

CONTENTS

Une Remarque a Propos de Fonctions Holomorphes et Du Front D'onde C^∞ 1323
J.-P. Rosay

An Elliptic Equation with Singular Nonlinearity 1333
J. I. Diaz, J. M. Morel, and L. Oswald

Contribution to the Study of the Uniqueness in the Analytic Linear Cauchy Problem 1345
X. Saint Raymond

Boundary Value Problems and Eigenvalue Problems for Second Order Equations with Nonnegative Characteristic Form 1365
H. Okumura

Existence and Nonuniqueness for the p-Laplacian: Nonlinear Eigenvalues 1389
J. P. García Azorero and I. Peral Alonso

Notes on Closed Billiard Ball Trajectories in Polygonal Domains. 1431
F. G. Friedlander

Index to Volume 12. 1457

Collective Contents to Volume 12 1461

AN ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

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1. Introduction.

The purpose of this paper is to study the problem

$$(P) \quad \begin{cases} -\Delta u + u^{-\alpha} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u^{-\alpha} \in L^1(\Omega), \quad u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth open set of \mathbb{R}^N , $f \geq 0$, $f \in L^1(\Omega)$ and $0 < \alpha < 1$.

When $\alpha \leq 0$ this problem corresponds to a monotone semilinear equation for which the existence and uniqueness of solutions are well-known (see e.g. Brezis-Strauss [7], Baras-Pierre [3], Gallouet-Morel [10]). On the other hand semilinear equations with non-monotone perturbations have also been considered in the literature by many different methods. For instance, the case of the equation $-\Delta u = j(u) + f$ where j is a convex function such that $j(0) = 0$ has been treated in Crandall-Rabinowitz [8], Mignot-Puel [13] and Baras-Pierre [4] among other authors. The originality of our study comes from the singularity of the nonlinearity in (P).

From a physical point of view, semilinear equations with a singular nonlinearity appear in the context of chemical heterogeneous catalysts

(Aris [2], Brauner-Nicolaenko [5], Perry [15], ...) as well as in non-Newtonian fluids (see e.g. Luning-Perry [12] and its references).

Our first result gives a necessary and a sufficient condition for the existence of solutions : if ϕ_1 denotes the first eigenfunction of $-\Delta$ with Dirichlet condition, we show that if $\int_{\Omega} f\phi_1$ is large enough then (P) has a solution, and that if $\int_{\Omega} f\phi_1$ is small enough no solution of (P) can exist. It turns out that such existence and nonexistence results hold without any assumption of convexity on the nonlinearity. In Theorem 2 we give a general framework under which the results of Theorem 1 remain true.

We also consider the variational formulation of (P) when f is assumed to be in $L^{\infty}(\Omega)$. In that case the energy functional

$$J(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{1-\alpha} \int u^{1-\alpha} - \int fu$$

is well defined on $H_0^1(\Omega)$. However, if $u \geq 0$ corresponds to a local minimum of the energy, it is not necessarily a solution of (P). Indeed, if $u = 0$ on a set of positive measure, u does not satisfy (P), but instead u is the solution of some kind of free boundary problem. Thus, it is interesting to examine whether the maximal solution of (P), when it exists, is a local minimum of J . We also point out that the condition $u > 0$ of (P) is not too restrictive. For example, it is easy to see, for $N = 1$ and f constant, that a local minimum $u \neq 0$ of J cannot vanish on a set of positive measure. This is in contrast with other works related to variational solutions u of the same equation with $f \equiv 0$ but satisfying $u = 1$ on the boundary $\partial\Omega$. (See Brauner-Nicolaenko [5], Phillips [16], [17] and Alt-Phillips [1]; see also the monograph Diaz [9]).

Our second result states that the maximal solution of (P), when it exists, is a local minimum of the energy. Moreover, it is the only local minimum which is positive on Ω . These results are similar to well-

known properties of semilinear equations with convex smooth nonlinearities $j(u)$ with $j(0) = 0$ (see [8], [13], [4]).

