Existence Theory and Qualitative Properties of the Solutions of Some First Order Quasilinear Variational Inequalities

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Introduction

This paper deals with the existence, the uniqueness and the qualitative properties of the solutions of the Cauchy problem

$$(\text{C.P.})_f \qquad \begin{aligned} &\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f \qquad \text{in } (0,+\infty) \times \mathbb{R}^N, \\ &u(0,.) = u_0(.) \qquad \text{in } \mathbb{R}^N, \end{aligned}$$

as well as those of the stationary equation

$$(\text{S.P.})_f \qquad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f \qquad \text{in } \mathbb{R}^N,$$

where $\Phi = (\Phi_1,\ldots,\Phi_N)$ is a continuous function from $\mathbb{R}$ into $\mathbb{R}^N$ and β a maximal monotone graph of $\mathbb{R}^2$ [6].

When β is a smooth function, the equation $(\text{C.P.})_0$ is known as a balance law equation (a conservation law if $\beta = 0$). It has been studied by numerous authors, namely: Burger, Hopf, Lax, Oleinik, Vol'pert, Kruzkov and it is well known that even if all the data are smooth, every solution of $(\text{C.P.})_0$ becomes in general discontinuous after a finite time [22]. Thus the equation must be taken in a weak sense. However, there may exist an infinite number of weak solutions of $(\text{C.P.})_0$ and an additional principle called the entropy condition is needed to select the unique physical weak solution. The form of the entropy condition which will be used here was first given by Vol'pert [29] and then improved by Kruzkov in his penetrating work [21].

As for the hypothesis on β, our assumption of monotonicity allows us to eliminate the assumption of Lipschitz continuity made in previous works, [2], [11] and [23]. We can even consider the case when β is a multivalued function, for example the evolution variational inequality of the type

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$$
(0.1)\qquad
\begin{aligned}
&\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\Phi_i(u)) - f \geq 0 && \text{in } (0,+\infty) \times \mathbb{R}^N,\\
&u \geq 0 && \text{in } (0,+\infty) \times \mathbb{R}^N,\\
&\left(\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\Phi_i(u)) - f\right) u = 0 && \text{in } (0,+\infty) \times \mathbb{R}^N,\\
&u(0,.) = u_0(.) && \text{in } \mathbb{R}^N.
\end{aligned}
$$

When Φ is linear such inequalities appear in Control Theory and they have been already intensively studied [4].

For the existence of the solutions of $(\mathrm{C.P.})_f$ we use the theory of nonlinear semigroups of contractions in Banach spaces. In fact, when $\beta = 0$, it has been noticed by Quinn [24] that the solutions of $(\mathrm{C.P.})_0$ define a semigroup of nonlinear contractions in $L^1(\mathbb{R}^N)$. A more complete analysis of that fact was made by Crandall [11] who constructed the generator of the semigroup. Crandall's results were improved by Benilan [1], [2] on whose ideas a part of this paper is based. We use some results of the theory of accretive operators via the generation theorem of Crandall and Liggett [12]. So we are led to the following stationary equation (with $\lambda > 0$)

$$
(0.2)\qquad u + \lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\Phi_i(u)) + \lambda\beta(u) \ni f \qquad \text{in } \mathbb{R}^N
$$

which is a special case of $(\mathrm{S.P.})_f$.

The qualitative properties of the solutions of $(\mathrm{C.P.})_f$ are various. For example, when $u_0(.)$ has a compact support in $\mathbb{R}^N$ we prove that the support of the solution $u(t,.)$ of $(\mathrm{C.P.})_0$ is also compact for $t > 0$ and we obtain an upper bound for its speed of propagation under one of the following hypotheses on Φ:

i) Φ is Lipschitz continuous,

ii) $\forall\, 1 \leq i \leq N$, $\int_{-1}^{1} d\sigma/|\Phi_i^{-1}(\sigma)| < +\infty$.

When the initial data u_0 is asymptotically vanishing in $\mathbb{R}^N$, Φ is Lipschitz continuous and $\int_{-1}^{1} d\sigma/|\beta^0(\sigma)| < +\infty$ we prove that the support of $u(t,.)$ is compact for any $t > 0$. We also obtain a time regularizing effect for the solution of $(\mathrm{C.P.})_0$, that is $u(t,.) \in L^\infty(\mathbb{R}^N)$ for any $t > 0$ even if u_0 is just an integrable function.

Some of those phenomena are deduced from previous qualitative properties of the solutions of (0.2). Moreover an important characteristic of all the qualitative properties given here is its unidirectional nature which is obtained thanks to a unidirectional principle of comparison whose idea is to majorize any bounded solution u of $(\mathrm{S.P.})_f$ by the solution of

$$
(0.3)\qquad
\begin{aligned}
&\frac{d}{dx_k} (\Phi_k(v)) + \beta(v) \ni \|f^+(x_k,.)\|_{L^\infty(\mathbb{R}^{N-1})} && \text{on } (\alpha,+\infty),\\
&v(\alpha) = \|u^+\|_{L^\infty(\mathbb{R}^N)},
\end{aligned}
$$

for any α and a suitable k.

The contents of the article is the following:

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1. Preliminaries and Statement of the Results Solving $(C.P.)_f$ and (0.2) in Kruzkov Sense

Throughout this article we always assume this fundamental hypothesis

(H.0) $\Phi = (\Phi_1, \ldots, \Phi_N)$ is a continuous function from $\mathbb{R}$ into $\mathbb{R}^N$ and β is a maximal monotone graph of $\mathbb{R}^2$ with $0 \in \beta(0)$.

We first define what is meant by a solution in Kruzkov sense.

Definition 1.1. Suppose Ω is some open subset of $\mathbb{R}^q$, $\Phi = (\Phi_1, \ldots, \Phi_q)$ a continuous function from $\mathbb{R}$ into $\mathbb{R}^q$ and β a multivalued function defined on $\mathbb{R}$ with values in $\mathcal{P}(\mathbb{R})$. A function $u \in L^\infty_{\text{loc}}(\Omega)$ is *a solution in Kruzkov sense in* Ω *of*

$$\sum_{i=1}^{q} \frac{\partial}{\partial x_i} (\Phi_i(u)) + \beta(u) \ni f, \tag{1.1}$$

where $f \in L^1_{\text{loc}}(\Omega)$, if there exists $h \in L^1_{\text{loc}}(\Omega), h(x) \in \beta(u(x))$ a.e. in Ω such that

$$\forall \zeta \in \mathcal{D}^+(\Omega) \quad \text{and for almost all } k \in \mathbb{R},$$

$$\int \text{sign}_0(u-k)\left\{\sum_{i=1}^{q} (\Phi_i(u) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i} + (f-h)\zeta\right\} dx \geq 0, \tag{1.2}$$

where $\mathcal{D}^+(\Omega)$ is the set of all nonnegative smooth functions with compact support in Ω and $\text{sign}_0(r) = -1$ if $r < 0$, 0 if $r = 0$, $+1$ if $r > 0$.

The definition means that u satisfies the equation

$$\sum_{i=1}^{q} \frac{\partial}{\partial x_i}(\Phi_i(u)) = f - h \tag{1.3}$$

in Kruzkov sense in Ω. Moreover (1.2) is valid for any $k \in \mathbb{R}$. Another useful definition is the following (see [1])

$$\forall \zeta \in \mathcal{D}^+(\Omega), \quad \forall k \in \mathbb{R},$$

$$\int_{u>k} \left\{ \sum_{i=1}^{q} (\Phi_i(u) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i} - (f-h)\zeta \right\} dx \geq 0 \tag{1.4}$$

$$\geq \int_{u<k} \left\{ \sum_{i=1}^{q} (\Phi_i(u) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i} - (f-h)\zeta \right\} dx.$$

It is easy to see that any solution of (1.3) in Kruzkov sense satisfies the equation in the sense of distributions in Ω, as well as it satisfies the entropy condition (E) of Oleinik [23].

In order to solve $(\text{C.P.})_f$ we need an additional assumption

$$\forall\, 1 \leq i \leq N, \quad \limsup_{r \to 0} \frac{|\Phi_i(r)|}{|r|^{(N-1)/N}} < +\infty. \tag{H.1}$$

Our main existence result which is proved in Section 2 with the results of Section 3 is the following

Theorem 1.1. *Suppose* (H.0) *and* (H.1) *hold. Then for any* $T > 0$,

$$u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

such that $u_0(x) \in \overline{D(\beta)}$ a.e. *in* $\mathbb{R}^N$ *and*

$$f \in L^1(0,T;L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)),$$

there exists a unique

$$u \in L^\infty(0,T;L^1(\mathbb{R}^N)) \cap L^\infty((0,T) \times \mathbb{R}^N)$$

such that $u(t,.) \to u_0(.)$ *in* $L^1_{\text{loc}}(\mathbb{R}^N)$ *when* $t \to 0$ *essentially, solution of* $(\text{C.P.})_f$ *in Kruzkov sense in* $(0,T) \times \mathbb{R}^N$. *Moreover* $u \in C([0,T];L^1(\mathbb{R}^N))$ *and the following estimate holds*

$$\|(u(t,.))^{(\pm)}\|_{L^p} \leq \|(u_0)^{(\pm)}\|_{L^p} + \int_0^t \|(f(s,.))^{(\pm)}\|_{L^p}\, ds, \tag{1.5}$$

(Under that formulation we mean that the estimate is true if we simultaneously take on each side of the inequality the couples (u,f), (u^+,f^+) or (u^-,f^-).) ($h^+ = \max(h,0)$, $h^- = \max(-h,0)$) *for all* $1 \leq p \leq +\infty$ *and* $t \in [0,T]$. *If* $\hat{u}$ *is the solution of* $(\text{C.P.})_{\hat{f}}$ *with the initial data* $\hat{u}_0$, *we have*

(1.6) $\|(u(t,.) - \hat{u}(t,.))^{(\pm)}\|_{L^1} \leq \|(u_0 - \hat{u}_0)^{(\pm)}\|_{L^1} + \int_0^t \|(f(s,.) - \hat{f}(s,.))^{(\pm)}\|_{L^1}\, ds.$

When $N = 1$ (H.1) reduces to Φ continuous. The estimate (1.6) gives us a first comparison principle between two solutions of (C.P.): $u_0 \leq \hat{u}_0$, $f \leq \hat{f}$ implies $u \leq \hat{u}$. We also deduce from (1.5) that for any $\tau \in \mathbb{R}^N$

(1.7) $\|u(t,. + \tau) - u(t,.)\|_{L^1}$

$$\leq \|u_0(. + \tau) - u_0(.)\|_{L^1} + \int_0^t \|f(s,. + \tau) - f(s,.)\|_{L^1}\, ds.$$

Hence if u_0 belongs to the space $\mathrm{BV}(\mathbb{R}^N)$ of functions with a bounded Tonelli-Cesàro variation and if f belongs to $L^1(0,T;\mathrm{BV}(\mathbb{R}^N))$, then u belongs to $C([0,T];\mathrm{BV}(\mathbb{R}^N))$.

The proof of Theorem 1.1. will be done in two steps. In the first one we solve $(\mathrm{C.P.})_f$ in the sense of accretive operator theory and get a *semigroup solution* (which satisfies (1.5), (1.6) and belongs to $C([0,T];L^1(\mathbb{R}^N))$). In the second one we prove that the semigroup solution of $(\mathrm{C.P.})_f$ satisfies the equation in Kruzkov sense in $(0,T) \times \mathbb{R}^N$. The first step is based on the solvability of the equation (0.2) in some appropriate class. The proof of the following theorem is given in Section 3.

Theorem 1.2. *Suppose* (H.0) *and* (H.1) *hold. Then for any* $\lambda > 0$ *and*

$$f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

there exists a unique $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *solution of* (0.2) *in Kruzkov sense in* $\mathbb{R}^N$. *More precisely there exists*

$$h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad h(x) \in \beta(u(x)) \quad \text{a.e. in } \mathbb{R}^N$$

such that (1.3) *with right-hand side* $(1/\lambda)(f - \lambda h - u)$ *is satisfied in Kruzkov sense in* $\mathbb{R}^N$. *Moreover, for every* $1 \leq p \leq +\infty$, *we have the estimate*

$$\|(u + \lambda h)^{(\pm)}\|_{L^p} \leq \|f^{(\pm)}\|_{L^p}. \tag{1.8}$$

If $\hat{u}$ *is the solution of* (0.2) *with right-hand side* $\hat{f}$, *we have*

$$\|(u + \lambda h - \hat{u} - \lambda\hat{h})^{(\pm)}\|_{L^1} \leq \|(f - \hat{f})^{(\pm)}\|_{L^1}, \tag{1.9}$$

where $\hat{h}$ *is the section of* $\{\beta(\hat{u})\}$ *corresponding to the equation satisfied by* $\hat{u}$.

Using the monotonicity of β we deduce from (1.8) and (1.9)

$$\|u^{(\pm)}\|_{L^p} \leq \|f^{(\pm)}\|_{L^p}, \tag{1.10}$$

$$\lambda\|h^{(\pm)}\|_{L^p} \leq \|f^{(\pm)}\|_{L^p}, \tag{1.11}$$

$$\|(u - \hat{u})^{(\pm)}\|_{L^1} + \lambda\|(h - \hat{h})^{(\pm)}\|_{L^1} \leq \|(f - \hat{f})^{(\pm)}\|_{L^1}. \tag{1.12}$$

We can notice that (1.12) gives us a simple comparison principle between two solutions of (0.2): $f \le \hat{f}$ implies $u \le \hat{u}$. More sophisticated principles will be given in Section 4. Theorem 1.2. also provides us a means to solve noncoercive stationary problems $(S.P.)_f$ (see Section 3).

The following result which is a generalization of Proposition 2 of [2] will be useful in the sequel.

Lemma 1.1. *Let $\Phi = (\Phi_1,\ldots,\Phi_q)$ be a continuous function from $\mathbb{R}$ into $\mathbb{R}^q$, β a maximal monotone graph of $\mathbb{R}^2$ and $f, \hat{f} \in L^1_{\text{loc}}(\Omega)$ (Ω being some open subset of $\mathbb{R}^q$). If u and $\hat{u}$ are solutions in Kruzkov sense in Ω of*

$$\sum_{i=1}^{q} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f \quad and \quad \sum_{i=1}^{q} \frac{\partial}{\partial x_i}(\Phi_i(\hat{u})) + \beta(\hat{u}) \ni \hat{f},$$

then for any $\zeta \in \mathcal{D}^+(\Omega)$ there exists $\alpha \in L^\infty(\Omega)$ with $\alpha(x) \in \operatorname{sign}^+(u(x) - \hat{u}(x))$ a.e. in Ω such that

$$\int_\Omega \alpha(.)\left\{\sum_{i=1}^{q} (\Phi_i(u) - \Phi_i(\hat{u}))\frac{\partial \zeta}{\partial x_i} + (f - \hat{f})\zeta\right\}dx \ge 0, \tag{1.13}$$

where $\operatorname{sign}^+(r) = 0$ *if* $r < 0$, $\operatorname{sign}^+(0) = [0,1]$ *and* $\operatorname{sign}^+(r) = 1$ *if* $r > 0$.

Proof. Since we follow Proposition 2 of [2] we will just sketch the proof. Take $\bar{\zeta} \in \mathcal{D}^+(\Omega)$, $\rho \in \mathcal{D}^+(\mathbb{R}^q)$ with support in $\{x : |x| \le 1\}$, $\int \rho\, dx = 1$ and we let $\rho_n(x) = n^q \rho(nx)$. For

$$n > (\operatorname{dist}(\partial\Omega, \operatorname{supp} \bar{\zeta}))^{-1}$$

we let $g_n(x,y) = \bar{\zeta}((x + y)/2)\,\rho_n((x - y)/2)$: we have $g_n \in \mathcal{D}^+(\Omega \times \Omega)$.

