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On a Nonlinear Degenerate Parabolic Equation in Infiltration or Evaporation through a Porous Medium

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We prove the uniqueness (as well as the existence and regularity) of solutions of the Cauchy problem and of the first and mixed boundary value problems for the equation

$$u_t = \phi(u)_{xx} + b(u)_x. \tag{E}$$

ϕ and b are assumed to belong to a large class of functions, including, in particular, cases $\phi(u) = u^m$, $b(u) = u^\lambda$, $m \geqslant 1$ and $\lambda > 0$. © 1987 Academic Press, Inc.

1. INTRODUCTION

This paper deals with the nonlinear parabolic equation

$$u_t = \phi(u)_{xx} + b(u)_x, \tag{E}$$

where ϕ and b are continuous real functions.

Equation (E), sometimes called the nonlinear Fokker–Planck equation, arises, for example, in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium. If $\theta(t, x, y, z)$ denotes the volumetric moisture content and $\mathbf{v}(t, x, y, z)$ the velocity then the continuity equation is

$$\frac{\partial \theta}{\partial t} + \operatorname{div} \mathbf{v} = 0,$$

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the density of the fluid being assumed constant. By the Darcy law

$$\mathbf{v} = -K(\theta) \cdot \operatorname{grad} \Phi,$$

where $K(\theta)$ is the hydraulic conductivity and Φ is the total potential. If absortion and chemical, osmotic and thermal effects are neglected, then, for unsaturated flows, Φ may be expressed as the sum of a hydrostatic potential due to capillary suction $\psi(0)$ and a gravitational potential [3, 31]. Thus, if we choose the (x, y, z) coordinate system in such a way that the z-coordinate is vertical and pointing upwards, we may write

$$\Phi = \psi(\theta) + z.$$

Then we obtain

$$\frac{\partial \theta}{\partial t} = \operatorname{div}\{D(\theta) \operatorname{grad} \theta\} + \frac{\partial}{\partial \theta} K(\theta), \tag{1.1}$$

where

$$D(\theta) = K(\theta) \cdot \frac{d\psi}{d\theta}(\theta). \tag{1.2}$$

If the fluid movement takes place in a vertical column of the medium, Eq. (1.1) takes the form

$$\theta_t = \phi(\theta)_{zz} + b(\theta)_z, \tag{1.3}$$

where

$$\phi(s) = \int_0^s D(r)\, dr, \qquad b(s) = K(s) \quad \text{for} \quad r \in \mathbb{R}. \tag{1.4}$$

If the fluid movement takes place in a horizontal column of the medium and x denotes distance along the column, (1.1) reduces to the equation

$$\theta_t = \phi(\theta)_{xx}. \tag{1.5}$$

As is well known, Eq. (1.5) also appears in many other contexts.

The functions D and K (and then ϕ and b) are usually determined empirically according to the nature of the flow problem, as well as the nature of the porous medium. In any case a reasonable choice for D and K is

$$D(u) = D_0 u^{m-1}, \qquad K(u) = k_0 u^{\lambda}$$

where D_0, k_0, m and λ are positive constants. After a suitable rescaling of the independent variables Eq. (1.3) yields (by changing z by x)

$$u_t = (u^m)_{xx} + (u^\lambda)_x. \qquad (\mathrm{E}_{m,\lambda})$$

The flow problem which has been treated more frequently in the mathematical literature corresponds to the phenomena of absorption and "downward infiltration" of a fluid (e.g., water) by the porous medium (e.g., soil). In those cases, some physical experiences show that the corresponding functions K and ψ are such that $\phi \in C^2([0, \infty))$, $\phi(0) = \phi'(0) = 0$, $\phi'(r) > 0$, $\phi''(r) > 0$ if $r > 0$ and $b \in C^2([0, \infty))$, $b(0) = 0$, $b'(r) > 0$, $b''(r) > 0$ if $r > 0$ (see [34, p. 220; 3, p. 511]). In terms of Eq. $(\mathrm{E}_{m,\lambda})$ those cases correspond to the assumptions $m > 1$ and $\lambda > 1$. (Some mathematical papers on such problems are [18, 15, 13, 26, 36, 5, 38].)

Nevertheless, there are other interesting flow problems that give rise to different elections of the functions K and ψ (and then of ϕ and b).

In particular, the physical problem of evaporation from bare soil when the surface is so dry that water loss is limited by the rate of soil–water movement upwards has been studied for many years (see, e.g., [30, 34] and the references therein). In such problems, the hydraulic conductivity function K is a regular *concave* function (see [23, p. 425; 30, p. 357; 34, p. 259] and D is a regular increasing function). An immediate change of variables shows that the values of m and λ for which Eq. $(\mathrm{E}_{m,\lambda})$ governs the evaporation problem are $m > 1$ and $0 < \lambda < 1$.

The main objective of this paper is to consider Eqs. (E) and $(\mathrm{E}_{m,\lambda})$ in a general framework, which includes the corresponding equations of evaporation problems as particular cases.

To be precise, we shall study the following three problems for Eq. (E):

$$\left.\begin{array}{ll} u_t = \phi(u)_{xx} + b(u)_x & \text{on } S = (0, T) \times (-\infty, \infty) \\ u(0, x) = u_0(x) & \text{on } (-\infty, \infty), \end{array}\right\} \qquad (\mathrm{CP})$$

$$\left.\begin{array}{ll} u_t = \phi(u)_{xx} + b(u)_x & \text{on } R = (0, T) \times (l_1, l_2) \\ u(t, l_1) = \psi_-(t),\ u(t, l_2) = \psi_+(t) & \text{on } (0, T) \\ u(0, x) = u_0(x) & \text{on } (l_1, l_2) \end{array}\right\} \qquad (\mathrm{FBVP})$$

and

$$\left.\begin{array}{ll} u_t = \phi(u)_{xx} + b(u)_x & \text{on } H = (0, T) \times (-\infty, l_2) \\ u(t, l_2) = \psi(t) & \text{on } (0, T) \\ u(0, x) = u_0(x) & \text{on } (-\infty, l_2). \end{array}\right\} \qquad (\mathrm{MBVP})$$

It is important to remark that the most interesting problem in evaporation (as well as in downward infiltration) corresponds to (MBVP) with $l_2 = 0$ (see [30, p. 359; 34, p. 229]).

Like the porous media equation, (E) is a degenerate parabolic equation. At points (t, x) where $u > 0$ it is parabolic, but at points where $u = 0$ it is not. Consequently, we cannot expect the above problems to have a classical solution (in fact, between a region where $u > 0$ and another one where $u = 0$, u need not be smooth). It is, therefore, necessary to generalize the notion of solutions of these problems. Among the different notions of solutions, we shall follow the one introduced in [19].

DEFINITION 1.1. A function $u(x, t)$ defined on $\bar{S}$ is said to be a generalized solution of the (CP) problem if

(i) u is bounded, continuous and nonnegative.[1]

(ii) u satisfies the integral identity

$$I(u, \zeta, P) \equiv \int_{t_0}^{t_1} \int_{x_1}^{x_2} [\phi(u)\, \zeta_{xx} + u\zeta_t - b(u)\, \zeta_x]\, dx\, dt$$
$$- \int_{x_1}^{x_2} u\zeta\, dx \,\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \phi(u)\, \zeta_x\, dt \,\Big|_{x_1}^{x_2} = 0$$

for all $P \equiv [t_0 t_1] \times [x_1, x_2]$ and for all $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t, x_1) = \zeta(t, x_2) = 0$ for any $t \in [t_0, t_1]$.

(iii) $u(0, x) = u_0(x)$ for all $x \in (-\infty, \infty)$.

DEFINITION 1.2. A function $u(x, t)$ defined on $\bar{R}$ is said to be a generalized solution of the (FBVP) problem if

(i) u is bounded, nonnegative and continuous on $\bar{R}$.

(ii) u satisfies the integral identity $I(u, \zeta, P) = 0$ for any $P = [t_0, t_1] \times [x_1, x_2] \subset \bar{R}$ and any $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta|_{x=x_1} = \zeta|_{x=x_2} = 0$.

(iii) $u(t, l_1) = \psi_{-(t)}$, $u(t, l_2) = \psi_+(t)$ for all $t \in [0, T]$ and $u(0, x) = u_0(x)$ for all $x \in [l_1, l_2]$.

DEFINITION 1.3. A function $u(x, t)$ defined on $\bar{H}$ is said to be a generalized solution of the (MBVP) problem if

(i) u is bounded, nonnegative and continuous on $\bar{H}$.

[1] We shall limit our attention to the physically reasonable case of nonnegative data.

(ii) u satisfies the integral identity $I(u, \zeta, P) = 0$ for any $P = [t_0, t_1] \times [x_1, x_2] \subset \bar{H}$ and any $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta|_{x=x_1} = \zeta|_{x=x_2} = 0$.

(iii) $u(t, l_2) = \psi(t)$ for all $t \in [0, T]$ and $u(0, x) = u_0(x)$ for all $x \in (-\infty, l_2)$.

To prove the existence of a generalized solution for each one of the three problems we shall follow the constructive method given initially by O. A. Oleinik, A. S. Kalashnikov and E. Yui Lin in [28] for the case of Eq. (1.5). To do this, we first obtain a sequence of classical solutions of (E) (which are strictly positive functions). We shall show that it tends (pointwise) to a function that we call the *limit solution.* (Such a function satisfies all the required properties except, perhaps, the continuity.) This will be done in Section 2.

In Section 3 we shall prove that under additional hypotheses the limit solution is more regular and, in particular, a generalized solution. Such results are well known when

$$b \in C^1([0, \infty)) \qquad \text{and} \qquad \int_x^1 \{|\phi''(r)| + |b''(r)|\}\, dr \in L^1(0, 1) \tag{1.6}$$

[13]. In the case of Eq. $(\mathrm{E}_{m,\lambda})$, this corresponds to the assumption $m \geqslant 1$ and $\lambda \geqslant 1$. The study of the regularity of its solutions is made in [18, 15]. In both cases, optimal estimates on the modulus of continuity of the solution are given; in fact, such estimates are independent of b. In consequence, the idea that the transport term $b(u)_x$ has no fundamental effect on the behaviour of the solution is defended in the previous literature. In contrast to that, here we shall show that if, for instance, $0 < \lambda \leqslant 1$, then $(u^{m-\lambda})_x \in L^\infty$ and so the modulus of continuity of the solution of $(\mathrm{E}_{m,\lambda})$ depends on λ. More generally, if the function J defined by

$$J(r) \equiv \int_0^r \frac{ds}{b(\phi^{-1}(s))}$$

is finite for $r > 0$ (this is the case, for instance, of $\phi(u) = u^m$, $b(s) = s^\lambda$ and $m > \lambda$) then we shall prove that $J(\phi(u))_x \in L^\infty$ for solutions u of (E).

In Section 4 the uniqueness of the generalized solutions is considered. The problem of uniqueness has been a polemic subject in the existing literature. Indeed, the first uniqueness result seems to be the one obtained in 1975 by A. S. Kalashnikov. In his paper [18], the uniqueness of a generalized solution of $(\mathrm{E}_{m,\lambda})$ is shown under the assumptions $m > 1$ and $\lambda \geqslant 1$. In 1976, B. H. Gilding and L. A. Peletier [15] made a systematic study of Eq. $(\mathrm{E}_{m,\lambda})$ independently of Kalashnikov's work. In fact they introduce a different notion of solution of the problem (CP): they substitute conditions (ii) and (iii) of Definition 1.1 by

(ii)* (u^m) has a bounded generalized derivative with respect to x in S,

(iii)* u satisfies the identity

$$\iint_S \{\phi_x[(u^m)_x + u^\lambda] - \phi_t u\}\, dx\, dt = \int_{-\infty}^{\infty} \phi(x, 0)\, u_0(x)\, dx$$

for all $\phi \in C^1(\bar{S})$ which for large $|x|$ and for $t = T$. The uniqueness result of [16] for such a class of solutions (called weak solutions) is obtained under the assumption $\lambda \geqslant \frac{1}{2}(m+1)$. The important work [15] has been the object of several generalizations in the last years. For instance, B. H. Gilding in [13] proved the uniqueness of weak solutions of (CP), (FBVP) and (MBVP) under the hypothesis

$$(b')^2\,(s) = \theta(\phi'(s)) \qquad \text{as } s \to 0^+. \tag{1.7}$$

More recently, Wu Dequan in [36] has proved the uniqueness of the generalized solution of (FBVP), assuming

$$\begin{gathered}(b')(s) = \theta((\phi')^\alpha\,(s)) \text{ as } s \to 0^+,\ \phi'(s) \geqslant Ks^\nu \text{ for } s > 0, \text{ and}\\ \alpha \geqslant \tfrac{1}{4} \text{ if } \nu < 2 \text{ and } \alpha > \tfrac{1}{2} - \tfrac{1}{2}\nu \text{ if } \nu \geqslant 2.\end{gathered} \tag{1.8}$$

We remark that in terms of Eq. $(\mathrm{E}_{m,\lambda})$ condition (1.7) is equivalent to $\lambda \geqslant \frac{1}{2}(m+1)$ and condition (1.8) is equivalent to $\lambda \geqslant \frac{1}{4}(m+3)$ if $m < 3$ and $\lambda > m/2$ if $m \geqslant 3$. (Other uniqueness results are given in [26, 28] for some variations of Eq. (E).) Finally, we point out some recent result obtained in [4] by a different approach.

