$, $v_2(t, \cdot) = v_2(t)$ and $f_2(t, \cdot) = f_2(t)$. One has that

$$\frac{dv}{dt}(t) - \frac{dv_2}{dt}(t) + \beta(-\Delta v(t)) - \beta(-\Delta v_2(t)) = -f_2(t).$$

We will now use the semi-inner product defined by

$$\tau(a, b) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (\|a + \varepsilon b\| - \|b\|) \quad \text{if } a, b \in L^\infty(\Omega).$$

Due to a result of Sato ([17], p. 433) one knows that

$$\tau(a, b) = \lim_{\varepsilon \downarrow 0} \text{ess sup} \{(\text{sign } a(x)) \cdot b(x) : x \in \Omega(a, \varepsilon)\} \\ \text{if } a \neq 0$$

being $\Omega(a, \varepsilon) = \{x \in \Omega : |a(x)| > \|a\|_\infty - \varepsilon\}$. Let us assume that

$$w(t') \equiv (v(t') - v_2(t'))^+ \neq 0 \quad \text{for some } t' > 0.$$

One has that a.e. $t \in (0, \infty)$

$$\tau(w(t), \beta(-\Delta v(t)) - \beta(-\Delta v_2(t))) \geq 0$$

because otherwise for some t and $\varepsilon > 0$ one would have $\beta(-\Delta v(t)) - \beta(-\Delta v_2(t)) < 0$ on $\Omega(w(t), \varepsilon)$. The monotonicity of β would lead to $\Delta(v(t) - v_2(t)) \geq 0$. As $(v(t) - v_2(t))^+ = \|w(t)\|_\infty - \varepsilon$ in the boundary of $\Omega(w(t), \varepsilon)$, the application of the maximum principle would lead to a contradiction. Hence

$$\tau(w(t), -f(t) - \frac{dv}{dt}(t) + \frac{dv_2}{dt}(t)) \geq 0.$$

Then, for all ε small enough

$$\frac{dv}{dt}(t) \leq \frac{dv_2}{dt}(t) \quad \text{a.e. in } \Omega(w(t), \varepsilon)$$

and, in particular, $v(t') - v_0 \leq v_2(t') - v_{0,2}$ a.e. in $\Omega(w(t'), \varepsilon)$ from what one deduces $\|w(t')\|_\infty < \varepsilon$. Letting $\varepsilon \downarrow 0$ one obtains the desired contradiction. #

The above lemma, with the mentioned duality, allows to state the following criterion of comparison for (1), in contrast with that given by (4)

Corollary 1. Let be $u_{0,i} \in H^{-1}(\Omega) \cap L^1(\Omega)$ such that $(-\Delta)^{-1} u_{0,i} \in \overline{D(\mathbb{E})}^L$. Let u_i be the solutions of the corresponding problems (1). Then

$$(-\Delta)^{-1} u_{0,1} \leq (-\Delta)^{-1} u_{0,2} \Rightarrow (-\Delta)^{-1} u_1(t,x) \leq (-\Delta)^{-1} u_2(t,x)$$

for all $t \in [0, \infty)$, a.e. $x \in \Omega$. #

2. The dual problem.

In this Section we will study problem (3). Such a equation appears in problems of elasticity with damping (see in Duvaut-Lions [11], the "problème semistatique", p. 175). The study of some problems of the Bellman-Dirichlet type leads also to a similar formulation (see the "parabolic-elliptic case" in Lions [14]).

With respect to the extinction one has:

Proposition 1. Let $v_0 \in H_0^1(\Omega)$ such that $\Delta v_0 \in L^\infty(\Omega)$. Let $v \in C([0, \infty): L^\infty(\Omega))$ the solution of (3). Then the neces-

nary and sufficient condition for the existence of a $T_0 > 0$ such that $v(t,x) = 0$, a.e. $x \in \Omega$, for every $t \geq T_0$ is the fulfillment of hypothesis (2).

