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ON THE EXISTENCE OF A FREE BOUNDARY FOR A CLASS OF REACTION-DIFFUSION SYSTEMS*

J. ILDEFONSO DIAZ[†] AND JESUS HERNANDEZ[‡]

Abstract. Some nonlinear stationary reaction-diffusion systems involving nonlinear terms which may be discontinuous are considered. Such systems occur, for instance, in the study of chemical reactions, and the discontinuities correspond to reactions of order zero. In such concrete models, the set where the reactant vanishes plays an important role. Here we prove the existence of solutions for a general class of such systems satisfying Dirichlet or nonlinear boundary conditions. Necessary and sufficient conditions are given assuring that the reactant component vanishes on a set of positive measure. Estimates on the location of such set are given.

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Introduction. Many papers have been devoted during recent years to the study of reaction-diffusion systems which arise very often in applications such as, mathematical biology, chemical reactions, and combustion theory.

Here we consider a system describing a single, irreversible, nonisothermic stationary reaction of the form

$$\text{(0.1)}\qquad \begin{aligned} -\Delta u+\mu^2F(u)e^{\gamma-\gamma/v}&=0 \quad \text{in } \Omega,\\ -\Delta v-\nu\mu^2F(u)e^{\gamma-\gamma/v}&=0 \quad \text{in } \Omega, \end{aligned}$$

$$\text{(0.2)}\qquad \begin{aligned} \frac{\partial u}{\partial n}+\varepsilon(u-1)&=0 \quad \text{on } \partial\Omega,\\ \frac{\partial v}{\partial n}+\zeta(v-1)&=0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, μ^2 is the Thiele number, ν is the Prater temperature and γ is the Arrhenius number (see [3]). Here ε and ζ (the Biot numbers) are positive, being in some cases infinity, in which case (0.2) is interpreted as the Dirichlet boundary conditions

$$\text{(0.3)}\qquad u=1, \qquad v=1 \quad \text{on } \partial\Omega.$$

The function $F(u)$ is assumed to be nondecreasing and it is also assumed to satisfy $F(0)=0$, $F(1)=1$ and $F(s)>0$ if $s>0$. The unknowns u and v are nonnegative and represent, respectively, the concentration and the temperature of the reactant.

Very often F takes the simple form $F(u)=u^p$, where $p\geq 0$ is the *order* of the reaction (see [3, Vol I]). In the case of a reaction of order zero F is given by $F(0)=0$ and $F(s)=1$ if $s>0$ (thus F is a *discontinuous* function).

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[†] Facultad de Mathemáticas, Universidad Complutense de Madrid, Madrid 3, Spain. The research of this author was sponsored by the U.S. Army under contract DAAG29-80-C-0041. This work was completed while this author was visiting the Mathematics Research Center at the University of Wisconsin-Madison, Madison, Wisconsin.

[‡] Division de Matematicas, Universidad Autonoma de Madrid, Madrid, Spain.

Existence and uniqueness results for the parabolic problem associated with (0.1), (0.2) (or (0.1),(0.3)) have been given by some authors (cf. e.g. [2],[4],[18]). Existence and, in some particular situations, uniqueness results for the elliptic problem can be found in [2], [21], [15] and [19] for $p\geq 1$. The case $0\leq p<1$ is considered in [3, p. 311] (see also [20]) but existence theorems are not given. It is shown in [3] and [20] that for $p=0$, if μ is large enough, no strictly positive solution can exist. It is also shown that, in some particular examples, the set $\Omega_0=\{x\in\Omega: u(x)=0\}$ (called the *dead core*) is not empty and has positive measure if $0\leq p<1$.

The main idea used in [20] and many other papers (cf. [3]) is to reduce (0.1), (0.2) to a nonlinear elliptic boundary value problem *for u alone*. Here we follow a different approach which allows us to obtain better results. Moreover, we are able to treat the case of nonlinear boundary conditions, which cannot be handled by the preceding device.

We shall consider the case of discontinuous functions $F(u)$ in the framework of maximal montone graphs in $\mathbb{R}^2$ (see [8]). For the reader's convenience, we recall that a maximal monotone graph α in $\mathbb{R}^2$ is always specified by a real nondecreasing function θ by $\alpha(r)=(-\infty,\theta(r-)]$ if $\theta(r-)=-\infty$, $\alpha(r)=[\theta(r-),\theta(r+)]$ if $-\infty<\theta(r-)\leq\theta(r+)<+\infty$ and $\alpha(r)=[\theta(r-),+\infty)$ if $\theta(r+)=+\infty$. We define $D(\alpha)=\{r\in R: \alpha(r)\neq\varnothing\}$ and the sections α^+ and α^- by

$$\begin{aligned} \alpha^+(r)&=\max\{z: z\in\alpha(r)\} \text{ if } r\in D(\alpha),\\ \alpha^-(r)&=\min\{z: z\in\alpha(r)\} \text{ if } r\in D(\alpha),\\ \alpha^+(r)&=\alpha^-(r)=+\infty \text{ if } r\notin D(\alpha),\ r\geq\sup D(\alpha),\\ \alpha^+(r)&=\alpha^-(r)=-\infty \text{ if } r\notin D(\alpha),\ r\leq\inf D(\alpha). \end{aligned}$$

Finally, we define $\alpha^0(r)$ as the element of $\alpha(r)$ with minimal absolute value.

Through the paper we shall study the following general formulation including the system (0.1) as a particular case:

$$\text{(NLS)}\qquad \begin{aligned} -\Delta u+\alpha(u)f(v)&\ni 0 \quad \text{in } \Omega,\\ -\Delta v-\beta(u)g(v)&\ni 0 \quad \text{in } \Omega \end{aligned}$$

with the boundary conditions

$$\text{(DBC)}\qquad u=\varphi_1, \qquad v=\varphi_2 \quad \text{on } \partial\Omega,$$

as well as the nonlinear boundary conditions

$$\text{(NBC)}\qquad \begin{aligned} Bu&\equiv\frac{\partial u}{\partial n}+b(u)=\psi_1 \quad \text{on } \partial\Omega,\\ Cv&\equiv\frac{\partial v}{\partial n}+c(v)=\psi_2 \quad \text{on } \partial\Omega \end{aligned}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. We also assume for the rest of the paper that

(0.4) α and β are maximal monotone graphs such that $0\in\alpha(0)\cap\beta(0)$.

(0.5) f and g are C^1 functions and $f(s)\geq 0$, $g(s)\geq 0$ if $s\geq 0$.

(0.6) $\varphi_1, \varphi_2, \psi_1$ and $\psi_2\in C^2(\partial\Omega)$

(0.7) b and c are C^2 nondecreasing real functions.

In particular, if α and β are single-valued (i.e. they are continuous real functions) then the set inclusion of (NLS) should be replaced by equality.

In the general situation, $(u,v)\in H^2(\Omega)\times H^2(\Omega)$ is a *solution* of (NLS) if there exists $a,d\in L^2(\Omega)$ such that $a(x)\in\alpha(u(x))$, $d(x)\in\beta(v(x))$ a.e. $x\in\Omega$ and

$$-\Delta u+af(u)=0, \qquad -\Delta v-dg(v)=0 \quad \text{in } \Omega.$$

We shall prove the following existence result, which extends in some sense those in [2], [21] and [15].