If we let $\operatorname{sign}_0^+(r) = \operatorname{sign}_0(r)\operatorname{sign}^+(r)$ and take $\zeta = g_n$, we deduce from (1.4)

$$\iint \operatorname{sign}_0^+(u(x) - \hat{u}(y))\left\{\sum_i (\Phi_i(u(x)) - \Phi_i(\hat{u}(y)))\frac{\partial \zeta}{\partial x_i} + (f(x) - h(x))\zeta\right\}dxdy \ge 0,$$

$$-\iint \operatorname{sign}_0^+(u(x) - \hat{u}(y))\left\{\sum_i (\Phi_i(\hat{u}(y)) - \Phi_i(u(x)))\frac{\partial \zeta}{\partial y_i} + (\hat{f}(y) - \hat{h}(y))\zeta\right\}dydx \ge 0,$$

where h, $\hat{h}$ belong to $L^1_{\text{loc}}(\Omega)$, $h(x) \in \beta(u(x))$ and $\hat{h}(y) \in \beta(\hat{u}(y))$ a.e. in Ω. If we sum and use the monotonicity of β we get

$$\iint \operatorname{sign}_0^+(u(x) - \hat{u}(y))\left\{\sum_i (\Phi_i(u(x)) - \Phi_i(\hat{u}(y)))\left(\frac{\partial \zeta}{\partial x_i} + \frac{\partial \zeta}{\partial y_i}\right) + (f(x) - \hat{f}(y))\zeta\right\}dxdy \ge 0.$$

We make the following change of variables: $x = \xi + \eta/n$, $y = \xi - \eta/n$ and we get

$$\int\left\{\int \operatorname{sign}_0^+\left(u\left(\xi+\frac{\eta}{n}\right)-\hat{u}\left(\xi-\frac{\eta}{n}\right)\right)\left\{\sum_i\left(\Phi_i\left(u\left(\xi+\frac{\eta}{n}\right)\right)\right.\right.\right.$$
$$\left.\left.\left.-\Phi_i\left(\hat{u}\left(\xi-\frac{\eta}{n}\right)\right)\right)\frac{\partial\bar{\zeta}}{\partial\xi_i}+\left(f\left(\xi+\frac{\eta}{n}\right)-f\left(\xi-\frac{\eta}{n}\right)\right)\bar{\zeta}\right\}d\xi\right\}\rho(\eta)\,d\eta \geq 0. \tag{1.14}$$

If we define (see [2]) the product τ^+ by

$$\tau^+(a,b) = \lim_{\lambda\downarrow 0}\lambda^{-1}(\|(a+\lambda b)^+\|_{L^1} - \|a^+\|_{L^1}) = \inf_{\lambda>0}\lambda^{-1}(\|(a+\lambda b)^+\|_{L^1} - \|a^+\|_{L^1}),$$

τ^+ is upper semicontinuous and we have

$$\tau^+(a,b) = \max\left\{\int \alpha\, b\, dx, \alpha \in L^\infty, \alpha(x) \in \operatorname{sign}^+ a(x) \text{ a.e.}\right\}. \tag{1.15}$$

We deduce

$$\int \tau^+\left(\left(u\left(.+\frac{\eta}{n}\right)-\hat{u}\left(.-\frac{\eta}{n}\right)\right)\bar{\zeta}(.),\sum_i\left(\Phi_i\left(u\left(.+\frac{\eta}{n}\right)\right)\right.\right.$$
$$\left.\left.-\Phi_i\left(\hat{u}\left(.-\frac{\eta}{n}\right)\right)\right)\frac{\partial\bar{\zeta}}{\partial\xi_i}+\left(f\left(.+\frac{\eta}{n}\right)-\hat{f}\left(.+\frac{\eta}{n}\right)\right)\bar{\zeta}\right)\rho(\eta)\,d\eta \geq 0. \tag{1.16}$$

Going to the limit as $n \to +\infty$ we deduce

$$\tau^+\left((u-\hat{u})\bar{\zeta},\sum_i(\Phi_i(u)-\Phi_i(\hat{u}))\frac{\partial\bar{\zeta}}{\partial\xi_i}+(f-\hat{f})\bar{\zeta}\right) \geq 0, \tag{1.17}$$

which implies (1.13).

2. Existence and Uniqueness for $(C.P.)_f$ via the Accretive Operator Theory

In this section we solve $(C.P.)_f$ by considering it as an abstract Cauchy problem of the following type

$$\frac{du}{dt} + A\,u \ni f \qquad \text{on } (0,T), \qquad u(0) = u_0, \tag{2.1}$$

where $u(t) = u(t,.)$ is a function from $[0,T]$ into the Banach space $L^1(\mathbb{R}^N)$. For that we define the operator A:

$$A(.) = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}(\Phi_i(.)) + \beta(.),$$

in the following way:

$$D(A) = \Big\{ u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) :$$

$$\exists h \in L^1_{\mathrm{loc}}(\mathbb{R}^N), h(x) \in \beta(u(x)) \text{ a.e. in } \mathbb{R}^N, \exists \Psi \in L^1(\mathbb{R}^N)$$

$$\text{and } \sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u)) = \Psi - h \text{ in Kruzkov sense in } \mathbb{R}^N \Big\},$$

and $A u = \Psi$ if $u \in D(A)$.

In order to construct a semigroup solution of (2.1) (see [1] and [12]) it suffices to show that the range (R) of $I + \lambda A$ satisfies

$$\overline{\mathrm{R}(I + \lambda A)}^{L^1} = L^1(\mathbb{R}^N) \qquad \text{for every } \lambda > 0, \tag{2.2}$$

and that A is T-accretive in $L^1(\mathbb{R}^N)$, that is

$$\begin{gathered} \forall \lambda > 0, \quad \forall v, \hat{v} \in D(A), \quad \forall w \in Av, \quad \forall \hat{w} \in Av, \\ \|(v + \lambda w) - (\hat{v} - \lambda \hat{w}))^+\|_{L^1} \geq \|(v - \hat{v})^+\|_{L^1}. \end{gathered} \tag{2.3}$$

The T-accretivity in L^1 implies the accretivity (which means that $(.)^+$ is replaced by (.) in (2.3)). For proving the T-accretivity of A we need the following result (see [2]).

Lemma 2.1. *Suppose* $\Phi = (\Phi_1, \ldots, \Phi_N)$ *is a continuous function from* $\mathbb{R}$ *into* $\mathbb{R}^N$ *satisfying* (H.1), $\zeta \in \mathfrak{D}(\mathbb{R}^N)$ *and let* $\zeta_n(x) = \zeta(x/n)$. *Then if* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, *we have*

i) $\left\| \sum_{i=1}^N \Phi_i(u) \frac{\partial \zeta_n}{\partial x_i} \right\|_{L^1} \leq C \|u\|_{L^1}^{(N-1)/N}$,

where C depends only on Φ, $\|u\|_{L^\infty}$, diam(supp ζ) *and* $\|\mathrm{grad}\, \zeta\|_{L^\infty}$.

ii) $\left\| \sum_{i=1}^N \Phi_i(u) \frac{\partial \zeta_n}{\partial x_i} \right\|_{L^1} \to 0$ *when* $n \to +\infty$.

Lemma 2.2. *Under the hypotheses* (H.0) *and* (H.1) *the operator A is T-accretive in* $L^1(\mathbb{R}^N)$.

Proof. Given u and $\hat{u}$ in $D(A)$, $\psi \in A u$ and $\hat{\psi} \in A \hat{u}$ we have

$$\sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni \psi \quad \text{and} \quad \sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(\hat{u})) + \beta(\hat{u}) \ni \hat{\psi}$$

in Kruzkov sense in $\mathbb{R}^N$. From Lemma 1.1, for any $\zeta \in \mathfrak{D}^+(\mathbb{R}^N)$ such that $\zeta(0) = 1$, there exists $\alpha \in L^\infty(R^N)$, $\alpha \in \mathrm{sign}^+(u - \hat{u})$ such that

$$\int_{\mathbb{R}^N} \alpha(.) \left\{ \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\hat{u})) \frac{\partial \zeta}{\partial x_i} + (\psi - \hat{\psi}) \zeta \right\} dx \geq 0.$$

For $n > 0$, we have, by replacing ζ with ζ_n as in Lemma 2.1,

$$\tau^+\left(u - \hat{u}, \sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(\hat{u}))\frac{\partial \zeta_n}{\partial x_i} + (\psi - \hat{\psi})\zeta_n\right) \geq 0.$$

Since τ^+ is upper semicontinuous in $L^1(\mathbb{R}^N)$ we deduce from Lemma 2.1 that

$$\text{(2.4)} \quad \tau^+(u - \hat{u}, \psi - \hat{\psi}) \geq \limsup_{n \to +\infty} \tau^+\left(u - \hat{u}, \sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(\hat{u}))\frac{\partial \zeta_n}{\partial x_i} + (\psi - \hat{\psi})\zeta_n\right).$$

Since the right-hand side is nonnegative we deduce that for any $\lambda > 0$

$$\text{(2.5)} \qquad \frac{1}{\lambda}(\|(u - \hat{u} + \lambda(\psi - \hat{\psi}))^+\|_{L^1} - \|(u - \hat{u})^+\|_{L^1}) \geq 0,$$

which ends the proof.

The following result will be useful for getting more precisely the existence of solutions of (2.1).

Proposition 2.1. *Under the assumptions* (H.0) *and* (H.1) *we have* $\overline{D(A)}^{L^1} = \{\varphi \in L^1(\mathbb{R}^N) : \varphi(x) \in \overline{D(\beta)}$ a.e. *in* $\mathbb{R}^N\}$.

Proof. We let $a = \inf D(\beta)$, $b = \sup D(\beta)$, $-\infty \leq a \leq 0 \leq b \leq +\infty$. We have

$$\overline{D(A)}^{L^1} \subset \{\varphi \in L^1(\mathbb{R}^N) : \varphi(x) \in \overline{D(\beta)} \text{ a.e. in } \mathbb{R}^N\}.$$

Given $w \in L^1(\mathbb{R}^N)$, $w \in \overline{D(\beta)}$ a.e. and n integer, we let

$$w_n(x) = \begin{cases} \min\left(\left(b - \frac{1}{n}\right)^+, n\right) & \text{if} \quad w(x) \geq \min\left(\left(b - \frac{1}{n}\right)^+, n\right), \\ w(x) & \text{if} \quad \max\left(-\left(a + \frac{1}{n}\right)^-, -n\right) \leq w(x) \leq \min\left(\left(b - \frac{1}{n}\right)^+, n\right), \\ \max\left(-\left(a + \frac{1}{n}\right)^-, -n\right) & \text{if} \quad w(x) \leq \max\left(-\left(a + \frac{1}{n}\right)^-, -n\right). \end{cases}$$

Hence $w_n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Thanks to Theorem 1.2, for any $\varepsilon > 0$ there exists a unique $u_\varepsilon \in D(A)$ such that $u_\varepsilon + \varepsilon Au_\varepsilon = w_n$, and we have for any $\tau \in \mathbb{R}^N$ and $1 \leq p \leq +\infty$

$$\text{(2.6)} \qquad \begin{aligned} \|u_\varepsilon^{(\pm)}\|_{L^p} &\leq \|w_n^{(\pm)}\|_{L^p}, \\ \|u_\varepsilon(. + \tau) - u_\varepsilon(.)\|_{L^1} &\leq \|w_n(. + \tau) - w_n(.)\|_{L^1}, \end{aligned}$$

which implies that $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^N)$ (see [17]). Hence there exists $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $u_{\varepsilon_k} \to u$ a.e. in $\mathbb{R}^N$ and in $L^1_{\text{loc}}(\mathbb{R}^N)$

as $\varepsilon_k \to 0$. From the definition of w_n and (2.6), $\{\beta(u_\varepsilon) : \varepsilon > 0\}$ remains bounded in $L^\infty(\mathbb{R}^N)$. Given $k \in \mathbb{R}$ and $\zeta \in \mathfrak{D}^+(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \operatorname{sign}_0(u_\varepsilon - k)\left\{\varepsilon \sum_{i=1}^N (\Phi_i(u_\varepsilon) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i} + (w_n - u - \varepsilon h_\varepsilon)\zeta\right\} dx \geq 0,$$

where h_ε is a section of $\{\beta(u_\varepsilon)\}$. If we take $k > \|w_n^+\|_{L^\infty}$ and $k < -\|w_n^+\|_{L^\infty}$ and go to the limit as $\varepsilon_k \to 0$, we deduce

$$\int_{\mathbb{R}^N} (w_n - u)\zeta dx = 0$$

which implies $w_n = u$. As $w_n \underset{n\to+\infty}{\longrightarrow} w$ in $L^1(\mathbb{R}^N)$, $w \in \overline{D(A)}^{L^1}$ which ends the proof.

Lemma 2.3. *Under the hypotheses* (H.0) *and* (H.1), *suppose* $\lambda > 0$, $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *is the solution of (0.2) in Kruzkov sense in* $\mathbb{R}^N$. *Let* $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *be the section of* $\{\beta(u)\}$ *corresponding to (0.2). If k is any real number greater than* $\|f^+\|_{L^\infty}$, *then*

$$(I + \lambda\beta)^{-1}k \geq \|u^+\|_{L^\infty} \quad \text{and} \quad \lambda^{-1}(k - (I + \lambda\beta)^{-1}k) \geq \|h^+\|_{L^\infty}. \tag{2.7}$$

Proof. We let $J_\lambda^\beta = (I + \lambda\beta)^{-1}$; J_λ^β is an increasing contraction defined on $\mathbb{R}$ and vanishing at 0. If we let $w = u + \lambda h$, then $w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and, as in the proof of Theorem 1.2 in Section 3, the following equation

$$w + \lambda \sum_{i=1}^N \frac{\partial}{\partial x_i} (\Phi_i \circ J_\lambda^\beta(w)) = f, \tag{2.8}$$

holds in Kruzkov sense in $\mathbb{R}^N$. From Theorem 1.2 (see also [1] and [2]) we deduce the following inequality

$$\|w^+\|_{L^\infty} \leq \|f^+\|_{L^\infty}, \tag{2.9}$$

which implies $u(x) + \lambda h(x) \leq k$ a.e. in $\mathbb{R}^N$. If we let $a = J_\lambda^\beta k$ and $b = \lambda^{-1}(k - a)$, then $b \in \beta(a)$. From the monotonicity of β, $u(x) \leq a$ and $h(x) \leq b$.

Proof of Theorem 1.1.

Part I: Uniqueness. Suppose u and $\hat{u}$ are two solutions of $(\text{C.P.})_f$ in Kruzkov sense in $(0,T) \times \mathbb{R}^N$, both belonging to $L^\infty(0,T;L^1(\mathbb{R}^N)) \cap L^\infty((0,T) \times \mathbb{R}^N)$ such that $u(t,.)$ and $\hat{u}(t,.)$ converge to $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ when t goes to 0 essentially. From Lemma 1.1, for any $\zeta \in \mathfrak{D}^+((0,T) \times \mathbb{R}^N)$ there exists $\alpha \in L^\infty((0,T) \times \mathbb{R}^N)$ with

$$\alpha(t,x) \in \operatorname{sign}^+(u(x,t) - \hat{u}(x,t)) \quad \text{a.e. in } (0,T) \times \mathbb{R}^N$$

such that

$$\int_0^T \int_{\mathbb{R}^N} \left((u - \hat{u})^+ \frac{\partial \zeta}{\partial t} + \alpha \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\hat{u})) \frac{\partial \zeta}{\partial x_i}\right) dx\, dt \geq 0. \tag{2.10}$$

In particular we take $\zeta(x,t) = \gamma(t)\,\mu(x)$ where $\gamma \in \mathcal{D}^+(0,T)$ and $\mu \in \mathcal{D}^+(\mathbb{R}^N)$.

If s and t $(0 \le s < t < T)$ are Lebesgue points of $\sigma \mapsto \|(u - u)^+(\sigma,.)\|_L{}^1$ we replace γ by some sequence $\{\gamma_n\}$ such that $d\gamma_n/dt \to \delta(s) - \delta(t)$ as $n \to +\infty$. Going to the limit in (2.10) yields

$$(2.11) \quad \|(u(t,.) - \hat{u}(t,.))^+ \mu\|_{L^1}$$

$$\le \|(u(s,.) - \hat{u}(s,.))^+ \mu\|_{L^1} + \int_s^t \int_{\mathbb{R}^N} \left| \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\hat{u}))\, \frac{\partial \mu}{\partial x_i} \right| dx\, dt.$$

If we make $s \to 0$ essentially we get

$$(2.12) \quad \|(u(t,.) - \hat{u}(t,.))^+ \mu\|_{L^1} \le \int_0^t \int_{\mathbb{R}^N} \left| \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\hat{u}))\, \frac{\partial \mu}{\partial x_i} \right| dx\, dt.$$

Now we replace $\mu(x)$ by $\mu_n(x) = \mu(x/n)$ with $\mu(0) = 1$. By Lemma 2.1 and Lebesgue's theorem we deduce from (2.12) that

$$(2.13) \qquad \|(u(t,.) - \hat{u}(t,.))^+\|_{L^1} \le 0,$$

which implies $u(t,x) \le \hat{u}(t,x)$. In the same way $\hat{u}(t,x) \le u(t,x)$ a.e. and $u = \hat{u}$.

Part II: Existence.