Proof. NECESSITY. Without loss of generality we can assume

$$\int_0^1 \frac{ds}{\beta(s)} = +\infty$$

and $v_0 \geq 0$, $v_0 \neq 0$. Due to lemma 1 it will enough to construct a subsolution \underline{v} , $\underline{v}(t,x) \neq 0$ for every $t \geq 0$. We will try $\underline{v}(t,x) = \phi(t) \psi(x)$. So, let $R > 0$, $k > 0$ and $x_0 \in \Omega$ be such that $v_0(x) > k$ for $x \in \Omega \cap B(x_0, 2R)$. Let $\psi \in C^2(\bar{\Omega})$ satisfying

$$\begin{aligned} \psi(x) &= 0 \quad \text{for } |x - x_0| > 2R \\ \frac{k}{2} &\leq \psi(x) \leq k \quad \text{and } -\Delta\psi(x) \geq 0 \quad \text{for } |x - x_0| \leq R, x \in \Omega \\ 0 &\leq \psi(x) < \frac{k}{2} \quad \text{and } -\Delta\psi(x) \leq 0 \quad \text{for } R < |x - x_0| \leq 2R, x \in \Omega \end{aligned}$$

On the other hand, let $q: (0,1] \rightarrow [0,\infty)$ be the strictly decreasing function

$$q(z) = \frac{k}{2} \int_z^1 \frac{ds}{\beta(\Theta s)},$$

where Θ is a positive constant to be determined. Then the function $\phi(t) = q^{-1}(t)$ satisfy: $\phi(t) > 0 \quad \forall t \geq 0$, $\phi(0) = 1$ and $\phi'(t) = -\beta(\Theta\phi(t)) \frac{2}{k}$ if $t > 0$.

Finally \underline{v} is a subsolution of (3) if Θ is large enough, because

$$a) \quad \underline{v}_t(t, x) + \beta(-\Delta \underline{v}(t, x)) = \phi'(t)\psi(x) + \beta(-\phi(t)\Delta\psi(x)) = f(t, x)$$

$$\text{being } f(t, x) \leq -\beta(\theta\phi(t)) \cdot \frac{2}{k} \frac{k}{2} + \beta(-\phi(t)\Delta\psi(x)) \leq 0$$

$$\text{in } |x-x_0| \leq R, \quad x \in \Omega, \quad \text{and } f(t, x) \leq \beta(-\phi(t)\Delta\psi(x)) \leq 0$$

$$\text{in } R < |x-x_0|, \quad x \in \Omega.$$

$$b) \quad \underline{v}(t, x) = 0 \quad \text{a.e. } (t, x) \in (0, \infty) \times \partial\Omega$$

$$c) \quad \underline{v}(0, x) = \phi(0)\psi(x) \leq k \leq v_0(x) \quad \text{a.e. } x \in \Omega.$$

SUFFICIENCY. 1st Step. Let $v_0 \in D(E)$. Again without loss of generality, by Lemma 1, one can $v_0 \geq 0$ and so it will suffice to construct a supersolution \bar{v} with the property of extinction. We will choose \bar{v} again in the form $\bar{v}(t, x) = \phi(t)\chi(x)$. Let $r > 0$ be such that $\bar{\Omega} \subset B(0, r)$ and let us define $\chi(x) = r^2 - |x|^2$. It is clear that $-\Delta\chi(x) = 2N$. Let us note by $M = r^2 = \max \chi$ and $m = \min \chi$ on $\bar{\Omega}$. On the other hand, let $q: [0, \infty) \rightarrow [0, \infty)$ be the strictly increasing function given by

$$q(z) = M \int_0^z \frac{ds}{\beta(s)} \quad \text{for all } z \geq 0.$$

In fact, we can assume without loss of generality that q is onto, because reasoning as in [4, Theoreme 2] one can modify as willing β out of the interval $(-\|Ev_0\|_\infty, \|Ev_0\|_\infty)$. It is clear that the inverse function q^{-1} satisfies that $(q^{-1}(z))' = \frac{\beta(q^{-1}(z))}{M}$ if $z \geq 0$. Now let us define