THEOREM A. *Assume*

(A.1) $D(\alpha)=D(\beta)=\mathbf{R}$,

(A.2) $f(s)\geq m_1\geq 0\ \forall s\in\mathbf{R}$ *and*

(A.3) $0\leq g(s)\leq m_2\ \forall s\in\mathbf{R}$.

Then there exists at least one solution (u,v) *of* (NLS) (DBC) (*resp.* (NLS) (NBC)). *Moreover* $u,v\in W^{2,p}(\Omega)$ *for any* p, $1\leq p<+\infty$.

We also consider the existence and nonexistence of a dead core Ω_0 where $u=0$ and consequently the existence of the free boundary $\partial\Omega_0$.[1] Roughly speaking, such a dead core for (0.1), (0.3) arises when it is impossible for diffusion to supply enough reactant from outside Ω to reach the central part of Ω. (cf. [20]). This may happen if the reaction rate $F(u)e^{\gamma-\gamma/v}$ remains high as the reactant concentration decreases. Thus (for (0.1), (0.3)) the existence of Ω_0 depends essentially on three things: the reaction order, the Thiele number and the size of Ω.

Our main result in this direction can be stated in the following general terms.

THEOREM B. *Assume that the hypotheses of Theorem A are satisfied. Then the following properties are true*:

i) *If* $\alpha(s)=\mu^2|s|^{p-1}s$ *and* (u,v) *is any solution of* (NLS) (DBC); *then a dead core may exist only if* $0\leq p<1$.[2]

ii) *Let* $\alpha(s)=\mu^2|s|^{p-1}s$ *with* $0<p<1$ *and let* (u,v) *be a solution of* (NLS) (DBC). *For* $\lambda>0$ *let*

$$\Omega_\lambda=\{x\in\Omega: f(v(x))\geq\lambda\}.$$

Then

$$\Omega_0\supset\left\{x\in\Omega_\lambda: d(x,\partial\Omega_\lambda-(\partial\Omega-\operatorname{supp}\varphi_1))\geq\left(\frac{M}{K_{\lambda,\mu}}\right)^{(1-p)/2}\right\} \tag{0.8}$$

where $M=\|\varphi_1\|_{L^\infty(\partial\Omega)}$ *and*

$$K_{\lambda,\mu}=\left[\frac{2N(1-p)+4p}{\lambda\mu^2(1-p)^2}\right]^{1/(p-1)}.$$

(iii) *Let* $\alpha(s)=\mu^2\cdot\operatorname{sign} s$. *Then the estimate* (0.8) *holds if we replace* M *by* $M^*=\|z\|_{L^\infty(\Omega)}$, *where* z *satisfies* $\Delta z=\mu^2 m_1$ *in* Ω *and* $z=\varphi_1$ *on* $\partial\Omega$. *Furthermore, if* Ω *is convex, the above results are still valid for* (NLS) (NBC) *in the sense that if* $0\leq p<1$, *then* Ω_0 *has a positive measure for* μ *large and it is possible to estimate* Ω_0 (*see* (2.22)).

[1] There is a large literature about this subject in the case of a *single* nonlinear equation. See, e.g. the systematic study of [12].

[2] By convention $|s|^{-1}s=\operatorname{sign} s$ ($=-1$ if $s<0$, $=[-1,1]$ if $s=0$ and $=1$ if $s>0$).

The above theorem is specially meaningful if m_1 in (A.2) is strictly positive (this is true in the case of the combustion system (0.1)), (0.3): indeed, in this case $v>0$ on $\bar\Omega$ and then we have $\Omega_\lambda=\Omega$ for any $\lambda\in(0,m_1]$. So the estimate (0.8) reads

$$\Omega_0\supset\left\{x\in\Omega: d(x,\partial\Omega)\geq\left(\frac{M}{K_{m_1,\mu}}\right)^{(1-p)/2}\right\}. \tag{0.9}$$

From the definition of $K_{\lambda,\mu}$ in Theorem B we deduce that $K_{\lambda,\mu}\searrow 0$ when $\lambda\searrow 0$ or $\mu\searrow 0$, and that $K_{\lambda,\mu}\nearrow+\infty$ if $\lambda\nearrow+\infty$ or $\mu\nearrow+\infty$. Therefore for a fixed bounded Ω the existence of a dead core Ω_0 may only be guaranteed (by estimate (0.9)) if

$$\delta(\Omega)\geq\left(\frac{M}{K_{m_1,\mu}}\right)^{(1-p)/2}$$

where $\delta(\Omega)$ is the radius of the largest ball contained in Ω, assuming $0\leq p<1$. Then a critical value μ_c of μ can be found such that Ω_0 has positive measure if $\mu>\mu_c$. In fact, direct computations show (when $N=1$) that function u is strictly positive in Ω if $\mu<\mu_c$ (for $\varphi_1\geq 0$) and u vanishes only at one point if $\mu=\mu_c$. (see the proof of Lemma 2.1 and also [20]). Estimate (0.8) of Theorem B can be also written independently of the function v for other systems in which it is not difficult to estimate the set Ω_λ (for instance $\Omega_\lambda=\Omega$ if in (NLS) we assume $f(s)=s$ and $\varphi_2>\delta>0$ for δ large (or λ small) enough).

Through the paper we also remark on other more general formulations of (NLS). The parabolic problem associated with (NLS) will be studied in a forthcoming paper by the authors. The case of Ω unbounded will be also treated elsewhere.

1. Existence results. Consider first the problem

$$\text{(DP)}\qquad \begin{aligned} &-\Delta u+\alpha(u)f(v)\ni 0 && \text{in } \Omega,\\ &-\Delta v-\beta(u)g(v)\ni 0 && \text{in } \Omega,\\ &u=\varphi_1,\ u=\varphi_2 && \text{on } \partial\Omega,\end{aligned}$$

where Ω is a bounded open subset of $\mathbf{R}^N$ with smooth boundary $\partial\Omega$ and $\alpha,\beta,f,g,\varphi_1,\varphi_2$ satisfy (0.4), (0.5) and (0.6). Set $X=(H^2(\Omega))^2$.

DEFINITION 1. We shall say that $(u,v)\in X$ is a *solution* of (DP) if there exist functions $a,b\in L^2(\Omega)$ such that $a(x)\in\alpha(u(x))$, $b(x)\in\beta(v(x))$ a.e. in Ω and

$$\begin{aligned} &-\Delta u(x)+a(x)\cdot f(v(x))=0 && \text{a.e. } x\in\Omega,\\ &-\Delta v(x)-b(x)g(v(x))=0 && \text{a.e. } x\in\Omega,\end{aligned}$$

and the boundary conditions (DBC) are satisfied.