First step: construction of the semigroup solution. Suppose $f \in L^1(0,T;L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$. There exist a sequence of partitions $P_n = \{0 = t_n^0 < t_n^1 < \ldots < t_n^{N(n)} = T\}$ and a sequence of step functions $\{f_n\}$ taking the value f_n^k on (t_n^{k-1}, t_n^k), such that

$$\lim_{n\to+\infty} \max_{1\le k\le N} (t_n^k - t_n^{k-1}) = 0 \quad \text{and} \quad \lim_{n\to+\infty} \int_0^T (\|f_n - f\|_{L^\infty})\, dt = 0.$$

For a fixed partition P_n we let u_n be the continuous piecewise linear function taking the value u_n^k at t_n^k $(0 \le k \le N(n))$, where the sequence $\{u_n^k\}$ is defined by the implicit scheme

$$(2.14) \qquad \begin{aligned} &\frac{u_n^k - u_n^{k-1}}{t_n^k - t_n^{k-1}} + A\,u_n^k \ni f_n^k, \quad k = 1, 2, \ldots, N, \\ &u_n^0 = u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{aligned}$$

If we let $\varepsilon_n^k = t_n^k - t_n^{k-1}$, the relation (2.14) is equivalent to

$$(2.15) \qquad u_n^k + \varepsilon_n^k A\,u_n^k \ni \varepsilon_n^k f_n^k + u_n^{k-1}, \qquad k = 1, 2, \ldots, N.$$

From Theorem 1.2 the sequence $\{u_n^k\}$ is well defined in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and we have the estimate $(1 \le p \le +\infty)$

$$(2.16) \qquad \|u_n^{(\pm)}\|_{L^\infty(0,T;L^p)} \le \|u_0^{(\pm)}\|_{L^p} + \int_0^T \|f_n^{(\pm)}\|_{L^p}\, dt.$$

Let $\hat{u}_n$ be the continuous piecewise linear function taking the value $\hat{u}_n^k$ $(0 \le k \le N(n))$ at t_n^k, where $\{\hat{u}_n^k\}$ is defined by

$$(2.17)\qquad \begin{gathered} \frac{\hat{u}_n^k - \hat{u}_n^{k-1}}{t_n^k - t_n^{k-1}} + A\,\hat{u}_n^k \ni \hat{f}_n^k, \quad k = 1, 2, \ldots, N, \\ \hat{u}_n^0 = \hat{u}_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{gathered}$$

We deduce from Theorem 1.2 that

$$(2.18)\qquad \|(u_n - \hat{u}_n)^{(\pm)}\|_{L^\infty(0,T;L^1)} \le \|(u_0 - \hat{u}_0)^{(\pm)}\|_{L^1} + \int_0^T \|(f_n - \hat{f}_n)^{(\pm)}\|_{L^1}\,dt.$$

Since $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$, the sequence $\{u_n\}$ converges in $C([0,T];L^1(\mathbb{R}^N))$ to some u belonging to $C([0,T];L^1(\mathbb{R}^N)) \cap L^\infty((0,T) \times \mathbb{R}^N)$ (see [18] for example), and the function u satisfies (1.5). Such a function is called the semigroup solution of $(\text{C.P.})_f$). If $\hat{u}$ is the semigroup solution of $(\text{C.P.})_{\hat{f}}$ with initial data $\hat{u}_0$, we get (1.6) from (2.18). Let h_n^k (respectively $\hat{h}_n^k$) be the section of $\{\beta(u_n^k)\}$ (respectively $\{\beta(\hat{u}_n^k)\}$) in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ corresponding to the equation (2.14) (respectively (2.17)) and h_n (respectively $\hat{h}_n$) the step function taking the value h_n^k (respectively $\hat{h}_n^k$) on (t_n^{k-1},t_n^k). We deduce

$$(2.19)\qquad \int_0^T \|h_n - \hat{h}_n\|_{L^1}\,dt \le \|u_0 - \hat{u}_0\|_{L^1} + \int_0^T \|f_n - \hat{f}_n\|_{L^1}\,dt,$$

from (1.12). We claim now that u satisfies $(\text{C.P.})_f$ in Kruzkov sense in $(0,T) \times \mathbb{R}^N$.

Second step: f is independent of t. We let $\{a_n^k\}$ and $\{b_n^k\}$ be the sequences defined by

$$(2.20)\qquad \begin{gathered} \frac{a_n^k - a_n^{k-1}}{t_n^k - t_n^{k-1}} + \beta(a_n^k) \ni \|f^+\|_{L^\infty}, \quad k = 1, 2, \ldots, N, \\ a_n^0 = \|u_0^+\|_{L^\infty}, \end{gathered}$$

$$(2.21)\qquad \begin{gathered} \frac{b_n^k - b_n^{k-1}}{t_n^k - t_n^{k-1}} + \beta(b_n^k) \ni -\|f^-\|_{L^\infty}, \quad k = 1, 2, \ldots, N, \\ b_n^0 = -\|u_0^-\|_{L^\infty}. \end{gathered}$$

If α_n^k (respectively γ_n^k) is the section of $\{\beta(a_n^k)\}$ (respectively $\{\beta(b_n^k)\}$) corresponding to (2.20) (respectively (2.21)), then

$$(2.22)\qquad \gamma_n^k \le -(h_n^k)^- \le 0 \le (h_n^k)^+ \le \alpha_n^k \quad \text{a.e. in } \mathbb{R}^N.$$

If α_n and γ_n are the step functions taking the values α_n^k and γ_n^k on (t_n^{k-1},t_n^k) then $\{\alpha_n\}$ and $\{\gamma_n\}$ remain bounded in $L^2_{\text{loc}}(0,T)$ (see Veron [27]) and it is the same for $\{h_n\}$ in $L^2_{\text{loc}}(0,T;L^\infty(\mathbb{R}^N))$.

We consider $\zeta \in \mathcal{D}^+((0,T) \times \mathbb{R}^N)$ and $\ell \in \mathbb{R}$. From (2.14) we get for $1 \leq k \leq N(n)$

$$\int_{\mathbb{R}^N} \operatorname{sign}_0(u_n^k - \ell)\left\{\sum_{i=1}^N (\Phi_i(u_n^k) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i}(t_n^k,.) + \left(f - h_n^k + \frac{u_n^{k-1} - u_n^k}{\varepsilon_n^k}\right)\zeta(t_n^k,.)\right\} dx \geq 0.$$

But we check easily that

$$\operatorname{sign}_0(u_n^k - \ell)(u_n^{k-1} - u_n^k) \leq |u_n^{k-1} - \ell| - |u_n^k - \ell|,$$

so

$$\sum_{k=1}^{N(n)} \int_{\mathbb{R}^N} (|u_n^{k-1} - \ell| - |u_n^k - \ell|)\,\zeta(t_n^k,.)\,dx + \sum_{k=1}^{N(n)} \varepsilon_n^k \int_{\mathbb{R}^N} \operatorname{sign}_0(u_n^k - \ell) \left\{\sum_{i=1}^N (\Phi_i(u_n^k) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i}(t_n^k,.) + (f - h_n^k)\,\zeta(t_n^k,.)\right\} dx \geq 0. \tag{2.23}$$

From Abel's transform we get

$$\sum_{k=1}^{N(n)} \int_{\mathbb{R}^N} (|u_n^{k-1} - \ell| - |u_n^k - \ell|)\,\zeta(t_n^k,.)dx = \sum_{k=1}^{N(n)-1} \int_{\mathbb{R}^N} |u_n^k - \ell|(\zeta(t_n^{k+1},.) - \zeta(t_n^k,.))dx + \int_{\mathbb{R}^N} (|u_0 - \ell|\,\zeta(t_n^1,.) - |u_n^N - \ell|\,\zeta(T,.))dx$$

and

$$\zeta(t_n^{k+1},.) - \zeta(t_n^k,.) = \frac{\partial \zeta}{\partial t}(t_n^k,.)\,\varepsilon_n^{k+1} + \int_{t_n^k}^{t_n^{k+1}} \frac{\partial^2 \zeta}{\partial t^2}(t,.)(t_n^{k+1} - t)\,dt.$$

Since $\{u_n\}$ remains bounded in $L^\infty((0,T) \times \mathbb{R}^N)$ and converges to u in $C([0,T];L^1(\mathbb{R}^N))$, we get

$$\lim_{n\to+\infty} \sum_{k=1}^{N(n)} \int_{\mathbb{R}^N} (|u_n^{k-1} - \ell| - |u_n^k - \ell|)\,\zeta(t_n^k,.)dx = \int_0^T \int_{\mathbb{R}^N} |u - \ell|\,\frac{\partial \zeta}{\partial t}\,dx\,dt.$$

Moreover

$$\lim_{n\to+\infty} \sum_{k=1}^{N(n)} \varepsilon_n^k \int_{\mathbb{R}^N} \operatorname{sign}_0(u_n^k - \ell) \sum_{i=1}^N (\Phi_i(u_n^k) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i}(t_n^k,.)\,dx$$

$$= \int_0^T \int_{\mathbb{R}^N} \operatorname{sign}_0(u-\ell) \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i} \, dx \, dt.$$

Since the sequence $\{h_n\}$ is relatively weakly compact in $L^2_{\text{loc}}(0,T;L^\infty(\mathbb{R}^N))$ there exists a sequence $\{h_{n_m}\}$ and $h \in L^2_{\text{loc}}(0,T;L^\infty(\mathbb{R}^N))$ such that $\{h_{n_m}\}$ is weakly convergent to h in $L^2_{\text{loc}}(0,T;L^\infty(\mathbb{R}^N))$. From the maximal monotonicity of β, $h(t,x) \in \beta(u(t,x))$ a.e. in $(0,T) \times \mathbb{R}^N$. Since the function $(t,x) \to u(t,x)$ is measurable on $(0,T) \times \mathbb{R}^N$, the subset of $k \in \mathbb{R}$ such that

$$\operatorname{meas}\{(t,x) \in (0,T) \times \mathbb{R}^N : u(t,x) = k\} > 0$$

is at most countable. So we can suppose that the measure of

$$\{(t,x) \in (0,T) \times \mathbb{R}^N : u(t,x) = \ell\}$$

is 0 and also that $\{u_{n_m}\}$ converges to u a.e. in $(0,T) \times \mathbb{R}^N$. Hence

$$\lim_{n_m \to +\infty} \operatorname{sign}_0(u_{n_m} - \ell) = \operatorname{sign}_0(u-\ell), \tag{2.24}$$

for almost all $(t,x) \in (0,T) \times \mathbb{R}^N$. Going to the limit in the last term of (2.23), we have

$$\lim_{n_m \to +\infty} \sum_{k=1}^{N(n)} \varepsilon^k_{n_m} \int_{\mathbb{R}^N} \operatorname{sign}_0(u^k_{n_m} - \ell)(f - h^k_{n_m})\zeta(t^k_{n_m},.)\,dx$$
$$= \int_0^T \int_{\mathbb{R}^N} \operatorname{sign}_0(u-\ell)(f-h)\zeta\,dxdt.$$

Hence we deduce from (2.30) that the following inequality holds for almost all $\ell \in \mathbb{R}$

$$\int_0^T \int_{\mathbb{R}^N} \operatorname{sign}_0(u-\ell)\left\{(u-\ell)\frac{\partial \zeta}{\partial t} + \sum_1^N (\Phi_i(u) - \Phi_i(\ell))\frac{\partial \zeta}{\partial x_i} + (f-h)\zeta\right\} dxdt \geq 0, \tag{2.25}$$

which means that u satisfies $(\text{C.P.})_f$ in Kruzkov sense in $(0,T) \times \mathbb{R}^N$.

Third step: f is a step function from $(0,T)$ *into* $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We suppose that there exists a partition $\{0 = t_0 < t_1 < \ldots < t_n = T\}$ such that $f = f_j$ (independant of t) on (t_j,t_{j+1}) $(0 \leq j \leq n-1)$. If u is the semigroup solution of $(\text{C.P.})_f$ on $(0,T)$, u is continuous from $[0,T]$ into $L^1(\mathbb{R}^N)$ and satisfies

$$\frac{\partial u}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f_j, \tag{2.26}$$

in Kruzkov sense in $(t_j,t_{j+1}) \times \mathbb{R}_N$ $(0 \leq j \leq n-1)$. From (2.19) and the second step there exists $h \in L^1(0,T;L^1(\mathbb{R}^N))$, $h(x,t) \in \beta(u(x,t))$ a.e. in $(0,T) \times \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) = f_j - h, \tag{2.27}$$

holds in Kruzkov in $(t_j,t_{j+1}) \times \mathbb{R}^N$.

We take $\zeta \in \mathcal{D}^+((0,T) \times \mathbb{R}^N)$ and for each $0 \le j \le n-1$ we consider a sequence $\{\zeta_j^p\}$, $\zeta_j^p \in \mathcal{D}^+(t_j,t_{j+1})$, such that $\zeta_j^p = 1$ on $[t_j + 1/p, t_{j+1} - 1/p]$ for $1 \le j \le n-2$, $\zeta_0^p = 1$ on $[0,t_1 - 1/p]$ (we can take $\zeta_0^p \in \mathcal{D}^+([0,t_1))$) and $\zeta_{n-1}^p = 1$ on $[t_{n-1} + 1/p, T]$ (we can take $\zeta_{n-1}^p \in \mathcal{D}^+((t_{n-1},T])$). We also suppose that for $1 \le j \le n-2$,

$$\lim_{p\to+\infty} \frac{\partial \zeta_j^p}{\partial t} = \delta(t_j) - \delta(t_{j+1}), \qquad \lim_{p\to+\infty} \frac{\partial \zeta_0^p}{\partial t} = -\delta(t_1)$$

and

$$\lim_{p\to+\infty} \frac{\partial \zeta_{n-1}^p}{\partial t} = \delta(t_{n-1}).$$

From (2.27), we have for any $\ell \in \mathbb{R}$ and $0 \le j \le n-1$

$$\int_0^{+\infty}\int_{\mathbb{R}^N}\Bigg(|u-\ell|\frac{\partial}{\partial t}(\zeta\zeta_j^p) + \operatorname{sign}_0(u-\ell)\Bigg\{\sum_{i=1}^{N}(\Phi_i(u) - \Phi_i(\ell))\frac{\partial}{\partial x_i}(\zeta\zeta_j^p) + (f_j - h)\zeta\zeta_j^p\Bigg\}\Bigg)\,dxdt \ge 0, \tag{2.28$_j$}$$

and

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^N}\operatorname{sign}_0(u-\ell)\left\{\sum_{i=1}^{N}(\Phi_i(u)-\Phi_i(\ell))\frac{\partial}{\partial x_i}((\zeta\zeta_j^p) + (f_j-h)\zeta\zeta_j^p\right\}dxdt$$

$$= \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^N}\operatorname{sign}_0(u-\ell)\left\{\sum_{i=1}^{N}(\Phi_i(u)-\Phi_i(\ell))\frac{\partial\zeta}{\partial x_i} + (f_j-h)\zeta\right\}dxdt.$$

We deduce from the continuity of $t \mapsto \int_{\mathbb{R}^N}|u(t,.)-\ell|\,dx$ that

$$\lim_{p\to+\infty}\int_0^{+\infty}\int_{\mathbb{R}^N}|u-\ell|\frac{\partial}{\partial t}(\zeta\zeta_j^p)\,dxdt = \int_{t_j}^{t_{j+1}}\int_{\mathbb{R}^N}|u-\ell|\frac{\partial\zeta}{\partial t}\,dxdt$$

$$+ \int_{\mathbb{R}^N}|u(t_j)-\ell|\frac{\partial\zeta}{\partial t}(t_j)\,dx - \int_{\mathbb{R}^N}|u(t_{j+1})-\ell|\frac{\partial\zeta}{\partial t}(t_{j+1})\,dx.$$

Going to the limit in $(2.28)_j$ as $p \to +\infty$ and summing the n inequalities yield

(2.29) $$\int_0^T \int_{\mathbb{R}^N} \left(|u - \ell| \frac{\partial \zeta}{\partial t} + \operatorname{sign}_0(u - \ell) \left\{ \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i} + (f - h)\zeta \right\} \right) dxdt \geq 0;$$

hence u satisfies $(\mathrm{C.P.})_f$ in Kruzkov sense in $(0,T) \times \mathbb{R}^N$.

Fourth step: end of the proof. We consider a general f belonging to $L^1(0,T;L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ and a sequence of step functions $\{f_n\}$ from $(0,T)$ into $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ converging to f in $L^1(0,T;L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$. If u_n is the solution of $(\mathrm{C.P.})_f$ with initial data u_0, then $\{u_n\}$ remains bounded and converges to u in $C([0,T];L^1(\mathbb{R}^N))$ as we have from (2.18)

(2.30) $$\|u_n - u\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} \leq \int_0^T \|f_n - f\|_{L^1} dt.$$

Moreover u_n satisfies $(\mathrm{C.P.})_{f_n}$ in Kruzkov sense in $(0,T) \times \mathbb{R}^N$. If h_n is the section of $\{\beta(u_n)\}$ in $L^1((0,T) \times \mathbb{R}^N)$ corresponding to the equation

(2.31) $$\frac{\partial u_n}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i} (\Phi_i(u_n)) = f_n - h_n$$

satisfied in Kruzkov sense, then we deduce from (2.19) that the sequence $\{h_n\}$ is a Cauchy sequence in $L^1((0,T) \times \mathbb{R}^N)$. Hence it converges to some $h \in L^1((0,T) \times \mathbb{R}^N)$ with $h(t,x) \in \beta(u(t,x))$ a.e. in $(0,T) \times \mathbb{R}^N$. It is easy to check that for almost all $\ell \in \mathbb{R}$ and any $\zeta \in \mathfrak{D}^+((0,T) \times \mathbb{R}^N)$ we have

(2.32) $$\int_0^T \int_{\mathbb{R}^N} \left\{ \operatorname{sign}_0(u - \ell) \frac{\partial \zeta}{\partial t} + \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i} + (f - h)\zeta \right\} dxdt \geq 0,$$

from (2.31), which ends the proof.

3. Existence and Uniqueness for (0.2) and $(\mathrm{S.P.})_f$

We first consider the coercive problem (0.2).

Proof of Theorem 1.2. *Uniqueness.* Suppose u and $\hat{u}$ are solutions of (0.2) in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with the same right-hand side $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. From Lemma 1.1, for any $\zeta \in \mathfrak{D}^+(\mathbb{R}^N)$ such that $\zeta(0) = 1$ there exists $\alpha \in L^\infty(\mathbb{R}^N)$, $\alpha(x) \in \operatorname{sign}^+(u(x) - \hat{u}(x))$ a.e. such that

(3.1) $$\int_{\mathbb{R}^N} \alpha \left\{ \sum_{i=1}^N (\Phi_i(u) - \Phi_i(\hat{u})) \frac{\partial \zeta}{\partial x_i} - (u - \hat{u})\zeta \right\} dx \geq 0.$$

If we replace $\zeta(.)$ by $\zeta_n(.) = \zeta(./n)$, we deduce from Lemma 2.1

$$\int_{\mathbb{R}^N} (u - \hat{u})^+ dx \leq 0; \tag{3.2}$$

hence $u \leq \hat{u}$. In the same way $\hat{u} \leq u$.

