ENERGY METHOD AND LOCALIZATION OF SOLUTIONS FOR CONTINUUM MECHANICS EQUATIONS

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ABSTRACT

The paper presents results on space and time localization of solutions for mathematical models in continuum mechanics obtained by the authors. Use is made of the method of integral energy estimates of solutions suggested and substantiated earlier by authors for equations of elliptic, parabolic and mixed type.

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A lot of actual mathematical models in continuum mechanics lead to investiggation of mixed type nonlinear systems of partial differential equations. In such systems the different components of the unknown solution vector (e.g., velocity, density, pressure, saturation, temperature etc.) can satisfy equations of different types (parabolic, hyperbolic, elliptic). The systems of equations may degenerate in the sense of type or order at certain values of the unknown solution or its derivatives. In this case the solutions themselves can have finite localization time (vanishing), finite velocity of propagation of disturbances on the initial data, space localization with inertia (metastable localization), etc. Such property of solutions of degenerating parabolic equations (say, nonlinear heat equation) as finite disturbance-propagation velocity has apparently been first noted and studied in [1 - 3]. For the degenerating elliptic equation a similar property seems to have been first noted in [4] in connection with the study of plane sonic jet flow. For one parabolic equation these problems were later investigated in a great number pf papers, a fairly comprehensive review of which is given in [5], [6]. At present the localization problems for solutions increasing infinitely in finite time are actively studied also for the quasilinear parabolic equations [7]. The results for one parabolic equation were obtained, as a rule, on the basis of theorems comparing the solution under investigation with an auxiliary (e.g. self-similar) one. However, similar methods are nor generally applicable to mixed-type systems of equations. In papers [8] - [10] an energy method was suggested and substantiated for the study of the nature of disturbances described by solutions of general elliptic, parabolic and mixed-type equations. The essence of the method is to derive and study ordinary differential inequalities for energy functions. The method was generalized and developed in papers [11] - [18] for the higher order equations as well. It turned out to be effective in the study of weak generalized solutions of mixed-type systems arising in continuum mechanics.

In papers [11], [19] - [23] the energy method was used to establish the finiteness of localization time and disturbance-propagation velocity on the initial data disturbances for a number of mathematical models in continuum mechanics (filtrational two-phase fluid flows, joint flows of surface and ground waters, flows in open channels, flows of incompressible nonhomogeneous visco-plastic media, uniform flows of viscous gas, etc.). In [21] an axisymmetric jet flowing with a sonic velocity along the symmetry axis was proved to align at a finite distance (as in the plane case [4]).

In the present study we establish the properties of finite localization time and the metastable localization (localization with inertia) of the solutions for some of the models mentioned above in the rpresence of "sources", i.e. prescribed right-hand parts of equations.

It should be noted that the paper does not consider the problems of existence of the corresponding solutions, it studies only their qualitative properties.

I. Incompressible Nonhomogeneous Non-Newtonian Fluids.

The system of equations representing the corresponding conservation laws has the mixed type and can ebe written as follows [24]-[26]:

(1.1) $\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0 \ , \quad \nabla \rho = \left(\frac{\partial \rho}{\partial x_1}, \dots , \frac{\partial \rho}{\partial x_n} \right)$

(1.2) $div \ \vec{v} = 0 \ , \quad \vec{v} = (v_1, \dots , v_n)$

(1.3) $\rho \frac{d\vec{v}}{dt} \equiv \rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla) \vec{v} \right) = div \ P + \rho \vec{f}$

(1.4) $P = - p E + F(D) \ , \quad D = \{ \mathcal{D}_{ij} \} = \left\{ \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right\}$

$$x \in \Omega \subset R^n , \ t \in (0,T) , \ Q = \Omega \times (0,T) \ .$$

Here $\vec{v}(t,x)$, $\rho(t,x)$ and $p(t,x)$ are the unknown velocity, density and fluid pressure, respectively; P, D are the tensors of stresses and deformation velocities; E is the unit tensor; $\vec{f}(t,x)$ is the prescribed mass force, i.e. the "source".

