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Boundary Behaviour of Solutions of the Signorini Problem.

I: The Elliptic Case.

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Sunto. – *Si studia il comportamento al contorno della soluzione del problema ellittico ad ostacoli, detto anche problema di Signorini. Si trova una condizione sui dati necessaria e «quasi» sufficiente affinchè si abbia un insieme di coincidenza non vuoto. Si danno inoltre delle valutazioni sulla collocazione di un tale insieme quando esso esiste. Si fanno anche ulteriori osservazioni sul comportamento delle soluzioni di altri problemi non lineari con valori al contorno.*

1. – Introduction.

Let Ω be a bounded set in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega = \Gamma$. We study the behaviour of the solution of the already classical Signorini problem:

(SP) « Given $\psi \in H^{\frac{1}{2}}(\Gamma)$, $g \in H^{-\frac{1}{2}}$, $f \in L^2(\Omega)$ and $\alpha \geqslant 0$, find $u \in H^1(\Omega)$ satisfying

$$u \in K_\psi \equiv \{v \in H^1(\Omega) : v \geqslant \psi \text{ on } \Gamma\}\,, \tag{1}$$

$$\int_\Omega \nabla u \nabla(v-u)\,dx + \alpha\int_\Omega u(v-u)\,dx \geqslant \int_\Omega f(v-u)\,dx + \int_\Gamma g(v-u)\,d\Gamma\,, \tag{2}$$

for all $v \in K_\psi$ »,

where we have used the notation

$$\int_\Omega g(v-u)\,d\Gamma = \langle g, v-u\rangle_{H^{-\frac{1}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)}\,. \tag{3}$$

The Signorini Problem arises in several different contexts. For instance, it models a specific phenomenon related to the linear elasticity which allows to describe the behaviour of the elastic body in contact to a perfectly rigid support. Physically, it is concerned with the finding of the displacement vector which satisfies the law of behaviour of the materials of visco-elastic type (linear in the sense established by such a law, i.e., defining a linear relationship between stress and deformation vectors). The first mathematical treatment of this problem was carryed out in Fichera [10]. After this work, many authors have used this kind of boundary constraints in order to model some other physical problems (see Duvaut-Lions [9], Baiocchi-Capelo [2], Kinderlehrer-Stampacchia [13], Friedman [12] etc.). Our formulation here concerns the case of a scalar unknown $u(x)$.

The existence and uniqueness for this problem are well know (see the above references). In the semi-coercive case ($\alpha = 0$) an extra condition is needed: that is

$$\int_\Omega f\,dx + \int_\Gamma g\,d\Gamma > 0 \tag{4}$$

for the special case $\psi \equiv 0$ (Lions-Stampacchia [16]). The regularity of the solution has also called the attention of many authors. In particular, we know that under additional assumptions

$$\psi \in H^{\frac{3}{2}}(\Gamma)\,, \qquad g \in H^{\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad f \in L^2(\Omega) \tag{5}$$

the solution u belongs to $H^2(\Omega)$ and satisfies the complementary formulation

$$(\mathrm{SP})^* \begin{cases} -\Delta u + \alpha u = f & \text{in } \Omega \\ -\dfrac{\partial u}{\partial n} + g \in \beta(u - \psi) & \text{on } \Gamma \end{cases}$$

where β is the maximal monotone graph of R^2 given by

$$\begin{cases} \beta(r) = 0,\ r > 0\,; \quad \beta(r) = [0, +\infty),\ r = 0\,; \\ \beta(r) = \text{empty set},\ r < 0 \end{cases} \tag{6}$$

and n is the unit outward normal to Γ; that is, on Γ we have the « unilateral type » boundary conditions

$$u \geqslant \psi\,, \qquad \frac{\partial u}{\partial n} \geqslant g\,, \qquad \left(-\frac{\partial u}{\partial n} - g\right)(u - \psi) = 0\,. \tag{7}$$

The main goal of this paper is to study the formation and location of te *coincidence set.*

$$I_\psi = \{\xi \in \Gamma : u(\xi) = \psi(\xi)\}.$$

We point out that the topological structure of the coincidence set was studied (for some special data and $N = 2$) in Lewy [14] and Athanasopoulos [1] but the existence and location of I_ψ was not analyzed in those works.