In this paper we give a general and unified answer to the problem of uniqueness of solutions of (CP), (FBVP) and (MBVP). Our assumptions on ϕ and b are weaker than those of the above papers. In particular, they are fulfilled if in Eq. $(\mathrm{E}_{m,\lambda})$ we assume $m \geqslant 1$ and $\lambda \geqslant 0$. On the other hand, in Section 3 the equivalence between the generalized and weak solutions is proved. Thus, the uniqueness of a weak solution is also ensured.

Our uniqueness result is a particular consequence of some L^1-estimates. These also show the continuous dependence of solutions with respect to the initial data as well as comparison results. Such estimates also show that the semigroup operator defined by the solution is a nonlinear semigroup of contractions on the space $L^1(-\infty, \infty)$, $L^1(l_1, l_2)$ or $L^1(-\infty, l_2)$, respectively.

In order to provide the reader with a summary collecting some of the results of this paper, we shall restrict ourselves to consideration of problem (PC) for Eq. $(\mathrm{E}_{m,\lambda})$. We can state the following result:

THEOREM 1.1. *Assume $m \geqslant 1$ and $\lambda \geqslant 0$. Let $u_0 \geqslant 0$ on $(-\infty, \infty)$ be such that u_0^β is Lipschitz continuous for some β such that*

$$\max\{(m-1), (m-\lambda)^+\} \leqslant \beta \qquad (h^+ = \max\{h, 0\}).$$

Then there exists a unique generalized solution u of the (CP) *problem for Eq.* $(\mathrm{E}_{m,\lambda})$. *In addition $(u^\nu)_x \in L^\infty(S)$ for $\nu = \min\{1, 1/\beta\}$ and u coincides with the unique weak solution of* (CP).

We point out that our results can be easily extended to a more general class of equations of the form

$$u_t = \phi(u)_x + b(x, t, u)_x + c(x, t, u),$$

where $\phi(u)$ is strictly increasing, $\phi(0) = 0$ and $\phi'(0) = 0$, and $b(\cdot, \cdot, u)$ and $c(\cdot, \cdot, u)$ are allowed to be not necessarily Lipschitz continuous at $u = 0$ (some additional hypotheses must be made on b and c, e.g., $c(\cdot, \cdot, u)$ nonincreasing in u, and so on).

A preliminary version of this paper appeared as MRC Technical Summary Report No. 2502 (1983). The qualitative behaviour of the solutions of $(\mathrm{E}_{m,\lambda})$ was considered in J. I. Diaz and R. Kersner, Non existence d'une des frontieres libres dans une équation dégénérée en théorie de la filtration, *C.R. Acad. Sci. Paris* **296** (1983), 505–508. See also G. Francsics, On the porous medium equations with lower order singular nonlinear terms, *Acta Math. Hung.* **45** (1985), 425–436. Some other references on the physical model and its treatement by semigroup theory may be found in N. I. Wolanski, Flow though a porous column, *J. Math. Anal. Appl.* **109** (1985), 140–159. Finally, we mention the works by M. Bertsch and D. Hilhorst, A density dependent diffusion equation in population dynamics: Stabilization to equilibrium, to appear, and M. Bertsch, R. Kersner, and L. A. Peletier, Positivity versus localization in degenerate diffusion problems, *Nonlinear Anal.* **9** (1985), 987–1008, where our uniqueness arguments are developed for some N-dimensional degenerate parabolic equations.

2. EXISTENCE OF A LIMIT SOLUTION

The basic idea in the study of degenerate equations like (E) consists in obtaining the solution as the limit of a sequence of functions which are solutions of some adequate nondegenerate parabolic equations approaching Eq. (E). This idea can be carried out in two different ways: (a) by considering the equations

$$u_{\varepsilon,t} = ((\phi'(u_\varepsilon) + \varepsilon)(u_\varepsilon)_x)_x + b_\varepsilon(u_\varepsilon)_x,$$

or (b) by replacing $u_0(x)>0$ by the sequence $u_{0,\varepsilon}(x)>\varepsilon>0$ and then showing (via the maximum principle) that the corresponding solutions u_ε satisfy $u_\varepsilon(t,x)\geqslant\varepsilon$, so they are solutions of the nondegenerate equations.

Method (a) is very useful if the signs of the data (for instance, u_0 for (CP)) are not "a priori" prescribed. However, the passage to the limit is often a difficult task (see the results of [7, 33] for the case $b\equiv 0$). Here we shall follow the method (b) introduced in [28]. Then we shall obtain a sequence of classical solutions and prove that they converge pointwise to a function that we call *limit solution*. In the next sections we shall prove that, under some supplementary hypotheses, the limit solution coincides with the unique generalized solution.

PROPOSITION 2.1. *Assume that there exists* $\alpha\in(0,1]$ *such that*

$$\phi\in C^{2+\alpha}((0,+\infty))\cap C^1([0,\infty)),\ \phi(0)=0,\text{ and } \phi'(r)>0 \text{ if } r>0 \quad (2.1)$$

$$b\in{}^{2+\alpha}((0,\infty). \quad (2.2)$$

Then:

(i) *For every* $u_0\in C_b(-\infty,\infty)$,[2] $u_0\geqslant 0$, *there exists at least one function* u *defined on* $\bar{S}$ *such that* $u\geqslant 0$, $u\in L^\infty(S)$ *and* u *satisfies* (ii) *and* (iii) *of Definition* 1.1.

(ii) *For every* $u_0\in C([l_1,l_2])$, $u_0\geqslant 0$, ψ_-, $\psi_+\in C([0,t])$, $\psi_-,\psi_+\geqslant 0$ *and* $\psi_-(0)=u_0(l_1)$, $\psi_+(0)=u_0(l_2)$ *there exists at least one function* u *defined on* $\bar{R}$ *such that* $u\geqslant 0$, $u\in L^\infty(R)$ *and* u *satisfies* (ii) *and* (iii) *of Definition* 1.2.

(iii) *For every* $u_0\in C_b((-\infty,l_2])$, $u_0\geqslant 0$, $\psi\in C([0,t])$, $\psi\geqslant 0$ and $\psi(0)=u_0(l_2)$ *there exists at least one function* u *defined on* $\bar{H}$ *such that* u *satisfies* (ii) *and* (iii) *of Definition* 1.3.

The proof of Proposition 2.1 is already standard after the work [28] and its generalizations (see, for instance, [18, 13, 26]). Nevertheless, in the next sections we shall need some properties of the function u which are obtained by using the proof of Proposition 2.1. This is the reason for our sketch.

We shall use the following result of the classical theory of quasilinear parabolic equations.

LEMMA 2.1 (see, e.g., [13]). *Let* $Q\equiv(\eta_1,\eta_2)\times(0,T]$, ε, $\alpha\in(0,1]$ *and* $M\in(0,\infty)$. *Suppose that* $u_0\in C^{2+\alpha}([\eta_1,\eta_2])$, $\psi_1,\psi_2\in C^{1+\alpha}[0,T]$ *and*

$$\varepsilon\leqslant u_0\leqslant M,\ \varepsilon\leqslant\psi_1,\psi_2\leqslant M$$

$$\psi_i(0)=u_0(\eta_i),\ \psi_i'(0)=\phi(u_0)''(\eta_i)+(b(u_0))'(\eta_i) \qquad \textit{for} \quad i=1,2.$$

[2] $C_b(\Omega)$ denotes the set of all the bounded continuous functions defined on Ω.

Then (under assumptions (2.1) *and* (2.2)) *there exists a unique function* $u(t, x)$ *such that*

$$u \in C^{1,2}_{t,x}(\bar{Q}), \qquad \phi(u)_x \in C^{1,2}_{x,t}(Q), \qquad \varepsilon \leqslant u \leqslant M \qquad in\ \bar{Q}$$

$$u_t = \phi(u)_{xx} + b(u)_x \qquad on\ Q$$

$$u(x, 0) = u_0(x) \qquad on\ [\eta_1 \eta_2]$$

$$u(t, \eta_i) = \psi_i(t) \qquad on\ [0, T], for\ i = 1, 2. \quad ▮$$

Proof of Proposition 2.1. We shall prove (i). We can always choose $M > 0$, $\{\varepsilon_k\}$ and $\{u_{0,k}\}$ such that

$$\begin{aligned}
&\varepsilon_k \in (0, 1],\ \varepsilon_k \to 0 \text{ as } k \to \infty,\\
&u_{0,k} \in C^{2+\alpha}(-\infty, \infty),\ \varepsilon_k \leqslant u_{0,k}(x) \leqslant M \text{ if } |x| < k \text{ and } u_{0,k}(x) = \varepsilon_k \text{ if } |x| \geqslant k\\
&u_{0,k+1}(x) \leqslant u_{0,k}(x) \text{ for all } x \in (-\infty, \infty)\\
&u_{0,k} \to u_0 \text{ as } k \to \infty \text{ uniformly on compact subsets of } (-\infty, \infty).
\end{aligned} \tag{2.3}$$

By classical theory [24], there exists a unique function $u_k \in C^{1,2}_{t,x}(S)$ such that (i) $(u_k)_x \in C^{1,2}_{t,x}(S)$ and $\varepsilon_k \leqslant u_k \leqslant M$ in $\bar{S}$, *(ii)* u_k satisfies (E) in S, and (iii) $u_k(0, x) = u_{0,k}(x)$, $x \in (-\infty, \infty)$. Then, by a standard application of the maximum principle we obtain that $u_{k+1}(t, x) \leqslant u_k(t, x)$ for all $(t, x) \in \bar{S}$. Hence, we can define

$$u(t, x) = \lim_{k \to \infty} u_k(t, x) \tag{2.4}$$

for all $(t, x) \in \bar{S}$. The function u is nonnegative and bounded and satisfies the integral condition (ii) in Definition 1.1. The proofs of (ii) and (iii) are similar. The natural modifications now being that $u_{0,k}$ are defined only in $[l_1, l_2]$ (in the proof of part (ii) of Proposition 2.1). Also, there exist $\{\psi_{-,k}\}$ and $\{\psi_{+,k}\}$ (sequences in $C^{1+\alpha}([0, T])$) such that $\varepsilon_k \leqslant \psi_{-,k}$, $\psi_{+,k} \leqslant M$, $\psi_{-,k+1} \leqslant \psi_{-,k}$, $\psi_{+,k+1} \leqslant \psi_{+,k}$, $\psi_{-,k}(0) = u_{0,k}(l_1)$, $\psi_{+,k}(0) = u_{0,k}(l_2)$, $(\psi_{-,k})'(0) = (\phi(u_{0,k}))''(l_1) + (b(u_{0,k})'(l_1)$, $(\psi_{+,k})'(0) = (\phi(u_{0,k}))''(l_2) + (b(u_{0,k}))'(l_2)$. Finally $\psi_{+,k} \to \psi_+$, $\psi_{-,k} \to \psi_-$ uniformly on $[0, T]$ when $k \to \infty$. ▮

Remark 2.1. Obviously, we can also consider more general quasilinear equations or choose data u_0, (u_0, ψ_-, ψ_+) and (u_0, ψ), not necessarily continuous (see [16, 4]). We remark that the result applies to Eq. $(E_{m,\lambda})$ when $0 < m < 1$. When $b \equiv 0$ such an equation arises in plasma physics (see the exact references in [29]).