Further the prescribed symmetric tensor F determining P is assumed to satisfy condition

(1.5) $\delta \cdot |D|^q \leq F : D = F_{ij} \cdot \mathcal{D}_{ij} \ , \quad 1 < q \ , \quad \delta = const > 0$.

For classic incompressible viscous fluid $P = - p E + 2\mu D$ and, respectively, $\delta = 2$, $q = 2$ in (1.5). For the visco-plastic fluids [25], [26]

$$P = - p E + 2 (\mu + \tau |D|^{\sigma - 1}) D \ , \quad 0 \leq \sigma < 1 \ .$$

The application of the Young inequality results in (1.5) with

(1.6) $\delta = \mu^{\frac{1}{\theta}} \cdot \tau^{\frac{\theta-1}{\theta}} \cdot \theta^{\frac{1}{\theta}} \left(\frac{\theta}{\theta-1} \right)^{\frac{\theta-1}{\theta}} , \quad q = \left(\frac{2}{\theta} + \frac{(\sigma+1)(\theta-1)}{\theta} \right) \in (\sigma+1, 2) \ .$

For $w = (\vec{v}, \rho, p)$ consider the following initial boundary-value problem

(1.7) $\vec{v}(t,x) = 0 \ , \quad (x,t) \in \Gamma_T = \partial\Omega \times (0,T)$

(1.8) $\vec{v}(0,x) = \vec{v}_0(x) \ , \quad \rho(0,x) = \rho_0(x) \ , \quad x \in \Omega$

Note that for problem (1.1),- (1.4), (1.7), (1.8), with in dependence relation (1.5) , the existence theorem had been proved in [27] for a weak generalized solution $w=(\vec{v},\rho,p)\in V_q$, where (in notation of [28])

$$V_q=\{w:\vec{v}\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W_0^{1,q}(\Omega)),\ \tfrac{1}{M}\le\rho\le M,\ \rho_t\in L^2(0,T;W^{-1,2}(\Omega))\},\ q=2\ .$$

In the case of homogeneous incompressible fluid ($\rho(t,x)\equiv$ const) the existence theorems for solution of systems (1.1) - (1.4) for the certain dependences (1.5) had bean proved in [28], [29]. We shall further study the qualitative properties of solutions $w\in V_q$ in system (1.1) - (1.4) assuming that Ω is a bounded domain with smooth boundary. Let condition (1.5) be satisfied, where

(1.9) $\qquad q\in\left(\dfrac{2n}{2+n},\ 2\right),\quad n\geq 2$

and, additionally,

(1.10) $\|\vec{v}_0(x)\|_{2,\Omega}\le C_v=const\ ,\quad \frac{1}{M}\le\rho_0\le M$

(1.11) $\|f(t,\cdot)\|_{2,\Omega}^{\frac{q}{q-1}}\le C_f\cdot(1-t/T_f)_+^{\frac{q}{2-q}},\ C_f=const,\ u_+=max(0,u),\ T_f\in(0,T)\,.$

<u>Theorem 1.</u> (Finiterlocalization time) Let $w=(\vec{v},\rho,p)\in V_q$ be the generalized solution of problem (1.1) - (1.4), (1.7), (1.8) and conditions (1.10), (1.11) be fulfilled. Then for any $T_f\in(0,T)$ there exist constants C_v , C_f (small with respect to δ , generally speaking) and C , such that

(1.12) $\qquad \|\vec{v}(t,\cdot)\|_{2,\Omega}^2\le C\cdot(1-t/T_f)_+^{\frac{q}{2-q}}$

and, in particular,

(1.13) $\qquad \vec{v}(t,x)\equiv 0\ ,\quad x\in\Omega\ ,\quad T_f\le t$.