We conclude this introduction with some notations. In the following, we denote by N the inverse on $L^1(\Omega)$ of the operator $-\Delta$ with Dirichlet condition. We denote by $\lambda_1(-\Delta + a \cdot)$ the first eigenvalue of the operator $u + -\Delta u + a(x)u$. A function \underline{u} (resp. \bar{u}) is called subsolution (resp. supersolution) of (P) if it satisfies $\underline{u} + N\underline{u}^{-\alpha} \leq Nf$ (resp. $\bar{u} + N\bar{u}^{-\alpha} \geq Nf$). Finally, we shall consider the problem (P_{λ}) which is (P) with λf instead of f , for $\lambda > 0$.

2. Existence results.

The main result of this section is the following :

Theorem 1.

(i) If (P) has a subsolution, it has a maximal solution.

There exist two constants $C_1, C_2 > 0$, depending only on Ω and α , such that

(ii) If (P) has a solution, then $\int f\phi_1 \geq C_1$.

(iii) If $\int f\phi_1 \geq C_2$, then (P) has a solution.

Corollary 1. Consider, for fixed $f \in L^1(\Omega)$, $f \geq 0$, the family of problems (P_{λ}) , $\lambda \geq 0$. Then there exists $\lambda^* > 0$ such that :

For $\lambda < \lambda^*$, (P_{λ}) has no solution

For $\lambda > \lambda^*$, (P_{λ}) has at least one solution.

Remark. We don't know, actually, whether P_{λ^*} has a solution or not.

Proof of Theorem 1.

Proof of (i). By the comparison principle, one sees that $\frac{Nf}{\lambda}$ is a supersolution of (P) greater than any subsolution of (P). Let v be such a subsolution, and consider (as in [4]) the operator A defined by

$$(2) \quad Au = Nf - Nu^{-\alpha}$$

Again by the comparison principle we have that A applied to a supersolution yields a supersolution, and that $u \geq v$ implies that $Au \geq v$.
 Not set $u_n = A^n Nf$. Thus

$$v \leq u_{n+1} \leq u_n \leq Nf$$

and

$$0 \leq u_n^{-\alpha} \leq v^{-\alpha} \in L^1(\Omega).$$

One deduces easily that u_n tends to a solution \bar{u} of (P) which is maximal among all solutions of (P).

Proof of (ii). Multiplying the equation by ϕ_1 , and integrating by parts, we get :

$$\int f\phi_1 = \int (u^{-\alpha+\lambda_1}u)\phi_1 \geq \min_{s \geq 0} (s^{-\alpha+\lambda_1}\theta) \int \phi_1 = C_1.$$

Proof of (iii). By (i) it is enough to construct a subsolution of (P) for $\int f\phi_1$ large enough. A good candidate is $v = \lambda\phi_1^{2/1+\alpha}$ with λ large enough.

Indeed one has

$$-\Delta v + v^{-\alpha} = -C\lambda\phi_1^{-\frac{2\alpha}{1+\alpha}} |\nabla\phi_1|^2 + C'\lambda\lambda_1\phi_1^{\frac{2\alpha}{1+\alpha}} + \lambda^{-\alpha}\phi_1^{-\frac{2\alpha}{1+\alpha}}$$

where C, C' only depend on α . Taking into account that $|\nabla\phi_1|^2$ is bounded from below near the boundary and choosing λ large enough, we get :

$$-\Delta v + v^{-\alpha} \leq C'' \left(\lambda\phi_1^{\frac{2}{1+\alpha}} \right)$$

Applying N to this inequality yields

$$(3) \quad v + Nv^{-\alpha} \leq CN\phi_1^{2/1+\alpha} \leq C'\phi_1.$$

We conclude by applying a uniform Hopf principle as formulated in [14] :

Lemma 1. There exists a constant C only depending on Ω such that for $f \geq 0$, $f \in L^1(\Omega)$ one has :

$$(4) \quad Nf \geq C \left(\int f\phi_1 \right) \phi_1.$$

From (3) and (4) we deduce the existence of the constant C_2 stated in Theorem 1 (iii).