3. Regularity of the Limit Solutions: Existence of Generalized Solutions

The continuity of the limit solutions is a consequence of the general results of [9, 33, 39], where in fact N-dimensional problems are considered. A modulus of continuity of the solutions is given in [9, 10], but, in general, it is not optimal. In the one-dimensional case, the optimal modulus of continuity of the solutions of $(\mathrm{E}_{m,\lambda})$, where $m \geqslant 1$ and $\lambda \geqslant 1$, was given in [18]. More generally, the modulus of continuity of solutions of (E) was obtained in [13] assuming that ϕ and b satisfy (2.1), (2.2) as well as

$$b \in C^1([0, \infty)) \tag{3.1a}$$

$$\int_s^1 \{|\phi''(r)| + |b''(r)|\}\, dr \in L^1(0, 1). \tag{3.1b}$$

Here we shall study some additional regularity for Eq. (E) under assumptions including Eq. $(\mathrm{E}_{m,\lambda})$ for $m \geqslant 1$ and $0 < \lambda < 1$.

An important tool in our study will be the following: if we define the improper integral

$$J(r) = \int_0^r \frac{ds}{b(\phi^{-1}(s))}$$

for every $r > 0$ (we can suppose, without loss of generality that $b(0) = 0$), then, when $\phi(s) = s^m$ and $b(s) = s^\lambda$, $J(r)$ is finite if and only if $m > \lambda$. Thus, our fundamental hypothesis will be $J(r) < +\infty$ for some $r \in (0, \infty)$.

We start by studying the general nondegenerate problem given in Lemma 2.1.

Proposition 3.1. *Given* $\delta \in (0, \frac{1}{2}(\eta_2 - \eta_1))$ *and* $\tau \in (0, T)$ *let* $Q_\delta = (0, T] \times (\eta_1 + \delta, \eta_2 - \delta)$, $Q(\tau) = (\tau, T] \times (\eta_1, \eta_2)$, $Q_\delta(\tau) = (\tau, T] \times (\eta_1 + \delta, \eta_2 - \delta)$. *Assume* (2.1), (2.2) *and that for every* $r \in (0, \varepsilon)$ *the following hypotheses hold*

$$J(r) < +\infty \tag{3.2}$$

$$\phi'(r)\, b'(r) \leqslant -C_1 b''(r)\, b(r) \tag{3.3}$$

$$|\phi''(r)| \leqslant C_2\, |b''(r)| \tag{3.4}$$

for some positive constants C_1 *and* C_2. *Then for any* u_0 *and* ψ_i, *given as in* Lemma 2.1, *the solution* u *satisfies: for any* δ *and* τ *there exists a constant* C (*depending only on* δ, τ *and* M) *such that*

$$|J(\phi(u))_x| \leqslant C \qquad in\ Q_\delta(\tau). \tag{3.5}$$

If in addition, u_0 verifies

$$\sup_{(\eta_1+\delta,\eta_2-\delta)} |J(\phi(u_0(x)))'| = L < +\infty, \tag{3.6}$$

then (3.5) *holds in* Q_δ.

Before proving the above result, let us explain some facts about the proof. The method we use is due to Bernstein. As is well known, the major difficulty of this method appears in the selection of the function of u to be estimated. The estimate

$$|(g(u))_x| \leqslant C \text{ in } Q_\delta(t) \tag{3.7}$$

has been obtained by different authors in the following cases:

(a) $g(s) = \phi(s)$. (see [1] for $b \equiv 0$ and [14] if b satisfies (3.1b).)

(b) $g(s) = \int_0^s (\phi'(s)/s)\, ds$, if such an integral converges and $b \equiv 0$ (see [1, 18]).

Estimate (3.6) is completely new. For Eq. $(\mathrm{E}_{m,\lambda})$, all the hypotheses of Proposition 3.1 are satisfied if

$$0 < \lambda < 1 \leqslant m.$$

In this case a single computation shows that $J(\phi(s)) = (m/(m-\lambda))\, s^{m-\lambda}$. More generally we can prove (using Proposition 3.1 if $0 < \lambda < 1$, or [18] if $\lambda \geqslant 1$) that

$$|(u^\beta)_x| \leqslant C \text{ in } Q_\delta(t) \tag{3.8}$$

for all $\beta \in R$ such that

$$\max\{(m-1), (m-\lambda)^+\} \leqslant \beta,$$

where $h^+ = \max\{h, 0\}$. Then, estimate (3.7) includes also the estimates of [1, 18, 15] for Eq. $(\mathrm{E}_{m,\lambda})$.

Proof of Proposition 3.1. Let f be the real function defined by $f(r) = \phi^{-1}(J^{-1}(r))$ for $r > 0$. Set $w = f^{-1}(u)$. From Eq. (E) we obtain

$$w_t = \frac{1}{f'(w)} [\phi(f(w))]_{ww} (w_x)^2 + \phi'(f(w))\, w_{xx} + b'(f(w))\, w_x. \tag{3.9}$$

Using the definition of J and f we have

$$[\phi(f(w))]_{ww} = [J^{-1}(w)]_{ww} = \frac{b'(f(w))\, b(f(w))}{\phi'(f(w))}$$

$$f'(w) = \frac{b(f(w))}{\phi'(f(w))}$$

and

$$\frac{[\phi(f(w))]_{ww}}{f'(w)} = b'(f(w)). \tag{3.10}$$

Then

$$w_t = \phi'(f(w))\, w_{xx} + b'(f(w))\, w_x^2 - b'(f(w))\, w_x. \tag{3.11}$$

Consider now a smooth function $\zeta(t, x)$ such that $\zeta = 1$ on $\bar{Q}_\delta(\tau)$, $\zeta = 0$ on the parabolic boundary of Q and $0 \leqslant \zeta \leqslant 1$ in $\bar{Q}$. Define the function $z = \zeta^2 p^2$, where $p = w_x$; at any point $(t_0, x_0) \in Q$, where z attains a positive maximum, one has

$$z_x = 0 \qquad \text{and} \qquad z_t - \phi'(f(w))\, z_{xx} \geqslant 0.$$

Hence, at (t_0, x_0) we have $\zeta p_x = -\zeta_x p$ and

$$\zeta^2 p(p_t - \phi'(f)\, p_{xx}) \geqslant (-\zeta\zeta_t + \phi'(f)\, \zeta\zeta_{xx} + 2\phi'(f)\, \zeta_x^2)\, p^2.$$

Differentiating (3.11) with respect to x, multiplying the result by $\zeta^2 p$ and using the former relations we obtain

$$\begin{aligned} -b''(f)\, f'\zeta^2 p^4 \leqslant \zeta p^3[&-\phi''(f)\, f'\zeta\zeta_x - 2b'(f)\, \zeta_x - b''(f)\, f'\zeta] \\ &+ p^2[\zeta\zeta_t - \phi'(f)\, \zeta\zeta_{xx} - 2\phi'(f)\, \zeta_x^2 - b'(f)\, \zeta_x\zeta]. \end{aligned} \tag{3.12}$$

Using the hypotheses (3.3) and (3.4) we can find two positive constants K_1 and K_2 depending only on ϕ, b, M, δ and τ, such that

$$2\zeta^2 p^2 \leqslant K_1 \zeta\, |p| + K_2 \qquad \text{at } (t_0, x_0). \tag{3.13}$$

By an elementary argument, (3.13) implies that

$$z(t_0, x_0) \leqslant K_1 + \tfrac{1}{4} K_2^2 = K_3$$

and hence

$$\sup_{\bar{Q}_\delta(\tau)} |w_x| \leqslant K_3{}^{1/2}.$$

To prove the second part, we note that $|w_x|$ is now bounded at $t = 0$. Hence we may take a cut-off function $\zeta(t, x) = \zeta(x)$ and allow z to attain its maximum at a point of the lower boundary of Q. Otherwise the proof is the same. ∎

Remark 3.1. Arguing as in [18] we can improve estimate (3.6) when $(J(\phi(\cdot)))'' \leqslant 0$. Indeed, in this case it is possible to show that, in fact, $u_x \in L^\infty$. In particular, using the results of [17] we can prove that if u is solution of $(\mathrm{E}_{m,\lambda})$ then $(u^\nu)_x$ is bounded for $\nu = \min\{1, 1/\beta\}$ and for any $\beta \geqslant \max\{(m-1), (m-\lambda)^+\}$.

A further regularity result is the following:

THEOREM 3.1. *Let ϕ and b satisfying the hypotheses of Proposition* 3.1. *Then*

(i) *For any $u_0 \geqslant 0$ such that $J(\phi(u_0))$ is bounded Lipschitz continuous on $(-\infty, \infty)$ there exists at least one generalized solution u of* (CP) *such that $J(\phi(u))_x \in L^\infty(S)$ (where the derivative is taken in the sense of distributions). In particular $\phi(u)_x \in L^\infty(S)$ and u satisfies*

$$\int_0^T \int_{-\infty}^{\infty} \{\theta_x[\phi(u)_x + b(u)] - \theta_t u\}\, dx\, dt = \int_{-\infty}^{\infty} \theta(0, x)\, u_0(x)\, dx \quad (3.14)$$

for all $\theta \in C^1(\bar{S})$ which vanish for large $|x|$ and $t = T$.

(ii) *For any u_0, ψ_- and ψ_+ nonnegative functions such that $\phi(u_0)$ is locally Lipschitz continuous on (l_1, l_2), and such that $\phi(\psi_-)$ and $\phi(\psi_+)$ are absolutely continuous on $[0, T]$ and $\psi_-(0) = u_0(l_1)$, $\psi_+(0) = u_0(l_2)$, there exists at least one generalized solution u of* (FBVP) *such that $\phi(u)_x \in L^2(R)$ (distributional derivative) and u satisfies the identity*

$$\iint_R \{\theta_x[\phi(u)_x + b(u)] - \theta_t u\}\, dx\, dt = \int_{l_1}^{l_2} \theta(0, x)\, u_0(x)\, dx \quad (3.15)$$

for all $\theta \in C^1(\bar{R})$ which vanish for $x = l_1$, $x = l_2$ and $t = T$.

(iii) *For any u_0, ψ nonnegative functions such that $\phi(u_0)$ is locally Lipschitz continuous and bounded on $(-\infty, l_2)$, $\phi(\psi)$ is absolutely continuous on $[0, T]$, and $\psi(0) = u_0(l_2)$, there exists at least one generalized solution u of* (MBVP) *such that $\phi(u)_x \in L^2_{\mathrm{loc}}(H)$ (distributional derivative) and such that*

$$\iint_H \{\theta_x[\phi(u)_x + b(u)] - \theta_t u\}\, dx\, dt = \int_{-\infty}^{l_2} \theta(0, x)\, u_0(x)\, dx. \quad (3.16)$$

for all $\theta \in C^1(\bar{H})$ which vanish for $x = l_2$, for large $|x|$ and for $t = T$.

The proof of (i) is a simple consequence of the fact that

$$|\phi(u_k)_x| = |\phi(f(f^{-1}(u_k)))_x| \leqslant C^* \cdot |f^{-1}(u_k)_x| \leqslant C^* \cdot C,$$

where $C^* = \max b(u_k(t, x))$ and $k \geqslant 1$. Otherwise the proof is standard (see, e.g., [13]).

In order to prove (ii) we need the following estimate near the boundary.

LEMMA 3.1. *Assume ϕ and b as in Proposition 3.1. Let u_0 and ψ_i satisfy the assumptions of Lemma 2.1, (3.6) and*

$$\int_0^T |(\phi(\psi_i(t)))'|\, dt \leqslant L^* \qquad \textit{for } i = 1, 2\ (L^* > 0). \tag{3.17}$$

Then, for any $\delta > 0$, there exists a constant C^*, which depends only on L, L^*, M, T and δ, such that

$$\iint_{Q(\tau) - Q_\delta(\tau)} \{\phi(u)_x\}^2\, dx\, dt \leqslant C^* \tag{3.18}$$

for any $\tau \in (0, T]$.