<u>Proof.</u>Først, for the solution w under consideration the foll lowing equality is proved by the energy estimate method [9], [11]-[13]:

$$(1.14)\quad \frac{1}{2}\,\frac{d\Pi}{dt} = (\operatorname{div} P, \vec{v})_\Omega + (\rho\vec{f}, \vec{v})_\Omega = -(F : D_\Omega + (\rho\vec{f}, \vec{v})_\Omega \equiv I$$

$$\Pi(t) = (\rho(t,\cdot)\,\vec{v}(t,\cdot), \ \vec{v}(t,\cdot))_\Omega , \quad (u,v)_\Omega = \int_\Omega u\cdot v\, dx .$$

Formally we derive the latter from (1.1), (1.2) and boundary condition (1.7) multiplying equation (1.3) by $\vec{v}(t,x)$ and subsequently integrating by parts.

Further we use the Korn inequality [26]

$$(1.15)\quad K\cdot \|\vec{v}\|_{m,\Omega} \le \|D(\vec{v})\|_{q,\Omega} , \quad m \le qn/(n-q)$$

at $m=2$, and estimate the right-hand part (1.14) (using (1.15), (1.11) as follows

$$(1.16)\quad \begin{aligned} I &\le -\delta \|D\|^{q}_{q,\Omega} + MK^{-1}\|\vec{f}\|_{2,\Omega}\cdot \|D\|_{q,\Omega} \le \\ &\le \tfrac{1}{2}\left(-a\,\Pi^{q/2} + b\cdot(1-t/T_f)_+^{q/2-q}\right), \end{aligned}$$

$$a = \delta\,(K/\sqrt{M})^{q}, \quad b = (M/q)^{q/q-1}\cdot \frac{q-1}{q}\cdot \left(\frac{\delta q}{2}\right)^{-\frac{1}{q-1}} .$$

Relations (1.14) and (1.16) yield the ordinary differential inequality for the energetic function $\Pi(t)$:

$$(1.17)\quad \frac{d\Pi}{dt} + a\cdot\Pi^{q/2} \le b\,(1-t/T_f)_+^{\frac{q}{2-q}}$$

$$\Pi(0) \le M\cdot C_v$$

All the non-negative solutions of inequality (1.17) are majorized by the function

$$\bar{\Pi}(t) = M\cdot C_v\cdot(1-t/T_f)_+^{\frac{2}{2-q}}$$

if the constants C_v , C_f , T_f , M , K , q , δ satisfy the relation

$$-2M\cdot C_v/(2-q)T_f + a\,(MC_v)^{q/2} \ge b .$$

An analysis of the latter completes the proof of the theorem.

Remark 1. Theorem 1 is interpreted in the physical saase as follows. The flow of non-Newtonian fluid (under conditions (1.5), (1.9) initiated by the initial data and mass forces (the "source") settles to rest $\vec{v}\equiv 0$ after time T_f , i.e. the moment when the "sources" are switched off.

Remark 2. Theorem 1 can be ales formulated as follows. For any constant $C_v \in (0,\infty)$ in (1.10) and small enough C_f in (1.11) there exists the value of $T_f \in (0,\infty)$, such that (1.12), (1.13) are valid. For $\vec{f} \equiv 0$ the analogous assertion is proved in [11].

Remark 3. The constant K in the Korn inequality (1.15) is independent of Ω, if

$$m = 2, \qquad q = 2n/(n-2), \qquad q > 1$$

Therefore, in this case the above assertions are also valid for the Cauchy problem

$$\vec{v}(0,x) = v_0(x), \quad \rho(0,x) = \rho_0(x), \quad x \in R^n.$$

for the system (1.1) - (1.4).

Let us now study local properties of solutions to the system (1.1) - (1.4) regardless the boundary conditions. We further confine ourselves to consideration of solutions in the following particular form:

$$\vec{v}(t,x) = (0,0,w(t,x_1,x_2)), \qquad \rho(t,x) = 1 \tag{1.18}$$

$$\vec{f}(t,x) = (0,0,f(t,x_1,x_2)), \qquad \frac{\partial p}{\partial x_3} = a(t)$$

assuming the pressure gradient to be prescribed. Such a solution may correspond to a flow in a tube. In this case the system of equations (1.1) - (1.4) has the form

$$\frac{\partial w}{\partial t} = div\,\vec{F}(\nabla w) - \frac{\partial p}{\partial x_3} + f \tag{1.19}$$

