

NEW RESULTS ON LOCALIZATION OF SOLUTIONS OF NONLINEAR ELLIPTIC AND PARABOLIC EQUATIONS OBTAINED BY THE ENERGY METHOD

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S. N. ANTONTSEV AND KH. I. DIAS [J. I. DÍAZ]

In this note, new results are presented on the spatial and temporal localization of solutions of general nonlinear elliptic and parabolic equations in the presence of "sources" given by the right-hand side. These results are obtained by the energy method previously presented and substantiated in the authors' papers [1]–[3], [9], and [10]. Such properties of the solutions as finite localization (vanishing) time and spatial localization for evolution equations are investigated, as well as vanishing on a set of positive measure for elliptic equations. For special parabolic equations without "sources", some of the above-indicated properties have been previously studied by numerous authors (see the references given in [3], [4], and [10]).

The basic goal of this note is the study of equations with given right-hand side, leading to inhomogeneous nonlinear ordinary differential inequalities for their energy functions. The method is also applied to systems of equations of composite type, including some which arise in continuum mechanics [2].

1. Parabolic equations. Consider in the region $Q = \Omega \times (0, T)$, $\Omega \subset R^n$, $t \in R_+^1$, the equation

$$(1) \quad \frac{\partial \psi(x, u)}{\partial t} - \operatorname{div} A(t, x, u, \nabla u) + B(t, x, u) = f(t, x) + \operatorname{div} g(t, x),$$
$$A = (A_1, \dots, A_n), \quad \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right), \quad \operatorname{div} A = \frac{dA_i}{dx_i},$$

assuming that Ω is a bounded region with smooth boundary $\Gamma = \partial\Omega$ and, for $(t, x, s, r) \in R_+^1 \times \Omega \times R \times R^n$, the following conditions are satisfied:

$$(2) \quad c_2 |r|^p \leq A(t, x, s, r) \cdot r \leq c_1 |r|^p, \quad 1 < p < \infty, \quad c_1, c_2 > 0;$$

$$(3) \quad c_3 |s|^{\sigma+1} \leq B(t, x, s) \cdot s, \quad \sigma > 0, \quad c_3 \geq 0;$$

$$(4) \quad c_5 |s|^{\beta+k} \leq \Psi(s, k) \leq c_4 |s|^{\beta+k}, \quad k > 0, \quad \beta > 0, \quad c_4, c_5 > 0,$$

$$\Psi(s, k) = |s|^{k-1} s \psi(x, s) - \int_0^s \psi(x, \tau) |\tau|^{k-1} \tau d\tau.$$

1a. Finite localization time of the solution. Let us begin by studying the localization (vanishing), after a finite time, of the solution of equation (1) satisfying the following initial-boundary value problem:

$$(5) \quad u(0, x) = u_0(x), \quad x \in \Omega; \quad u(t, x) = 0, \quad (t, x) \in \Gamma_T = (0, T) \times \Gamma.$$

We shall consider the weak generalized solution $u(t, x)$ of problem (1), (5) such that

$$(6) \quad v(t, x) \equiv |u(t, x)|^{(k-1)/p} \cdot u(t, x) \\ \in L_{\text{loc}}(0, T; \dot{W}_p^1(\Omega)) \cap L^\infty(0, T; L^{p(\beta+k)/(p+k-1)}(\Omega)), \\ k = \begin{cases} 1, & \text{if } n \leq p \text{ or } \beta + 1 \leq np/(n-p), \\ \beta(n-p) - n(p-1)/p, & \text{if } n < p \text{ and } 1 + \beta > np/(n-p). \end{cases}$$

A detailed definition of generalized solution is given in [2], [9], and [10]. It is assumed that

$$(7) \quad f(t, x) \in L^{(p+k-1)/(p+k-2)}(0, T; L^{(\beta+k)/\beta}(\Omega)), \\ g(t, x) \in L^{(p+k-1)/(p-1)}(0, T; L^{p/(p-1) \cdot (k+\beta)/(\beta+1)}(\Omega)).$$