The first analysis on the existence of the coincidence set is due to Friedman [11] for the case $\psi \equiv g \equiv 0$ and $\alpha \equiv 0$. He proved that if $u = 0$ a.e. on an open set Γ_0 of Γ then necessarily $\tilde{f}(\xi) \geq 0$ on Γ_0, where

$$\tilde{f}(\xi) = \int_\Omega f(x) \frac{\partial G}{\partial n_\xi}(x, \xi)\, dx \tag{8}$$

with $G(x, \xi)$ the Green function for the Dirichlet problem. The main contribution of Friedman to the existence and location of I_0 (i.e. I_ψ with $\psi \equiv 0$) establishes that if there exists ξ^* and ξ^0 in Γ such that $f(\xi^*) = 0$ and $\tilde{f}(\xi^0) < 0$, then either i) in every neighbourhood Γ^* of ξ^* exists a subset Γ^* of positive measure such that $u(\xi) > 0$ for all $\xi \in \Gamma^*$ or ii) u vanishes at ξ^0.

A more explicit result was given in Díaz [4] for the case $g = \psi \equiv 0$: if $f < -\varepsilon(\varepsilon > 0)$ in a neihborhood of a region $\Gamma_\varepsilon \subset \Gamma$ then there exist at least a subregion $\Gamma_0 \subset \Gamma_\varepsilon$ (which is explicitly estimated) such that $\Gamma_0 \subset I_0$.

The results of this paper, extend the approach in both works to the general formulation (SP) and improve (in some sense which is precised later) the mentioned necessary and sufficient conditions for the existence of I_ψ.

In the last section of this paper we also study the boundary behaviour of solutions of other nonlinear boundary value problems such as

$$\left| \begin{array}{ll} -\Delta u + \alpha u = 0 & \text{in } \Omega \\ -\dfrac{\partial u}{\partial n} + g = b(u) & \text{on } \Gamma \end{array} \right. \tag{9}$$

where $b(u)$ is a continuons nondecreasing function such that $b(0) = 0$. Problem (9) has many points in commun with the Signorini problem (for instance the existence of solutions for (SP) is obtained via the approximation of (SP) by a family of problems of the type (9)). Nevertheless, in constrast with other free foundaries problems (see e.g. Díaz [6]) the vanishing of the solution (at the boundary) is only specific of multivalued maximal monotone graphs β and not of other cases: even if b is not Lipschitz continuous at the origin the trace of the solution on Γ may be a strictly positive function.

Finally, we mention that a preliminar announcement of our results was presented in Díaz-Jiménez [7]. The study of the parabolic problem will be the subject of a forthcoming paper (see also Díaz-Jiménez [8]).

2. – The necessary condition.

In that section we shall find a necessary condition on the data ψ, g and f in order to have a positively measured coincidence set I_ψ.

For the sake of simplicity, we will use the complementary formulation (SP)*. Due to that, we shall always assume the regularity assumption (5). Nevertheless, we point out that no important difference occurs in the treatment of the general formulation (SP).

The following result gives a necessary condition in order to have $|I_\psi| = 0$. It is remarkable that this condition is formulated in terms of a function collecting the different influence of the data ψ, g and f on the behaviour of the solution on the boundary Γ.

THEOREM 1. – *Let us assume the coincidence set I_ψ to be smooth and such that $|I_\psi| > 0$. consider the function $\tilde{g}$ defined on Γ by*

$$\tilde{g} = g - \frac{\partial u_0}{\partial n} \tag{10}$$

where u_0 is defined as the unique solution of

$$\begin{cases} -\Delta u_0 + \alpha u_0 = f & \text{in } \Omega \\ u_0 = \psi & \text{on } \Gamma . \end{cases} \tag{11}$$

Then, necessary $\tilde{g} \leq 0$ on I_ψ.