Proof. We shall only prove that

$$\int_0^T \int_{\eta_1}^{\eta_1 + \delta} \{\phi(u)_x\}^2\, dx\, dt \leqslant \frac{C^*}{2}$$

(the estimate $\int_0^T \int_{\eta_2 - \delta}^{\eta_2} \{\phi(u)_x\}^2\, dx\, dt \leqslant C^*/2$ is obtained in a similar way). The key idea is due to Gilding [13]. Let $\chi(t, x) = \phi(u(t, x)) - \phi(\psi_1(t))$. If we take Eq. (E), multiply it by χ and integrate by parts we obtain

$$\begin{aligned}
&\int_0^T \int_{\eta_1}^{\eta_1 + \delta} \{\phi(u)_x\}^2\, dx\, dt \\
&\quad = \int_0^T \{(\phi(u)_x)(t, \eta_1 + \delta) + b(u(t, \eta_1 + \delta))\}\, \chi(t, \eta_1 + \delta)\, dt \\
&\qquad - \int_0^T \int_{\eta_1}^{\eta_1 + \delta} b(u)(\phi(u))_x\, dx\, dt - \int_0^T \int_{\eta_1}^{\eta_1 + \delta} u_t \phi(u)\, dx\, dt \\
&\qquad + \int_0^T \int_{\eta_1}^{\eta_1 + \delta} u_t(t, x)\, \phi(\psi_1(t))\, dx\, dt.
\end{aligned} \tag{3.19}$$

We denote the four integrals on the right-hand side of (3.19) by I_1, I_2, I_3 and I_4, respectively. The only difficult term to estimate is I_1 (see [13]). But by Proposition 3.1 we know that

$$|f^{-1}(u)_x\, (t, \eta_1 + \delta)| \leqslant C \qquad \text{for any } t \in [0, T].$$

Then

$$|\phi(u)_x\, (t, \eta_1 + \delta)| = |\phi' f'(f^{-1}(u))_x\, (t, \eta_1 + \delta)| \leqslant C'$$

for some $C > 0$. So

$$|I_1| \leqslant 2\phi(M)\, T(C' + \sup_{s \in (0, M]} b(s)). \quad ▮$$

Proof of (ii) *of Theorem* 3.1. Now the functions $u_{0,k}$, $\psi_{-,k}$ and $\psi_{+,k}$ can be assumed to satisfy the conditions given in the proof of Proposition 2.1 as well as

$$\text{for any } \delta \in (0, 1) \text{ there exists a constant } L(\delta) \text{ such that}$$
$$|\phi(u_{0,k})'(x)| \leqslant L(\delta) \text{ for all } x \in (l_1 + \delta, l_2 - \delta)$$

and

$$\int_0^T |\phi(\psi_{-,k}(t))'|\, dt, \qquad \int_0^T |\phi(\psi_{+,k}(t))'|\, dt \leqslant L^*.$$

Then, by Lemma 3.1, there exists a constant C^* (which depends only on $L(\beta)$, L^*, M and $T(\beta = (l_2 - l_1)/4)$) such that

$$\int_{l_1}^{l_1+\beta} \int_0^T \{\phi(u_k)_x\}^2\, dx\, dt + \int_{l_2-\beta}^{l_2} \int_0^T \{\phi(u_k)_x\}^2\, dx\, dt \leqslant C^* \tag{3.20}$$

for all $k \geqslant 1$. On the other hand, by Proposition 3.1 there exists a constant C_1 which depends only on $L(\beta)$ and M such that

$$|(\phi(u_k))_x(t, x)| \leqslant C_1 \text{ for all } (t, x) \in [0, T] \times [l_1 + \beta, l_2 - \beta]. \tag{3.21}$$

From (3.20) and (3.21) we obtain that $\|\phi(u_k)_x\|_{L^2(H)}$ is uniformly bounded and by using the fact that u_k is a classical solution it is easy to see that the weak limit $v \in L^2(H)$ of $\{\phi(u_k)_x\}$ can only be $\phi(u)_x$. The proof of (iii) is analogous. ▮

Remark 3.2. By using a generalization of the Nash theorem [24 p. 204], it is not difficult to show that, under the assumptions of Theorem 3.1, the generalized solution obtained in the above result is a classical solution of (E) in a neighborhood of any interior point (t_0, x_0) where $u(t_0, x_0) > 0$ (see, e.g., [1] or [13]).

Remark 3.3. Suppose, for instance, that $b(s) \geqslant 0$ for any $s \geqslant 0$. Given $l \in R$, we define the stationary function

$$U(t, x) \equiv f((l-x)^+) = \begin{cases} \phi^{-1}(J^{-1}(l-x)) & \text{if } x \leqslant l \\ 0 & \text{if } x > l. \end{cases} \tag{3.22}$$

It is easy to see that u is a generalized solution of (MBVP) and satisfies $u(0, x) = f((l-x)^+)$ for $0 \leqslant x < \infty$ and $u(t, 0) = f(l)$ for $t \in [0, T]$.

Moreover

$$(f^{-1}(u))_x = ((l-x)^+)_x$$

and hence $(f^{-1}(u))_x = 0$ if $x > l$ but $(f^{-1}(u))_x \nearrow -1$ when $x \nearrow l$. Thus, the estimate $J(\phi(u))_x \in L^\infty$ is exact and cannot be improved.

Remark 3.4. In some previous works (see [15, 13]) a different notion of solution of (CP) (respectively (FBVP) and (MBVP)) is introduced by means of the integral equality (3.14) (respectively (3.15) and (3.16)). Thus, following [13], a function u defined on $\bar{S}$ is said to be a *weak solution* of (CP) if u satisfies (i) and (iii) of the Definition 1.1 as well as the condition

$$\int_0^T \int_{-\infty}^{\infty} \{\Theta_x[\phi(u)_x + b(u)] - \Theta_t u\}\, dx\, dt = \int_{-\infty}^{\infty} \Theta(0, x)\, u_0(x)\, dx \qquad (3.23)$$

for every $\Theta \in C^1(\bar{S})$ such that Θ vanishes for large $|x|$ and $t = T$. Analogously, the notions of weak solutions of (FBVP) and (MBVP) are defined by substituting the integral conditions of Definitions 1.2 and 1.3 by the conditions (3.15) and (3.16), respectively. Theorem 3.1 states that, under some natural assumptions, every limit solution is also a weak solution. The following result shows the equivalence between both notions of solution.

THEOREM 3.2. *Assume $\phi \in C^1([0, \infty))$ and $b \in C^0([0, \infty))$. Then every weak solution of* (CP) (*resp.* (FBVP) *and* (MBVP)) *is a generalized solution of* (CP) (*resp.* (FBVP) *and* (MBVP)).

Proof. We shall follow an idea suggested to the first author of this paper by M. G. Crandall.

Let u be a weak solution of (CP) and let $P = [t_0, t_1] \times [x_1, x_2]$ and $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t, x_1) = \zeta(t, x_2) = 0$ for any $t \in [t_0, t_1]\ C[0, T]$. Let $\eta \in C^2(R)$ be such that

(a) $\eta(r) = 1$ if $r \leqslant -1$ and $\eta(r) = 0$ if $r \geqslant 0$;

(b) $\eta'(0) = \eta'(-1) = 0$.

For every $\varepsilon > 0$, we define the test function $\Theta_\varepsilon(t, x)$ as

$$\Theta_\varepsilon(t, x) = \begin{cases} \zeta(t, x)\, \eta\left(\dfrac{t - t_1}{\varepsilon}\right) \eta\left(\dfrac{t_0 - t}{\varepsilon}\right) \eta\left(\dfrac{x - x_2}{\varepsilon}\right) \eta\left(\dfrac{x_1 - x}{\varepsilon}\right) & \text{if } (t, x) \in P \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of η, it is immediate that $\Theta_\varepsilon \in C^2(\bar{S})$ (and support $\Theta_\varepsilon) \subseteq P$. By assumption we have

$$
\begin{aligned}
0 &= -\int_0^T \int_{-\infty}^{\infty} \Theta_{\varepsilon,t} u \, dx \, dt + \int_0^T \int_{-\infty}^{\infty} \Theta_{\varepsilon,x} \phi(u)_x \, dx \, dt \\
&\quad + \int_0^T \int_{-\infty}^{\infty} \Theta_{\varepsilon,x} b(u) \, dx \, dt \qquad (3.24) \\
&= I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon}.
\end{aligned}
$$

One has

$$
\begin{aligned}
-I_{1,\varepsilon} &= \iint_P \zeta_x u \eta \left(\frac{t-t_1}{\varepsilon} \right) \eta \left(\frac{t_0 - t}{\varepsilon} \right) \eta \left(\frac{x-x_2}{\varepsilon} \right) \eta \left(\frac{x_1 - x}{\varepsilon} \right) dx \, dt \\
&\quad + \iint_P \frac{\zeta u}{\varepsilon} \left[\eta' \left(\frac{t-t_1}{\varepsilon} \right) \eta(\cdot) \, \eta(\cdot) \, \eta(\cdot) \right] dx \, dt \\
&\quad - \iint_P \frac{\zeta u}{\varepsilon} \left[\eta(\cdot) \, \eta' \left(\frac{t_0 - t}{\varepsilon} \right) \eta(\cdot) \, \eta(\cdot) \right] dx \, dt \\
&= \iint_P \zeta_t u \eta(\cdot) \, \eta(\cdot) \, \eta(\cdot) \, \eta(\cdot) \, dx \, dt \\
&\quad + \int_{x_1}^{x_2} \int_{(t_0 - t_1)/\varepsilon}^{0} \zeta(\varepsilon\tau + t_1, x) \, u(\varepsilon\tau + t_1, x) \, \eta \left(\frac{t_0 - t_1}{\varepsilon} + \tau \right) \\
&\quad \times \eta \left(\frac{x - x_2}{\varepsilon} \right) \eta \left(\frac{x_1 - x}{\varepsilon} \right) \eta'(\tau) \, d\tau \\
&\quad - \int_{x_1}^{x_2} \int_{(t_0 - t_1)/\varepsilon}^{0} \zeta(t_0 - \varepsilon\tau, x) \, u(t_0 - \varepsilon\tau, x) \, \eta \left(\frac{t_0 - t_1}{\varepsilon} - \tau \right) \\
&\quad \times \eta \left(\frac{x - x_2}{\varepsilon} \right) \eta \left(\frac{x_1 - x}{\varepsilon} \right) \eta'(\tau) \, d\tau.
\end{aligned}
$$

Then, when ε converges to zero, we obtain

$$
\begin{aligned}
-I_{1,\varepsilon} &\to \iint_P \zeta_t u \, dx \, dt - \int_{x_1}^{x_2} \zeta(t_1, x) \, u(t_1, x) \, dx \\
&\quad + \int_{x_1}^{x_2} \zeta(t_0, x) \, u(t_0, x) \, dx.
\end{aligned}
$$

Analogously

$$I_{2,\varepsilon}=\iint_P \Theta_{\varepsilon,x}\phi(u)_x\,dx\,dt=\int_{t_0}^{t_1}\Theta_{\varepsilon,x}\phi(u)\,|_{x_1}^{x_2}\,dt-\iint_P \Theta_{\varepsilon,xx}\phi(u)\,dx\,dt$$

$$=\int_{t_0}^{t_1}\{(\zeta_x\eta(\cdot)\,\eta(\cdot)\,\eta(\cdot)\,\eta(\cdot)\,\phi(u))+\zeta\eta\left(\frac{t-t_1}{\varepsilon}\right)\eta\left(\frac{t_0-t}{\varepsilon}\right)$$

$$\times\left[\frac{1}{\varepsilon}\eta'\left(\frac{x-x_2}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\right.$$

$$\left.\left.-\frac{1}{3}\eta\left(\frac{x-x_2}{\varepsilon}\right)\eta'\left(\frac{x_1-x}{\varepsilon}\right)\right]\right|_{x_1}^{x_2}dt$$

$$-\iint_P \Theta_{\varepsilon,xx}\phi(u)\,dx\,dt=I^1_{2,\varepsilon}+I^2_{2,\varepsilon}.$$