Existence. Since J_λ^β is an increasing contraction vanishing at 0, the N functions $\Phi_i \circ J_\lambda^\beta$ satisfy (H.0) and (H.1). By Proposition 2.8 of [1] there exists a unique $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfying

$$v + \lambda \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\Phi_i \circ J_\lambda^\beta(v)) = f, \tag{3.3}$$

in Kruzkov sense in $\mathbb{R}^N$, so

$$\int_{\mathbb{R}^N} \operatorname{sign}_0(v - k)\left\{\lambda \sum_{i=1}^{N} (\Phi_i \circ J_\lambda^\beta(v) - \Phi_i \circ J_\lambda^\beta(k)) \frac{\partial \zeta}{\partial x_i} + (f - v)\zeta\right\} dx \geq 0, \tag{3.4}$$

holds for any $\zeta \in \mathfrak{D}^+(\mathbb{R}^N)$ and $k \in \mathbb{R}$. Moreover, for any $1 \leq p \leq +\infty$, we have (see [1] and [2])

$$\|v^{(\pm)}\|_{L^p} \leq \|f^{(\pm)}\|_{L^p}. \tag{3.5}$$

If $\hat{v}$ is the solution in Kruzkov sense of (3.3) with right-hand side $\hat{f}$, we have

$$\|(v - \hat{v})^{(\pm)}\|_{L^1} \leq \|(f - \hat{f})^{(\pm)}\|_{L^1}. \tag{3.6}$$

If we let $u = J_\lambda^\beta v$, $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $u(x) \in D(\beta)$ a.e. in $\mathbb{R}^N$. We also have that $u(x) + \lambda\beta(u(x)) \ni v(x)$ a.e. so there exists $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $h(x) \in \beta(u(x))$ a.e., such that $u(x) + \lambda h(x) = v(x)$ a.e. Since J_λ^β is from $\mathbb{R}$ onto $D(\beta)$, we have for any $\ell \in D(\beta)$ ($\ell = J_\lambda^\beta(k)$)

$$\int_{\mathbb{R}^N} \operatorname{sign}_0(v - k)\left\{\lambda \sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i} + (f - u - \lambda h)\zeta\right\} dx \geq 0. \tag{3.7}$$

Moreover, except for a countable number of values of ℓ, we have $\operatorname{sign}_0(v - k) = \operatorname{sign}_0(J_\lambda^\beta v - J_\lambda^\beta k) = \operatorname{sign}_0(u - \ell)$ a.e.; so

$$\int_{\mathbb{R}^N} \operatorname{sign}_0(u - \ell)\left\{\lambda \sum_{i=1}^{N} (\Phi_i(u) - \Phi_i(\ell)) \frac{\partial \zeta}{\partial x_i} + (f - u - \lambda h)\zeta\right\} dx \geq 0 \tag{3.8}$$

holds for almost all $\ell \in D(\beta)$. Since the equation

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\Phi_i(u)) + u + \lambda h = f, \tag{3.9}$$

is satisfied in $\mathfrak{D}'(\mathbb{R}^N)$, (3.8) also holds for any $\ell > \sup D(\beta)$ and for any $\ell < \inf D(\beta)$; so (0.2) holds in Kruzkov sense in $\mathbb{R}^N$. Using the fact that J_λ^β is an increasing contraction, we deduce (1.8) and (1.9) from (3.5) and (3.6).

Remark 3.1. Notice that there is a one-to-one correspondence between the solutions (in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$) of (0.2) and (3.3) taken in Kruzkov sense in $\mathbb{R}^N$: u is the solution of (0.2) and h the corresponding unique section of $\{\beta(u)\}$ if and only if $v = u + \lambda h$ is the solution of (3.1).

Proposition 3.1. *Suppose* (H.0) *and* (H.1) *hold. If* $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* β *are such that* $f(x) \in R(\beta)$ a.e. *in* $\mathbb{R}^N$ *and* β^{-1} *is continuous on the interval* $[-\|f^-\|_{L^\infty}, \|f^+\|_{L^\infty}]$, *there exists at least one* $u \in L^\infty(\mathbb{R}^N)$ *such that*

$$(\text{S.P.})_f \qquad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f,$$

holds in Kruzkov sense in $\mathbb{R}^N$.

Proof. For any $\varepsilon > 0$ let $u_\varepsilon \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be the solution of

$$(3.8) \qquad \varepsilon u_\varepsilon + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u_\varepsilon)) + \beta(u_\varepsilon) \ni f,$$

in Kruzkov sense in $\mathbb{R}^N$. If h_ε is the section of $\{\beta(u_\varepsilon)\}$ corresponding to (3.8), we have for any $1 \leq p \leq +\infty$

$$(3.9) \qquad \varepsilon \|u_\varepsilon\|_{L^p} \leq \|f\|_{L^p} \quad \text{and} \quad \|h_\varepsilon^{(\pm)}\|_{L^p} \leq \|f^{(\pm)}\|_{L^p}.$$

From unicity and (1.9), we get for any $\tau \in \mathbb{R}^N$

$$(3.10) \qquad \|h_\varepsilon(. + \tau) - h_\varepsilon(.)\|_{L^1} \leq \|f(. + \tau) - f(.)\|_{L^1}.$$

Hence the set $\{h_\varepsilon : \varepsilon > 0\}$ is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^N)$ and there exist some $h \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and a sequence $\{\varepsilon_n\}$ going to 0 such that $\{h_{\varepsilon_n}\}$ converges to h in $L^1_{\text{loc}}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. From (3.9) and the continuity of β^{-1}, $\{u_{\varepsilon_n}\}$ converges to some u a.e. in $\mathbb{R}^N$ and $h(x) \in \beta(u(x))$ a.e. in $\mathbb{R}^N$; hence $u \in L^\infty(\mathbb{R}^N)$ as it is the same with h. For any $k \in \mathbb{R}$ and $\zeta \in \mathcal{D}^+(\mathbb{R}^N)$ we have

$$(3.11) \qquad \int_{\mathbb{R}^N} \operatorname{sign}_0(u_{\varepsilon_n} - k)\left\{\sum_{i=1}^{N} (\Phi_i(u_{\varepsilon_n}) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i} + (f - h_{\varepsilon_n} - \varepsilon_n u_{\varepsilon_n})\zeta\right\} dx \geq 0.$$

Going to the limit as $\varepsilon_n \to 0$, we conclude that (1.2) holds except for a countable number of values k, which ends the proof.

Remark 3.2. We do not know whether u is the unique solution of $(\text{S.P.})_f$ in $L^\infty(\mathbb{R}^N)$ or not, but we notice that $\{h_\varepsilon\} \to h$ and $\{u_\varepsilon\} \to u$ (not only some sequence $\{\varepsilon_n\}$). Otherwise we would have for two limit functions u and $\hat{u}$, $u = \beta^{-1}(h)$ and $\hat{u} = \beta^{-1}(\hat{h})$, $\|h - \hat{h}\|_{L^1} = 0$ as in (1.9), which implies $u = \hat{u}$ since β is strictly increasing on the interval which contains almost all the values of $u(x)$ and $\hat{u}(x)$.

As for the assumption of continuity on β^{-1}, it can be avoided in some cases, in particular when Φ is linear.

Remark 3.3. A natural extension of Proposition 3.1 is to suppose that β^{-1} is continuous, (H.0) holds and

$$\limsup_{r\to 0} |\Phi_i \circ \beta^{-1}(r)|/|r|^{(N-1)/N} < +\infty.$$

Then for any $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists at least one $u \in L^\infty(\mathbb{R}^N)$ satisfying $(\mathrm{S.P.})_f$. Moreover the corresponding section h of $\{\beta(u)\}$ belongs to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

4. Comparison Principles

4.1. The stationary case. We introduce the following hypotheses on Φ_i $(1 \le i \le N)$

$$(\mathrm{H.2}_i) \qquad \begin{aligned} &\Phi_i(0) = 0, \quad \Phi_i \text{ is strictly increasing} \\ &\text{on } \mathbb{R}^+ \text{ and } \quad R(\Phi_i) \cup R(\beta) \supset \mathbb{R}^+. \end{aligned}$$

Under the hypotheses (H.0) and $(\mathrm{H.2}_1)$, the operator $\beta \circ \Phi_1^{-1}$ is maximal monotone with respect to $\mathbb{R}^+$, so for every $\alpha \in \mathbb{R}$, every $g \in L^{1+}_{\mathrm{loc}}([\alpha,+\infty))$ and every $v_0 \in D(\beta) \cap \mathbb{R}^+$ there exists a unique absolutely continuous nonnegative function w defined on $(\alpha,+\infty)$ satisfying

$$(4.1) \qquad \begin{aligned} &\frac{dw}{dx_1} + \beta \circ \Phi_1^{-1}(w) \ni g \quad \text{a.e. in } (\alpha,+\infty), \\ &w(\alpha) = \Phi_1(v_0). \end{aligned}$$

Moreover $w(x) \in D(\beta) \cap R(\Phi_1)$ a.e. and $w(x) \in R(\Phi_1)$ for any $x > \alpha$.

Definition 4.1. Under hypotheses (H.0) and $(\mathrm{H.2}_1)$, we say that v *satisfies the evolution equation*

$$(4.2) \qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(v)) + \beta(v) \ni g \qquad \text{in} \quad (\alpha,+\infty), \\ &v(\alpha) = v_0 \in \overline{D(\beta)} \cap \mathbb{R}^+, \qquad g \in L^{1+}_{\mathrm{loc}}([\alpha,+\infty)), \end{aligned}$$

if $v = \Phi_1^{-1}(w)$ where w satisfies (4.1).

Lemma 4.1. *Suppose* (H.0) *and* $(\mathrm{H.2}_1)$ *are satisfied and v is the solution of the evolution equation (4.2); then*

$$(4.3) \qquad \frac{d}{dx_1}(\Phi_1(v)) + \beta(v) \ni g,$$

holds in Kruzkov sense in $(\alpha,+\infty)$.

Proof. Let $\mathcal{P}$ be the set of all nondecreasing Lipschitz continuous functions on $\mathbb{R}$ having derivative with compact support and let $p_n \in \mathcal{P}$ such that $-1 \le$

$p_n \leq 1$, $p_n(0) = 0$, $p_n' \leq Cn$ and supp $p_n' \subset [-1/n, 1/n]$. For any $k \in \mathbb{R}^+$ and any $\zeta \in \mathfrak{D}^+(\alpha, +\infty)$ we have

$$(4.4) \qquad \int_\alpha^{+\infty} \left\{ \frac{dw}{dx_1} \zeta p_n(w-k) + (g-h) p_n(w-k)\zeta \right\} dx_1 = 0,$$

where $h = g - dw/dx_1 \in \beta \circ \Phi_1^{-1}(w)$ ($h \in L^1_{\mathrm{loc}}([\alpha, +\infty))$). Hence

$$\int_\alpha^{+\infty} p_n(w-k) \left\{ (w-k) \frac{d\zeta}{dx_1} + (g-h)\zeta \right\} dx_1$$

$$= -\int_\alpha^{+\infty} \zeta \frac{dw}{dx_1} (w-k) p_n'(w-k)\, dx_1.$$

Since $|\zeta(dw/dx_1)(w-k)p_n'(w-k)| \leq 2C\zeta|dw/dx_1|$, we deduce by Lebesgue's theorem that

$$(4.5) \qquad \int_\alpha^{+\infty} \mathrm{sign}_0(w-k) \left\{ (w-k) \frac{d\zeta}{dx_1} + (g-h)\zeta \right\} dx_1 = 0.$$

Moreover $\Phi_1^{-1}(w) \in D(\beta)$ a.e. in $(\alpha, +\infty)$. If we let $v = \Phi_1^{-1}(w)$ and take $k = \Phi_1(\ell)$ with $\ell \in \mathbb{R}^+$ then

$$(4.6) \qquad \int_\alpha^{+\infty} \mathrm{sign}_0(v-\ell) \left\{ (\Phi_1(v) - \Phi_1(\ell)) \frac{d\zeta}{dx_1} + (g-h)\zeta \right\} dx_1 = 0,$$

and $h(x) \in \beta(v(x))$ a.e. Since $v \geq 0$ and (4.3) holds in $\mathfrak{D}'(\alpha, +\infty)$ we can take $\ell < 0$ in (4.6).

Remark 4.1. When $g = 0$, the hypothesis (H.2_1) can be weakened since we no longer have to suppose that $\mathbb{R}^+ \subset R(\Phi_i) \cup R(\beta)$.

Remark 4.2. Under the hypotheses of Lemma 4.1, if we suppose $v_0 \in \mathbb{R}^+ \cap \beta^{-1}(0)$ and define $\tilde{v}$ and $\tilde{g}$ on $\mathbb{R}$ by

$$\tilde{v}(x) = v(x) \quad \text{on} \quad [\alpha, +\infty), \qquad \tilde{v}(x) = v_0 \quad \text{on} \quad (-\infty, \alpha),$$

$$\tilde{g}(x) = g(x) \quad \text{on} \quad (\alpha, +\infty), \qquad \tilde{g}(x) = 0 \quad \text{on} \quad (-\infty, \alpha),$$

then the equation $(d/dx_1)(\Phi_1(\tilde{v})) + \beta(\tilde{v}) \ni \tilde{g}$ holds in Kruzkov sense in $\mathbb{R}$.

In order to get an x_1-directional comparison principle we need weaker assumptions on $(\Phi_2, \ldots, \Phi_N)$ than (H.1). So, for $i = 1, 2, \ldots, N$ we consider

$$(\mathrm{H}.3_i) \qquad \forall j \in \{1, \ldots, N\} - \{i\}, \quad \limsup_{r \to 0} \frac{|\Phi_j(r)|}{|r|^{(N-2)/(N-1)}} < +\infty.$$

Theorem 4.1. *Suppose $N \geq 2$ and* (H.0), (H.2_1) *and* (H.3_1) *hold. If $f \in L^1_{\mathrm{loc}}(R, L^\infty(\mathbb{R}^{N-1}))$ and $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ satisfies in Kruzkov sense in $\mathbb{R}^N$*

$$(\text{S.P.})_f \qquad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni f,$$

then for any $\alpha \in \mathbb{R}$ *and almost all* $(x_1,x') \in (\alpha,+\infty) \times \mathbb{R}^{N-1}$, $u(x_1,x')$ *is majorized by* $u_\alpha(x_1)$, *where* u_α *is the solution of the evolution equation*

$$(4.7) \qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(u_\alpha)) + \beta(u_\alpha) \ni \|f^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \quad \text{in} \quad (\alpha,+\infty), \\ &u_\alpha(\alpha) = \|u^+\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

Proof. For $\lambda > 0$ we let $\beta_\lambda = \lambda^{-1}(I - J_\lambda)$; β_λ is a $(1/\lambda)$-Lipschitz continuous nondecreasing function defined on $\mathbb{R}^+$ and $R(\Phi_1) \cup R(\beta_\lambda) \supset \mathbb{R}^+$. Let v_λ be the solution of the evolution equation

$$(4.8) \qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(v_\lambda)) + \beta_\lambda(v_\lambda) = \lambda + \|f^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \quad \text{in} \quad (\alpha,+\infty), \\ &v_\lambda(\alpha) = u_\alpha(\alpha) + \lambda. \end{aligned}$$

The function v_λ is absolutely continuous and strictly positive on $[\alpha,+\infty)$. Since $\lambda \to \beta_\lambda \circ \Phi_1^{-1}(r)$ is nonincreasing and converges to the projection of 0 onto $\beta \circ \Phi_1^{-1}(r)$ as λ goes to 0, $v_\lambda(x_1)$ converges by decreasing to $u_\alpha(x_1)$ when λ goes to 0. So it suffices to prove that $\|u^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \le v_\lambda(x_1)$ on $(\alpha,+\infty)$. If we define v_λ on $[\alpha,+\infty) \times \mathbb{R}^N$ by $\tilde{v}_\lambda(x_1,x') = v_\lambda(x_1)$, we have for any $k \in \mathbb{R}$, any $\zeta \in \mathcal{D}^+((\alpha,+\infty) \times \mathbb{R}^{N-1})$ and any $x_1 > \alpha$ that

$$(4.9) \qquad \int_{\mathbb{R}^{N-1}} \sum_{i=2}^{N} (\Phi_i(\tilde{v}) - \Phi_i(k)) \frac{\partial \zeta}{\partial x_i}(x_i,x')\, dx' = 0.$$

Since $\text{sign}_0(\tilde{v}_\lambda - k)$ does not depend on x', we deduce from Lemma 4.1 that the following equation holds in Kruzkov sense in $(\alpha,+\infty) \times \mathbb{R}^N$

$$(4.10) \qquad \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(\tilde{v}_\lambda)) + \beta_\lambda(\tilde{v}_\lambda) = \|f^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} + \lambda.$$

Since the support of $(u - \tilde{v}_\lambda)^+$ is some closed subset of $(\alpha,+\infty) \times \mathbb{R}^{N-1}$, we let

$$R = \inf\{x_1 \ge \alpha : \exists z \in \mathbb{R}^{N-1} \text{ and } (x_1,z) \in \text{supp}(u - \tilde{v}_\lambda)^+\}$$

and we suppose $R < +\infty$.