The definition of $\tilde{g}$ is motivated by the following trivial lemma.

LEMMA 1. – *Let u be the solution of* (SP: ψ, f, g) *i.e. with data ψ, f and g respectively. Let u_0 be the solution of* (11). *Then the function $\tilde{u} = u - u_0$ satisfies* (SP: $0, 0, \tilde{g}$) *with $\tilde{g}$ given by* (10). ∎

REMARK 1. – The function $\tilde{g}$ can be made explicit by using the Green function associated to (11). Indeed, if we assume $\psi \in H^2(\Omega)$ (which is no restriction by assumption (5) and trace theorems) then

$$u_0(x) = \psi(x) + \int_\Omega f^*(\xi)\, G(x, \xi)\, d\xi \qquad x \in \bar{\Omega}\,,$$

where $f^* \equiv f + \Delta\psi - \alpha\psi$, and $G(x, y)$ is the Green function for the Dirichlet problem associated to the operator $-\Delta u + \alpha u$ on Ω (see e.g. Stakgold [18]). In particular, if $x \in \Gamma$ we have

$$\tilde{g}(x) = g(x) - \frac{\partial\psi}{\partial n}(x) - \int_\Omega f^*(\xi)\, \frac{\partial G}{\partial n_\xi}(x, \xi)\, d\xi\,. \tag{12}$$

Note that in Friedman's notation, $\tilde{g} \equiv \tilde{f}$ in the special case $\psi \equiv 0$ and $g \equiv 0$. We also recall that, by the weak and strong maximum principles, the functions $G(x, \xi)$ and $\partial G/\partial n_\xi(x, \xi)$ are respectively positive and negative functions.

PROOF of THEOREM 1. – From (2) we deduce that for every $v \in K_0^*\colon = \{v \in H^2(\Omega)\colon v \geqslant 0 \text{ on } \Gamma\}$ the function $w = v - u + u_0 \in \in H^2(\Omega)$ and satisfies

$$-\int_\Omega (u - u_0)\, \Delta w\, dx + \alpha \int_\Omega (u - u_0)\, w\, dx + \int_\Gamma (u - u_0)\, \frac{\partial w}{\partial n}\, d\Gamma \geqslant \int_\Gamma \tilde{g} w\, d\Gamma\,. \tag{13}$$

Now, suppose I_ψ is positively measured and smooth. Then there exists $\theta \in C^2(\Gamma)$ be such that $\theta \geqslant 0$ on I_ψ and $\theta = 0$ on $\Gamma - I_\psi$: Define $w_0 \in H^2(\Omega)$ such that $-\Delta w_0 + \alpha w_0 = 0$ in Ω and $w_0 = \theta$ on Γ. Taking $w = w_0$ in (13) (this is possible because $w_0 + u \in K_0^*$), we deduce that

$$0 \geqslant \int_\Gamma \tilde{g} w_0\, d\Gamma = \int_\Gamma \tilde{g}\theta\, d\Gamma\,.$$

Since θ is taken arbitrarly out of I_ψ, it follows that $\tilde{g} \leqslant 0$ on I_ψ. ∎

In contrast with the result above, the nonnegativity of $\tilde{g}$ on a part Γ_0 of Γ is not enough to ensure that $I_0 \cup \Gamma_0$ is not the empty set. In fact, let consider the following

COUNTEREXAMPLE. – Given $R > 0$, define

$$\Omega = \{(x, y) \in \mathbf{R}^2 \colon y > 0,\ x^2 + y^2 < R^2,\ y - \sqrt{3x} < 0\}\,.$$

Inspired in Shamir [17], define $u(x, y) = \operatorname{Re}(z^{\frac{3}{2}})$, $z = x + iy$, i.e.

$$u(x, y) = \varrho^{\frac{3}{2}} \cos\frac{2}{3\theta}\,, \qquad x = \varrho\cos\theta\,; \qquad y = \varrho \operatorname{sen}\theta\,. \tag{14}$$

It is not difficult to check that $u \in H^2(\Omega)$ and $\Delta u = 0$ on Ω. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ (see figure 1).