Proof of Corollary 1.

Let us first note that if \bar{u}_λ is a solution for (P_λ) then \bar{u}_λ is a subsolution for $(P_{\lambda'})$ for $\lambda' \geq \lambda$. Indeed

$$-\Delta \bar{u}_\lambda + \bar{u}_\lambda^{-\alpha} = \lambda f \leq \lambda' f$$

By (i) we find that the set of the values of λ for which P_λ has a solution is an interval and that $u_\lambda < u_{\lambda'}$, if $\lambda < \lambda'$.

The conclusions of Theorem 1 can be extended to the case of more general nonnecessarily convex nonlinearities :

Theorem 2. Let $f \geq 0$, $f \in L^1(\Omega)$. Let g be a continuous nonincreasing function on $(0, \infty)$ such that

$$(5) \quad \int_0^1 g(s) ds < +\infty$$

and

$$(6) \quad \int_0^1 \left(\int_0^t g(s) ds \right)^{-1/2} dt < +\infty.$$

Consider the problem

$$(Pg) \quad \begin{cases} -\Delta u + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ g(u) \in L^1(\Omega), u > 0 & \text{in } \Omega, \end{cases}$$

Then the conclusions of Theorem 1 hold for solutions of (Pg).

Proof. We follow the same arguments as in the proof of Theorem 1. The main point is to find a subsolution of (Pg). To do that, let $H : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be such that

$$\begin{aligned} H''(t) &= g(H(t)) \\ H(0) &= H'(0) = 0. \end{aligned}$$

Such a function exists by assumption (6). Now we look for a subsolution of the form $v = \lambda H(\phi_1)$. We have

$$-\Delta v + g(v) = \lambda\lambda_1 H'(\phi_1)\phi_1 + g(\lambda H(\phi_1)) - \lambda g(H(\phi_1)) |\nabla\phi_1|^2.$$

Now we take $\lambda \geq 1$. Since g is nonincreasing :

$$g(\lambda H(\phi_1)) \leq g(H(\phi_1)) .$$

Thus, for λ fixed large enough the right hand side of the above inequality is bounded :

$$-\Delta v + g(v) \leq C .$$

Therefore

$$v + N(g(v)) \leq NC \leq C'\phi_1$$

and the conclusion follows as in the proof of Theorem 1.

Remark 1. An alternative proof of Theorem 1 can be given by using the results of Baras-Pierre [4]. In that paper the authors obtain a necessary and sufficient condition in order to solve semilinear problems of the type $-\Delta u = j(x,v) + \tilde{f}$, where $j(x, \cdot)$ is assumed convex, l.s.c., increasing and $j(x,0) < \infty$ a.e. $x \in \Omega$. It is clear that (P) is not included in such a framework. Nevertheless, setting $v = Nf - u$, (P) is transformed into $v = N(Nf-u)^{-\alpha}$ and setting $j(x,r) = (Nf(x)-r)^{-\alpha}$ if $r \leq Nf(x)$ and $j(x,r) = +\infty$ otherwise, (P) is reformulated as in the framework of Baras-Pierre [4] with $\tilde{f} = 0$. Their necessary and sufficient condition is rather difficult to check and it seems easier to seek directly for a subsolution of (P), as done in our proof. Moreover, our technique of proof does not involve any convexity assumption on the nonlinearity.

3. Variational characterization of the maximal solution.

In this paragraph we assume $f \in L^\infty(\Omega)$. By the following lemma, any solution of (P) belongs to $H^1_0(\Omega)$ and so we can consider its energy

$$J(u) = \frac{1}{2} \int |\nabla u|^2 + \int \frac{u^{1-\alpha}}{1-\alpha} - \int fu$$

Lemma 2. Let $g \in L^1(\Omega)$, $g \geq 0$ and $f \in L^\infty(\Omega)$. Let $u \geq 0$ be a solution of

$$\begin{aligned} -\Delta u + g &= f & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega . \end{aligned}$$

Then $u \in H^1_0(\Omega)$.