If $R = \alpha$, then $\|u^+\|_{L^\infty(\mathbb{R}^N)} \ge u_\alpha(\alpha) + \lambda$ which is a contradiction, so $R > \alpha$.

For any $\gamma > 0$ we consider $\zeta_1 \in \mathcal{D}^+(\mathbb{R})$ with support $[-\gamma,\gamma]$, $\zeta' \in \mathcal{D}^+(\mathbb{R}^{N-1})$ with support $\{x' : |x'| \le \gamma\}$ and we let $\zeta = \zeta_1\zeta'$. We can suppose

$$\frac{\partial \zeta}{\partial x_1} < 0 \quad \text{on} \quad \{x = (x_1,x') : 0 < x_1 < \gamma, |x'| < \gamma\}.$$

For $n \in \mathbf{N}^*$ and $a' \in \mathbb{R}^{N-1}$ we let

$$\zeta_n(x_1,x') = \zeta_1(x_1 - R)\zeta'\left(\frac{x' - a'}{n}\right).$$

If h is a section of $\{\beta(u)\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ corresponding to $(\mathrm{S.P.})_f$, there exists $\eta \in L^\infty((\alpha,+\infty) \times \mathbb{R}^{N-1})$, $\eta \in \operatorname{sign}^+(u - \tilde{v}_\lambda)$ a.e., such that

$$\int_{\mathbf{R}^N} \eta\left\{\sum_{i=1}^N (\Phi_i(u) - \Phi_i(\tilde{v}_\lambda))\frac{\partial \zeta_n}{\partial x_i} + (f - \lambda - \|f^+(,.)\|_{L^\infty(R^{N-1})} + \beta_\lambda(\tilde{v}_\lambda) - h)\zeta_n\right\}dx \geq 0. \tag{4.11}$$

But $\beta_\lambda(u) \leq h \in \beta(u)$ on $\{x : u(x) \geq \tilde{v}_\lambda(x)\}$, so we deduce that

$$\int_{u>\tilde{v}_\lambda} (\Phi_1(u) - \Phi_1(\tilde{v}_\lambda))\frac{\partial \zeta_n}{\partial x_1}dx + \sum_{i=2}^N \int_{u>\tilde{v}_\lambda} (\Phi_i(u) - \Phi_i(\tilde{v}_\lambda))\frac{\partial \zeta_n}{\partial x_i}dx \geq 0. \tag{4.12}$$

When $n \to +\infty$, $(\partial\zeta_n/\partial x_1)(x_1,x') \to \zeta'(0)(d\zeta_1/dx_1)(x_1 - R)$; so if

$$\operatorname{meas}\{x = (x_1,x') : R < x_1 < R + \gamma \quad \text{and} \quad u(x) > \tilde{v}_\lambda(x)\} > 0, \tag{4.13}$$

we would have that

$$\limsup_{n\to+\infty} \int_{u>v_\lambda} (\Phi_1(u) - \Phi_1(\tilde{v}_\lambda))\frac{\partial \zeta_n}{dx_1}dx < 0 \tag{4.14}$$

since Φ_1 is strictly increasing on $\mathbb{R}^+$.

Now let

$$A_i^n = \int_{u>v_\lambda} \Phi_i(u)\frac{\partial \zeta_n}{\partial x_i}dx \qquad \text{and} \qquad B_i^n = \int_{u>\tilde{v}_\lambda} \Phi_i(\tilde{v}_\lambda)\frac{\partial \zeta_n}{\partial x_i}dx, \quad i \geq 2.$$

$$|A_i^n| \leq \int_R^{R+\gamma} \zeta_1(. - R)\int_{u>\tilde{v}_\lambda}\left|\Phi_i(u)\frac{\partial}{\partial x_i}\left(\zeta'\left(\frac{. - a'}{n}\right)\right)\right|dx'\,dx_1.$$

From Lemma 2.1

$$\int_{\mathbf{R}^{N-1}}\left|\Phi_i(u)\frac{\partial}{\partial x_i}\left(\zeta'\left(\frac{. - a'}{n}\right)\right)\right|dx' \leq C\|u(x_1,.)\|_{L^1(\mathbf{R}^{N-1})}^{(N-2)/(N-1)}.$$

Moreover

$$\lim_{n\to+\infty}\int_{\mathbf{R}^{N-1}}\left|\Phi_i(u)\frac{\partial}{\partial x_i}\left(\zeta'\left(\frac{. - a'}{n}\right)\right)\right|dx' = 0$$

for almost all x_1. Hence by Lebesgue's theorem $\lim_{n\to+\infty} A_i^n = 0$.

Since $u \in L^1(\mathbb{R}^N)$ and $v_\lambda(x) \geq \varepsilon > 0$ on $[R,R+\gamma]$, the measure defined in (4.13) is finite. Since $|\partial\zeta_n/\partial x_i| \leq C/n$ for $i \geq 2$, $\lim_{n\to+\infty} B_i^n = 0$.

With those two estimates we deduce that

$$\liminf_{n\to+\infty} \int_{u>\tilde{v}_\lambda} (\Phi_1(u) - \Phi_1(\tilde{v}_\lambda)) \frac{\partial\zeta_n}{\partial x_1}\, dx \geq 0, \tag{4.15}$$

which contradicts (4.14). So $R = +\infty$ and $v_\lambda(x_1) \geq u(x_1,x')$ on $(\alpha,+\infty) \times \mathbb{R}^{N-1}$, which ends the proof.

In order to obtain more complete information on any solution u of $(\text{S.P.})_f$, we introduce the following hypotheses $(1 \leq i \leq N)$:

(H.2$_i'$) $\Phi_i(0) = 0$, Φ_i is strictly decreasing on $\mathbb{R}^-$ and $R(\Phi_i) \cup R(-\beta) \supset \mathbb{R}^+$,

(H.2$_i''$) $\Phi_i(0) = 0$, Φ_i is strictly decreasing on $\mathbb{R}^+$ and $R(\Phi_i) \cup R(-\beta) \supset \mathbb{R}^-$,

(H.2$_i'''$) $\Phi_i(0) = 0$, Φ_i is strictly increasing on $\mathbb{R}^-$ and $R(\Phi_i) \cup R(\beta) \supset \mathbb{R}^-$.

As an easy adaptation of Theorem 4.1, we have

Corollary 4.1. *Suppose* $N \geq 2$, $f \in L^1_{\text{loc}}(\mathbb{R};L^\infty(\mathbb{R}^{N-1}))$ *and* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *is a solution of* $(\text{S.P.})_f$ *in Kruzkov sense in* $\mathbb{R}^N$. *Then for any fixed* $\alpha \in \mathbb{R}$ *we have*

i) under hypotheses (H.0), (H.2$_1'$) *and* (H.3$_1$), *if* u_α *is the solution of*

$$\begin{aligned} &\frac{d}{dx_1}(\Phi_1(u_\alpha)) + \beta(u_\alpha) \ni -\|f^-(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \quad \text{in} \quad (-\infty,\alpha), \\ &u_\alpha(\alpha) = -\|u^-\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{4.16}$$

then $u_\alpha(x_1) \leq -u^-(x_1,x') \leq 0$, a.e. *in* $(-\infty,\alpha) \times \mathbb{R}^{N-1}$;

ii) under hypotheses (H.0), (H.2$_1''$) *and* (H.3$_1$), *if* u_α *is the solution of*

$$\begin{aligned} &\frac{d}{dx_1}(\Phi_1(u_\alpha)) + \beta(u_\alpha) \ni \|f^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \quad \text{in} \quad (-\infty,\alpha), \\ &u_\alpha(\alpha) = \|u^+\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{4.17}$$

then $0 \leq u^+(x_1,x') \leq u_\alpha(x_1)$, a.e. *in* $(-\infty,\alpha) \times \mathbb{R}^{N-1}$;

iii) under hypotheses (H.0), (H.2$_1'''$) *and* (H.3$_1$), *if* u_α *is the solution of*

$$\begin{aligned} &\frac{d}{dx_1}(\Phi_1(u_\alpha)) + \beta(u_\alpha) \ni -\|f^-(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \quad \text{in} \quad (\alpha,+\infty), \\ &u_\alpha(\alpha) = -\|u^-\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{4.18}$$

then $u_\alpha(x_1) \leq -u^-(x_1,x') \leq 0$, a.e. *in* $(\alpha,+\infty) \times \mathbb{R}^{N-1}$.

Remark 4.3. In equations (4.7), (4.16), (4.17) and (4.18) the initial condition on u_α can be slightly modified in order to obtain a more accurate estimate: for example Theorem 4.1 is still valid if

$$u_\alpha(\alpha) = \lim_{t \downarrow \alpha} \|u^+\|_{L^\infty((-\infty,t)\times\mathbb{R}^{N-1})}.$$

Remark 4.4. The condition of strict monotonicity assumed on Φ_i in (H.2_i) can be relaxed in some cases. For example, when Φ_i is not increasing on $\mathbb{R}^+$ but there exists $(a_j) \in \mathbb{R}^N$, $a_1 \neq 0$ such that $\sum_{j=1}^N a_j\Phi_j$ is strictly increasing on $\mathbb{R}^+$ and $\sum_{j=1}^N a_j\Phi_j(0) = 0$, we can make the change of variables $X_1 = \sum_{j=1}^N a_jx_j$ and $X_i = x_i$ for $i \geq 2$. The function v defined by $v(X_1,\ldots,X_N) = u(x_1,\ldots,x_N)$ belongs to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and satisfies

$$\frac{\partial}{\partial X_1}\left(\sum_{i=1}^N a_i\Phi_i(v)\right) + \sum_{i=2}^N \frac{\partial}{\partial X_i}(\Phi_i(v)) + \beta(v) \ni f, \tag{4.19}$$

in Kruzkov sense in $\mathbb{R}^N$.

When $\beta^{-1}(0) \cap \mathbb{R}^+ = \{0\}$, we have a *nonpropagation phenomenon*

Proposition 4.1. *Under hypotheses* (H.0), (H.2_1) *and* (H.3_1), *we suppose that* $\beta^{-1}(0) \cap \mathbb{R}^+ = \{0\}$ *and that* $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *is such that* $f^+ = 0$ *in* $(-\infty,\delta) \times \mathbb{R}^{N-1}$. *If* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *is a solution in Kruzkov sense in* $\mathbb{R}^N$ *of* (S.P.)$_f$, *then* $u^+ = 0$ *in* $(-\infty,\delta) \times \mathbb{R}^{N-1}$.

Proof. For any α, the solution u_α of

$$(4.20)\qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(u_\alpha)) + \beta(u_\alpha) \ni 0 \qquad \text{in } (\alpha,+\infty),\\ &u_\alpha(\alpha) = \|u^+\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

goes to 0 as x_1 goes to $+\infty$, uniformly with respect to α (in the sense that $u_\alpha(x_1)$ only depends on $x_1 - \alpha$). For any $\varepsilon > 0$ and $\lambda < \delta$ there exists $\alpha < \lambda$ such that $u_\alpha(x_1) < \varepsilon$ on $[\lambda,\delta]$; hence $u(x_1,x') < \varepsilon$ in $(\lambda,\delta) \times \mathbb{R}^{N-1}$. Since ε and λ are arbitrary, $u^+ = 0$ in $(-\infty,\delta) \times \mathbb{R}^{N-1}$.

As a consequence we have the triviality of the kernel of A.

Corollary 4.2. *Under hypotheses* (H.0), (H.2_i), (H.$2'_i$) (*or* (H.$2'''_i$)) *and* (H.3_i) *for some* $1 \leq i \leq N$, *if we suppose that* $0 = \beta^{-1}(0)$, *then the only* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *which satisfies*

$$\text{(S.P.)}_0 \qquad \sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) \ni 0,$$

in Kruzkov sense in $\mathbb{R}^N$ *is the zero function.*

4.2. The evolution case. Our first global comparison result is a consequence of Lemma 2.3 and the first step of the proof of Theorem 1.1.

Proposition 4.2. *Assume* (H.0) and (H.1). *For* $f \in L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ *and* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$, *let* u *be the semigroup (and also Kruzkov) solution of* $(\text{C.P.})_f$ *in* $\mathbb{R}^+ \times \mathbb{R}^N$ *with initial data* u_0. *Let* v^+ *(respectively* v^-*) be the solution of*

$$\frac{dv}{dt} + \beta(v) \ni g \qquad \text{in } (0,+\infty), \qquad v(0) = v_0, \tag{4.21}$$

with $v_0 \geq \|u_0^+\|_{L^\infty}$ *and* $g(t) \geq \|f^+(t,.)\|_{L^\infty}$ *in* $(0,+\infty)$ *(respectively* $v_0 \leq -\|u_0^-\|_{L^\infty}$ *and* $g(t) \leq -\|f^-(t,.)\|_{L^\infty}$*). Then* $u(t,x) \leq v^+(t)$ *in* $(0,+\infty) \times \mathbb{R}^N$ *(respectively* $u(t,x) \geq v^-(t)$*).*

The next result has a unidirectional character.

Proposition 4.3. *We suppose that* (H.0), (H.1) *and* $(\text{H}.2_1)$ *hold. Let* $f \in L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, $g \in L^1_{loc}(\mathbb{R}^+; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$, $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *and* $v_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $v_0(x_1) \in \overline{D(\beta)}$ a.e. *in* $\mathbb{R}$. *We call* $v(t,x_1)$ *the solution of*

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(v)) + \beta(v) \ni g \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}, \qquad v(0,x_1) = v_0(x_1) \qquad \text{in } \mathbb{R}, \tag{4.22}$$

and $u(t,x_1,x')$, *the solution of* $(\text{C.P.})_f$. *If* $u_0^+(x_1,x') \leq v_0(x_1)$ a.e. *in* $\mathbb{R} \times \mathbb{R}^{N-1}$ *and* $f^+(t,x_1,x') \leq g(t,x_1)$ a.e. *in* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$, *then* $u^+(t,x_1,x') \leq v(t,x_1)$ a.e. *in* $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^{N-1}$.

Before proving this result we need

Lemma 4.2. *Suppose* $N \geq 2$ *and* (H.0), (H.1) *and* $(\text{H}.2_1)$ *hold. Let* $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *and* $v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *be such that*

$$\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u)) + \beta(u) + u \ni f, \tag{4.23}$$

and

$$\frac{d}{dx_1}(\Phi_1(v)) + \beta(v) + v \ni g, \tag{4.24}$$

hold in Kruzkov sense in $\mathbb{R}^N$ *and* $\mathbb{R}$ *respectively. If* $g(x_1) \geq f^+(x_1,x')$ a.e. *in* $\mathbb{R} \times \mathbb{R}^{N-1}$, *then* $v(x_1) \geq u^+(x_1,x')$ a.e. *in* $\mathbb{R} \times \mathbb{R}^{N-1}$.

Proof. From Theorem 1.2 u and v are the unique solutions of (4.23) and (4.24) respectively. Since they depend continuously on f and g we can suppose that

$$\operatorname{supp} f \subset \{x = (x_1,x') : x_1^2 + |x'|^2 \leq R^2\} \quad \text{and} \quad \operatorname{supp} g \subset [-R,R].$$

Moreover v is a nonnegative absolutely continuous function since (4.24) holds in $\mathfrak{D}'(\mathbb{R})$; so, for any α, it coincides on $(\alpha,+\infty)$ with the solution of

$$\frac{d}{dx_1}(\Phi_1(v_\alpha)) + \beta(v_\alpha) + v_\alpha \ni g \qquad \text{on } (\alpha,+\infty), \tag{4.25}$$

$$v_\alpha(\alpha) = v(\alpha).$$

If w_α is the solution of (4.25) with initial data $\max(v(\alpha),\|u^+\|_{L^\infty})$, we deduce from Theorem 4.1 and classical comparison theorems that

$$w_\alpha(x_1) \geq v(x_1) \quad \text{and} \quad w_\alpha(x_1) \geq \|u^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})}, \tag{4.26}$$

a.e. in $(\alpha,+\infty)$. Since g vanishes outside $[-R,R]$, for any $x_1 < -R$, we have $\lim\limits_{\alpha\to-\infty} w_\alpha(x_1) = 0 = v(x_1)$. From the contraction property, we get for any $x_1 > \tilde{x}_1 > \alpha$ that

$$|w_\alpha(x_1) - v(x_1)| \leq |w_\alpha(\tilde{x}_1) - v(\tilde{x}_1)|. \tag{4.27}$$

If we take $\alpha < \tilde{x}_1 < -R$ and make $\alpha \to -\infty$ we get $\lim\limits_{\alpha\to-\infty} w_\alpha(x_1) = v(x_1)$, and we deduce the result from (4.26).