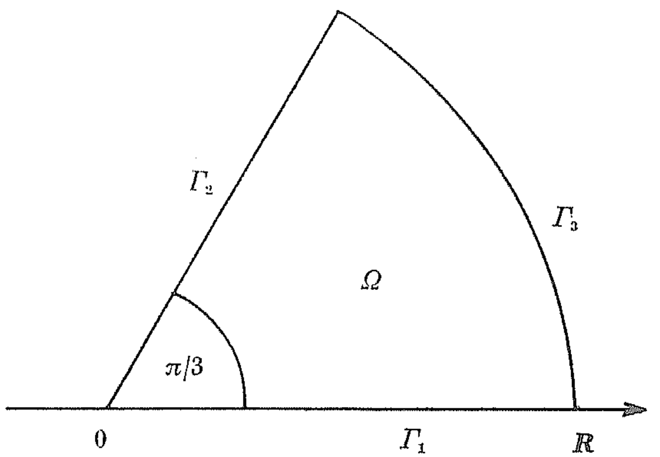

Figure 1

Then

$$\text{on } \Gamma_1\colon \qquad -\frac{\partial u}{\partial n} = 0\,, \qquad u(x, y) = u(x, 0) = \varrho^{\frac{3}{2}} > 0$$

$$\text{on } \Gamma_2\colon \qquad -\frac{\partial u}{\partial n} = \frac{3}{4}\,\varrho^{\frac{1}{2}} > 0\,, \qquad u(x, y) = 0$$

$$\text{on } \Gamma_3\colon \qquad -\frac{\partial u}{\partial n} = \frac{3}{2}\,R^{\frac{1}{2}} \cos\frac{3\theta}{2}\bigg|_0^{\pi/3} < 0\,, \qquad u(x, y) > 0\,.$$

We conclude that, for all $R > 0$, the function $u = \varrho^{\frac{3}{2}} \cos 3\theta/2$ is the solution of $-\Delta u = 0$ in Ω, $-\partial u/\partial n + g \in \beta(u)$ on Γ with $g \equiv 0$ on Γ_1, thus $\tilde{g} = 0$ on Γ_1, however $u > 0$ on Γ_1. This shows that the condition $\tilde{g} = 0$ leads to an indetermination on the vanishing of u on Γ because in our example $u = 0$ on Γ_2 but $u > 0$ on Γ_1 being $\tilde{g} = 0$ on $\Gamma_1 \cup \Gamma_2$.

3. – The existence and location of the coincidence set.

The following result shows that the necessary condition is « almost » sufficient for the formation of the coincidence set. In this case we shall use an additional geometric property on Ω.

THEOREM 2. – *Let Ω be a convex open bounded set of R^N and let $u \in H^2(\Omega) \cup L^\infty(\Omega)$ be a solution of* (SP). *Assume $f \in L^2(\Omega)$ and $g \in H^{\frac{1}{2}}(\Gamma)$ and $\psi \in H^{\frac{3}{2}}(\Gamma)$ satisfying that there exist $\varepsilon > 0$ and $\Gamma_\varepsilon \subset \Gamma$ such that*

$$\tilde{g}(\xi) \leqslant -\varepsilon \quad \text{on } \Gamma_\varepsilon \tag{15}$$

with $\tilde{g}$ defined by (10). *Then we have the estimate*

$$I_0 \supset \{\xi \in \Gamma_\varepsilon : d(\xi, \Gamma - \Gamma_\varepsilon) \geqslant R\} \tag{16}$$