DEFINITION 2. The pair $[(u_0,v_0),(u^0,v^0)]\in X\times X$ is a *sub-supersolution* of (DP) if $u_0\leq u^0$, $v_0\leq v^0$ a.e. on Ω and

$$-\Delta u_0+\alpha^-(u_0)f(v)\leq 0\leq-\Delta u^0+\alpha^+(u^0)f(v)\ \forall v\in[v_0,v^0], \tag{1.1}$$

$$-\Delta v_0-\beta^0(u)g(v_0)\leq 0\leq-\Delta v^0-\beta^0(u)g(v^0)\ \forall u\in[u_0,u^0], \tag{1.2}$$

$$(1.3)\qquad u_0\le\varphi_1\le u^0 \quad \text{on } \partial\Omega,$$

$$(1.4)\qquad v_0\le\varphi_2\le v^0 \quad \text{on } \partial\Omega,$$

where $[K,l]=\{h\in L^2(\Omega)|K(x)\le h(x)\le l(x)$ a.e. on $\Omega\}$ if $K,l\in L^2(\Omega)$.

Our main existence result for (DP) is the following

THEOREM 1.1. *Suppose that* $[(u_0,v_0),(u^0,v^0)]$ *is a sub-supersolution satisfying*

$$(\mathrm{H}_1)\quad u_0,v_0,u^0,v^0\in L^\infty(\Omega)$$

and that

$$(\mathrm{H}_2)\quad D(\alpha)=D(\beta)=\mathbb{R}.$$

Then there exists at least one solution (u,v) *of* (DP) *such that* $u_0\le u\le u^0$, $v_0\le v\le v^0$. *In addition* $u,v\in W^{2,p}(\Omega)$ *for any* p, $1\le p<+\infty$.

Remark 1.1. This theorem generalizes results of [2], [15], [16] and [20].

To prove Theorem 1.1 we define $E=[L^2(\Omega)]^2$ and $K=[u_0,u^0]\times[v_0,v^0]$. It is clear that K is a convex, closed and bounded subset of E. Now we define a nonlinear operator $T\colon K\to E$ in the following way: for $(\bar u,\bar v)\in K$, $T(\bar u,\bar v)=(w,z)$ is the *unique* solution of the uncoupled system

$$\begin{array}{lll}(1.5) & -\Delta w+\alpha(w)f(\bar v)+w\ni\bar u & \text{in } \Omega,\\ (1.6) & w=\varphi_1 & \text{on } \partial\Omega,\\ (1.7) & -\Delta z+M\cdot z=\beta^0(\bar u)g(\bar v)+M\cdot\bar v & \text{in } \Omega,\\ (1.8) & w=\varphi_2 & \text{on } \partial\Omega.\end{array}$$

Here $M>0$ is such that the right-hand side of (1.7) is increasing in $\bar v$ (we can choose such a M because g is C^1 and (H_1) has been assumed). Indeed by (H_2) we can apply the results of [10] to obtain the existence of a unique solution w of (1.5), (1.6). Moreover, by (H_1), (H_2) and the L^p-regularity results (see e.g. [14]) $w\in W^{2,p}(\Omega)$ for any p, $1\le p<+\infty$. A similar argument works for z.

The proof of Theorem 1.1 will follow from Schauder's fixed point theorem applied to the operator T. It is sufficient to check that T is compact and that it sends K into itself.

LEMMA 1.1. *T is compact.*

Proof. As K is bounded it is easy to show that

$$\|w\|_{H^1(\Omega)}\le C$$

with C independent of $(\bar u,\bar v)\in K$. Thus it is sufficient to recall the compactness of the imbedding $H^1(\Omega)\hookrightarrow L^2(\Omega)$ to see that T sends bounded subsets into relatively compact ones (the same for z). To prove that T is continuous, suppose that $(u_n,v_n)\to(u,v)$ in E. Then

$$\begin{array}{ll}-\Delta(w-w_n)+\alpha(w)f(v)-\alpha(w_n)f(v_n)+w-w_n\ni u-u_n & \text{in } \Omega,\\ w-w_n=0 & \text{on } \partial\Omega.\end{array}$$

Multiplying by $w-w_n$ and integrating by parts we obtain (for the case α single-valued for simplicity)

$$\begin{aligned}&\int_\Omega|\nabla(w-w_n)|^2+\int_\Omega[\alpha(w)f(v)-\alpha(w_n)f(v_n)](w-w_n)+\int_\Omega|w-w_n|^2\\ &\quad=\int_\Omega|\nabla(w-w_n)|^2+\int_\Omega[\alpha(w)f(v)-\alpha(w_n)f(v)](w-w_n)\\ &\qquad+\int_\Omega\alpha(w_n)(f(v)-f(v_n))(w-w_n)+\int_\Omega|w-w_n|^2\\ &\quad=\int_\Omega(u-u_n)(w-w_n)\\ &\quad\ge\int_\Omega|\nabla(w-w_n)|^2+\int_\Omega\alpha(w_n)(f(v)-f(v_n))(w-w_n),\end{aligned}$$

and by the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}\|w-w_n\|^2_{H^1(\Omega)}\le{}&\|\alpha(w_n)\|_{L^\infty(\Omega)}\|f(v)-f(v_n)\|_{L^2(\Omega)}\|w-w_n\|_{L^2(\Omega)}\\ &+\|u_n-u\|_{L^2(\Omega)}\|w-w_n\|_{L^2(\Omega)}.\end{aligned}$$

Now it is easy to conclude that $w_n\to w$ in $H^1(\Omega)$. A similar argument can be used for z. $\square$

LEMMA 1.2. $T(K)\subset K$.

Proof. We first prove $u_0\le w$, i.e. $(u_0-w)^+=0$, with $h^+=\max(h,0)$. For $v=\bar v$, (1.1) yields

$$0\ge-\Delta(u_0-w)+\alpha(u_0)f(\bar v)-\alpha(w)f(\bar v)+u_0-w.$$

(We again suppose α single-valued for simplicity in the notation.) Multiply this inequality by $(u_0-w)^+$, integrate over Ω and use Green's formula to obtain

$$\begin{aligned}0&\ge\int_\Omega-\Delta(u_0-w)(u_0-w)^++\int_\Omega(\alpha(u_0)-\alpha(w))f(\bar v)(u_0-w)^+\\ &\qquad+\int_\Omega(u_0-w)(u_0-w)^+\\ &\ge\int_\Omega|\nabla(u_0-w)^+|^2\end{aligned}$$

by the monotonicity of α. This gives $(u_0-w)^+=0$. A similar argument shows that $w\le u^0$.

For the second component v we have, with $u=\bar u$ in (1.2),

$$0\ge-\Delta(v_0-z)+\beta^0(\bar u)g(\bar v)-\beta^0(\bar u)g(v^0)+M(v_0-z).$$

Multiplying by $(v_0-z)^+$ and integrating yields

$$0\geq -\int_\Omega \Delta(v_0-z)(v_0-z)^+ + \int_\Omega[\beta^0(\bar u)g(v_0)+Mv_0-\beta^0(\bar u)g(z)-Mz](v_0-z)^+$$

$$\geq \int_\Omega |\nabla(v_0-z)^+|^2$$

(by the choice of M the second integral is positive).

Then T has at least one fixed point (u,v) in K which is a solution of (DP). Moreover $u,v\in L^\infty(\Omega)$ and this implies that $u,v\in W^{2,p}(\Omega)$ for any p, $1\leq p<\infty$.