THEOREM 1. *Let $u(t, x)$ be a generalized solution of problem (1), (5) and, in addition to conditions (6), (7), and (2)–(4) with $c_3 \geq 0$, assume the following condition is satisfied:*

$$(8) \quad p - 1 < \beta.$$

Then, for $T_f \in (0, T)$ there exist positive constants c_6, c_7 , and c_8 (where c_6 and c_7 are small compared to c_2) such that, if

$$(9) \quad (\|f(t, \cdot)\|_{(\beta+k)/\beta, \Omega}^{(p+k-1)/(p+k-2)}; \|g(t, \cdot)\|_{\lambda p/(p-1), \Omega}^{(p+k-1)/(p-1)}) \\ \leq c_6(T_f - t)_+^{\alpha/(1-\alpha)}, \quad \forall t \in (t_1, T);$$

$$(10) \quad \|u(t_1, \cdot)\|_{\beta+k, \Omega}^{\beta+k} \leq c_7(T_f - t_1)^{1/(1-\alpha)}, \quad v_+ = \max(0, v), \\ \alpha = (p - 1 + k)/(\beta + k), \quad \lambda = (k + \beta)/(\beta + 1), \quad \|\cdot\|_{q, \Omega} = \|\cdot\|_{L^q(\Omega)},$$

then

$$(11) \quad \|u(t, \cdot)\|_{\beta+k, \Omega}^{\beta+k} \leq c_8(T_f - t)_+^{1/(1-\alpha)}, \quad \forall t \in (t_1, T)$$

and, in particular,

$$(12) \quad u(t, x) \equiv 0, \quad \forall x \in \Omega, \forall t \geq T_f.$$

PROOF. We present a sketch of the proof, considering for simplicity only the case $k = 1$. According to [2] and [9], for the solution $u(t, x)$ of problem (1), (5), the energy relation

$$(13) \quad \frac{dy}{dt} + (A, \nabla u)_\Omega + (B, u)_\Omega = (f, u)_\Omega - (g, \nabla u)_\Omega$$

holds, where

$$(u, v)_\Omega = \int_\Omega u \cdot v \, dx, \quad y(t) = (\Psi(x, u(t, x)); 1)_\Omega.$$

This may be derived formally by multiplying (1) by $|u|^{k-1} \cdot u$ and integrating by parts. Further, we use Young's and Hölder's inequalities, as well as a corollary of the Sobolev imbedding theorem to obtain

$$(14) \quad c_2 c_9 \|v\|_{q, \Omega}^p \leq c_2 \|\nabla v\|_{p, \Omega}^p \leq (A(v, \nabla v), \nabla v)_\Omega, \\ 1 \leq q \leq np/(n-p), \quad \forall v \in \dot{W}_p^1(\Omega).$$

From (13), taking into account (2)–(4), (6)–(8) and (14) for $q = \beta + 1$, we obtain the ordinary, inhomogeneous differential inequality

$$(15) \quad \frac{dy}{dt} + c_{10} y^\alpha \leq c_{11}(T_f - t)_+^{\alpha/(1-\alpha)}, \quad 0 < \alpha = \frac{p}{\beta + 1} < 1,$$

where

$$c_{10} = \left(1 - \frac{2\varepsilon^p}{p}\right) \frac{c_9 c_2}{c_4^\alpha}, \quad c_{11} = \frac{\varepsilon^{-p'}}{p'} (1 + (c_9)^{-p'/p}) c_6 c_2^{-p'/p},$$

$$\forall \varepsilon > 0, \quad p' = p/(p-1).$$

For solutions of (15), the estimate

$$(16) \quad y(t) \leq c_8 (T_f - t)_+^{1/(1-\alpha)}, \quad c_7 \leq c_8 (c_7, c_{10}, c_{11})$$

(see [3] and [8]), holds, if the initial data (the constant c_7), the cutoff time T_f of the "sources", and the capacity of the "source" c_6 (i.e. the constant c_{11}) satisfy the relation

$$c_{11} + c_7/(1-\alpha) \leq c_{10} c_7^\alpha.$$

The result of the theorem can be interpreted as follows: the solution $u(t, x)$ tends to zero, starting at the cutoff time T_f of the "sources" f and g .

REMARK 1. The property of the solution $u(t, x)$ proved above also holds when condition (8) is changed to $\beta \leq p-1$, if $c_3 > 0$ and $\sigma < \beta$ in (3). In this case, instead of (5) the more general boundary condition

$$\nu_1(t, x) \cdot \mathbf{A} \cdot \mathbf{n} + \nu_2(t, x) \cdot u = 0, \quad x \in \Gamma,$$

can be considered, where $\nu_1 \nu_2 \geq 0$, $\nu_1^2 + \nu_2^2 = 1$, and \mathbf{n} is the normal vector to Γ .