Proof of Proposition 4.3. There exists a sequence of partitions $\{P_n\} = 0 = t_n^0 < t_n^1 < \ldots < t_n^{N(n)} = T\}$ and two sequences of step functions $\{f_n\}$ and $\{g_n\}$ taking the values $f_n^k \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $g_n^k \in L^1(\mathbb{R} \cap L^\infty(\mathbb{R}$ on (t_n^{k-1},t_n^k) such that

$$\lim_{n\to+\infty} \max_k (t_n^k - t_n^{k-1}) = 0$$

and

$$\lim_{n\to+\infty} \|f_n - f\|_{L^1(0,T;L^1\cap L^\infty)} = 0 = \lim_{n\to+\infty} \|g_n - g\|_{L^1(0,T;L^1\cap L^\infty)}.$$

Moreover since $f^+ \in L^1(0,T;L^1(\mathbb{R};L^1(\mathbb{R};L^\infty(\mathbb{R}^{N-1})))$ we can suppose that $f_n^k \in L^1(\mathbb{R};L^\infty(\mathbb{R}^{N-1}))$. Since $f^+(t,x_1,x') \leq g(t,x_1)$ we can assume that

$$(f_n^k)^+(x_1,x') \leq g_n^k(x_1) \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^{N-1}.$$

Let u_n and v_n be the continuous piecewise linear approximations of u and v defined through the implicit schemes

$$\frac{u_n^k - u_n^{k-1}}{t_n^k - t_n^{k-1}} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(\Phi_i(u_n^k)) + \beta(u_n^k) \ni f_n^k \qquad \text{in } \mathbb{R}^N, \tag{4.28}$$

$$u_n^0 = u_0 \qquad \text{in } \mathbb{R}^N,$$

$$\frac{v_n^k - v_n^{k-1}}{t_n^k - t_n^{k-1}} + \frac{d}{dx_1}(\Phi_1(v_n^k)) + \beta(v_n^k) \ni g_n^k \qquad \text{in } \mathbb{R}^N;$$

(4.29)

$$v_n^0 = v_0 \qquad \text{in } \mathbb{R}.$$

Since $\{u_n\}$ and $\{v_n\}$ converge to u and v in $C([0,T];L^1(\mathbb{R}^N))$ and $C([0,T];L^1(\mathbb{R}))$, it is sufficient to prove $(u_n^k)^+ \leq v_n^k$, which by induction reduces to $(u_n^1)^+ \leq v_n^1$. As the first step we have

$$(t_n^1 - t_n^0)\sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u_n^1)) + (t_n^1 - t_n^0)\beta(u_n^1) + u_n^1 \ni u_0 + (t_n^1 - t_n^0)f_n^1,$$

(4.30)

$$(t_n^1 - t_n^0)\frac{d}{dx_1}(\Phi_1(v_n^1)) + (t_n^1 - t_n^0)\beta(v_n^1) + v_n^1 \ni v_0 + (t_n^1 - t_n^0)g_n^1.$$

Since $v_0 \geq u_0^+$ and $g_n^1 \geq (f_n^1)^+$, we are finished because of Lemma 4.2.

The next result is a unidirectional version of Theorem 1 of [21] with weaker hypotheses.

Proposition 4.4. *Under hypotheses* (H.0) *and* (H.3_1), *we suppose that for* $j = 1, 2$ *and* $f_j \in L^1_{\mathrm{loc}}(0,T;L^1(\mathbb{R}^N))$, *the functions*

$$u_j \in L^1((0,T) \times \mathbb{R}^N) \cap L^\infty((0,T) \times \mathbb{R}^N)$$

are solutions of

$$\frac{\partial u_j}{\partial t} + \sum_{i=1}^N \frac{\partial}{\partial x_i}(\Phi_i(u_j)) + \beta(u_j) \ni f_j, \tag{4.31}$$

in Kruzkov sense in $(0,T) \times \mathbb{R}^N$. *We suppose moreover that* Φ_1 *is Lipschitz continuous on the set*

$$\{r \in \mathbb{R} : |r| \leq \max_{j=1,2} \|u_j\|_{L^\infty((0,T)\times\mathbb{R}^N)}\}$$

with constant of Lipschitz C and, for $a \in \mathbb{R}^N$ *and* $t \in (0,T)$, *we let*

$$S_1(t) = \{x = (x_1,x') : |x_1 - a_1| \leq C(T-t)\}. \tag{4.32}$$

Then for any $0 < s \leq t < T$, *we have*

$$\int_{S_1(t)} (u_1 - u_2)^+(t,x)\,dx \leq \int_{S_1(s)} (u_1 - u_2)^+(s,x)\,dx + \int_s^t \int_{S_1(\tau)} (f_1 - f_2)^+(\tau,x)\,dx d\tau. \tag{4.33}$$

Proof. For the sake of simplicity we take $a = 0$. Let $\gamma \in \mathfrak{D}^+(0,T)$, $\zeta \in \mathfrak{D}^+(\mathbb{R}^{N-1})$ and $\rho \in \mathfrak{D}^+(\mathbb{R})$ be such that $\zeta(0) = 1$ $\int_{-\infty}^{+\infty}(\rho(s)\,ds = 1$. For m and $n \in \mathbb{N}$ we define

$$X_m(t,x_1) = \int_{m(|x_1|+C(t-T))}^{+\infty} \rho(s)\,ds$$

and

$$\zeta_{m,n}(t,x_1,x') = \gamma(t)X_m(t,x_1)\,\zeta\left(\frac{x'}{n}\right).$$

For m large enough $\zeta_{m,n} \in W_0^{1,1}((0,T) \times \mathbf{R}^N)$. With a slight extension of Lemma 1.1, there exists $\alpha \in L^\infty((0,T) \times \mathbf{R}^N)$, $\alpha \in \operatorname{sign}^+(u_1 - u_2)$ such that

$$\iint_{(0,T)\times\mathbf{R}^N} \alpha\left\{(u_1 - u_2)\left(\frac{d\gamma}{dt} X_m \zeta\left(\frac{\cdot}{n}\right) + \gamma\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial t}\right) + (\Phi_1(u_1) - \Phi_1(u_2))\gamma\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial x_1} + \sum_{i=2}^{N} (\Phi_i(u_1) - \Phi_i(u_2))\gamma X_m \frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right) + (f_1 - f_2)\zeta_{m,n}\right\} dxdt \geq 0.$$

But

$$\alpha\left\{(u_1 - u_2)\gamma\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial t} + (\Phi_1(u_1) - \Phi_1(u_2))\gamma\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial x_1}\right\} = -m\rho(m|x_1| + C(t - T))\zeta\left(\frac{\cdot}{n}\right)\gamma\left\{C(u_1 - u_2)^+ + \alpha\operatorname{sign}(x_1)(\Phi_1(u_1) - \Phi_1(u_2))\right\} \leq 0.$$

So we get

$$\iint_{(0,T)\times\mathbf{R}^N} \alpha\left\{(u_1 - u_2)\frac{d\gamma}{dt} X_m \zeta\left(\frac{\cdot}{n}\right) + \sum_{i=2}^{N} (\Phi_i(u_1) - \Phi_i(u_2))\gamma X_m \frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right) + (f_1 - f_2)\zeta_{m,n}\right\} dxdt \geq 0. \tag{4.34}$$

Using Lemma 2.1 as we did in Theorem 4.1 we have

$$\lim_{n\to+\infty} \iint_{(0,T)\times\mathbf{R}^N} \alpha \sum_{i=2}^{N} (\Phi_i(u_1) - \Phi_i(u_2))\gamma X_m \frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right) dxdt = 0, \tag{4.35}$$

so we get

$$\iint_{(0,T)\times\mathbf{R}^N} \left\{(u_1 - u_2)^+ X_m \frac{d\gamma}{dt} + (f_1 - f_2)^+ X_m \gamma\right\} dxdt \geq 0. \tag{4.36}$$

When $m \to +\infty$, $X_m(t)$ converges to the characteristic function of $S_1(t)$. If we fix s and t $(0 < s < t < T)$ and replace $d\gamma/dt$ by some sequence converging to $\delta(s) - \delta(t)$, we get (4.33).

Remark 4.5. Proposition 4.4 gives us a new way for getting uniqueness of solutions of $(\text{C.P.})_f$. When $N = 1$ the assumption on u_j can be weakened and replaced by $u_j \in L^\infty_{\text{loc}}((0,T) \times \mathbb{R})$, $j = 1, 2$.

5. Qualitative Properties of Solutions of $(\text{S.P.})_f$

In this section we obtain many qualitative properties for solutions of $(\text{S.P.})_f$ which can be compared to those of solutions of second order nonlinear elliptic equations (see [16] and [25]). The greatest difference lies in the unidirectional character of our properties

Proposition 5.1. (Uniform boundedness principle). *Under hypotheses* (H.0), (H.2_1) *and* (H.3_1), *if we suppose moreover that*

$$\int_1^{+\infty} \frac{d\sigma}{\beta^0 \circ \Phi_1^{-1}(\sigma)} < +\infty, \tag{5.1}$$

then there exists a nonincreasing function $\theta : (0,+\infty) \to \mathbb{R}^+$ *such that for all* $T \in R$, *all* $f \in L^1_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{R}^{N-1}))$ *which are nonpositive in* $(T,+\infty) \times \mathbb{R}^{N-1}$ *and all* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *solutions of* $(\text{S.P.})_f$ *in Kruzkov sense in* $\mathbb{R}^N$, *we have* a.e. *in* $(T,+\infty) \times \mathbb{R}^{N-1}$

$$u(x_1,x') \le \theta(x_1 - T). \tag{5.2}$$

Proof. From (5.1) $\mathbb{R}^+ \subset R(\Phi_1)$. Since (5.2) is obvious when $\sup D(\beta) < +\infty$, we assume $\mathbb{R}^+ \subset D(\beta)$ and let v_R be the solution of

$$\begin{gathered} \frac{d}{dx_1}(\Phi_1(v_R)) + \beta(v_R) \ni 0 \qquad \text{in } (T,+\infty), \\ v_R(T) = R. \end{gathered} \tag{5.3}$$

v_R can be computed with the formula

$$x_1 - T = \int_{\Phi_1(v_R(x_1))}^{\Phi_1(R)} \frac{d\sigma}{\beta^0 \circ \Phi_1^{-1}(\sigma)}, \qquad \text{for } x_1 > T. \tag{5.4}$$

If $R \ge \|u^+\|_{L^\infty(\mathbb{R}^N)}$, then $v_R(x_1) \ge \|u^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})}$. If $R \to +\infty$, $v_R(x_1)$ increases and converges to $\theta(x_1 - t)$ where θ is defined by

$$r = \int_{\Phi_1(\theta(r))}^{+\infty} \frac{d\sigma}{\beta^0 \circ \Phi_1^{-1}(\sigma)}, \tag{5.5}$$

which yields (5.2.).

We obtain a global uniform boundedness property by introducing the following hypotheses for $1 \leq i \leq N$

One of the following pairs of hypotheses is satisfied:

(H.2$_i^*$) {(H.2$_i$),(H.2$_i'$)}, {(H.2$_i$),(H.2$_i'''$)}, {(H.2$_i''$),(H.2$_i'$)} and {(H.2$_i''$),(H.2$_i'''$)}.

We let $K = \{(x_i) \in \mathbb{R}^N : a_i \leq x_i \leq b_i, \forall\, 1 \leq i \leq N\}$ and $L_0^\infty(K)$ be the space of all measurable and bounded functions with support in K. As a consequence of Proposition 5.1, Proposition 4.1 and Corollary 4.1 we have

Corollary 5.1. *Under hypotheses* (H.0), (H.2$_i^*$) *and* (H.3$_i$) *for any* $1 \leq i \leq N$, *if we suppose moreover that* $0 = \beta^{-1}(0)$ *and*

$$\int_{|\sigma|>1} \frac{d\sigma}{|\beta^0 \circ \Phi_i^{-1}(\sigma)|} < +\infty \qquad \text{if } \Phi_i \text{ is monotone,}$$

$$(5.6) \qquad \int_1^{+\infty} \frac{d\sigma}{|\beta^0 \circ \Phi_i^{-1}(\sigma)|} < +\infty \qquad \text{if } \Phi_i \text{ is decreasing on } \mathbb{R}^-, \text{ increasing on } \mathbb{R}^+,$$

$$\int_{-\infty}^{-1} \frac{d\sigma}{|\beta^0 \circ \Phi_i^{-1}(\sigma)|} < +\infty \qquad \text{if } \Phi_i \text{ is increasing on } \mathbb{R}^-, \text{ decreasing on } \mathbb{R}^+,$$

then there exists a function Θ *defined in* $\mathbb{R}^N - K$, *bounded in the complement of any neighbourhood of* K, *such that for any* $f \in L_0^\infty(K)$ *and any* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *satisfying* (S.P.)$_f$ *in Kruzkov sense in* $\mathbb{R}^N$ *we have* a.e. *in* $\mathbb{R}^N - K$

$$(5.7) \qquad u(x) \leq \Theta(x).$$

Proposition 5.2. (Compact support property). *Under hypotheses* (H.0), (H.2$_1$) *and* (H.3$_1$), *if we suppose moreover that*

$$(5.8) \qquad \int_0^1 \frac{d\sigma}{\beta^0 \circ \Phi_1^{-1}(\sigma)} < +\infty,$$

then for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}; L^\infty(\mathbb{R}^{N-1}))$ *with* supp $f^+ \subset (-\infty, T) \times \mathbb{R}^{N-1}$ *and any* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *solution of* (S.P.)$_f$ *in Kruzkov sense in* $\mathbb{R}^N$, *there exists* $T' > T$ *such that* supp $u^+ \subset (-\infty, T') \times \mathbb{R}^{N-1}$.

The proof is essentially the same as the one for Proposition 5.1 in noticing that the solution v_R of (5.3) vanishes for $x_1 > T'$ where

$$(5.9) \qquad T' = T + \int_0^{\Phi_1(R)} \frac{d\sigma}{\beta^0 \circ \phi_1^{-1}(\sigma)}.$$

As a consequence we have a global compact support property.

Corollary 5.2. *Under hypotheses* (H.0), (H.2$_i^*$) *and* (H.3$_i$) *for all* $1 \leq i \leq N$ *if*

$$\int_{-1}^{1} \frac{d\sigma}{|\beta^0 \circ \phi_i^{-1}(\sigma)|} < +\infty \qquad \textit{if } \Phi_i \textit{ is monotone,}$$

$$(5.10) \quad \int_{0}^{1} \frac{d\sigma}{|\beta^0 \circ \Phi_i^{-1}(\sigma)|} < +\infty \qquad \textit{if } \Phi_i \textit{ is decreasing on } \mathbb{R}^-, \textit{ increasing on } \mathbb{R}^+,$$

$$\int_{-1}^{0} \frac{d\sigma}{|\beta^0 \circ \Phi_i^{-1}(\sigma)|} < +\infty \qquad \textit{if } \Phi_i \textit{ is increasing on } \mathbb{R}^-, \textit{ decreasing on } \mathbb{R}^+,$$

then for any $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *with compact support, any solution* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of* (S.P.)$_f$ *in Kruzkov sense in* $\mathbb{R}^N$ *also has a compact support.*

Remark 5.2. We can obtain other compactness results if we suppose that Int $\beta(0) = (\beta^-(0),\beta^+(0))$ is nonempty. *If* (H.0), (H.2$_i^*$) *and* (H.3$_i$) *hold for any* $1 \le i \le N$ *and if there exists a function* $\varepsilon : [R,+\infty) \mapsto \mathbb{R}^+$ *such that* $\int_R^{+\infty} \varepsilon(s)\, ds = +\infty$, *then for any* $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *such that*

$$(5.11) \qquad \beta^-(0) + \varepsilon(|x|) \le -f^-(x) \le f^+(x) \le \beta^+(0) - \varepsilon(|x|)$$

holds for $|x| > R$, *any* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *solution of* (S.P.)$_f$ *in Kruzkov sense in* $\mathbb{R}^N$ *has a compact support.*

6. Qualitative Properties of Solutions of (C.P.)$_f$

6.1. Finite propagation speed. The finite propagation speed of the support of the solution of (C.P.)$_f$ is a characteristic phenomenon of first and second order hyperbolic equations and it appears in equation (C.P.)$_0$ under two very different hypotheses on Φ

i) Φ is locally Lipschitz continuous,

ii) $\forall\, 1 \le i \le N, \displaystyle\int_{-1}^{1} \frac{d\sigma}{|\Phi_i^{-1}(\sigma)|} < +\infty.$

Under the first hypothesis our results are just a generalization of the results of [21], but under the second hypothesis they are completely new.

Proposition 6.1. *Suppose* (H.0) *and* (H.1) *hold and* Φ_1 *is Lipschitz continuous with constant* C *on* $\{r : 0 \le r \le \|u_0^+\|_{L^\infty}\}$ *for some* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$. *If* $u_0^+(x_1,x')$ *vanishes outside* $\{(x_1,x') : |x_1 - a_1| \le R\}$ *and if* u *is the solution of* (C.P.)$_0$ *with initial data* u_0, *then* $u^+(t,x_1,x')$ *vanishes outside* $\{(x_1,x') : |x_1 - a_1| \le R + Ct\}$.

Proof. Under our hypotheses the solution of (C.P.)$_0$ exists and belongs to $C(\mathbb{R}^+;L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^N)$. The equation is satisfied in Kruzkov sense in $(0,+\infty) \times \mathbb{R}^N$ and $t \mapsto \|u^+(t,.)\|_{L^\infty}$ is nonincreasing. For $n \in \mathbb{N}^*$ and $0 \le t \le n/C$ we define

$$S_n(t) = \{(x_1,x') : |R + a_1 + n - x_1| \le n - Ct\}.$$

In particular $R + a_1 + Ct \le x_1 \le R + a_1 + 2n - Ct$. By Proposition 4.4 we have

$$\int_{S_n(t)} u^+(t,x)\,dx \le \int_{S_n(0)} u_0^+(x)\,dx. \tag{6.1}$$

But $(x_1,x') \in S_n(0) \Leftrightarrow R + a_1 \le x_1 \le R + a_1 + 2n$. Hence

$$\int_{S_n(t)} u^+(t,x)\,dx = 0.$$

If we make $n \to +\infty$, $u^+(t,x)$ vanishes on $\{(x_1,x') : x_1 \ge R + a_1 + Ct\}$. In the same way $u^+(t,x)$ vanishes on $\{(x_1,x') : x_1 \le -R + a_1 - Ct\}$.