The proof of this lemma will be given later. In order to study the variational characterization of the maximal solution of (P) we shall use the problem (P_λ) :

Theorem 3. Given $f \in L^\infty(\Omega)$, $f \geq 0$, consider problem (P_λ) for $\lambda > \lambda^*$.

Then

- i) The maximal solution \bar{u}_λ of (P_λ) is a strict local minimum for the energy. More precisely $\lambda_1(-\Delta - \alpha \bar{u}_\lambda^{-\alpha-1}) > 0$.
- ii) Among all the solutions of (P_λ) , \bar{u}_λ is the only local minimum of the energy.

Proof of Theorem 3.

Proof of (ii). Assume by contradiction that (P_λ) has another solution u which is a local minimum for the energy.

Then, for every $\phi \in H^1_0(\Omega)$, $\phi \geq 0$, and for every $t > 0$ small enough one has $J(u+t\phi) - J(u) \geq 0$. Applying pointwise the Taylor formula to $r \rightarrow |r|^{-\alpha+1}$ and integrating, we get

$$\int |\nabla \phi|^2 - \alpha \int |u+t\phi|^{-\alpha-1} \phi^2 \geq 0 , \text{ with } 0 \leq \theta(x) \leq 1 .$$

Letting $t \rightarrow 0$, by the Beppo-Levi Theorem we obtain

$$(8) \quad \int |\nabla \phi|^2 - \alpha \int u^{-\alpha-1} \phi^2 \geq 0 ,$$

for every $\phi \in H^1_0(\Omega)$, $\phi \geq 0$. On the other hand, if we set $\bar{u} = u + w$ with $w \geq 0$, then

$$-\Delta u + u^{-\alpha} = f$$

$$-\Delta(u+w) + (u+w)^{-\alpha} = f$$

and so

$$-\Delta w = u^{-\alpha} - (u+w)^{-\alpha} \leq \alpha u^{-\alpha-1} w .$$

Following an idea of Brezis-Oswald [6], we note that, since $w \neq 0$, the above inequality is strict on a set of positive measure. Therefore multiplying by w and integrating on Ω we get $\int |\nabla w|^2 < \alpha \int u^{-\alpha-1} w^2$, which contradicts (8).

Proof of (i).

Step 1. The minimum of J on $[\bar{u}, Nf]$ is achieved at \bar{u} .

(A similar result can be found in Puel [18]).

Let $v \in [\bar{u}, Nf]$ and consider the sequence $A^n v$ where A is defined in (2). By the comparison principle we have $A^n v \in [\bar{u}, A^n Nf]$ and therefore $A^n v \rightarrow \bar{u}$ in $L^\infty(\Omega)$.

Moreover, by lemma 2, for v in $H^1_0(\Omega)$ we have $A^n v \in H^1_0(\Omega)$ and so $J(A^n v)$ makes sense.

We can easily see that $J(\bar{u}) \leq \liminf J(A^n v)$. In order to get $J(\bar{u}) \leq J(v)$, we shall prove that $J(A^n v)$ is nonincreasing. This is due to the fact that A is a steepest descent method for J . More precisely, if we denote by $J'(v)$ the gradient of J in $H^1_0(\Omega)$, then $J'(v) = N(-\Delta v + v^{-\alpha} - f)$ and therefore $Av = v - J'(v)$.

As in the proof of (ii), using Taylor formula we get

$$J(Av) = J(v) - \langle J'(v), J'(v) \rangle_{H^1_0} + \frac{1}{2} \int |\nabla J(v)|^2 - \frac{\alpha}{2} \int_\Omega |v - \theta J(v)|^{-\alpha-1} (J'(v))^2$$

with $0 \leq \theta(x) \leq 1$. Thus

$$J(Av) \leq J(v) - \frac{1}{2} \|J'(v)\|_{H^1_0}^2$$

which gives the result.