REMARK 2. In the case when $f = g = 0$, the finite localization time for the solution of problem (1), (5) was proved using the energy method in [1] and [5]-[7].

REMARK 3. The constants c_{10} and c_{11} in (13) are independent of Ω if $\beta > n(p-1)/(n-p)$. Consequently, in this case the Cauchy problem for equation (1) can be treated analogously, as well.

REMARK 4. Theorem 1 can also be formulated as follows: for any

$$y(0) = (\Psi(x, u(0, x)), 1)_\Omega > 0$$

and sufficiently small constant $c_6 > 0$, there exists $T_f < \infty$ such that (11) and (12) hold.

The approach presented here can also be used for systems of equations of composite type. In such equations, the components of the unknown vector, the solution $\mathbf{w}(t, x)$, may satisfy equations of different types. Consider the following initial-boundary value problem:

$$(17) \quad \mathbf{B}(t, x, \mathbf{w}, \mathbf{w}_t) = \mathbf{A}(t, x, \mathbf{w}), \quad (x, t) \in Q = \Omega \times (0, T),$$

$$\mathbf{w} = (\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_m, v_1, \dots, v_q),$$

$$(18) \quad P\mathbf{w} = 0, \quad x \in \Gamma, \quad t \in (0, T); \quad D\mathbf{w}|_{t=0} = \mathbf{g}, \quad x \in \Omega.$$

Assume that problem (17), (18) has the generalized solution $\mathbf{w}(t, x)$ in some class $V(Q) \subset L^\infty(0, T; L^2(\Omega))$, and that for all $\mathbf{w} \in V(Q)$ and almost all $t \in (0, T)$

$$(19) \quad (\mathbf{A}, \mathbf{w})_\Omega \leq -\varphi(c \|\mathbf{u}\|_{2,\Omega}^2) + F((T_f - t)_+);$$

$$(20) \quad (\mathbf{B}, \mathbf{w})_\Omega = \frac{dy}{dt}, \quad \frac{1}{c} \|\mathbf{u}\|_{2,\Omega}^2 \leq y \leq c \|\mathbf{u}\|_{2,\Omega}^2.$$

Here, $\varphi(s) > 0$, $s > 0$, $\varphi(0) = 0$, is a nondecreasing continuous function such that

$$(21) \quad \int_0^1 \frac{d\tau}{\varphi(\tau)} \leq c < \infty$$

and there exists $\bar{\mu} < 1$ such that

$$(22) \quad 0 \leq F(s) \leq (1 - \bar{\mu})\varphi(\eta_{\bar{\mu}}(s)), \quad \eta_{\bar{\mu}}[\theta_{\bar{\mu}}^{-1}(s)] \equiv s, \quad \theta_{\bar{\mu}}(\tau) = \int_0^\tau \frac{ds}{\mu\varphi(s)}.$$

The following abstract theorem holds.

THEOREM 2. Let $w(t, x)$ be a generalized solution of problem (17), (18) for $t \in (t_1, \infty)$, where conditions (19)–(22) and

$$(23) \quad T_f - t_1 \geq \theta_{\bar{\mu}}(y(t_1))$$

are satisfied. Then

$$\|u(t, \cdot)\|_{2, \Omega} = 0, \quad \forall t \geq T_f.$$

PROOF. Theorem 2 follows from the energy inequality

$$(24) \quad \frac{dy}{dt} + \varphi(y) \leq F((T_f - t)_+)$$

and the following auxiliary statement.

LEMMA 1. Let the function $y(t) \geq 0$ satisfy (24) for any $t \in (t_1, \infty)$, and let conditions (21)–(23) hold for φ, F, T_f , and t_1 . Then $y(t) \equiv 0$ for all $t \geq T_f$.

1b. Spatial localization. In contrast to the foregoing, we shall now consider the local properties of weak solutions of equation (1) without any information on the values of $u(t, x)$ on Γ . Let $B_\rho(x_0) = \{x \in \Omega: |x - x_0| < \rho\}$. The basic result of this subsection is the following (for $k = 1$).