Remark 1.2. It follows easily from Morrey's theorem $(W^{2,p}(\Omega)\hookrightarrow C^{1,r}(\bar\Omega)$ if $p>N$ with $r=1-N/p)$ that $u,v\in C^{1,\delta}(\bar\Omega)$ for any $0<\delta<1$. On the other hand, if we suppose for instance that α and β are C^1 then $u,v\in C^{2,\delta}(\bar\Omega)$ for every $0<\delta<1$. Indeed, in this case $\alpha(u)f(v)$, $\beta(u)g(v)\in C^\delta(\bar\Omega)$ and we can apply Schauder theory ([14]).

The main conclusion of Theorem A (for the Dirichlet problem) follows from the next lemma.

LEMMA 1.3. *Suppose* (H_1), (H_2) *and*

(H_3) $0\leq m_1\leq f(s)\ \forall s\in\mathbf{R}$.

Then if $u_0, v_0, u^0, v^0\in H^2(\Omega)$ *satisfy*

(1.9) $\quad -\Delta u_0+m_1\alpha^-(u_0)\leq 0\leq -\Delta u^0+m_1\alpha^+(u^0)$ *in* Ω,

(1.10) $\quad u^0\leq\varphi_1\leq u^0$ *on* $\partial\Omega$,

(1.11) $\quad -\Delta v_0-\beta^0(-\|\varphi_1\|_{L^\infty(\partial\Omega)})g(v_0)\leq 0\leq -\Delta v^0-\beta^0(\|\varphi_1\|_{L^\infty(\partial\Omega)})g(v^0)$ *in* Ω,

(1.12) $\quad v_0\leq\varphi_2\leq v^0$ *on* $\partial\Omega$

the couple $[(u_0,v_0),(u^0,v^0)]$ *is a sub-supersolution for* (DP).

Proof. Let $u_0\leq u^*\leq u^0$, $v_0\leq v^*\leq v^0$. By the maximum principle we have

$$-\|\varphi_1\|_{L^\infty(\partial\Omega)}\leq u_0\leq 0\leq u^0\leq\|\varphi_1\|_{L^\infty(\partial\Omega)}.$$

Then, by (1.9)

$$-\Delta u_0+\alpha^-(u_0)f(v^*)\leq -\Delta u_0+m_1\alpha^-(u_0)\leq 0,$$
$$-\Delta u^0+\alpha^+(u^0)f(v^*)\geq -\Delta u^0+m_1\alpha^+(u^0)\geq 0,$$

and also by (1.11)

$$-\Delta v_0-\beta^0(u^*)g(v_0)\leq -\Delta v_0-\beta^0(-\|\varphi_1\|_{L^\infty(\partial\Omega)})g(v_0)\leq 0,$$
$$-\Delta v^0-\beta^0(u^*)g(v^0)\geq -\Delta v^0-\beta^0(\|\varphi_1\|_{L^\infty(\partial\Omega)})g(v^0)\geq 0.$$

Moreover, a simple argument gives $v_0\leq 0\leq v^0$. $\square$

Now the problem is to find u_0, u^0, v_0, $v^0\in L^\infty(\Omega)$ satisfying (1.9)–(1.12). The fact that such u_0, u^0 exist follows from the results of [10] applied to α. It is easy to check that v_0 and v^0 can be taken as the (unique) solutions of the problems

$$-\Delta w=am_2 \quad \text{in } \Omega,$$
$$w=\varphi_2 \quad \text{on } \partial\Omega$$

and

$$-\Delta w=a^*m_2 \quad \text{in } \Omega,$$
$$w=\varphi_2 \quad \text{on } \partial\Omega$$

respectively, being $a=\beta^0(\|\varphi_1\|_\infty)$ and $a^*=\beta^0(-\|\varphi_1\|_\infty)$. This proves Theorem A.

Remark 1.3. It is clear that assumption (A.3) of Theorem A is only used to find v_0 and v^0. If for example, g is such that the nonlinear problem

$$-\Delta w=\beta^0(\|\varphi_1\|_{L^\infty(\partial\Omega)})g(w) \quad \text{in } \Omega,$$
$$w=\varphi_2 \quad \text{on } \partial\Omega$$

has a solution, then we can remove (A.3). There is a very extensive literature for this kind of problem with different assumptions on g, but we do not want to consider this point here (cf. e.g. [1] and the survey [17]).

Remark 1.4. If α is assumed to be single-valued and C^1 the hypothesis $f(s)\geq m_1\geq 0$ is not necessary (cf. [15]). On the other hand, if α and β are single-valued and α,β,f and g are C^1 with sufficiently "small" Lipschitz constants, then it can be shown (cf. [2], [15]) that the solution is unique.

It is very easy now to apply the preceding results to the particular example (0.1), (0.3) considered at the beginning of this paper. It is sufficient to take $f(v)=g(v)=e^{\gamma-\gamma/v}$, $\alpha(u)=\mu^2u^p$, $\beta(u)=\nu\mu^2u^p$, $p>0$ and $\varphi_1=\varphi_2=1$. A sub-supersolution is given by $u_0\equiv 0$, $u^0\equiv 1$, $v_0\equiv 0$ and v^0 the unique solution of

$$-\Delta v^0=\nu\mu^2e^\gamma \quad \text{in } \Omega, \qquad v^0=1 \quad \text{on } \partial\Omega.$$

The case of nonlinear boundary conditions can be handled in a very similar way. We only point out some differences. First, the definition of sub-supersolution is the same except that the boundary conditions

$$Bu_0\leq\psi_1\leq Bu^0, \qquad Cv_0\leq\psi_2\leq Cv^0$$

should be satisfied instead of (1.10), (1.12).

The main existence result is

THEOREM 1.2. *Suppose that* $[(u_0,v_0),(u^0,v^0)]$ *is a sub-supersolution satisfying* (H_1), (H_2). *Then there exists at least one solution* (u,v) *of*

$$-\Delta u+\alpha(u)f(v)\ni 0 \quad \text{in } \Omega,$$
$$-\Delta v-\beta(u)g(v)\ni 0 \quad \text{in } \Omega,$$
$$Bu=\psi_1,\ Cv=\psi_2 \quad \text{on } \partial\Omega,$$

such that $u_0\leq u\leq u^0$, $v_0\leq v\leq v^0$. *Moreover*, $u,v\in W^{2,p}(\Omega)$ *for any* p, $1\leq p<+\infty$.

Proof (sketch). We just give the definition of the nonlinear operator T; the other details are very similar to those for the Dirichlet problem. For $(\bar u,\bar v)\in K$, define $T(\bar u,\bar v)=(w,z)$ to be the unique solution of the system

$$-\Delta w+\alpha(w)f(\bar v)+w\ni\bar u \quad \text{in } \Omega,$$
$$Bw=\psi_1 \quad \text{on } \partial\Omega,$$
$$-\Delta z+z=\beta^0(\bar u)g(\bar v)+\bar v \quad \text{in } \Omega,$$
$$Cz=\psi_2 \quad \text{on } \partial\Omega.$$

The existence and uniqueness of w and z follows from [6, Thm. II.1] (for z we can also use the results of [10]). $\square$

A result very close to Lemma 1.3 can also be proved for the boundary conditions (NBC).