Step 2. $\lambda_1(-\Delta \cdot -\alpha \bar{u}^{-\alpha-1}) \geq 0$.

From step 1, for every ϕ in $\mathcal{D}(\Omega)$, $\phi \geq 0$, and for every $t \geq 0$ small enough we have

$$J(\bar{u} + t\phi) \geq J(\bar{u})$$

The conclusion follows as in the proof of Theorem 3 (ii).

Step 3. $\lambda_1(-\Delta \cdot -\alpha \bar{u}_\lambda^{-\alpha-1})$ is an increasing function of λ .

Set $E_\lambda(\phi) = \int |\nabla \phi|^2 - \alpha \int \bar{u}_\lambda^{-\alpha-1} \phi^2$. Then

$$\lambda_1 = \text{Min}_{\phi \in H^1_0} E_1(\phi) \quad ; \quad \int \phi^2 = 1$$

and it suffices to prove that if $\lambda' < \lambda$,

$$E_\lambda(\phi) - E_{\lambda'}(\phi) \geq c \int \phi^2 \quad \text{with } c > 0.$$

Now

$$E_\lambda(\phi) - E_{\lambda'}(\phi) = \int (\bar{u}_\lambda^{-\alpha-1} - \bar{u}_{\lambda'}^{-\alpha-1}) \phi^2$$

We know (cf. Corollary 1) that $\bar{u}_{\lambda'} < \bar{u}_\lambda$.

By the Strong Maximum Principle (cf. Lemma 1) there exists $\epsilon > 0$ such that

$$\bar{u}_{\lambda'} + \epsilon \phi_1 < \bar{u}_\lambda.$$

On the other hand

$$\bar{u}_\lambda \leq \lambda Nf \quad \text{and} \quad \bar{u}_{\lambda'} \leq \lambda' Nf \leq c \phi_1.$$

So we can easily prove that

$$\text{Inf}_\Omega (\bar{u}_{\lambda'}^{-\alpha-1} - \bar{u}_\lambda^{-\alpha-1}) > 0.$$

In conclusion, from steps 2 and 3 we get

$$\lambda_1(-\Delta \cdot -\alpha \bar{u}^{-\alpha-1}) \text{ is positive for } \lambda > \lambda^*.$$

Proof of Lemma 2. Set $g_k = \inf(g, k)$, $k \geq 0$. Then the sequence of functions $(u_k)_k \in \mathbb{N}$ defined by

$$\begin{cases} -\Delta u_k + g_k = f & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

verifies $u_k \in H^1_0(\Omega)$ and $u_k \downarrow u \geq 0$ in $L^2(\Omega)$. Multiplying by u_k and integrating by parts we conclude that u_k is bounded in $H^1_0(\Omega)$, and therefore $u \in H^1_0(\Omega)$.

Remark 2. By the regularity results of Giaquinta-Giusti [11] and Theorem 3, we know that the maximal solution \bar{u}_λ of (P_λ) is such that $\bar{u}_\lambda \in C^{1,\beta}(\Omega)$ with $\beta = (1-\alpha)/(1+\alpha)$.

Remark 3. If $N = 1$ and $f = C$, a positive constant, one can see that any nonnegative local minimum u of J must satisfy $u > 0$ on Ω .

Remark 4. Theorem 3 shows that the maximal solution \bar{u}_λ obtained via the algorithm A is the only stable solution. Nevertheless it suffices to consider the onedimensional problem to see that (P) can have only one solution, or can have two, three and may be more solutions depending on f . For instance, when f is constant, a second solution appears for a sequence of isolated values λ_n .

However, it is possible to prove, by using an argument of Brezis-Oswald [6] that (P) cannot have more than two ordered solutions.

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