THEOREM 3. Let $u(t, x)$ be a weak generalized solution of equation (1) with $g = 0$, and let conditions (2)–(4) with $c_3 \geq 0$ and (8) be satisfied. Assume, additionally, that there exist $\rho_0 > 0$, $T_f > 0$, $\delta > 0$, and $\varepsilon > 0$ (ε sufficiently small) such that

$$\left(\|u(0, \cdot)\|_{\beta+1, B_\rho(x_0)}^{\beta+1}; \left(\int_0^{T_f} \|f(\tau, \cdot)\|_{r/(r-1), B_\rho(x_0)}^{p/(p-\theta)} d\tau \right)^{(p-\theta)/p(1-\lambda)} \right) \leq \varepsilon(\rho - \rho_0)_+^{1/(1-\alpha)}$$

for almost all $\rho \in (0, \rho_0 + \delta)$, where

$$\alpha = \frac{p}{p-1} \left(1 - \left(\frac{\theta}{p} + \frac{1-\theta}{\beta+1} \right) \right), \quad \theta = \left(\frac{1}{\beta+1} - \frac{1}{r} \right) / \left(\frac{1}{\beta+1} - \frac{n-p}{np} \right),$$

$$\lambda = \frac{1-\theta}{\beta+1} + \frac{\theta}{p}, \quad \beta+1 \leq r \leq \frac{np}{n-p}.$$

Then there exists $t^* > 0$, $t^* \leq T_f$, such that

$$u(t, x) \equiv 0, \quad \forall x \in B_{\rho_0}(x_0), \quad t \in [0, t^*].$$

PROOF. For the energy functions

$$E(t, \rho) = \int_0^t \int_{B_\rho(x_0)} A \cdot \nabla u \, dx \, dt, \quad b(t, \rho) = \sup_{0 \leq \tau \leq t} \int_{B_\rho(x_0)} \Psi(x, u(\tau, x)) \, dx$$

analogously to [1]–[3] and [10], we obtain the inhomogeneous differential inequality

$$(E + b)^\alpha \leq ct^w \frac{\partial E}{\partial \rho} + c(\rho - \rho_0)_+^{\alpha/(1-\alpha)}, \quad w > 0, \quad 0 < \alpha < 1.$$

Then it suffices to use the following lemma.

LEMMA 2. Let the function $y(t, \rho) \geq 0$ for $t \leq T_f$, and for some $w > 0$, $c > 0$, and $\delta > 0$ let it satisfy the inequality

$$\varphi(y(t, \rho)) \leq ct^w \frac{\partial y}{\partial \rho} + F((\rho - \rho_0)_+), \quad \rho \in (0, \rho_0 + \delta),$$

where φ and F obey the conditions of Lemma 1. Then there exists $t^* \leq T_f$ such that

$$y(t, \rho) = 0, \quad \forall \rho \in [0, \rho_0], \quad \forall t \in [0, t^*].$$

REMARK 5. The statement of the theorem remains true if (8) is changed to the inequality $\beta \leq p - 1$ and it is additionally required that $c_3 > 0$ and $\sigma < p - 1$.

2. **Elliptic equations.** The local energy method used in §1 can be applied to the study of nonlinear elliptic equations of the form [8]

$$(25) \quad -\operatorname{div} A(x, u, \nabla u) + B(x, u) = f(x),$$

where A and B satisfy conditions (2) and (3).

THEOREM 4. Let $u(x)$ be a weak generalized solution of equation (25), where $c_3 > 0$ and $\sigma < p - 1$. Assume there exist $\rho_0 > 0$, $\delta > 0$, and $\varepsilon_i > 0$, $i = 1, 2$ (ε_i sufficiently small), such that

$$\|\nabla u\|_{\rho, B_{\rho_1}(x_0)} \leq \varepsilon_1, \quad \rho_1 = \rho_0 + \delta,$$

$$\|f\|_{r/(r-1), B_{\rho}(x_0)}^{1/(1-\lambda)} \leq \varepsilon_2(\rho - \rho_0)_+^{1/(1-q)}$$

for almost all $\rho \in (0, \rho_0 + \delta)$, where

$$\sigma + 1 \leq r \leq \frac{np}{n-p}, \quad \theta = \left(\frac{1}{\sigma+1} - \frac{1}{r} \right) / \left(\frac{1}{\sigma+1} - \frac{n-1}{np} \right),$$

$$\lambda = \frac{1-\theta}{\sigma+1} + \frac{\theta}{p}, \quad q = \frac{p}{p-1}(1-\lambda), \quad 0 < \lambda < 1.$$

Then $u(x) = 0$ for all $x \in B_{\rho_0}(x_0)$.

The method of energy estimates used here can also be applied to the investigation of systems of equations of continuum mechanics.

Lavrent'ev Institute of Hydrodynamics

Siberian Branch, Academy of Sciences of the USSR

Novosibirsk

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Universidad Complutense

Madrid

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