Remark 1.6. The operator $-\Delta$ in (NLS) can be replaced by two (possible different) elliptic second order differential operators or even by nonlinear operators of the form

$$-\Delta_q u \equiv -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{q-2}\frac{\partial u}{\partial x_i}\right)$$

with $1<q<\infty$. Indeed, in this case one can define a nonlinear operator T by using again [6] (cf. also [13]). The more involved situation of b and c maximal monotone graphs can also be studied by similar methods.

2. Existence of a "dead core". In this section we shall consider the existence of a "dead core" for solutions u of (NLS), i.e., we shall prove that the set $\Omega_0=\{x\in\Omega: u(x)=0\}$ has a strictly positive measure under adequate hypotheses on α and eventually on $\|\varphi_1\|_{L^\infty(\partial\Omega)}$ or $|\Omega|$. In fact much more precise information is obtained about Ω_0.

Our study will be carried out by using results concerning a single nonlinear equation but arguing in a different way than usual for the combustion example. Indeed, if (u,v) is *any* solution of (NLS) (DBC) [resp. (NLS) (NBC)] then u satisfies

$$(2.1)\qquad -\Delta u+\tilde{f}(x)\alpha(u)\ni F(x) \quad \text{in } \Omega,^3$$

$$(2.2)\qquad u=\varphi_1 \quad \text{on } \partial\Omega,$$

[respectively,

$$(2.3)\qquad \frac{\partial u}{\partial n}+b(u)=\psi_1 \quad \text{on } \partial\Omega],$$

where $F\equiv 0$ and $\tilde{f}(x)=f(v(x))$ a.e. on Ω.

The study of the subset Ω_0 corresponding to solutions of (2.1), (2.2) (or (2.1), (2.3)) has occupied the attention of many authors, but, as far as we know, all these results are given for the simplest case $\tilde{f}(x)\equiv$ constant. We recall the two different approaches in the literature:

a) $\Omega=\mathbb{R}^N$ [7] or Ω being an unbounded set [11];

b) α being multivalued at the origin [9], [5], [13], [22].

More recently, a systematic study has been made in [12] giving a unified view of both situations, but always for $\tilde{f}(x)$ constant. Our results, in this section, follow the ideas of [12].

2.1. Dirichlet problem. We now prove parts i), ii) and iii) of Theorem B. For this we begin with some useful lemmas.

LEMMA 2.1. *Let $F\in L^\infty(\Omega)$, $\varphi\in C^2(\partial\Omega)$ and suppose that $u\in H^2(\Omega)$ satisfies*

$$(2.4)\qquad -\Delta u(x)+\mu^2\tilde{f}(x)|u(x)|^p \operatorname{sign} u(x)\ni F(x) \quad \text{in } \Omega,$$

$$(2.5)\qquad u=\varphi \quad \text{on } \partial\Omega$$

where $\tilde{f}\in L^\infty(\Omega)$, $\tilde{f}\geq 0$ on Ω and $p\geq 0$.[4] *If $0\leq p<1$ and Ω_λ denotes the set*

$$\Omega_\lambda=\{x\in\Omega: \tilde{f}(x)\geq\lambda\}, \qquad \lambda>0,$$

we have the estimate

$$(2.6)\qquad \Omega_0\equiv\{x\in\Omega: u(x)=0\}$$
$$\supset\left\{x\in\Omega_\lambda-\operatorname{supp}F: d(x,\partial(\Omega_\lambda-\operatorname{supp}F)-(\partial\Omega-\operatorname{supp}\varphi))\geq\left(\frac{\tilde{M}}{K_{\lambda,\mu}}\right)^{(1-p)/2}\right\}.$$

Here

$$\tilde{M}=\max\left\{\left(\frac{\|F\|_{L^\infty(\Omega)}}{\lambda\mu^2}\right)^{1/p}, \|\varphi\|_{L^\infty(\partial\Omega)}\right\}$$

for $p>0$ and $\tilde{M}=\|z\|_{L^\infty(\Omega)}$ (with $\Delta z=\mu^2\lambda$ in Ω, $z=\varphi$ on $\partial\Omega$) for $p=0$. $K_{\lambda,\mu}$ is given by

$$(2.7)\qquad K_{\lambda,\mu}=\left[\frac{2N(1-p)+4p}{\lambda\mu^2(1-p)^2}\right]^{1/(p-1)}.$$

Proof. If we denote by u_+ (resp. u_-) the solutions of (2.4), (2.5) corresponding to the data F^+, φ^+ (resp. F^-, φ^-) then by well-known comparison theorems we have $u_+\geq 0$ (resp. $u_-\leq 0$) and also $u_-(x)\leq u(x)\leq u_+(x)$ a.e. $x\in\Omega$. Hence it is clear that $\Omega_0\supset\{x\in\Omega: u_-(x)=0 \text{ and } u_+(x)=0\}$. For the sake of simplicity we shall only consider the case $F=F^+$, $\varphi=\varphi^+$, the other case being analogous. Let $u_\lambda\in H^2(\Omega)$ such that

$$(2.8)\qquad \begin{aligned} &-\Delta u_\lambda+\lambda\mu^2|u_\lambda|^p\geq F && \text{in } \Omega_\lambda,\\ &u_\lambda\geq\varphi && \text{on } \partial\Omega_\lambda\cap\partial\Omega,\\ &u_\lambda\geq\|u\|_{L^\infty(\Omega)} && \text{on } \partial\Omega_\lambda-\partial\Omega.\end{aligned}$$

We claim that $0\leq u(x)\leq u_\lambda(x)$ a.e. on Ω_λ. Indeed,[5] taking $\tilde{F}(x)=-\Delta u+\lambda\mu^2u^p$, it is clear that $\tilde{F}(x)=F(x)+\lambda\mu^2u^p-\tilde{f}(x)\mu^2u^p$ and hence $\tilde{F}\leq F$ on Ω_λ. Moreover, $-\Delta u+\lambda\mu^2u^p=\tilde{F}$ on Ω_λ and thus by the comparison results (cf. e.g. [12]) one has $0\leq u\leq u_\lambda$. Therefore the conclusion of the lemma will follow by *constructing* one of such functions u_λ and the set $\{x\in\Omega_\lambda: u_\lambda(x)=0\}$ will give the estimate (2.6) for Ω_0. We will choose $u_\lambda(x)=h(|x-x_0|)$ for some $x_0\in\Omega_\lambda$. First, note that for $h\in C^2(\mathbb{R})$ and any $\eta\in(0,1)$ we have

$$\begin{aligned} -\Delta h(|x-x_0|)&+\lambda\mu^2h(|x-x_0|)^p\\ &=-h''(|x-x_0|)-\left(\frac{N-1}{|x-x_0|}\right)h'(|x-x_0|)+\lambda\mu^2h(|x-x_0|)^p\\ &=-h''(|x-x_0|)+\eta\lambda\mu^2h(|x-x_0|)^p\\ &\quad+(1-\eta)\lambda\mu^2h(|x-x_0|)^p-\frac{(N-1)}{|x-x_0|}h'(|x-x_0|).\end{aligned}$$

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[3] Equation (2.1) also appears in the study of a stationary isothermical single reaction (see [3, Chap. 3]).