$$\phi(t) = \begin{cases} q^{-1}((T_0-t)2N) \cdot 1/2N & \text{if } 0 \leq t \leq T_0 \\ 0 & \text{if } t > T_0 \end{cases}$$

where T_0 is a positive constant to be determined. The function $\bar{v} \in W^{2,\infty}((0,\infty) \times \Omega)$ satisfies

$$\begin{aligned} \text{i) } \bar{v}_t + \beta(-\Delta \bar{v}) &\geq \phi'(t)M + \beta(-\phi(t)\Delta \chi(x)) \geq \\ &\geq -[(q^{-1})'((T_0-t)2N)]M + \beta(q^{-1}((T_0-t) \cdot 2N)) = 0 \\ &\text{in } (0,\infty) \times \Omega \end{aligned}$$

$$\text{ii) } \bar{v}(t,x) \geq 0 \quad \text{a.e. } (t,x) \in (0,\infty) \times \partial\Omega$$

$$\text{iii) } \bar{v}(0,x) = \phi(0) \cdot \chi(x) \geq (1/2N)q^{-1}(2N \cdot T_0) \cdot m \quad \text{a.e. } x \in \Omega.$$

Hence, in order that \bar{v} be a supersolution of (3) it will be enough to have $\bar{v}(0,x) \geq v_0(x)$, which is true taking

$$(5) \quad T_0 \geq (1/2N)q\left(\frac{2N}{m} \|v_0\|_\infty\right).$$

2nd Step. Let us assume $v_0 \in H_0^1(\Omega)$ such that $\Delta v_0 \in L^\infty(\Omega)$ and β eventually multivalued. We will use now an argument of approximation of β and a further pass to the limit. Let β_ε be a family of increasing, lipschitzian, real functions with $\beta_\varepsilon(0)=0$ and such that $\beta_\varepsilon(r) \rightarrow \beta(r)$ when $\varepsilon \downarrow 0$ uniformly on compacts. Due to the lipschitz condition on β one has that $v_0 \in D(E_\varepsilon)$ where E_ε is the operator defined in Section 1 taking β_ε instead of β (note that $\Delta v_0 \in L^\infty(\Omega) \Rightarrow v_0 \in L^\infty(\Omega) \cap H^2(\Omega)$). The previous step says that the solution v_ε of the corresponding problem (3) vanishes after a $T_0^\varepsilon \geq (1/2N)q_\varepsilon\left(\frac{2N}{m} \cdot \|v_0\|_\infty\right)$.

The convergence of v_ϵ to v , solution of (3), is assured by some technical results (Proposition I.24 of [2] and Theorem I.17 of [12]). Finally, it is clear that $q_\epsilon \uparrow q$ and so $T_0^\epsilon \leq T_0$, T_0 given by (5). #

Remark 1. As a matter of fact, the following property has also been obtained in the proof

Corollary 2. Let β does not satisfy hypothesis (2). Let v and v_0 are as in Proposition 1 and Ω' be an open set of Ω of positive measure such that $v_0(x) > 0$ (resp. < 0) for a.e. $x \in \Omega'$. Then $v(t,x) > 0$ (resp. < 0) for a.e. $x \in \Omega'$ and for all $t > 0$. #

3. The main result.

The result announced in the Introduction is the following

Theorem. Let $u_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $(-\Delta)^{-1} u_0 \in L^\infty(\Omega)$. Let u be the solution of (1). Then, the hypothesis (2) is the necessary and sufficient condition for the existence of a $T_0 > 0$ such that $u(t,x) = 0$, a.e. $x \in \Omega$, for all $t \geq T_0$.