For a multidirectional version of Proposition 6.1 we let

$$|x|_\infty = \max_{1 \le i \le N} |x_i|, \qquad x = (x_1,\ldots,x_N).$$

Corollary 6.1. *Suppose* (H.0) *and* (H.1) *hold and* Φ *is Lipschitz continuous with constant* C *on* $\{r : |r| \le \|u^0\|_{L^\infty}\}$ *for some* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$. *If* u_0 *has its support in* $\{x \in \mathbb{R}^N : |x|_\infty \le R\}$, *then the solution* $u(t,.)$ *of* (C.P.)$_0$ *with initial data* u_0 *has its support in* $\{x \in \mathbb{R}^N : |x|_\infty \le R + Ct\}$.

Proposition 6.2. *Suppose* (H.0), (H.1) *and* (H.2$_1$) *hold and*

$$\int_0^1 \frac{ds}{\Phi_1^{-1}(s)} < +\infty. \tag{6.2}$$

Let $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *be such that*

$$\operatorname{supp} u_0^+ \subset \{(x_1,x') : a \le x_1 \le b\}.$$

Then if u *is the solution of* (C.P.)$_0$ *with initial data* u_0, *we have for* $t \ge 0$.

$$\operatorname{supp} u^+(t,.) \subset \left\{(x_1,x') : a \le x_1 \le b + t \int_0^{\Phi_1(\|u_0^+\|_{L^\infty})} \frac{ds}{\Phi_1^{-1}(s)}\right\}.$$

Proof. From Proposition 4.3, $u^+(t,x_1,x')$ is majorized by $v(t,x_1)$ where v is the solution of

$$\begin{aligned} &\frac{\partial v}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(v)) + \beta(v) \ni 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}, \\ &v(0,x_1) = v_0(x_1) = \|u_0^+\|_{L^\infty} \chi_{(a,b)}(x_1), \end{aligned} \tag{6.3}$$

where $\chi_{(a,b)}$ is the characteristic function of $[a,b]$. So it suffices to prove that for any $t \ge 0$

$$\text{supp}\, v(t,.) \subset \left[a, b + t\int_0^{\Phi_1(\|u_0^+\|_{L^\infty})} \frac{ds}{\Phi_1^{-1}(s)}\right]. \tag{6.4}$$

For $n \in \mathbf{N}^*$ we let v_n be the continuous piecewise linear function taking the value v_n^k at kt/n $(0 \leq k \leq n)$ where the sequence $\{v_n^k\}$ is defined by

$$\begin{gathered} \frac{n}{t}(v_n^k - v_n^{k-1}) + \frac{d}{dx_1}(\Phi_1(v_n^k)) + \beta(v_n^k) \ni 0 \qquad \text{in } \mathbf{R}, \\ v_n^0 = v_0, \end{gathered} \tag{6.5}$$

the equation being taken in Kruzkov sense in $\mathbf{R}$. In particular for $k = 1$ we have

$$\frac{d}{dx_1}(\Phi_1(v_n^1)) + \beta(v_n^1) + \frac{n}{t}v_n^1 \ni \frac{n}{t}v_n^0. \tag{6.6}$$

Since $(\beta + n/t)^{-1}(0) = 0$ and $\text{supp}\, v_n^0 \subset [a,+\infty)$, $\text{supp}\, v_n^1 \subset [a,+\infty)$. Moreover for $x_1 \geq b$, v_n^1 coincides with the solution w_n^1 of the following evolution equation

$$\begin{gathered} \frac{d}{dx_1}(\Phi_1(w_n^1)) + \beta(w_n^1) + \frac{n}{t}w_n^1 \ni 0 \qquad \text{on } (b,+\infty), \\ w_n^1(b) = v_n^1(b). \end{gathered} \tag{6.7}$$

Since we have

$$x_1 - b = \int_{\Phi_1(w_n^1(x_1))}^{\Phi_1(w_n^1(b))} \frac{ds}{\beta^0 \circ \Phi_1^{-1}(s) + (n/t)\Phi_1^{-1}(s)},$$

$\Phi_1(w_n^1)$ has a compact support. Since $\Phi_1(w_n^1(b)) \leq \Phi_1(\|v_0\|_{L^\infty})$ we deduce that

$$\text{supp}\, w_n^1 \subset \left[b, b + \int_0^{\Phi_1(\|v_0\|_{L^\infty})} \frac{ds}{\beta^0 \circ \Phi_1^{-1}(s) + (n/t)\Phi_1^{-1}(s)}\right]. \tag{6.8}$$

We define the sequences $\{b_k\}$ and $\{w_n^k\}$ $(k = 1,\dots,n)$ such that

$$b_k = b_{k-1} + \int_0^{\Phi_1(\|v_0\|_{L^\infty})} \frac{ds}{\beta^0 \circ \Phi_1^{-1}(s) + (n/t)\Phi_1^{-1}(s)}, \quad b_0 = b,$$

and w_n^k is the solution of the following evolution equation

$$\begin{gathered} \frac{d}{dx_1}(\Phi_1(w_n^k)) + \beta(w_n^k) + \frac{n}{t}w_n^k \ni 0 \qquad \text{in } (b_{k-1},+\infty), \\ w_n^k(b_{k-1}) = v_n^k(b_{k-1}). \end{gathered} \tag{6.9}$$

Suppose we have proved that for $1 \leq j \leq k-1$ that $\text{supp}\, v_n^j \subset [a,b_j]$. For $x_1 \geq b_{k-1}$, v_n^k coincides with w_n^k and w_n^k is given by

$$(6.11)\qquad x_1 - b_{k-1} = \int_{\Phi_1(w_n^k(x_1))}^{\Phi_1(v_n^k(b_{k-1}))} \frac{ds}{\beta^0\circ\Phi_1^{-1}(s) + (n/t)\,\Phi_1^{-1}(s)}.$$

since $\Phi_1(v_n^k(b_{k-1})) \le \Phi_1(\|v_0\|_{L^\infty})$, supp $w_n^k \subset [b_{k-1},b_k]$. Since v_n^{k-1} is zero in $(-\infty,a)$, it is the same with v_n^k and we get supp $v_n^k \subset [a,b_k]$. Hence

$$(6.12)\qquad \text{supp } v_n^n \subset \left[a, b + n\int_0^{\Phi_1(\|u_0\|_{L^\infty})} \frac{ds}{\beta^0\circ\Phi_1^{-1}(s) + (n/t)\,\Phi_1^{-1}(s)}\right].$$

By Lebesgue's theorem

$$\lim_{n\to+\infty} n\int_0^{\Phi_1(\|u_0\|_{L^\infty})} \frac{ds}{\beta^0\circ\Phi_1^{-1}(s) + (n/t)\,\Phi_1^{-1}(s)} = t\int_0^{\Phi_1(\|u_0\|_{L^\infty})} \frac{ds}{\Phi_1^{-1}(s)}.$$

Since v_n^n converges to $v(t,.)$ in $L^1(\mathbb{R})$, we get (6.4).

Remark 6.1. We can make $a = -\infty$ in Proposition 6.2. For that we replace the initial data u_0 by $\chi_{(R,b)}$, apply the previous result to the corresponding solution u_R of (C.P.)$_0$ and make $R \to -\infty$.

When $\int_0^{+\infty} ds/\Phi_1^{-1}(s) < +\infty$ we obtain a propagation speed independant of u_0.

Remark 6.2. Under hypotheses (H.0), (H.1) and (H.2$_1^*$) we can give nonpropagation results for the support of $u(t,.)$. If we suppose supp $u_0^+ \subset [a,b] \times \mathbb{R}^{N-1}$ and supp $u_0^- \subset [\tilde a,\tilde b] \times \mathbb{R}^{N-1}$, then

$[(\text{H}.2_1),(\text{H}.2_1')] \Rightarrow \text{supp } u^+(t,.) \subset [a,+\infty)\times\mathbb{R}^{N-1}$, $\text{supp } u^-(t,.) \subset (-\infty,\tilde b]\times\mathbb{R}^{N-1}$,

$[(\text{H}.2_1),(\text{H}.2_1''')] \Rightarrow \text{supp } u^+(t,.) \subset [a,+\infty)\times\mathbb{R}^{N-1}$, $\text{supp } u^-(t,.) \subset [\tilde a,+\infty)\times\mathbb{R}^{N-1}$,

$[(\text{H}.2_1''),(\text{H}.2_1')] \Rightarrow \text{supp } u^+(t,.) \subset (-\infty,b]\times\mathbb{R}^{N-1}$, $\text{supp } u^-(t,.) \subset (-\infty,\tilde b]\times\mathbb{R}^{N-1}$,

$[(\text{H}.2_1''),(\text{H}.2_1''')] \Rightarrow \text{supp } u^+(t,.) \subset (-\infty,b)\times\mathbb{R}^{N-1}$, $\text{supp } u^-(t,.) \subset [\tilde a,+\infty)\times\mathbb{R}^{N-1}$.

The multidirectional version of Proposition 6.2 is the following

Corollary 6.2. *Under hypotheses* (H.0), (H.1) *and* (H.2$_i^*$) *for all* $1 \le i \le N$, *if we suppose moreover that*

$$\int_{-1}^{1}\frac{ds}{|\Phi_i^{-1}(s)|} < +\infty \qquad \text{if } \Phi_i \text{ is monotone},$$

$$(6.13)\qquad \int_0^1 \frac{ds}{|\Phi_1^{-1}(s)|} < +\infty \qquad \text{if } \Phi_i \text{ is decreasing on } \mathbb{R}^-, \text{ increasing on } \mathbb{R}^+,$$

$$\int_{-1}^{0}\frac{ds}{|\Phi_i^{-1}(s)|} < +\infty \qquad \text{if } \Phi_i \text{ is increasing on } \mathbb{R}^-, \text{ decreasing on } \mathbb{R}^+,$$

then for any $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *with support in* $\{x \in \mathbb{R}^N : |x - a|_\infty \le \mathbb{R}\}$, *the solution* $u(t,.)$ *of* (C.P.)$_0$ *with initial data* u_0 *has its support in* $\{x \in \mathbb{R}^N : |x -$

$a|_\infty \le R + Ct\}$ *for* $t \ge 0$, *where* C *depends on* Φ *and* $\|u_0\|_{L^\infty}$.

In some cases it is useful to exchange the role of t and one of the x_i; for example

Proposition 6.3. *Suppose* (H.0) *and* (H.1) *hold,* Φ_1 *is strictly increasing on* $\mathbb{R}^+$ *and* Φ_1^{-1} *is Lipschitz continuous with constant* C *on* $\{r : 0 \le r \le \|u_0^+\|_{L^\infty}\}$ *where* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$. *If* u *is the solution of* (C.P.)$_0$ *with initial data* u_0 *and if* $\operatorname{supp} u_0^+ \subset \{(x_1,x') : x_1 \ge a\}$, *then for any* $t \ge 0$ $\operatorname{supp} u^+(t,.) \subset \{(x_1,x') : x_1 \ge a + t/C\}$.

Proof. We let $w = \Phi_1(u)$ and, as we did in Proposition 4.4, we consider for $\varepsilon > 0$, ρ and γ belonging to $\mathcal{D}^+(\mathbb{R})$ such that $\operatorname{supp}\gamma \subset [a - \varepsilon/2, +\infty)$, $\int_{-\infty}^{+\infty}\rho(s)\,ds = 1$, $\eta \in \mathcal{D}^+([0,+\infty))$ such that $\eta = 1$ on $[0,T]$ and $d\eta/dt \le 0$, and $\zeta \in \mathcal{D}^+(\mathbb{R}^{N-1})$ such that $\zeta(0) = 1$. Let

$$X_m(t,x_1) = \int_{m(x_1+\varepsilon-a-t/C)}^{+\infty} \rho(s)\,ds \quad \text{and} \quad \zeta_{m,n}(t,x_1,x') = \eta(t)\gamma(x_1)X_m(t,x_1)\zeta\left(\frac{x'}{n}\right).$$

For $t/C < \varepsilon/3$, $x_1 + \varepsilon - a - t/C > x_1 - a + 2\varepsilon/3$; so $x_1 - a - 2\varepsilon/3 > \varepsilon/6$ on the support of γ and $\gamma X_m \in \mathcal{D}^+((0,T) \times \mathbb{R})$ for m large enough. Hence $\zeta_{m,n} \in \mathcal{D}^+((0,T) \times \mathbb{R}^N)$ and we have from Lemma 1.1 that

$$\int\int_{u>0}\left(u\frac{\partial}{\partial t}\zeta_{m,n} + \sum_{i=1}^{N}\Phi_i(u)\frac{\partial}{\partial x_i}\zeta_{m,n}\right)dx\,dt \ge 0, \tag{6.14}$$

(for the sake of simplicity we suppose $N \ge 2$ so H.1 implies $\Phi(0) = 0$). (6.14) can be written as

$$\begin{aligned}\int\int\int_{w>0}\Bigg\{& w\left(\frac{d\gamma}{dx_1}\eta X_m\zeta\left(\frac{\cdot}{n}\right) + \gamma\eta\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial x_1}\right) + \Phi_1^{-1}(w)\left(\gamma\eta\zeta\left(\frac{\cdot}{n}\right)\frac{\partial X_m}{\partial t}\right.\\ &\left. + \gamma X_m\zeta\left(\frac{\cdot}{n}\right)\frac{d\eta}{dt}\right) + \sum_{i=2}^{N}\Phi_i\circ\Phi_1^{-1}(w)\gamma X_m\eta\frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right)\Bigg\}dx_1\,dt\,dx' \ge 0.\end{aligned} \tag{6.15}$$

But

$$w\left(\frac{\partial X_m}{\partial x_1}\right) + \Phi_1^{-1}(w)\left(\frac{\partial X_m}{\partial t}\right) \le 0 \quad \text{and} \quad \Phi_1^{-1}(w)\,\gamma X_m\zeta\left(\frac{\cdot}{n}\right)\left(\frac{d\eta}{dt}\right) \le 0$$

on $\{(t,x) : w(t,x) > 0\}$. When $m \to +\infty$, $X_m(t,x_1)$ converges to the characteristic function $X_\varepsilon(t,x_1)$ of $\{(t,x_1) : x_1 < a - \varepsilon + t/c\}$, so we get

$$\begin{aligned}\int\int\int_{w>0}\Bigg\{& w\frac{d\gamma}{dx_1}\eta X_\varepsilon\zeta\left(\frac{\cdot}{n}\right)\\ & + \sum_{i=2}^{N}\Phi_i\circ\Phi_1^{-1}(w)\gamma X_\varepsilon\eta\frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right)\Bigg\}dx_1\,dt\,dx' \ge 0.\end{aligned} \tag{6.16}$$

If we make η go to the characteristic function of $(0,T)$ and γ go to $\delta(a - \varepsilon/2) - \delta(\alpha)$ for some $\alpha > a$, we obtain (for almost all ε)

$$(6.17)\quad \int_0^T \int w^+(t,\alpha,x')X_\varepsilon(t,\alpha)\zeta\left(\frac{x'}{n}\right) dx'\,dt$$
$$\leq \int_0^T \int w^+\left(t,a-\frac{\varepsilon}{2},x'\right)X_\varepsilon\left(t,a-\frac{\varepsilon}{2}\right)\zeta\left(\frac{x'}{n}\right) dx'\,dt$$
$$+ \int_0^T \int_{a-\varepsilon/2}^{\alpha} \int_{w>0} \sum_{i=2}^{N} \Phi_i \circ \Phi_1^{-1}(w) X_\varepsilon \frac{\partial}{\partial x_i}\left(\zeta\left(\frac{x'}{n}\right)\right) dx'\,dx_1\,dt.$$

From Remark 6.2 the support of u^+ does not propagate on the left of a so $w^+(t,a - \varepsilon/2,x') = 0$ a.e. Moreover, arguing as we did in Proposition 4.3 and Theorem 4.1, we have

$$(6.18)\quad \lim_{n\to+\infty} \int_0^T \int_{a-\varepsilon/2}^{\alpha} \int_{w>0} \sum_{i=2}^{N} \Phi_i \circ \Phi_1^{-1}(w) X_\varepsilon \frac{\partial}{\partial x_i}\left(\zeta\left(\frac{\cdot}{n}\right)\right) dx'\,dx_1\,dt = 0,$$

which yields

$$(6.19)\quad \limsup_{n\to+\infty} \int_0^T \int w^+(t,\alpha,x')X_\varepsilon(t,\alpha)\zeta\left(\frac{x'}{n}\right) dx'\,dt \leq 0.$$

Hence $w^+(t,\alpha,x') = 0$ a.e. for $x' \in \mathbb{R}^{N-1}$ and $T/C > t/C > \alpha - a + \varepsilon$. Making $\varepsilon \to 0$ and $T \to +\infty$ we obtain the result.