with R given by

$$R = \frac{2MN}{\varepsilon} \tag{17}$$

and $M > 0$ such that

$$\|u\|_{L^\infty(\Omega)} \leqslant M . \tag{18}$$

PROOF. – Again, by Lemma 1, is suffices to show the estimate (16) for u solution of $(\mathrm{SP}: 0, 0, \tilde{g})$ i.e. such that

$$(\mathrm{SP}: 0, 0, \tilde{g}) \begin{cases} -\Delta u + \alpha u = 0 & \text{in } \Omega \\ -\dfrac{\partial u}{\partial n} + \tilde{g} \in \beta(u) & \text{on } \Gamma . \end{cases}$$

where $\tilde{g}$ is defined in (10). Notice that in that case I_ψ reduces to the set I_0. Without loose of generality, we can suppose that $\tilde{g} = -\varepsilon$ on Γ_ε and $\tilde{g} \leqslant -\varepsilon$ on $\Gamma - \Gamma_\varepsilon$. It is enough to see that if u_ε is the solution of

$$(P_\varepsilon) \begin{cases} -\Delta u_\varepsilon + \alpha u_\varepsilon = 0 & \text{in } \Omega \\ -\dfrac{\partial u_\varepsilon}{\partial n} + g_\varepsilon \in \beta(u_\varepsilon) & \text{on } \Gamma \end{cases}$$

with $g_\varepsilon = \tilde{g}$ on $\Gamma - \Gamma_\varepsilon$ and $g_\varepsilon = \varepsilon$ on Γ_ε, then by the comparison theorems (see eg. Brezis [3]) we deduce that $u \leqslant u_\varepsilon$ on $\bar{\Omega}$ and in particular $0 \leqslant u(\xi) \leqslant u_\varepsilon(\xi)$ a.e. $\xi \in \Gamma$ and thus if $u_\varepsilon = 0$ on Γ_ε we have the same for u.

Now, let $x_0 \in \Gamma_\varepsilon$ be such that $d(x_0, \Gamma - \Gamma_\varepsilon) = R$. Let $D = \Omega \cap \cap B(x_0, R)$ and define $\partial_1 D = \partial D \cap \Gamma$ and $\partial_2 D = \partial D - \Gamma$. For $C > 0$, to be chosen later, we shall construct $U \in H^2(D)$ such that $U \geqslant 0$ and

$$\begin{cases} -\Delta U + \alpha U \geqslant C & \text{in } D \\ U = 0 & \text{on } \partial_1 D \\ \dfrac{\partial U}{\partial n} = -\varepsilon & \text{on } \partial_1 D\,. \end{cases}$$

To do that, let V_δ^- be a tubular semineighbourhood of Γ_ε defined by the usual parametric representation

$$x = x(\xi, s) = \xi + s n(\xi)\,, \qquad \xi \in \Gamma_\varepsilon\,, \qquad s \in \,]-\delta, 0[\,, \qquad \delta \geqslant R\,,$$

where $n(\xi)$ is the outward normal unit vector to Γ at ξ and $\delta > 0$ such that $V_\delta^- \supset D$ (see Figure 2)