[4] If $p=0$, (2.4) should be interpreted in the sense that there exists $w\in L^2(\Omega)$ such that $w(x)\in\operatorname{sign}(u(x))$ a.e. $x\in\Omega$ and $-\Delta u+\mu^2\tilde{f}w=F$ in Ω.

[5] We shall only prove this inequality for $p>0$. If $p=0$, a natural adaptation of the argument leads to the claim.

Now, for a fixed η, let h_η be a solution of the Cauchy problem

$$h_\eta''(r) \in \eta\lambda\mu^2|h_\eta(r)|^p \operatorname{sign}(h_\eta(r)), \quad h_\eta(0)=h_\eta'(0)=0. \tag{2.9}$$

It is easy to check (recall that $0\le p<1$) that

$$h_\eta(r)=L_\eta r^{2/(1-p)} \tag{2.10}$$

where

$$L_\eta=\left(\frac{2(1+p)}{\eta\lambda\mu^2(1-p)^2}\right)^{1/(p-1)} \tag{2.11}$$

is a solution of (2.9). We have

$$(1-\eta)\lambda\mu^2 h_\eta(r)^p-\frac{(N-1)}{r}h_\eta'(r)=L_\eta r^{2p/(1-p)}\left[(1-\eta)\lambda\mu^2L_\eta^{p-1}-\frac{2(N-1)}{1-p}\right];$$

choosing η such that

$$\eta\le\frac{p+1}{1+p+(N-1)(1-p)} \tag{2.12}$$

leads to

$$-\Delta h_\eta(|x-x_0|)+\lambda\mu^2 h_\eta(|x-x_0|)^p\ge 0$$

for any $x\in\Omega_\lambda$.

Finally, consider the set $\tilde\Omega=\Omega_\lambda-\operatorname{supp}F$. The considerations made above show that the function

$$u_\lambda(x)=K_{\lambda,\mu}|x-x_0|^{2/(1-p)}$$

with $K_{\lambda,\mu}$ given by (2.7) satisfies

$$\begin{aligned} -\Delta u_\lambda+\lambda\mu^2u_\lambda^p\ge 0=F(x) \quad &\text{in } \tilde\Omega,\\ u_\lambda\ge 0=\varphi \quad &\text{on } \partial\tilde\Omega\cap(\partial\Omega-\operatorname{supp}\varphi).\end{aligned}$$

Hence it is sufficient to have

$$u_\lambda\ge\max\{\varphi,\|u\|_{L^\infty(\Omega)}\} \quad \text{on } \partial\tilde\Omega-(\partial\tilde\Omega\cap(\partial\Omega-\operatorname{supp}\varphi)) \tag{2.13}$$

to obtain

$$0\le u(x)\le u_\lambda(x) \quad \text{on } \tilde\Omega.$$

But, by the maximum principle we know that $u(x)\le\tilde M$ on Ω and this implies that (2.13) is satisfied if we choose x_0 such that

$$|x-x_0|\ge\left(\frac{\tilde M}{K_{\lambda,\mu}}\right)^{(1-p)/2} \tag{2.14}$$

for every $x\in\partial\tilde\Omega-(\partial\tilde\Omega\cap(\partial\Omega-\operatorname{supp}\varphi))$. The conclusion now follows trivially from (2.13) and (2.14) (we recall that $u_\lambda(x_0)=0$). □

Statements ii) and iii) of Theorem B follow immediately from the above lemma. We remark that the constant $K_{\lambda,\mu}$ given in (2.7) is such that $K_{\lambda,\mu}\searrow 0$ when $\lambda\searrow 0$ or $\mu\searrow 0$ and that $K_{\lambda,\mu}\nearrow+\infty$ if $\lambda\nearrow+\infty$ or $\mu\nearrow+\infty$. Then, if Ω_λ is bounded and not empty, estimate (2.6) shows that the measure of Ω_0 is positive at least if

$$\delta(\Omega_\lambda-\operatorname{supp}F)>\left(\frac{\tilde M}{K_{\lambda,\mu}}\right)^{(1-p)/2}$$

where $\delta(\Omega_\lambda-\operatorname{supp}F)$ is the radius of the largest ball contained in $\Omega_\lambda-\operatorname{supp}F$ (assuming $0\le p<1$). Therefore, if Ω_λ is given, Ω_0 "exists" if μ is large enough or $\tilde M$ is sufficiently small. In the simple case of problem (0.1), (0.3) with $f(s)=s^p$, $0\le p<1$, it is easy to find a critical value μ_c of μ (now depending on p,γ and Ω) such that $\hat\Omega_0$ is not empty if $\mu>\mu_c$. When $N=1$ direct computations show that, for p,γ and Ω fixed, the function u is strictly positive if $\mu<\mu_c$ (see e.g. [3] and [20]).

We shall prove part i) of Theorem B. Indeed, we shall prove that if $p\ge 1$ then for any value of λ,μ and $\delta(\Omega)$ there exist functions (u,v) satisfying (DP) (with $\alpha(s)=|s|^{p-1}s$) such that $u(x)>0$ on Ω. To do this we shall consider the worst case, i.e., when $\delta(\Omega)=+\infty$ (for instance $N=1$ and $\Omega=(0,\infty)$) and even for a larger class of nonlinearities α.

Lemma 2.2. *Let $u\in H^2(0,\infty)$ satisfying*

$$-u''(x)+\tilde f(x)\alpha(u(x))\ni 0, \quad x\in(0,\infty), \qquad u(0)=1, \tag{2.15}$$

where α is a maximal monotone graph such that $0\in\alpha(0)$ and the function $j(s)=\int_0^s\alpha^0(r)\,dr$ satisfies

$$\int_0^1\frac{ds}{\sqrt{j(s)}}=+\infty. \tag{2.16}$$

(These hypotheses are satisfied when $\alpha(s)=|s|^{p-1}s$, $p\ge 1$.) Assume that $\tilde f\in L^\infty(0,\infty)$ and $0\le\tilde f(x)\le m_2$ a.e. $x\in(0,\infty)$, for some $m_2>0$. Then $u(x)>0$ for any $x\in[0,\infty)$.