Because of the fact that $\zeta(t,x_1)=\zeta(t,x_2)=0$, one has

$$I^1_{1,\varepsilon}\to\int_{t_0}^{t_1}(\zeta_x(t,x_2)\,\phi(u(t_1,x_2))-\zeta_x(t,x_1)\,\phi(u(t,x_1)))\,dt.$$

On the other hand

$$-I^2_{2,\varepsilon}=\iint_P \zeta_{xx}\eta(\cdot)\,\eta(\cdot)\,\eta(\cdot)\,\eta(\cdot)\,\phi(u)\,dx\,dt$$

$$+2\iint_P \zeta_x\eta(\,)\,\eta(\,)\left[\frac{1}{\varepsilon}\eta'\left(\frac{x-x_2}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\right.$$

$$\left.-\frac{1}{\varepsilon}\eta\left(\frac{x-x_2}{\varepsilon}\right)\eta'\left(\frac{x_1-x}{\varepsilon}\right)\right]\phi(u)\,dx\,dt$$

$$+\iint_P \zeta\eta(\cdot)\,\eta(\cdot)\left[\frac{1}{\varepsilon^2}\eta''\left(\frac{x-x_2}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\right.$$

$$-\frac{2}{\varepsilon^2}\eta'\left(\frac{x-x_2}{\varepsilon}\right)\eta'\left(\frac{x_1-x}{\varepsilon}\right)$$

$$\left.+\frac{1}{\varepsilon^2}\eta\left(\frac{x-x_2}{\varepsilon}\right)\eta''\left(\frac{x_1-x}{\varepsilon}\right)\right]\phi(u))\,dx\,dt.$$

Arguing similarly as in integral $I_{1,\varepsilon}$, we obtain

$$\iint \zeta_x \eta(\cdot)\,\eta(\cdot)\left[\frac{1}{\varepsilon}\eta'\left(\frac{x-x_2}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\right.$$

$$\left.-\frac{1}{\varepsilon}\eta'\left(\frac{x-x_2}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\right]\phi(u)\,dx\,dt \to \int_{t_0}^{t_1}(\zeta_x(t,x_2)\,\phi(u(t,x_2))$$

$$-\zeta_x(t,x_1)\,\phi(u(t,x_1))\,dt,$$

when $\varepsilon \to 0$. Moreover

$$I_{2,\varepsilon}^{2,*} = \frac{1}{\varepsilon^2}\iint_P \zeta\eta\left(\frac{t-t_1}{\varepsilon}\right)\eta\left(\frac{t_0-t}{\varepsilon}\right)\eta\left(\frac{x_1-x}{\varepsilon}\right)\eta''\left(\frac{x-x_2}{\varepsilon}\right)\phi(w)\,dx\,dt$$

$$=\int_{t_0}^{t_1}\int_{(x_1-x_2)/\varepsilon}^{0}\zeta\left(\frac{t,x_2+\varepsilon\tau}{\varepsilon\tau}\right)\eta\left(\frac{t_0-t}{\varepsilon}\right)\eta\left(\frac{x_1-x_2}{\varepsilon}\right)+\tau\Bigg)$$

$$\times\,\tau\eta''(t)\,\phi(u(t,x_2+\varepsilon\tau))\,dx\,dt$$

and then

$$I_{2,\varepsilon}^{2,*} \xrightarrow{\varepsilon\to 0} \int_{t_0}^{t_1}\left[\zeta_x(t,x_2)\,\phi(u(t,x_2))\int_{-1}^{0}\tau\eta''(\tau)\,d\tau\right]dt$$

$$=\int_{t_0}^{t_1}\zeta_x(t,x_2)\,\phi(u(t,x_2)$$

(we recall that $\eta'(0)=\eta'(-1)=0$). We also remark that

$$\frac{1}{\varepsilon^2}\iint_P \zeta(t,x)\,\eta\left(\frac{t-t_1}{\varepsilon}\right)\eta\left(\frac{t_0-t}{\varepsilon}\right)\eta'\left(\frac{x-x_2}{\varepsilon}\right)$$

$$\times\,\eta'\left(\frac{x_1-x}{\varepsilon}\right)\phi(u(t,x))\,dx\,dt=0$$

for every $\varepsilon>0$ such that $0<\varepsilon<(x_2-x_1)/2$. Then

$$I_{2,\varepsilon} \xrightarrow{\varepsilon\to 0} \int_{t_0}^{t_1}\zeta_x\phi(u)\,|_{x=x_1}^{x=x_2}\,dt$$

$$-\iint_P \zeta_{xx}\phi(u)\,dx\,dt - 2\int_{t_0}^{t_1}\zeta_x\phi(u)\,|_{x=x_1}^{x=x_2}\,dt$$

$$=\iint_P \zeta_{xx}\phi(u)\,dx\,dt-\int_{t_0}^{t_1}\zeta_x\phi(u)\,|_{x=x_1}^{x=x_2}\,dt.$$

Finally, in a similar way we obtain

$$I_{3,\varepsilon} \xrightarrow{\varepsilon \to 0} \iint_P \zeta_x b(u)\, dx\, dt.$$

Then, making $\varepsilon \to 0$ in (3.24) we obtain that $-I(u, \zeta, P) = 0$ and then u is a generalized solution of (CP). The cases of the problems (FBVP) and (MBVP) are similar. ∎

4. Uniqueness, Comparison Results and Continuous Dependence

In this section we prove that the generalized solution obtained in Proposition 2.1 and Theorem 3.1 (i.e., the limit solution) is the unique generalized solution.

Our uniqueness result will be a consequence of some L^1-estimates that also prove the continuous dependence of the solutions on the data. Other important consequences of these L^1-estimates are the comparison results showing the monotone dependence of solutions with respect to the data.

To formulate general results about comparison of solutions we introduce the following definition:

Definition 4.1. Let G be a closed set of $\bar{S}$. A function $v(t, x)$ defined on G is a generalized supersolution (resp. subsolution) of Eq. (E) in G if

(a) v is nonnegative, bounded and continuous, and

(b) v satisfies the integral inequality $I(v, \zeta, P) \leqslant 0$ (*resp.* $\geqslant 0$).

(I given in Definition 1.1 for any rectangle $P = [t_0, t_1] \times [x_1, x_2]$, $P \subset G^0$ and for all $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t, x_1) = \zeta(t, x_2) = 0$ for any $t \in [t_0, t_1]$, $\zeta \geqslant 0$ on P.)

In this section we shall assume the following hypotheses:

$$\left.\begin{array}{l}\phi \in C^1([0,\infty)) \cap C^2((0,\infty)),\ \phi(0) = \phi'(0) = 0 \text{ and there exists a}\\ \text{convex function } \mu \in C^0([0,\infty)) \cap C^2((0,\infty)) \text{ such that } \mu(0) = 0\\ \text{and } 0 < \mu'(r) \leqslant \phi'(r) \text{ for } r > 0.\end{array}\right\} \quad (\mathrm{H}_\phi)$$

$$\left.\begin{array}{l} b \in C^0([0,\infty)) \cap C^2((0,\infty)),\ \liminf_{r\to 0^+} b'(r) > -\infty \text{ and}\\ \limsup_{r \to 0^+} b''(r) < +\infty \text{ if } \limsup_{r\to 0^+} b'(r) = +\infty.\end{array}\right\} \quad (\mathrm{H}_b)$$

We remark that (H_ϕ) obviously holds if ϕ is a convex function and (H_b) is trivially verified if $b \in C^1([0, \infty))$ (no condition on b'' is requested in that case). On the other hand, if $b(s) = s^\lambda$, $\lambda \in R$, then (H_b) is satisfied if $\lambda > 0$.

We start considering the (CP) problem. The main result in this section is the following:

THEOREM 4.1. *Assume* (H_ϕ) *and* (H_b) *or* (H_{-b}). *Let* u *be a limit solution of* (CP) *continuous on* $\bar{S}$ *and let* $\bar{u}$ *(resp.* $\underline{u}$*) be a generalized supersolution (resp. subsolution) of* (E) *on* $G=\bar{S}$. *Then for every* $0 \leqslant t \leqslant T$ *we have*

$$\int_{-\infty}^{\infty} (u(t,x)-\bar{u}(t,x))^+ \, dx \leqslant \int_{-\infty}^{\infty} (u(0,x)-\bar{u}(0,x))^+ \, dx \tag{4.1}$$

(resp. $\int_{-\infty}^{\infty} (\underline{u}(t,x)-u(t,x))^+ \, dx \leqslant \int_{-\infty}^{\infty} (\underline{u}(0,x)-u(0,x))^+ \, dx$*), where* $r^+ = \max\{r, 0\}$.

As a first consequence of the above result we can state our main result about uniqueness.

THEOREM 4.2. *Assume* (H_ϕ) *and* (H_b) *or* (H_{-b}). *Let* $u_0 \in C_b(-\infty, \infty)$, $u_0 \geqslant 0$. *Then under any of the following hypotheses there exists a unique generalized solution of* (CP):

(1) *Assumption or Condition* (3.1) *is satisfied and* $\phi(u_0)$ *is Lipschitz continuous.*

(2) *Assumptions or Conditions* (3.2), (3.3), *and* (3.4) *are satisfied and* $J(\phi(u_0))$ *is Lipschitz continuous.*

Before giving the proof of the above result let us make some remarks. First, we recall that, by Theorem 3.3, every "weak solution" (see the definition in Remark 3.4) is a generalized solution. Then, by Theorem 3.2, Theorem 4.2 gives automatically the uniqueness of weak solutions, improving the knowledge in the literature about Eq. (E) (see the Introduction). Second, if we consider the particular case of $\phi(s)=s^m$ and $b(s)=s^\lambda$ (i.e., (E) coincides with $(\mathrm{E}_{m,\lambda})$) then, for adequate data, Theorem 4.2 shows uniqueness of generalized (and weak) solutions under the following restrictions:

$$m \geqslant 1, \qquad \lambda > 0.$$

In particular, uniqueness of solutions for the evaporation type problems ($\lambda \in (0, 1)$) follows.

Other consequences of Theorem 4.1 will be commented upon later.

Proof of Theorem 4.2. Under the assumptions of the theorem, we know the existence of at least one limit solution of the problem. Moreover, this limit solution is continuous (see [13] if (1) or [9] if (2). Then, if $\hat{u}$ is another generalized solution of (CP), we can obviously apply the estimate (4.1) and then $u \leqslant \hat{u}$ on $\bar{S}$. Analogously, $\hat{u}$ is also a generalized subsolution of (E) on $\bar{S}$ and the dual estimate of (4.1) implies that $\hat{u} \leqslant u$ on $\bar{S}$. In conclusion, $u=\hat{u}$.

Proof of Theorem 4.1. Let $u=\lim_{j\to\infty} u_j$ be the limit solution of (CP) obtained in Proposition 2.1.