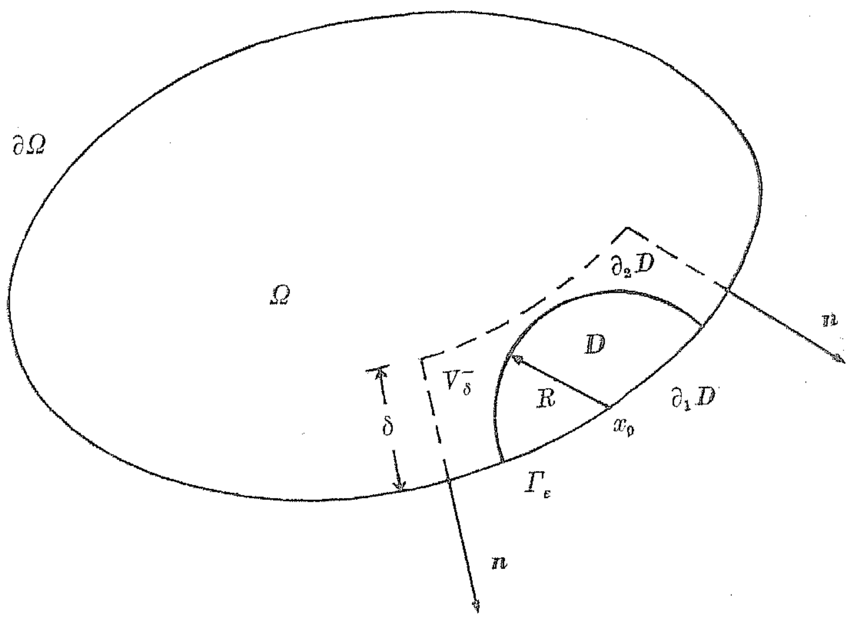

Figure 2

Taking $U(x) = U(ns) = \varphi(s)$ and recalling the expression of the Laplacian operator on V_δ^- (see Sperb [18]), the construction of such a U is reduced to find $\varphi(s)$ such that

$$\varphi''(s) + (N-1)H\varphi'(s) \leqslant -C$$
$$\varphi(0) = 0$$
$$\varphi'(0) = -\varepsilon ,$$

where H is the mean curvature of Γ. Notice that $H \geqslant 0$ due to the assumption of convexity on Ω. We shall determine C in such a way that $\varphi(s) \geqslant 0$ for any $-R \leqslant s \leqslant 0$.

Among the multiple choices, we shall take $\varphi(s) = -\varepsilon s(s/2R + 1)$ and then $C = \varepsilon/R$, which is possible because $H \geqslant 0$.

Now we introduce into D the auxiliary function

$$\overline{u}(x) = U(x) + \frac{C}{2N}|x - x_0|^2$$

where C is taken as mentioned above. We have

$$-\Delta\overline{u} + \alpha\overline{u} = -\Delta U + \alpha U - C + \frac{\alpha C}{2N}|x - x_0|^2 \geqslant 0 \qquad \text{in } D .$$

Moreover, on $\partial_1 D$ it is clear that $\overline{u} \geqslant 0$ and that

$$\frac{\partial \overline{u}}{\partial n}(\xi) = \frac{\partial U}{\partial n}(\xi) + \frac{C}{N}|x - x_0| \cos(n(\xi), \xi - x_0) \geqslant -\varepsilon$$

because of the convexity of Ω. On the other hand, in $\partial_2 D$

$$\overline{u}(x) \geqslant \frac{C}{2N} R^2 \geqslant M \geqslant u(x)$$

if $R \geqslant (2MN/C)^{\frac{1}{2}}$. In conclusion, by the comparison theorems, we deduce that $u \leqslant \overline{u}$ on $\overline{D}$ and in particular

$$0 \leqslant u(x) \leqslant \frac{C}{2N}|\xi - x_0|^2 , \qquad \xi \in \Gamma \cap B(x_0, R)$$

which proves the result. ∎

REMARK 2. – The condition $u \in L^\infty(\Omega)$, as well as, some explicit estimates of the constant M can be found in Brezis [3]. These results hold from maximum principle (for f, g and ψ bounded) or by elliptic regularization properties ($f \in L^p(\Omega)$, $p > N$, $\psi \equiv g \equiv 0$).