Remark 6.3. Thanks to Propositions 6.1 and 6.3, if we suppose moreover that all the Φ_i and Φ_i^{-1} are Lipschitz continuous with constants C_i and $\bar{C}_i$ on $\{r : 0 \leq r \leq \|u_0^+\|_{L^\infty}\}$ for some u_0 in $\overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} u_0^+ \subset \prod_{i=1}^{N} [a_i,b_i]$, then if u is the solution of $(\mathrm{C.P.})_0$ with initial data u_0 we have

$$\operatorname{supp} u^+(t,.) \subset \prod_{i=1}^{N} \left[a_i + \frac{t}{\bar{C}_i}, b_i + tC_i\right].$$

6.2. Localization. In this section we give a sufficient condition on Φ and β such that the support of the solution $u(t,.)$ of $(\mathrm{C.P.})_0$ remains bounded (in some direction) independently of $t > 0$.

Proposition 6.4. *Under hypotheses* (H.0), (H.1) *and* (H.2_1), *we suppose moreover that*

$$(6.20)\quad \int_0^1 \frac{ds}{\beta^0 \circ \Phi_1^{-1}(s)} < +\infty.$$

If $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *is such that* $\operatorname{supp} u_0^+ \subset \{(x_1,x') : a \leq x_1 \leq b\}$ *and if*

u is the solution of (C.P.)$_0$ *with initial data* u_0, *there exists* $b^* > b$ *such that* supp $u^+(t,.) \subset \{(x_1,x') : a \le x_1 \le b^*\}$ *for any* $t > 0$.

Proof. From Remark 6.2 supp $u^+(t,.) \subset [a,+\infty) \times \mathbb{R}^{N-1}$ and from Proposition 4.3, $u^+(t,x_1,x')$ is majorized by the solution v of

$$(6.21)\qquad \begin{aligned} &\frac{\partial v}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(v)) + \beta(v) \ni 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R},\\ &v(0,x_1) = v_0(x_1) = \|u_0^+\|_{L^\infty}\chi_{(a,b)}(x_1). \end{aligned}$$

For ε and $\lambda > 0$ we let w_ε be the solution of the evolution equation

$$(6.22)\qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(w_\varepsilon)) + \beta_\lambda(w_\varepsilon) \ni 0 \qquad \text{in } (b,+\infty),\\ &w_\varepsilon(b) = \|u_0^+\|_{L^\infty} + \varepsilon, \end{aligned}$$

and we define $\tilde{w}_\varepsilon$ in $\mathbb{R}^+ \times (b,+\infty)$ by $\tilde{w}_\varepsilon(t,x_1) = w_\varepsilon(x_1)$. As we have seen in Proposition 4.1, $\tilde{w}_\varepsilon$ satisfies

$$(6.23)\qquad \frac{\partial \tilde{w}_\varepsilon}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(\tilde{w}_\varepsilon)) + \beta_\lambda(\tilde{w}_\varepsilon) \ni 0,$$

in Kruzkov sense in $(0,+\infty) \times (b,+\infty)$ and there exists $\eta > 0$ such that $\tilde{w}_\varepsilon(t,x_1) > v(t,x_1)$ a.e. in $[0,+\infty) \times [b,b+\eta)$. We consider $\gamma \in \mathcal{D}^+(0,+\infty)$ and $\zeta \in \mathcal{D}^+(b,+\infty)$ such that $d\zeta/dx_1 \le 0$ in $[b+\eta/2,+\infty)$. From Proposition 2 of [2] we have

$$(6.24)\qquad \int_{v>\tilde{w}_\varepsilon} \left\{(v-\tilde{w}_\varepsilon)\zeta\frac{d\gamma}{dt} + (\Phi_1(v) - \Phi_1(\tilde{w}_\varepsilon))\gamma\frac{d\zeta}{dx_1}\right\} dx_1\,dt \ge 0.$$

If we make $d\gamma/dt \to \delta_0 - \delta_t$ we deduce that

$$(6.25)\qquad \begin{aligned} &\int \zeta(v-\tilde{w}_\varepsilon)^+(t,x_1)\,dx_1\\ &\qquad \le \int \zeta(v_0 - w_\varepsilon)^+(x_1)\,dx_1 + \int_0^t\int_{v>\tilde{w}_\varepsilon}(\Phi_1(v) - \Phi_1(\tilde{w}_\varepsilon))\frac{d\zeta}{dx_1}\,dx_1\,dt, \end{aligned}$$

and the right-hand side is nonpositive. Going to the limit as ε and λ go to 0 and ζ to the characteristic function of $[b,+\infty)$, we get $v(t,x_1) \le w(x_1)$ in $\mathbb{R}^+ \times (b,+\infty)$ where w is the solution of

$$(6.26)\qquad \begin{aligned} &\frac{d}{dx_1}(\Phi_1(w)) + \beta(w) \ni 0 \qquad \text{in } (b,+\infty),\\ &w(b) = \|u_0^+\|_{L^\infty}. \end{aligned}$$

If we compute w we deduce

$$\text{(6.27)} \qquad \operatorname{supp} v(t,.) \subset \left[a, b + \int_0^{\Phi_1(\|u_0^+\|_{L^\infty})} \frac{ds}{\beta^0 \circ \Phi_1^{-1}(s)}\right],$$

for any $t \geq 0$.

Remark 6.4. By combining Propositions 6.3 and 6.4 we obtain a compact support property in $\mathbb{R}^+ \times \mathbb{R}$ for $(t,x_1) \mapsto \|u^+(t,x_1,.)\|_{L^\infty}$ where u is the solution of $(\text{C.P.})_0$ with initial data $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} u_0^+ \subset \{(x_1,x') : a \leq x_1 \leq b\}$.

The multidirectional version of Proposition 6.4 is

Corollary 6.3. *Under hypotheses* (H.0), (H.1) *and* (H.2_i^*) *for all* $1 \leq i \leq N$, *if we suppose moreover that condition (5.10) holds, then for any* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *with compact support there exists a compact K of $\mathbb{R}^N$ such that, if u is the solution of* $(\text{C.P.})_0$ *with initial data* u_0, $\operatorname{supp} u(t,.) \subset K$ *for any* $t > 0$.

6.3 Asymptotic behaviour. Our first result is a time-regularizing effect with a uniform boundedness principle.

Proposition 6.5. *Assume* (H.0), (H.1) *and for some* $R > 0$

$$\text{(6.28)} \qquad \int_R^{+\infty} \frac{ds}{\beta^0(s)} < +\infty;$$

then there exists a continuous decreasing function $h : (0,+\infty) \to [0,+\infty)$ *such that for any* $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$, *the solution u of* $(\text{C.P.})_0$ *satisfies* $u(t,x) \leq h(t)$ a.e. *in* $(0,+\infty) \times \mathbb{R}^N$.

Proof. First we notice that (6.28) implies $\max \beta^{-1}(0) < +\infty$. We let

$$R^+ = \max \beta^{-1}(0) \qquad \text{and} \qquad T^+ = \int_{R^+}^{\infty} \frac{ds}{\beta^0(s)},$$

$T^+ \in (0,+\infty]$. From (6.28), $t \mapsto \int_t^{+\infty} ds/\beta^0(s)$ is a decreasing bijection from $[R^+,+\infty)$ to $(0,T^+]$. Let $\tilde{h}$ be the inverse mapping. We define h on $(0,+\infty)$ by $h(t) = \tilde{h}(t)$ on $(0,T^+]$ and $h(t) = R^+$ on $(T^+,+\infty)$. The function h is continuous on $(0,+\infty)$ and it satisfies

$$\text{(6.29)} \qquad \begin{cases} \dfrac{dh}{dt} + \beta(h) \ni 0 \quad \text{a.e. in } (0,+\infty), \\ h(0) = \sup D(\beta) \in \mathbb{R}^+ \cup \{+\infty\}. \end{cases}$$

By Proposition (4.2) $u(t,x) \leq h(t)$ a.e. in $(0,+\infty) \times \mathbb{R}^N$.

Remark 6.4. If we suppose that the following inequality

$$(6.30)\qquad \int_R^{+\infty} \frac{ds}{\beta^0(s)} - \int_{-\infty}^{-R} \frac{ds}{\beta^0(s)} < +\infty,$$

holds for some $R > 0$, there exists $\bar{h}:(0,+\infty) \mapsto \mathbb{R}^+$ such that $\|u(t,.)\|_{L^\infty} \leq \bar{h}(t)$ for any $t > 0$ and any u solution of $(\mathrm{C.P.})_0$ with initial data $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$. Other regularizing effects can be found in [3] and [26].

Using the same ideas we have the following compact support property for $t \mapsto u(t,.)$.

Proposition 6.6. *Assume* (H.0), (H.1) *and*

$$(6.31)\qquad \int_0^{1} \frac{ds}{\beta^0(s)} - \int_{-1}^{0} \frac{ds}{\beta^0(s)} < +\infty;$$

then for any $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ *there exists* $T^* > 0$ *such that the solution* u *of* $(\mathrm{C.P.})_0$ *with initial data* u_0 *has its support in* $[0,T^*] \times \mathbb{R}^N$.

Remark 6.5. When Int $\beta(0) = (\beta^-(0),\beta^+(0))$, and is nonempty, any solution $u(t,.)$ of $(\mathrm{C.P.})_f$ with initial data $u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N)$ vanishes for t large enough provided (H.0) and (H.1) hold and f which belongs to $L^1_{loc}([0,+\infty);L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$ satisfies for $t \geq T_0 > 0$

$$(6.32)\qquad \beta^-(0) + \rho(t) \leq -f^-(t,.) \leq 0 \leq f^+(t,.) \leq \beta^+(0) - \rho(t),$$

where $\rho:[T_0,+\infty) \mapsto \mathbb{R}^+$ is such that $\int_{T_0}^{+\infty} \rho(s)ds = +\infty$.

6.4. Instantaneous shrinking. The instantaneous shrinking of the support of the solutions of $(\mathrm{C.P.})_0$ is a simple consequence of the hyperbolic nature of the equations and of the comparison theorems previously obtained.

Proposition 6.7. *Under hypotheses* (H.0), (H.1) ($\mathrm{H.2}_1$) *and*

$$(6.33)\qquad \int_0^{1} \frac{ds}{\beta^0(s)} < +\infty,$$

we suppose moreover that Φ_1 *is Lipschitz continuous on* $\{r: 0 \leq r \leq \|u_0^+\|_{L^\infty}\}$ *where*

$$u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R};L^\infty(\mathbb{R}^{N-1}))$$

and that

$$\lim_{|x_1|\to+\infty} \|u_0^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} = 0.$$

If u *is the solution of* $(\mathrm{C.P.})_0$ *with initial data* u_0, *then*

i) *there exists* $T^* < +\infty$ *such that* $u^+(t,x) = 0$ *in* $[T^*,+\infty) \times \mathbb{R}^N$,

ii) *there exist two monotone real functions* b^+ *and* b^- *defined in* $(0,+\infty)$ *such that* supp $u^+(t,.) \subset \{(x_1,x'): b^-(t) \leq x_1 \leq b^+(t)\}$ *for any* $t > 0$.

Proof. Part i) was proved in Proposition 6.6. Thanks to Proposition 4.3 it is sufficient to prove ii) for the solution v of

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(v)) + \beta(v) \ni 0 \qquad \text{in} \quad (0,+\infty) \times \mathbb{R}, \tag{6.34}$$

$$v(0,x_1) = v_0(x_1) = \|u_0^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})}$$

since $v(t,x_1) \geq u^+(t,x_1,x')$ a.e. We call C the constant of Lipschitz of Φ_1 on $\{r : 0 \leq r \leq \|u_0^+\|_{L^\infty}\}$. For $\varepsilon > 0$ we define

$$\delta^+(\varepsilon) = \inf\{x_1 : v_0(x \leq \varepsilon \quad \text{a.e. in } (x_1,+\infty)\},$$

$$\delta^-(\varepsilon) = \sup\{x_1 : v_0(x) \leq \varepsilon \quad \text{a.e. in } (-\infty,x_1)\}, \tag{6.35}$$

$$t(\varepsilon) = \int_0^\varepsilon \frac{ds}{\beta^0(s)},$$

and we claim that $v(t,x_1)$ is almost everywhere zero on the set

$$\{(t,x_1) : t \geq t(\varepsilon), x_1 \geq \delta^+(\varepsilon) + Ct\} \cup \{(t,x_1) : t \geq t(\varepsilon), x_1 \leq \delta^-(\varepsilon) - Ct\}. \tag{6.36}$$

For that we call h_ε the solution of the evolution equation

$$\frac{dh_\varepsilon}{dt} + \beta(h_\varepsilon) \ni 0 \qquad \text{in } (0,+\infty), \tag{6.37}$$

$$h_\varepsilon(0) = \varepsilon.$$

h_ε is given by inversion from $t = \int_{h_\varepsilon(t)}^\varepsilon ds/\beta^0(s)$, for $0 \leq t \leq t(\varepsilon)$, $h_\varepsilon = 0$ for $t > t(\varepsilon)$. Moreover we have

$$\frac{\partial \tilde{h}_\varepsilon}{\partial t} + \frac{\partial}{\partial x_1}(\Phi_1(\tilde{h}_\varepsilon)) + \beta(\tilde{h}_\varepsilon) \ni 0, \tag{6.38}$$

in Kruzkov sense in $(0,+\infty) \times \mathbb{R}$ where $\tilde{h}_\varepsilon(t,x_1) = h_\varepsilon(t)$. For $n \in \mathbb{N}$ and $0 \leq t \leq n/C$, we set $S_n(t) = \{x_1 : |x_1 - \delta(\varepsilon) - n| < C(n/C - t)\}$. From Proposition 4.4 we have

$$\int_{S_n(t)} (v - \tilde{h}_\varepsilon)^+ dx \leq \int_{S_n(0)} (v_0 - \varepsilon)^+ dx. \tag{6.39}$$

If we make $n \to +\infty$ and use the fact that $h_\varepsilon = 0$ on $[t(\varepsilon),+\infty)$ we deduce that v vanishes on

$$\{(t,x_1) : t \geq t(\varepsilon), x_1 \geq \delta^+(\varepsilon) + Ct\}.$$

In the same way v vanishes on

$$\{(t,x_1) : t \geq t(\varepsilon), x_1 \leq \delta^-(\varepsilon) - Ct\}.$$

Moreover for any $t > 0$ small enough we can find $\varepsilon > 0$ such that $t = t(\varepsilon)$; we let $\varepsilon = \varepsilon(t)$, define $b^+(t) = \delta^+(\varepsilon(t)) + Ct$, $b^-(t) = \delta^-(\varepsilon(t)) - Ct$ and we have supp $v(t,.) \subset [b^-(t),b^+(t)]$ for any $t > 0$.

Remark 6.6. When $N = 1$ the assumption (H.2_1) is unnecessary and (H.0) reduces to Φ_1 is continuous.

The multidirectional version of Proposition 6.7 is the following

Corollary 6.4. *Under hypotheses* (H.0), (H.1) *and* (H.2_i^*) *for any* $1 \leq i \leq N$ *and*

$$\int_0^1 \frac{ds}{\beta^0(s)} - \int_{-1}^0 \frac{ds}{\beta^0(s)} < +\infty, \tag{6.40}$$

we suppose moreover that Φ *is Lipschitz continuous on* $\{r : |r| \leq \|u_0\|_{L^\infty}\}$ *where*

$$u_0 \in \overline{D(A)}^{L^1} \cap \left\{ \bigcap_{i=1}^{N} L^1(\mathbb{R}_{x_i} : L^\infty(\mathbb{R}^{N-1})) \right\}$$

and $\lim_{|x| \mapsto +\infty} u_0(x) = 0$. *If* u *is the solution of* (C.P.)$_0$ *with initial data* u_0, *then*

i) there exists $T^* > 0$ *such that* $u(t,x) = 0$ *in* $[T^*,+\infty) \times \mathbb{R}^N$,

ii) there exists a monotone function $b : (0,+\infty) \to [0,+\infty)$ *such that* supp $u(t,.) \subset \{x : |x|_\infty \leq b(t)\}$ *for any* $t > 0$.

When Int $\beta(0)$ is not empty we also have an instantaneous shrinking of the support of the solution of (C.P.)$_f$. For example

Proposition 6.8. *Under hypotheses* (H.0), (H.1) *and* (H.2_1), *we suppose moreover that* Φ_1 *is locally Lipschitz continuous on* $\mathbb{R}^+$ *and* Int $\beta^{-1}(0) = (0,\beta^+(0))$ *with* $\beta^+(0) > 0$. *Assume*

$$u_0 \in \overline{D(A)}^{L^1} \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}; L^\infty(\mathbb{R}^{N-1}))$$

and

$$f \in L^1_{\text{loc}}([0,+\infty); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$$

such that $\lim_{|x_1| \mapsto +\infty} \|u_0^+(x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} = 0$,

$$f^+ \in L^1_{\text{loc}}([0,+\infty); L^1(\mathbb{R}; L^\infty(\mathbb{R}^{N-1})))$$

and

$$\|f^+(t,x_1,.)\|_{L^\infty(\mathbb{R}^{N-1})} \leq \beta^+(0) - \alpha(t), \tag{6.41}$$

for $t \geq T$ *and* $|x_1| \geq R$, *where* $\alpha : [T,+\infty) \mapsto \mathbb{R}^+$ *is such that* $\int_T^{+\infty} \alpha(s)\,ds = +\infty$. *if* u *is the solution of* (C.P.)$_f$ *with initial data* u_0 *then the conclusions of Proposition 6.7 remain valid.*

The proof follows the one of Proposition 6.7 and is left to the reader as well as the proof of the multidirectional version of Proposition 6.8. Other examples of instantaneous shrinking for solutions of parabolic variational inequalities can be found in [7] and [19].

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