We prove estimate (4.1) by showing that inequality

$$\int_{-\infty}^{\infty} (u(t,x)-\bar{u}(t,x))^{+}\,\omega(x)\,dx \leqslant \int_{-\infty}^{\infty} (u(0,x)-\bar{u}(0,x))^{+}\,dx \tag{4.2}$$

holds for every $\omega \in C_0^{\infty}(R)$, $0 \leqslant \omega \leqslant 1$. To do this, we suppose that $(\operatorname{supp}\omega) = [-L, L]$. For every $t^* \in (0, T]$, let $P \equiv (0, t^*) \times (-r, r)$, where $r > L+1$. Let $\zeta \in C_{t,x}^{1,2}(P)$, $\zeta \geqslant 0$, such that $\zeta(t,-r) = \zeta(t,r) = 0$ for all $t \in [0, t^*]$. Then $I(u_j, \zeta, P) - I(\bar{u}, \zeta, P) \geqslant 0$, i.e.,

$$\begin{aligned}
&\int_{-r}^{r} (u_j(t^*,x) - \bar{u}(t^*,x)\,\zeta(t^*,x)\,dx \\
&\quad\leqslant \int_{-r}^{r} (u_j(0,x)-\bar{u}(0,x))\,\zeta(0,x)\,dx \\
&\qquad + -\int_0^{t^*} [\phi(u_j(t,r)) - \phi(\bar{u}(t,r))]\,\zeta_x(t,r)\,dt \\
&\qquad + \int_0^{t^*} [\phi[u_j(t,-r)) - \phi(\bar{u}(t,-r))]\,\zeta_x(t,-r)\,dt \\
&\qquad + \iint_P (u_j - \bar{u})(\zeta_t + A^j\zeta_{xx} - B^j\zeta_x)\,dx\,dt,
\end{aligned} \tag{4.3}$$

where

$$A^j = A^j(t,x) = \int_0^1 \phi'(\Theta u_j(t,x) + (1-\Theta)\,\bar{u}(t,x))\,d\Theta \tag{4.4}$$

and

$$B^j = B^j(t,x) = \int_0^1 b'(\Theta u_j(t,x) + (1-\Theta)\,\bar{u}(t,x))\,d\Theta. \tag{4.5}$$

Hypothesis (H_ϕ) and the properties of u_j imply

$$0 < \frac{1}{\varepsilon_j}\,\mu(\varepsilon_j) \leqslant A^j(t,x) \leqslant M_1 \tag{4.6}$$

for every $(t,x) \in P$ and for some M_1 independent on j. On the other hand, due to hypothesis (H_b), there exist two real numbers M_2 and M_3 (M_2 independent of j), such that

$$M_2 \leqslant B^j(t,x) \leqslant M_3(j) \tag{4.7}$$

for every $(t, x) \in P$. Indeed, if $-\infty < \liminf_{s \to 0^+} b'(s) \leqslant \limsup_{s \to 0^+} b'(s) < +\infty$, there exist M_2 and M_3 (both independent of j) such that $M_2 \leqslant b'(s) \leqslant M_3$ for every $s \in [0, M]$ and then (4.9) is obvious. If $\limsup_{s \to 0^+} b'(s) = +\infty$, then, by the second part of (H_b), there exist M_2 and M_3^* (both independent of j) such that

$$M_2 \leqslant b'(s) \qquad \text{and} \qquad b''(s) < M_3^*$$

for every $s \in (0, M]$. Therefore,

$$M_2 \leqslant \int_0^1 b'(\Theta u_j + (1-\Theta)\,\bar{u})\, d\Theta \leqslant \int_0^1 b'(\Theta \varepsilon_j)\, d\Theta + \int_0^1 M_3^*(\Theta(u_j - \varepsilon_j)$$

$$+ (1-\Theta)\,\bar{u})\, d\Theta \leqslant \frac{1}{\varepsilon_j} b(\varepsilon_j) + |M_3^*|, M.$$

Then (4.7) holds with $M_3(j) = (1/\varepsilon_j)\, b(\varepsilon_j) + |M_3^*|\, M$.

Analogously, if we suppose (H_{-b}) we can find two real numbers $M_2(j)$ and M_3 (M_3 independent of j) such that $M_2(j) \leqslant B^j(t, x) \leqslant M_3$ for every $(t, x) \in P$. Hence, in any case, we can assume that $M_2(j) \leqslant B^j(t, x) \leqslant M_3(j)$ for every $(t, x) \in P$.

Define now two sequences of smooth functions, on $P = P_r$, $\{A_n^{j,r}\}_{n=1}^{\infty}$ and $\{B_n^{j,r}\}_{n=1}^{\infty}$, satisfying

$\{A_n^{j,r}\}$ is monotonically decreasing on n and converges uniformly to A^j, on P_r (when $n \to +\infty$).

$\{B_n^{j,r}\}$ is, e.g., monotonically increasing on n and converges uniformly to B^j, on P_r (when $n \to +\infty$).

Then, by (4.6) and (4.7) we have

$$0 < \frac{1}{\varepsilon_j} \mu(\varepsilon_j) \leqslant A_n^{j,r} \leqslant M_1$$

and

$$M_2 \leqslant B_n^{j,r} \leqslant M_3(j).$$

On the other hand, inequality (4.3) can be written in the following way:

$$\int_{-r}^{r} (u_j(t^*, x) - \bar{u}(t^*, x))\, \zeta(t^*, x)\, dx$$

$$\leqslant \int_{-r}^{r} (u_j(0, x) - \bar{u}(0, x))\, \zeta(0, x)\, dx$$

$$+\int_0^{t^*} [\phi(u_j(t,-r))-\phi(\bar{u}(t,-r))]\,\zeta_x(t,-r)\,dt$$

$$-\int_0^{t^*} [\phi(u_j(t,r))-\phi(\bar{u}(t,r))]\,\zeta_x(t,r)\,dt$$

$$+\iint_{P_r} (A^j-A_n^{j,r})(u_j-\bar{u})\,\zeta_{xx}\,dx\,dt$$

$$+\iint_{P_r} (B_n^{j,r}-B^j)(u_j-\bar{u})\,\zeta_x\,dx\,dt$$

$$+\iint_{P_r} (A_n^{j,r}\zeta_{xx}+\zeta_t-B_n^{j,r}\zeta_x)(u_j-\bar{u})\,dx\,dt. \tag{4.8}$$

Now, let $\zeta=\zeta_n^{j,r}$ be the classical solution of the linear parabolic problem

$$\begin{aligned} &\mathscr{L}\zeta\equiv A_n^{j,r}\zeta_{xx}-B_n^{j,r}\zeta_x+\zeta_t=0 && \text{on } P_r\\ &\zeta(t^*,x)=\omega(x)\,\chi(x) && \text{on } (-r,r)\\ &\zeta(t,-r)=\zeta(t,r)=0 && \text{on } (0,t^*),\end{aligned} \tag{4.9}$$

where χ is a given function such that $\chi\in C_0^\infty(R)$ and $0\leqslant\chi\leqslant 1$ (The existence and uniqueness of ζ is a well-known result (see [24]).) One of the crucial points in the present proof is based on the following estimates of the solution of (4.9).

LEMMA 4.1. *Let ζ be the solution of* (4.9). *Then*

(i) $0\leqslant\zeta(t,x)\leqslant\max|\omega(x)\,\chi(x)|\leqslant 1$, *for all* $(t,x)\in P_r$.

(ii) *There exists* $M_4=M_4(j)$ *such that* $0\leqslant\zeta(t,x)\leqslant M_4(j)\,e^{-|x|}$, *for all* $(t,x)\in\overline{P_r}$.

(iii) *There exists* $M_5=M_5(j)$ *such that* $\max\{|\zeta_x(t,r)|,|\zeta_x(t,-r)|\}\leqslant M_5(j)\,e^{-r}$, *for all* $t\in[0,t^*]$.

(iv) *There exists* $M_6=M_6(j)$ *such that* $|\zeta_x(t,x)|\leqslant M_6(j)$ *for all* $(t,x)\in P_r$.

(v) *There exists* $M_7=M_7(j,r,t^*)$ *such that* $\int_0^{t^*}\int_{-r}^r(\zeta_{xx})^2\,dx\,dt\leqslant M_7(j,r,t^*)$ *for all* $(t,x)\in P_r$.

Proof of Lemma 4.1. We follow some of the ideas introduced in [27]. Part (i) is a consequence of the maximum principle. To prove (ii), let us consider the function $w=z-\zeta$, where

$$z(t,x)=C\exp(-x+\beta(t^*-t)),$$

where C and β will be chosen later. Let $P_r^+ = (0, t^*) \times (0, r)$. Then we have

$$\begin{aligned} \mathscr{L}w &\equiv C \exp(-x + \beta(t^* - t))\{A_n^{j,r} + B_n^{j,r} - \beta\} \\ &\leqslant \exp(-x + \beta(t^* - t))\{M_1 + M_3(j) - \beta\} < 0 \end{aligned}$$

if $\beta > M_1 + M_3(j)$.

$$w(t^*, x) = Ce^{-x} - \omega(x)\,\chi(x) \geqslant 0, \qquad \text{for every } x \in [0, r]$$

if $Ce^{-L} - 1 \geqslant 0$, i.e., $C \geqslant e^L$.

$$\begin{aligned} w(t, 0) &= Ce^{\beta(t^* - t)} - \zeta(t, 0) \geqslant 0, && \text{for every } t \in [0, t^*], \\ w(t, r) &= C \exp(-r + \beta(t^* - t)) \geqslant 0, && \text{for every } t \in [0, t^*]. \end{aligned}$$

Hence, by using the maximum principle we have

$$0 \leqslant \zeta(t, x) \leqslant e^{\beta(t^* - t)} e^{-x} \leqslant M_4^1(j)\, e^{-x}$$

on P_r^+, where

$$M_4^1(j) = e^L e^{t^*(M_1 + M_3(j) + 1)}.$$

On the set $P_r^- = [0, t^*] \times [-r, 0]$ we use the auxiliary function $w = z - \zeta$, where now

$$z(t, x) = C \exp(x + \beta(t^* - t)).$$

Then we obtain

$$0 \leqslant \zeta(t, x) \leqslant M_4^2(j)\, e^x$$

on P_r^-, with

$$M_4^2(j) = e^L e^{t^*(M_1 - M_2(j) + 1)}.$$

This proves (ii) for $M_4(j) = \max\{M_4^1(j), M_4^2(j)\}$. (We remark that if M_2 and M_3 are independent of j, the same holds for M_4.)

In order to prove (iii), we define the function

$$w(t, x) = e^{-r+1} \exp \beta(x - r + 1) - \zeta(t, x)$$

for some β to be chosen. Consider the cylinder $P(r-1, r) = (0, t) \times (r-1, r)$. Then we have

$$\begin{aligned} \mathscr{L}w &\equiv e^{-r+1} \exp \beta(x - r + 1)\{\beta^2 A_n^{j,r} - \beta B_n^{j,r}\} \\ &\geqslant e^{-r+1} \exp \beta(x - r + 1)\left\{\beta \frac{2\mu(\varepsilon_j)}{\varepsilon_j} - \beta M_3(j)\right\} > 0 \end{aligned}$$

if $\beta > \max\{M_3(j)\,\varepsilon_j/\mu(\varepsilon_j), 1\}$,

$$w(t, r-1) = e^{-r+1} - \zeta(t, r-1) \leqslant e^{-r+1}$$

$$w(t, r) = e^{-r+1} e^{\beta}$$

$$w(t^*, x) = e^{-r+1} e^{\beta(x-r+1)}$$

(we recall that $r > L+1$). Then, $w(t, x)$ attains the positive maximum $e^{-r+1}e^{\beta}$ at (t, r). Hence

$$\zeta_x(t, r-0) \leqslant e^{-r+1}\beta e^{\beta} = M_5^1(j)\, e^{-r} \qquad \text{for} \quad M_5^1 = \beta(j)\, e^{\beta(j)+1}.$$

Now, if we consider the function

$$w(t, x) = e^{-r+1} \exp \beta(x-r+1) + \zeta(t, x),$$

we have $\zeta_x(t, r-0) \geqslant -M_5^1(j)\, e^{-r}$. Finally, by using the auxiliary functions

$$w(t, x) = e^{-r+1} \exp \beta(x+r-1) \pm \zeta(t, x)$$

on the set $P(-r, -r+1) = (0, t^*) \times (-r, -r+1)$, for some suitable β we obtain $|\zeta_x(t, -r+0)| \leqslant M_5^2(j)\, e^{-r}$ for some $M_5^2(j)$. This proves (iii) for $M_5(j) = \max\{M_5^1(j), M_5^2(j)\}$.

Part (iv) is a consequence of the fact that the coefficients $A_n^{j,r}$ and $B_n^{j,r}$ are bounded independently of n and r. Indeed, in these circumstances we can apply the results of the classical theory of linear parabolic equations (see [24]).