Before giving the proof, we state, as an example, the following

Corollary 3. The conclusion of the above Theorem remains valid for $u_0 \in L^p(\Omega)$ with p such that

$$\max \{2N/N+2, N/2\} < p \leq +\infty.$$

Proof of the Corollary 3. By the imbedding theorem, one has that if $p \geq \frac{N2}{N+2}$ then $L^p(\Omega) \subset H^{-1}(\Omega)$ and by L^p -estimates $(-\Delta)^{-1} u_0 \in L^\infty(\Omega)$ if $u_0 \in L^p(\Omega)$ for $p > \frac{N}{2}$. #

Proof of the Theorem. 1st Step. Let $u_0 \in L^\infty(\Omega)$. Due to the fact that the operator A is the subdifferential of a convex functional, l.s.c on $H^{-1}(\Omega)$, one has a smoothing effect for (1) in the sense in which for all $t > 0$, $u(t, \cdot) \in D(A)$ (see Brezis [6]). Also, from the maximum theorems it is known that $u(t, \cdot) \in L^\infty(\Omega)$ for all $t > 0$. We use these facts to get the sufficiency. Fixed $\delta > 0$, the function $u_1(x) = u(\delta, x)$ satisfies $u_1 \in L^\infty(\Omega)$ and $\beta(u_1) \in H_0^1(\Omega)$. Let $v(t, x)$ be the solution of (3) corresponding to the datum $v_0 = (-\Delta)^{-1} u_1$. It is clear that $v_0 \in H_0^1(\Omega)$, $\Delta v_0 \in L^\infty(\Omega)$ and also $v_0 \in D(C)$ so one has $v(t, \cdot) \in D(C)$ for all $t > 0$. Hence the function $\tilde{u}(t, \cdot) = -\Delta v(t, \cdot)$ verifies (1) for the initial datum u_1 and vanishes (as v does). Finally, by uniqueness, $\tilde{u}(t, \cdot) = u(t+\delta, \cdot)$ for all $t > 0$, being u the solution of (1) for u_0 .

With respect to the necessity, we observe that if $u_0 \neq 0$, then $u_1(x) = u(\delta, x)$ is different from zero, for δ small enough. Being $-\Delta$ an isomorphism, the function $v_0 = (-\Delta)^{-1} u_1$ is not zero and by Corollary 2 $v(t, \cdot) \neq 0$ for all $t \geq 0$. Finally $\tilde{u}(t, \cdot) = -\Delta v(t, \cdot) (= u(t+\delta, \cdot))$ is different from zero.

2nd Step. Let $u_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $(-\Delta)^{-1} u_0 \in L^\infty(\Omega)$. We can assume $u_0 \geq 0$, for convenience. Proceeding by truncation, we define $u_{0,n}$, $n \in \mathbb{N}$ by

$$u_{0,n}(x) = \begin{cases} n & \text{if } u_0(x) \geq n \\ u_0(x) & \text{if } 0 \leq u_0(x) \leq n. \end{cases}$$

One has $u_{0,n} \in L^\infty(\Omega)$ and then, since $0 \leq u_{0,n} \leq u_0$, the necessity follows from the previous step.

On the other hand, it is clear that $u_{0,n} \rightarrow u_0$ in $L^1(\Omega)$ and that $\|(-\Delta)^{-1} u_{0,n}\|_\infty \leq \|(-\Delta)^{-1} u_0\|_\infty$. Then, by standard arguments, $u_n(t, \cdot) \rightarrow u(t, \cdot)$ in $L^1(\Omega)$ for all $t > 0$. From (5) we observe that the $T_{0,n}$ depend increasingly on $\|(-\Delta)^{-1} u_{0,n}\|_\infty$, and the conclusion holds. #

The above result had been obtained for $\beta \in C^2(\mathbb{R})$, $\beta' > 0$, $\beta'' < 0$ and $u_0 \in W_0^{1,\infty}$ by Sabinina [15] for the case $N = 1$ and in [16] for $N > 1$. In this situation, ($N = 1$) the behavior of u near the extinction time has been considered by Berryman and Holland [5]. On the other hand, if $\beta(r) = |r|^m \text{sign } r$ and $u_0 \in L^\infty(\Omega)$, the techniques of [3] can be applied to obtain the extinction if $0 < m < 1$. (See also Evans [10]).