REMARK 3. – Theorem 2 can be generalized or improved in several different ways. For instance, sharper estimates on the location of I_ψ indicating the dependence on α ($\alpha > 0$) can be found by using more «ad hoc» supersolutions (see Díaz [5]). On the other hand, Theorem 2 remains true when the differential operator $-\Delta u + \alpha u$ is replaced by a general linear second order elliptic operator

$$Lu = -\sum_{ij=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i} + b_j(x)\,u\right) + c(x)\,u$$

or some quasilinear second order elliptic operator, as, for instance

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \qquad 1 < p < \infty .$$

The modifications in the definition of the supersolution $\bar{u}$, needed to consider both generalizations follow some related results in Díaz [6] (Theorems 1.13 and 1.9 respectively). Finally, we point out that the proof of Theorem 2 has a local character and so it can also be applied to the case of mixed boundary conditions of the type

$$-\frac{\partial u}{\partial n} + g \in \beta(u-\psi) \quad \text{on } \Gamma_1, \qquad u = h \quad \text{on } \Gamma_2, \qquad \Gamma = \Gamma_1 \cap \Gamma_2 .$$

4. – Boundary behaviour for other nonlinear boundary value problems.

To the light of the above results, a natural question arises: for which maximal monotone graphs β the coincidence set I_ψ (defined through the solution u of (SP)*) is not empty for suitable data f, g and ψ? It tourns out that among the different peculiarities of the special graph β associated to the Signorini problem (see (6)), the fact that $D(\beta) = [0, \infty)$ is the crucial one in order to allow the apparition of the coincidence set. To explain that we shall consider the case in which $D(\beta) = \mathbf{R}$. More precisely, consider the problem

$$(19) \qquad \begin{cases} -\Delta u + \alpha u = 0 & \text{in } \Omega , \\ -\dfrac{\partial u}{\partial n} + g = b(u) & \text{on } \Gamma \end{cases}$$

where $\alpha \geqslant 0$ and b is a nondecreasing continuous function with $b(0)=0$. Assume that $g(x) \leqslant 0$ on Γ. By the comparison results $u \leqslant 0$ in $\bar{\Omega}$ and from the strong maximun principle $u<0$ in Ω. So if $u(x_0)=0$ for some $x_0 \in \Gamma$ then $(\partial u/\partial n)(x_0)>0$ which contradicts the assumption $g \leqslant 0$. Then if g does not changes of sign the trace $u|_\Gamma$ cannot vanish.

When g is changing of sign Theorem 2 can be useful in order to estimate the parts of Γ where u is positive or negative. Indeed, by the comparison results (Brezis [3]) we know that if u is the solution of (19) then $u \leqslant v$ on $\bar{\Omega}$, where v satisfies

$$(20) \qquad \begin{cases} -\Delta v + \alpha v = 0 & \text{in } \Omega \\ -\dfrac{\partial v}{\partial n} + g \in \beta(v) & \text{on } \Gamma \end{cases}$$

with β given by

$$\beta(0) = (-\infty, 0]$$

$$\beta(r) = \{b(r)\} \text{if } r>0 \text{ and } \beta(r) = \text{the empty set if } r<0\,.$$

In particular we have that

$$0 \leqslant (u|_\Gamma)^+ \leqslant v \qquad \text{on } \Gamma\,.$$

Now, Theorem 2 can be easily generalized to the solution of (20) and, in conclusion, the estimates on the location of the coincidence set $\{x \in \Gamma : v(x) = 0\}$ gives automatically estimates on the region of Γ where $u \leqslant 0$.

REMARK 4. – Introducing the pseudo-differential operator $A: H^{\frac12}(\Gamma) \to H^{-\frac12}(\Gamma)$ given by $Aw = \partial v/\partial n$, where v is the solution of $-\Delta v + \alpha v = 0$ in Ω and $v = w$ on $\partial\Omega$ problem (19) way be equivalently formulated in terms of

$$Aw + b(w) = g\,,$$

where now w represents the trace on Γ of the solution of (19) (Lions [15]). The above considerations show that a free boundary (the boundary of the coincidence set) is formed when $b=\beta$ is the multivalued graph given by (6) (the Signorini problem). Nevertheless, in contrast with many cases in which A is a local operator (e.g. $A = -\Delta$ plus boundary conditions) the free boundary is not formed for others choices of b, even if b is not Lipschitz continuous at the origin.

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