Finally, to show (v) we multiply the equation in (4.9) by ζ_{xx} and integrate. Then

$$\int_0^{t^*} \int_{-r}^{r} A_n^{jr} (\zeta_{xx})^2 \, dx\, dt = -\int_0^{t^*} \int_{-r}^{r} \zeta_t \zeta_{xx} \, dx\, dt + \int_0^{t^*} \int_{-r}^{r} B_n^{jr} \zeta_x \zeta_{xx} \, dx\, dt$$

$$= I_1 + I_2. \tag{4.10}$$

Integrating by parts,

$$I_1 = \int_0^{t^*} \int_{-r}^{r} \zeta_{tx} \zeta_x = \frac{1}{2} \int_{-r}^{r} \left(\frac{d}{dx}(\omega\chi(x))\right)^2 dx - \frac{1}{2} \int_r^r (\zeta_x(0, x))^2 \, dx$$

$$\leqslant \frac{1}{2} \int_{-L}^{L} \frac{d}{dx}(\omega\chi(x))^2 \, dx = M_7^1.$$

On the other hand,

$$I_2 \leqslant \left[\int_0^{t^*}\int_{-r}^{r} (B_n^{jr}\zeta_x)^2\,dx\,dt\right]^{1/2}\left[\int_0^{t^*}\int_{-r}^{r} (\zeta_{xx})^2\,dx\,dt\right]^{1/2}$$

$$\leqslant (t^*2r)^{1/2}\max\{|M_2(j)|, |M_3(j)|\}\,M_6(j)\left[\int_0^{t^*}\int_{-r}^{r} (\zeta_{xx})^2\,dx\,dt\right]^{1/2}$$

$$= M_7^2(j, r, t^*)\left[\int_0^{t^*}\int_{-r}^{r} (\zeta_{xx})^2\,dx\,dt\right]^{1/2}.$$

Therefore, from (4.10) we deduce

$$\int_0^{t^*}\int_{-r}^{r} (\zeta_{xx})^2\,dx\,dt \leqslant \frac{\varepsilon_j}{\mu(\varepsilon_j)}\left[M_7^1 + M_7^2(j, r, t^*)\left(\int_0^{t^*}\int_{-r}^{r} (\zeta_{xx})^2\,dx\,dt\right)^{1/2}\right].$$

This finishes the proof.

Proof of Theorem 4.1 (*continued*). By substituting the solution $\zeta = \zeta_n^{j,r}$ of (4.9) in the expression (4.8) and applying Lemma 4.1, we have

$$\begin{aligned}\int_{-r}^{r} (u_j(t^*, x) - \bar{u}(t^*, x))\,\omega(x)\,\chi(x)\,dx &\leqslant \int_{-r}^{r} (u_j(0, x) - \bar{u}(0, x))^+\,dx \\ &+ t^*M_6(j)\,e^{-r}\{\max_{0\leqslant t\leqslant t^*} |\phi(u_j(t, r)) - \phi(\bar{u}(t, r))| + \max_{0\leqslant t\leqslant t^*} |\phi(u_j(t, -r)) \\ &- \phi(\bar{u}(t, -r))|\} + \max_{\bar{P}_r} |A^j - A_n^{j,r}| \max_{\bar{P}_r} |u_j - u|\,M_8(j, r, t^*) \\ &+ \max_{\bar{P}_r} |B_n^{j,r} - B^j| \max_{\bar{P}_r} |u_j - \bar{u}|\,2t^*rM_6(j). \end{aligned} \tag{4.11}$$

By taking limits, first with respect to n ($n \to +\infty$) and then with respect to r ($r \to \infty$), we obtain

$$\int_{-\infty}^{\infty} (u_j(t^*, x) - \bar{u}(t^*, x))\,\omega(x)\,\chi(x)\,dx \leqslant \int_{-\infty}^{\infty} (u_j(0, x) - \bar{u}(0, x))^+\,dx \tag{4.12}$$

(we recall that $|u_j - u| \leqslant M$ and that $|\phi(u_j) - \phi(u)| \leqslant \phi(M)$). Now letting j diverge to infinity, we obtain

$$\int_{-\infty}^{\infty} (u(t^*, x) - \bar{u}((t^*, x))\,\omega(x)\,\chi(x)\,dx \leqslant \int_{-\infty}^{\infty} (u(0, x) - \bar{u}(0, x))^+\,dx. \tag{4.13}$$

Finally, relation (4.13) is also true for the function χ given by $\chi(x)=1$ on the set $\{x: u(t^*, x) > \bar{u}(t^*, x)\}$ and $\chi(x)=0$ otherwise. (Indeed, it suffices to approximate the function χ by $\chi_m \in C_0^\infty(R)$ and then pass to the limit on m.) This concludes the proof of (4.2). Finally, if $\underline{u}$ is a subsolution of (E) on $\bar{S}$, by a similar argument we obtain

$$\int_{-\infty}^{\infty} (\underline{u}, (t, x) - u(t, x))^+ \, \omega(x) \, dx \leqslant \int_{-\infty}^{\infty} (\underline{u}(0, x) - u(0, x))^+ \, dx$$

for every $\omega \in C_0^\infty(\mathrm{R})$, $0 \leqslant \omega \leqslant 1$, and the proof of Theorem 4.1 is finished. ∎

For problems (MBVP) and (FBVP), our answers are similar to Theorem 4.1 but the proof is somewhat more delicate and therefore we need another assumption on ϕ:

ϕ satisfies (H_ϕ) as well as $\phi(r) \leqslant C\mu(r)$ on $r > 0$, for some $C > 0$. (H_ϕ^*)

THEOREM 4.3. *Assume* (H_ϕ^*) *and* (H_b) *or* (H_{-b}).

(a) *Let u be a limit solution of* (FBVP) *continuous on* $\bar{R}$. *Let* $\underline{u}$ *(resp.* u*) be a generalized supersolution (resp. subsolution) of* (E) *on* $G = \bar{R}$ *such that*

$$\psi_-(t) \leqslant \underline{u}(t, l_1), \qquad \psi_+(t) \leqslant \bar{u}(t, l_2)$$

(resp. $\psi_-(t) \geqslant \underline{u}(t, l_1)$, $\psi_+(t) \geqslant \underline{u}(t, l_2)$*) for every* $t \in [0, T]$. *Then*

$$\int_{l_1}^{l_2} (u(t, x) - \bar{u}(t, x))^+ \, dx \leqslant \int_{l_1}^{l_2} (u(0, x) - \bar{u}(0, x))^+ \, dx \tag{4.14}$$

(resp. $\int_{l_1}^{l_2} (\underline{u}(t, x) - u(t, x))^+ \, dx \leqslant \int_{l_1}^{l_2} (u(0, x) - u(0, x))^+ \, dx$*).*

(b) *Let u be a generalized solution of* (MBVP) *continuous on* $\bar{H}$. *Let* $\bar{u}$ *(resp. u be a generalized supersubsolution (resp. subsolution) of* (E) *on* $G = \bar{H}$ *such that*

$$\psi(t) \leqslant \bar{u}(t, l_2)$$

(resp. $\psi(t) \geqslant \underline{u}(t, l_2)$*) for every* $t \in [0, T]$. *Then*

$$\int_{-\infty}^{l_2} (u(t, x) - \bar{u}(t, x))^+ \, dx \leqslant \int_{-\infty}^{l_2} ((u(0, x) - \bar{u}(0, x))^+ \, dx \tag{4.15}$$

(resp. $\int_{-\infty}^{l_2} (\underline{u}(t, x) - u(t, x))^+ \, dx \leqslant \int_{-\infty}^{l_2} (\underline{u}(0, x) - u(0, x))^+ \, dx$*).*

About the uniqueness question we have.

THEOREM 4.4. *Assume* (H^*_ϕ) *and* (H_b) *or* (H_{-b}).

(a) *Let* $u_0 \in C_b([l_1, l_2])$, $u_0 \geqslant 0$ *and* $\psi_-, \psi_+ \in C([0, T])$, $\psi_-, \psi_+ \geqslant 0$, *satisfy* $\psi_-(0) = u_0(l_1)$, $\psi_+(0) = u_0(l_2)$. *Then, under one of the following hypotheses there exists a unique generalized solution of* (FBVP):

(1) *Assumption* (3.1) *is satisfied and* $\phi(u_0)$ *is locally Lipschitz continuous on* (l_1, l_2) *and* $\phi(\psi_+)$, $\phi(\psi_-)$ *are absolutely continuous on* $[0, T)$.

(2) *Assumptions* (3.2), (3.3), *and* (3.4) *are satisfied and* $f^{-1}(u_0)$ *is locally Lipschitz continuous on* (l_1, l_2).

(b) *Let* $u_0 \in C_b((-\infty, l_2])$, $u_0 \geqslant 0$, *and* $\psi \in C([0, T])$, $\psi \geqslant 0$, *satisfy* $\psi(0) = u_0(l_2)$. *Then under any of the following assumptions there exists a unique generalized solution of* (MBVP):

(1) *Assumption* (3.1) *is satisfied and* $\phi(u_0)$ *is Lipschitz continuous on* $(-\infty, l_2 - \delta)$ *for every* $\delta > 0$ *and* $\phi(\psi)$ *is absolutely continuous on* $[0, T]$.

(2) *Assumptions* (3.2), (3.3), *and* (3.4) *are satisfied and* $f^{-1}(u_0)$ *is Lipschitz continuous on* $(-\infty, l_2 - \delta)$ *for every* $\delta > 0$.

Proof of Theorem 4.3. (a) Let u be a limit solution of (FBVP) continuous on $\bar{R}$ and let $\bar{u}$ be a generalized supersolution of (E) on $G = \bar{R}$ such that

$$\psi_-(t) \leqslant \bar{u}(t, l_1) \qquad \text{and} \qquad \psi_+(t) \leqslant \bar{u}(t, l_2)$$

for every $t \in [0, T]$. Let $P \equiv (0, t^*) \times (l_1, l_2)$. Then, if $u = \lim u_j$, we obtain, as in (4.8),

$$\begin{aligned}\int_{l_1}^{l_2} (u_j(t^*, x) - \bar{u}(t^*, x))\, \zeta(t^*, x)\, dx &\leqslant \int_{l_1}^{l_2} (u_j(0, x) - \bar{u}(0, x))\, \zeta(0, x)\, dx \\ &+ \int_0^{t^*} (\phi(u_j(t, l_1)) - \phi(\bar{u}(t, l_1))\, \zeta_x(t, l_1)\, dt \\ &- \int_0^{t^*} (\phi(u_j(t, l_2)) - \phi(\bar{u}(t, l_2)))\, \zeta_x(t, l_2)\, dt \\ &+ \iint_P (A^j - A^j_n(u_j - \bar{u})\, \zeta_{xx}\, dx\, dt + \iint_P (B^j_n - B^j)(u_j - \bar{u})\, \zeta_x\, dx\, dt \\ &+ \iint_P (A^j_n \zeta_{xx} + \zeta_t - B^j_n \zeta_x)(u_j - \bar{u})\, dx\, dt, \end{aligned} \tag{4.16}$$

where $\{A^j_n\}$ and $\{B^j_n\}$ are two sequences of smooth functions as in the proof of Theorem 4.1. Now define $\zeta = \zeta^j_n$ to be the classical solution of (4.9) after substituting $A_n^{j,r}$, $B_n^{j,r}$ and P_r by A_n^j, B_n^j and P, respectively. Our intention is to pass to the limit in (4.16) first with respect to n and afterwards with respect to j. To do this we have to distinguish two different cases[3]:

(a_1) The supersolution is such that

$$\bar{u}(t, l_1) \geqslant \psi_{-,j}(t) \quad \text{and} \quad \bar{u}(t, l_2) \geqslant \psi_{+,j}(t) \quad \text{for every } t \in [0, T], \tag{4.17}$$

for some approximations of $\psi_-(t)$ and $\psi_+(t)$ as in the proof of Proposition 2.1.

(a_2) The supersolution does not satisfy (4.17) (as, for instance, if $\bar{u}(t_0, l_1) = 0$ or $\bar{u}(t_0, l_2) = 0$ for some $t_0 \in [0, T]$).

If (a_1) holds, then by observing that $\zeta_x(t, l_1) \geqslant 0$ and $\zeta_x(t, l_2) \leqslant 0$ for every $t \in [0, T]$, we obtain the conclusion after passing to the limit in n and j, respectively, as in the proof of Theorem 4.1.