Proof. We shall use some ideas of [7] and [13]. By reasoning as in the proof of Lemma 2.1 we can always suppose without loss of generality that $\alpha^{-1}(0)=0$ and that α is single-valued. By a comparison argument completely analogous to the ones in the proof of Lemma 2.1 we show that if $\underline{u}\in H^2(0,\infty)$ satisfies

$$-\underline{u}''(x)+m_2\alpha(\underline{u}(x))=0 \quad \text{on } (0,\infty), \qquad \underline{u}(0)=1$$

then $\underline{u}(x)\le u(x)$ for any $x\in(0,+\infty)$. Thus it suffices to prove that $\underline{u}(x)>0$ for any $x\in(0,\infty)$. Suppose that $\underline{u}$ has compact support and we shall obtain a contradiction. The maximum principle implies $0\le\underline{u}(x)\le 1$ and hence $\underline{u}''\in L^\infty(0,\infty)$. Thus $\underline{u}\in C^1([0,\infty))$ with $\underline{u}''\ge 0$. Let $R=\sup\{x: \underline{u}(x)\ne 0\}$ ($R>0$ and finite by assumption). As $u'(R)=0$ it is not difficult to see that $\underline{u}'(x)<0$ and $\underline{u}(x)>0$ on $(0,R)$ (it is a consequence of $\underline{u}''\ge 0$). But (2.16) yields

$$\int_0^{\underline{u}(R)}\frac{ds}{\sqrt{j(s)}}=-\int_0^R\frac{\underline{u}'(r)}{\sqrt{j(\underline{u}(r))}}\,dr=+\infty$$

and we will get a contradiction by estimating $\underline{u}'(r)/\sqrt{j(\underline{u}(r))}$ on $(0,R)$. Defining $w(x)=(\underline{u}'(x))^2$ we have

$$(j(\underline{u}))'=\alpha(\underline{u})\underline{u}'=\frac{1}{m_2}\underline{u}'\underline{u}''=\left[\frac{1}{2m_2}(\underline{u}')^2\right]'.$$

But $w(R)=0$ and $j(\underline{u}(R))=0$. By integrating we get $j(\underline{u})=w/2m_2$, and, finally,

$$\int_0^R \frac{\underline{u}'(r)}{\sqrt{j(\underline{u}(r))}}\,dr=\sqrt{2m_2}\int_0^R ds<+\infty,$$

a contradiction. □

Remark 2.1. By arguing in a similar way as in [11] we can prove that if Ω is an unbounded subset of $\mathbf{R}^N$, the maximal monotone graph α satisfies

$$(2.17)\qquad \int_0^1\frac{ds}{\sqrt{j(s)}}<+\infty \qquad \left(j(s)=\int_0^s\alpha^0(r)\,dr\right),$$

and u satisfies

$$\begin{aligned} -\Delta u+\bar{f}(x)\alpha(u)&\ni F \quad \text{in } \Omega \quad (\bar{f}\geq\lambda),\\ u&=\varphi \quad \text{on } \partial\Omega, \end{aligned}$$

where F and φ are assumed with compact support, then u has compact support. We point out that the improper integral (2.17) converges when $\alpha(s)=|s|^p\operatorname{sign}s$ if and only if $0\leq p<1$ and hence the compactness of the support of u is an obvious consequence of Lemma 2.1.

Remark 2.2. Lemma 2.1 (and then Theorem B) can also be obtained when the operator $-\Delta$ in (NLS) is replaced by other elliptic second order differential operators as in Remark 1.6. The new definition of the functions $u_\lambda(x)=h(|x-x_0|)$ in the proof of Lemma 2.1 can be found by the methods of [12].

2.2. Nonlinear boundary conditions. Statement iv) of Theorem B will follow as in the preceding section by considering the nonlinear equation

$$(2.18)\qquad \begin{aligned} -\Delta u+\mu^2\bar{f}(x)|u|^p\operatorname{sign}u&\ni F \quad \text{in } \Omega,\\ Bu=\frac{\partial u}{\partial n}+b(u)&=\psi \quad \text{on } \partial\Omega, \end{aligned}$$

where $\bar{f}\in L^\infty(\Omega)$, $\bar{f}\geq0$, $0\leq p<1$, b is C^1 nondecreasing with $b(0)=0$, $F\in L^\infty(\Omega)$ and $\psi\in C^2(\partial\Omega)$.

First, we remark that "interior estimates" for Ω_0 can be obtained as in Lemma 2.1. More precisely, we have

$$(2.19)\qquad \Omega_0\supset\left\{x\in\Omega_\lambda-\operatorname{supp}F:\ d(x,\partial(\Omega_\lambda-\operatorname{supp}F))\geq\left(\frac{M^*}{K_{\lambda,\mu}}\right)^{(1-p)/2}\right\}$$

where now[6] $M^*=\|u\|_{L^\infty(\Omega)}$. To show this it is sufficient to choose x_0 in such a way that $u_\lambda\geq M^*$ on $\partial\tilde{\Omega}$, being $\tilde{\Omega}=\Omega_\lambda-\operatorname{supp}F$ in the proof of Lemma 2.1. It is clear that (2.19) does not give any information about the behavior of Ω_0 near the boundary of $\Omega_\lambda-\operatorname{supp}F$.

[6] One obtains estimates for M^* by means of comparison theorems (see e.g. Lemma 3 in [12]).

To improve the estimate (2.19), we introduce the following notation: given a smooth curve Γ in $\mathbf{R}^N$ and $x_0\in\mathbf{R}^N$, we define

$$(2.20)\qquad O(x_0,\Gamma)=\inf\{\cos(\overrightarrow{n(x)},\overrightarrow{x-x_0}):x\in\Gamma\},$$

where $\vec{n}(x)=(n_1(x),\cdots,n_N(x))$ is the unitary outward normal vector to Γ at x and $(\overrightarrow{n(x)},\overrightarrow{x-x_0})$ denotes the angle between the vectors $\vec{n}(x)$ and $\overrightarrow{x-x_0}$. It is clear that the value of $O(x_0,\Gamma)$ depends essentially on the "geometry" of Γ. If for instance $\Gamma=\partial\Omega$ and Ω is a convex bounded set of $\mathbf{R}^N$ it is easy to see that $O(x_0,\Gamma)>0$ when $x_0\in\Omega$.

LEMMA 2.3. *Assume that $u\in H^2(\Omega)\cap L^\infty(\Omega)$ satisfies (2.18). For $\lambda>0$, let $\Omega_\lambda=\{x\in\Omega:\bar{f}(x)\geq\lambda\}$. Moreover, suppose $0\leq p<1$ and*

$$(2.21)\qquad O(x_0,\partial(\Omega_\lambda-\operatorname{supp}F)\cap\partial\Omega)\geq0\ \ \forall x_0\in\Omega_\lambda-\operatorname{supp}F.$$

Define

$$\Gamma=\partial(\Omega_\lambda-\operatorname{supp}F)\cap\partial\Omega\cap\operatorname{supp}\psi.$$

Then

$$(2.22)\qquad \Omega_0\supset\left\{x\in\Omega_\lambda-\operatorname{supp}F:\ d(x,\Gamma)\geq\left[\frac{(1-p)\|\psi\|_{L^\infty(\partial\Omega)}}{2K_{\lambda,\mu}O(x_0,\Gamma)}\right]^{(1-p)/(1+p)}\ \text{and}\ \ d(x,\partial(\Omega_\lambda-\operatorname{supp}F)-\partial\Omega)\geq\left(\frac{M^*}{K_{\lambda,\mu}}\right)^{(1-p)/2}\right\},$$

where $M^*=\|u\|_{L^\infty(\Omega)}$.