If $\Omega = \mathbb{R}^N$, $N \geq 3$, the situation is different, because Benilan and Crandall [3] the property of extinction is obtained for solutions in $L^1(\mathbb{R}^N)$ and $\beta(r) = |r|^m \text{sign } r$ when $0 < m < (N-2)/2$. In this case, a result by Benilan and Aronson shows the non extinction for $\frac{N-2}{N} < m \leq 1$.

Remark 2. When studying the phenomenon of extinction for the non-homogeneous equation and β , in general, a maximal graph of \mathbb{R}^2 , i.e. the problem

$$\begin{aligned} u_t(t,x) - \Delta\beta(u(t,x)) &\ni f(t,x) \quad \text{a.e. } (t,x) \in (0,\infty) \times \Omega \\ (6) \quad \beta(u(t,x)) &\ni 0 \quad \text{a.e. } (t,x) \in (0,\infty) \times \partial\Omega \\ u(0,x) &= u_0(x) \quad \text{a.e. } x \in \Omega \end{aligned}$$

one serves that a necessary condition is

$$(7) \quad -\Delta\beta(0) \ni f(t,\cdot) \quad \text{for } t \text{ large enough.}$$

Assuming $\beta(0) = 0$, this implies that there exist $t_0 \geq 0$ such that $f(t,\cdot) = 0$ for all $t \geq t_0$ and then the Theorem is still valid for (6), because one can argue for $t \geq t_0$.

If β is multivalued in 0, i.e. $\beta(0) = [\beta^-, \beta^+]$, condition (7) is "almost sufficient", and we have the

Proposition 2. Let $u_0 \in H^{-1}(\Omega) \cap L^1(\Omega)$, $(-\Delta)^{-1} u_0 \in L^\infty(\Omega)$, $f \in L^1_{loc}((0,\infty); L^\infty(\Omega))$ and u be the solution of (6). Assuming there exists $t_0 \geq 0$ and $\varepsilon > 0$ such that

$$\begin{aligned} \beta^- + \varepsilon \leq (-\Delta)^{-1} f(t,x) \leq \beta^+ - \varepsilon, \\ \text{a.e. } (t,x) \in (t_0,\infty) \times \Omega \end{aligned}$$

then there exists a finite time of extinction for u .

Proof. Proceeding as in the Theorem, one can assume $u_0 \in L^\infty(\Omega)$ and then the function $v(t, \cdot) = (-\Delta)^{-1} u(t, \cdot)$ satisfies

$$v_t(t, \cdot) + \beta(-\Delta v(t, \cdot)) \ni (-\Delta)^{-1} f(t, \cdot) \quad \text{a.e. } t \in (0, \infty), \text{ in } L^\infty(\Omega)$$

$$v(t, x) = 0 \quad \text{a.e. } (t, x) \in (0, \infty) \times \partial\Omega$$

$$v(0, x) = (-\Delta)^{-1} u_0(x) \quad \text{a.e. } x \in \Omega$$

and hence it suffices to apply Corollary 1 of I. Diaz [9]. #

Remark 3. It is interesting to observe how equation (1) exhibits an "almost" curious alternative, depending on β : either the solution has a finite time of extinction (hypothesis (2)), or there exists a finite speed of propagation of signals (which can be seen in I. Diaz [8] under the hypothesis (2) for β^{-1}), except obviously if β and β^{-1} do not satisfy (2), which includes the corresponding problem of the linear heat equation.

ACKNOWLEDGEMENTS

The authors wish to express their thanks to Professor Haim Brezis for the stimulating conversations during his visit to the Universidad Complutense de Madrid.

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Received May 1979