To treat case (a_2) we obtain sharper estimates on $\zeta_x(t, l_1)$ and $\zeta_x(t, l_2)$:

LEMMA 4.2. *Assume* (H_ϕ^*) and (H_b) or (H_{-b}). *Let* ζ *be the solution of*

$$\begin{aligned} &\mathscr{L}\zeta \equiv A_n^j \zeta_{xx} - B_n^j \zeta_x + \zeta_t = 0 && \text{on } P \\ &\zeta(t^*, x) = \omega(x)\, \chi(x) && \text{on } (l_1, l_2) \\ &\zeta(t, l_1) = \zeta(t, l_2) = 0 && \text{on } (0, t^*), \end{aligned} \tag{4.18}$$

where χ *is a given function such that* $\chi \in C_0^\infty(l_1, l_2)$ *and* $0 \leqslant \chi \leqslant 1$. *Then there exist two constants* $M_8(j)$ *and* $M_9(j)$ *such that*

$$0 \leqslant \zeta_x(t, l_1) \leqslant M_8(j), \tag{4.19}$$

$$M_9(j) \leqslant \zeta_x(t, l_2) \leqslant 0 \tag{4.20}$$

for every $t \in (0, t^*)$ *Moreover,*

$$\phi(\varepsilon_j)\, M_8(j) \to 0 \qquad \text{and} \qquad \phi(\varepsilon_j)\, M_9(j) \to 0$$

as $j \to \infty$.

Proof. We first remark that the first inequality in (4.19) and the second one in (4.20) are trivial because by the maximum principle ζ is nonnegative in P. Now we shall prove the first inequality in (4.20) (the second one in (4.19) is obtained in a similar way). To do this, we construct an adequate function $\sigma_j(x)$ in such a way that the function $W_j(t, x) = \sigma_j(x) + \zeta(t, x)$ has a positive maximum at (t, l_2). Then we shall deduce that $\zeta_x(t, l_2) \geqslant -\sigma_j'(l_2)$,

[3] The authors thank M. Bertsch for some remarks on this point.

which proves (4.20). Consider the cylinder $P(l_2-\delta, l_2)=(0, t)\times(l_2-\delta, l_2)$, where $\delta>0$ will be fixed later. Define the function

$$\sigma_j(x)=C_j\left(\alpha_j e^{x/\alpha_j}-\frac{K_j^2}{M_3(j)}x+L_j\right),$$

where

$$\alpha_j=\frac{\mu(\varepsilon_j)}{M_3(j)\,\varepsilon_j},\qquad C_j=\frac{2e^{-l_2/\alpha_j}}{\alpha_j},$$

and K_j and L_j will be suitably chosen. Assuming $\sigma_j'\geqslant 0$ we have

$$\mathscr{L}W_j=A_n^j\sigma_j''(x)-B_n^j\sigma_j'(x)\geqslant C_j\left\{\frac{\mu(\varepsilon_j)}{\alpha_j\varepsilon_j}e^{x/\alpha_j}\right.$$

$$\left.-M_3(j)\left(e^{x/\alpha_j}-\frac{K_j^2}{M_3(j)}\right)\right\}=C_jK_j^2>0.$$

Note that without loss of generality we may assume that $l_2-\delta>0$. On the parabolic boundary of $P(l_2-\delta, l_2)$ we have

$$W_j(t, l_2)=\sigma_j(l_2)$$
$$W_j(t, l_2-\delta)\leqslant\sigma_j(l_2-\delta)+1$$
$$W_j(t^*, x)=\sigma_j(x)\qquad (\text{if } \delta \text{ is such that supp } \omega\chi\in(l_1, l_2-\delta)).$$

But $W_j(t, x)\geqslant 0$ on $P(l_2-\delta, l_2)$ if L_j is large enough (take $L_j\geqslant\sup_{l_2-\delta<x\leqslant l_2}\{(K_j^2/M_3(j))\,x-\alpha_je^{x/\alpha_j}\}$). By the maximum principle max W_j is attained on the parabolic boundary of $P(l_2-\delta, l_2)$.

If

$$\sigma_j(l_2-\delta)+1\leqslant\sigma_j(l_2) \tag{4.21}$$

and $\sigma_j'\geqslant 0$ then

$$W_j(t^*, x)\leqslant W_j(t^*, l_2)\qquad\text{for}\quad x\in(l_2-\delta, l_2), \tag{4.22}$$

therefore max W_j is attained on $x=l_2$.

Condition (4.21) is equivalent to

$$\frac{K_j^2\,\delta}{M_3(j)}\leqslant F(\alpha_j), \tag{4.23}$$

where $F(z)=ze^{l_2/z}(\frac{1}{2}-e^{-\delta/z})$. From the definition of α_j and $M_3(j)$ (recall that $M_3(j)=b(\varepsilon_j)/\varepsilon_j+\bar{M}$ for some $\bar{M}>0$) we observe that $\alpha_j=\mu(\varepsilon_j)/(b(\varepsilon_j)+\bar{M}\varepsilon_j)$ and then $\alpha_j\to 0$ as $\varepsilon_j\to 0$: indeed, if

$\lim \operatorname{supp}_{r \to 0^+} b'(r) = C$ for some $C < \infty$ then we have $\lim \alpha_j = \lim (\mu'(\varepsilon_j)/(C + \bar{M})) \leqslant \lim (\phi'(\varepsilon)/(C + M)) = 0$. On the other hand, if $\lim \operatorname{supp} b'(r) = +\infty$ then $\lim \alpha_j = \lim (\mu'(\varepsilon_j)/(b'(\varepsilon_j) + \bar{M})) = 0$. Consequently we can choose j_0 such that $\alpha_j < z_0$ for any $j \geqslant j_0$, where z_0 is the first positive zero of the equation $F(z) = 0$. Note that such a z_0 exists because $F(z) \nearrow +\infty$ when $z \searrow 0$ and $F(z) \to -\infty$ when $z \to +\infty$. Then condition (4.23) makes sense because $F(\alpha_j) > 0$ if $j \geqslant j_0$.

With respect to (4.22) we remark that $\sigma_j'(x) \geqslant 0$ holds on $(l_2 - \delta, l_2)$ if K_j is chosen such that $K_j^2/M_3(j) \leqslant e^{(l_2 - \delta)/\alpha_j}$. It is clear that it is possible to choose K_j satisfying the above condition as well as (4.23) because $M_3(j) \geqslant \bar{M}$.

The proof of Lemma 4.2 concludes by noting that

$$\begin{aligned}
\phi(\varepsilon_j)\, \sigma_j'(l_2) &= \phi(\varepsilon_j)\, C_j \left(e^{l_2/\alpha_j} - \frac{K_j^2}{M_3(j)} \right) = \phi(\varepsilon_j) \frac{2e^{-l_2/\alpha_j}}{\alpha_j} \left(e^{l_2/\alpha_j} - \frac{K_j^2}{M_3(j)} \right) \\
&= \frac{2\phi(\varepsilon_j)\, M_3(j)\, \varepsilon_j}{\mu(\varepsilon_j)} \left(1 - \frac{K_j^2}{M_3(j)} e^{-l_2/\alpha_j} \right) \\
&\leqslant 2 \frac{\phi(\varepsilon_j)}{\mu(\varepsilon_j)} (b(\varepsilon_j) + \bar{M}\varepsilon_j) \to 0
\end{aligned}$$

when $\varepsilon_j \to 0$ (see (H_ϕ^*)). ∎

Proof of Theorem 4.3. (*continued*). Suppose that a_2 holds. Then if we denote $I_j = \{ t \in (0, t^*) \colon \phi(\psi_{-,j}(t)) > \phi(\bar{u}(t, l_1)) \}$ we have

$$\begin{aligned}
\int_0^{t^*} (\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t, l_1))\, \zeta_x(t, l_1)\, dt \leqslant \int_{I_j} (\phi(\psi_{-,j}(t)) \\
- \phi(\bar{u}(t, l_1))\, \zeta_x(t, l_1)\, dt \leqslant t^* \phi(\varepsilon_j)\, M_B(j)
\end{aligned}$$

(here we use the fact that $\psi_{-,j}$ can be always chosen such that $\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t, l_1)) \leqslant \phi(\varepsilon_j)$ if $t \in I_j$). Now, by Lemma 4.2 we have

$$\int_0^{t^*} (\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t, l_1))\, \zeta_x(t, l_1)\, dt \to 0$$

when j diverges to infinity. Similarly

$$\int_0^{t^*} \phi(\psi_{+,j}(t)) - \phi(\bar{u}(t, l_2))\, \zeta_x(t, l_2)\, dt \to 0$$

when j diverges to infinity. Then the conclusion follows by passing to the limit in (4.16) in n and then in j.

We remark that in order to prove the conclusion for subsolutions it is not necessary to use Lemma 4.2, because in all cases we may choose $\psi_{-,j}$ and $\psi_{+,j}$ satisfying

$$\underline{u}(t, l_1) \leqslant \psi_{-,j}(t) \qquad \text{and} \qquad \underline{u}(t, l_2) \leqslant \psi_{+,j}(t)$$

for every $t \in [0, T]$.

The proof of part (b) is an easy modification of the proofs of Theorem 4.1 and part (a), above. ∎

The proof of Theorem 4.4 is analogous to that of Theorem 4.2.

Other important consequences of Theorems 4.1 and 4.3 are included in the following theorem, which shows continuous and monotone dependence of generalized solutions with respect to the initial data. (We shall consider only the (CP) problem, analogous statements holding for the others.)

THEOREM 4.5. *Assume the hypotheses of Theorem* 4.2.

(i) *Let u, $\hat{u}$ be generalized solutions of* (CP) *corresponding to the initial data u_0 and $\hat{u}_0$, respectively. Then*

$$\|u(t, \cdot) - \hat{u}(t, \cdot)\|_{L^1(R)} \leqslant \|u_0 - \hat{u}_0\|_{L^1(R)} \tag{4.24}$$

for every $t \in [0, T]$.

(ii) *Let u be a generalized solution of* (CP) *and $\bar{u}$, $\underline{u}$ generalized super- and subsolutions of* (E) *on $G = \bar{S}$. Then if $\underline{u}(0, x) \leqslant u_0(x) \leqslant \bar{u}(0, x)$ on $(-\infty, \infty)$ it follows that*

$$\underline{u}(t, x) \leqslant u(t, x) \leqslant \bar{u}(t, x) \tag{4.25}$$

for every $(t, x) \in \bar{S}$.

Proof. The assertion (i) follows from part (a) of Theorem 4.1 by applying the estimates to $\bar{u} = \hat{u}$ and $\underline{u} = \hat{u}$. Part (ii) is also a trivial consequence of such estimates. ∎

Other estimates giving the continuous dependence on the initial data as well as the numerical treatment of Eq. (E) for $b \in C^1([0, \infty))$ can be found in [32].

We end this section by making several comments on the obtained results.

Remark 4.1. The conclusions of Theorem 4.1 are true even under more general hypotheses. So, for the (CP) problem, e.g., it is enough that u, $\bar{u}$, $\underline{u}$ be in the function space $C([0, T]: L^1_{\text{loc}}(R))$. The existence of solutions of (CP) in such a function space is not difficult and some hypotheses on ψ and b made in Theorem 3.1 can be weakened (see, e.g., the approach made in [2] considering a different nonlinear degenerated parabolic equation).

Remark 4.2. If we denote by $S(t)\, u_0 = u(t, \cdot)$ the generalized solution of

(CP) corresponding to the initial datum u_0, then, by the uniqueness of solutions of (CP), $S(t)$ is a semigroup. The estimate (4.24) shows that it is a semigroup of contractions on the space $X = L^1(R)$. Our conclusion, then, coincides with the one obtained by the abstract theory of accretive operators on Banach spaces and evolution equations. Such an approach has been applied to the concrete case of Eq. (E) by different authors (see [38, 37, 5]).

Remark 4.3. There exists a vast literature about the existence and uniqueness of solutions of (CP) when function ϕ is not assumed to be strictly increasing on R^+. It is clear that the approach is very different from ours. Indeed, such an approach includes the case $\phi \equiv 0$ and then Eq. (E) reduces to the "conservation law" equation

$$u_t - b(u)_x = 0,$$

for which the existence of discontinuous solutions is well known. The uniqueness of solutions is then found by introducing a different notion of generalized solutions of (CP) (see, e.g., [35, 5, 11, 37, 38, 25]).

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