Proof. Arguing as in Lemma 2.1 we only consider the case $F\geq0$ and $\psi\geq0$. Let $\tilde{\Omega}=\Omega_\lambda-\operatorname{supp}F$. By again using comparison results (cf. e.g. [12]) it is not difficult to see that if u_λ satisfies

$$\begin{aligned} &(2.23)\qquad -\Delta u_\lambda+\lambda\mu^2u_\lambda^p\geq0 && \text{in } \tilde{\Omega},\\ &(2.24)\qquad u_\lambda\geq M^* && \text{on } \partial\tilde{\Omega}-\partial\Omega,\\ &(2.25)\qquad \frac{\partial u_\lambda}{\partial n}\geq\|\psi\|_{L^\infty(\partial\Omega)} && \text{on } \Gamma=\partial\tilde{\Omega}\cap\partial\Omega\cap\operatorname{supp}\psi,\\ &(2.26)\qquad \frac{\partial u_\lambda}{\partial n}\geq0 && \text{on } \partial\tilde{\Omega}\cap(\partial\Omega-\operatorname{supp}\psi), \end{aligned}$$

then $0\leq u(x)\leq u_\lambda(x)$ for $x\in\tilde{\Omega}$. From the proof of Lemma 2.1 we know that the function

$$u_\lambda(x)=K_{\lambda,\mu}|x-x_0|^{2/(1-p)},$$

satisfies

$$-\Delta u_\lambda+\lambda\mu^2u_\lambda^p\geq0 \quad \text{on } \tilde{\Omega}$$

for any $x_0 \in \tilde{\Omega}$. Condition (2.24) is satisfied if

$$|x-x_0| \geq \left(\frac{M^*}{K_{\lambda,\mu}} \right)^{(1-p)/2} \quad \forall x \in \partial\tilde{\Omega} - \partial\Omega. \tag{2.27}$$

On the other hand,

$$\frac{\partial u_\lambda}{\partial n}(x) = \sum_{i=1}^{N} \frac{\partial u_\lambda}{\partial x_i}(x) \cdot n_i(x) = K_{\lambda,\mu}\left(\frac{2}{1-p}\right)|x-x_0|^{(1+p)/(1-p)}\cos\left(\overrightarrow{n(x)},\overrightarrow{x-x_0}\right)$$

$$\geq K_{\lambda,\mu}\left(\frac{2}{1-p}\right)|x-x_0|^{(1+p)/(1-p)} O\left(x_0, \partial\tilde{\Omega} \cap \partial\Omega\right).$$

Thus (2.26) is a consequence of (2.21), and (2.25) holds if we choose $x_0 \in \tilde{\Omega}$ satisfying

$$|x-x_0| \geq \left(\frac{(1-p)\|\psi\|_{L^\infty(\partial\Omega)}}{2K_{\lambda,\mu} O(x_0,\Gamma)} \right)^{(1-p)/(1+p)} \quad \forall x \in \Gamma.$$

This completes the proof. □

Remark 2.3. Part iv) of Theorem B follows from Lemma 2.3 if we set $F \equiv 0$; (2.21) holds easily if, for instance, $f(s) \geq m_1 > 0 \; \forall s \in \mathbb{R}$ (as in the combustion system) and Ω is a convex set.

Addendum. After the completion of this work, the authors learned that C. Bandle, R. P. Sperb and I. Stakgold have recently obtained, in the paper *Diffusion-reaction with monotone kinetics*, results similar to our Lemma 2.1, by using different methods. Some results related to Remark 2.1 can be found in a paper (to appear) by M. Schatzman, *Stationary solutions and asymptotic behaviour of a quasilinear degenerate parabolic equation*.

REFERENCES

[1] H. AMANN, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), pp. 620–709.

[2] ______, *Existence and stability of solutions for semilinear parabolic systems and applications to some diffusion-reaction equations*, Proc. Roy. Soc., Edinburgh, 81A (1978), pp. 35–47.

[3] R. ARIS, *The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts*, Clarendon Press, Oxford, 1975.

[4] J. BEBERNES, K. N. CHUEH AND W. FULKS, *Some applications of invariance to parabolic systems*, Indiana Univ. Math. J., 28 (1979), pp. 269–277.

[5] A. BENSOUSSAN, H. BREZIS AND A. FRIEDMAN, *Estimates on the free boundary for quasi-variational inequalities*, Comm. Partial Differential Equations, 2 (1977), pp. 297–321.

[6] PH. BENILAN, *Operateurs accretifs et semi-groupes dans les espaces $L^p(1 \leq p \leq +\infty)$*, in Functional Analysis and Numerical Analysis, H. Fujita, ed., Japan Society for the Promotion of Sciences, 1978, pp. 15–53.

[7] PH. BENILAN, H. BREZIS AND M. G. CRANDALL, *A semilinear elliptic equation in $L^1(\mathbb{R}^N)$*, Ann. Scuola Norm. Sup. Pisa., Serie IV–II (1975), pp. 523–555.

[8] H. BREZIS, *Operateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[9] ______, *Solutions of variational inequalities with compact support*, Uspekhi Mat. Nauk., 129 (1974), pp. 103–108. (In Russian.)

[10] H. BREZIS AND W. STRAUSS, *Semilinear second-order elliptic equations in L^1*, J. Math. Soc. Japan, 25 (1973), pp. 565–590.

[11] J. I. DIAZ, *Soluciones con soporte compacto para algunos problemas semilineales*, Collect. Math. 30 (1979), pp. 141–179.

[12] ______, *Tecnica de supersoluciones locales para problemas estacionarios no lineales: applicacion al estudio de flujos subsonicos*, Memorias no. 14 de la Real Academia de Ciencias, Madrid, 1980.

[13] J. I. DIAZ AND M. A. HERRERO, *Estimates on the support of the solutions of some nonlinear elliptic and parabolic problems*, Proc. Roy. Soc. Edinburgh, 39A (1981), pp. 249–258.

[14] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1977.

[15] J. HERNANDEZ, *Some existence and stability results for solutions of reaction-diffusion systems with nonlinear boundary conditions*, in Nonlinear Differential Equations: Invariance, Stability and Bifurcation, P. de Mottoni and L. Salvadori, eds., Academic Press, New York, 1981, pp. 161–173.

[16] ______, *Existence and global stability results for reaction-diffusion equations with nonlinear boundary conditions*, to appear.

[17] P. L. LIONS, *On the existence of positive solutions of semilinear elliptic equations*, Tech. Summ. Rep. #2209, Mathematics Research Center, University of Wisconsin, Madison, 1981.

[18] T. NAGAI, *Estimates for the coincidence sets of solutions of elliptic variational inequalities*, Hiroshima Math. J., 9 (1979), pp. 335–346.

[19] C. V. PAO, *Asymptotic stability of reaction-diffusion systems in chemical reactor and combustion theory*, J. Math. Anal. Appl., to appear.

[20] I. STAKGOLD, *Estimates for some free boundary problems*, in Ordinary and Partial Differential Equations Proceedings, Dundee, Scotland, 1980, W. N. Everitt and B. D. Sleeman, eds., Lecture Notes in Mathematics 846, Springer-Verlag, New York, 1981.

[21] L. V. TSAI, *Existence of solutions of nonlinear elliptic systems*, Bull. Inst. Math. Acad. Sinica, 8 (1980), pp. 111–126.

[22] N. YAMADA, *Estimates on the support of solutions of elliptic variational inequalities in bounded domains*, Hiroshima Math. J., 9 (1979), pp. 7–16.