The vector $\vec{F}(\nabla w)$ is assumed to satisfy the condition

$$\delta \cdot |\nabla w|^q \le \vec{F}(\nabla w)\cdot \nabla w \le \delta^{-1} |\nabla w|^q, \quad 2 < q \tag{1.20}$$

Introduce notation

$$B_\rho(x_0) = \{x : x \in \Omega, \quad |x - x_0| < \rho\}$$

and in the domain $B_{\rho_1} \times (0,T)$ consider the solution of equation (1.19) with the boundary

(1.21) $$w(0,x) = w_0(x) , \quad x \in B_{\rho_1} .$$

In this case it is supposed that

(1.22) $$\left(\| w_0 \|^2_{2,B_\rho} + \int_0^T \| f(\tau,\cdot) \|^2_{2,B_\rho} d\tau \right) \leq C \cdot (\rho - \rho_0)_+^{\frac{1}{1-\alpha}} ,$$

$$\rho \in (0, \rho_1) , \quad 0 < \rho_0 < \rho_1 , \quad \alpha = (3q-2)/(q-1) ,$$

(1.23) $$\left(|a(t)| + \| w(t,\cdot) \|^2_{2,B_{\rho_1}} \right) \leq M .$$

Theorem 2. (Metastable localization). Let $w(t,x) \in V_q$ be the generalized solution of equation (1.19) in $B_{\rho_1} \times (0,T)$ with the initial condition (1.21) and assume the conditions (1.20) - (1.23) to be fulfilled. Then there exists $t_0 = t_0(M, q, \rho_1, \delta) \in (0,T)$, such that

(1.24) $$u(t,x) = \left(w(t,x) + \int_0^t a(\tau) d\tau \right) = 0, \quad x \in B_{\rho_0} , \quad 0 \leq t \leq t_0 .$$

Proof. Let us introduce the energy functions

(1.25) $$\Pi(t,\rho) = (u(t,\cdot), u(t,\cdot))_{B_\rho} , \quad \mathcal{E}(t,\rho) = \sup_{0 \leq \tau \leq t} \Pi(\tau,\rho)$$

$$E(t,\rho) = \int_0^t (F(\nabla u), \nabla u)_{B_\rho} d\tau$$

which possess, as is easily checked up, the following properties

(1.26) $$\frac{\partial E}{\partial \rho} = \int_0^t (F(\nabla u), \nabla u)_{\partial B_\rho} d\tau \geqslant \delta \cdot \int_0^t \| \nabla u \|^q_{q,\partial B_\rho} d\tau$$

$$\frac{\partial E}{\partial t} = (F, \nabla u)_{B_\rho} \geqslant \delta \cdot \| \nabla u \|^q_{q, B_\rho} .$$

For function $u(t,x) = w(t,x) + \int_0^t a(\tau) d\tau$, according to (1.19), we derive the following energy equality

(1.27) $$\frac{1}{2} \left(\Pi(t,\rho) - \Pi(0,\rho) \right) + E(t,\rho) = I_1 + I_2$$

where $I_1 = \int_0^t (\vec{F} \cdot \vec{n}, u)_{\partial B_\rho} d\tau$, $I_2 = \int_0^t (f, u)_{B_\rho} d\tau$

and $\vec{n}$ is the vector of a normal to ∂B_ρ. In accordance with [11]-[13], the summands in the right-hand part of equation (1.27) can be estimated as follows

$$(1.28)\quad |I_1| \le \frac{\varepsilon}{2}\cdot T\cdot \mathscr{B}(t,\rho) + \frac{1}{2\varepsilon}\cdot\int_0^t \|f(\tau,\cdot)\|^2_{2,B_\rho}\, d\tau\ ,\quad \varepsilon>0$$

$$|I_2| \le \frac{1}{\delta}\int_0^t \|\nabla u\|^q_{q,\partial B_\rho}\cdot \|u\|_{q,\partial B_\rho}\, d\tau \le \varepsilon(E+\mathscr{B}) + C\cdot t^{\omega}\left(\frac{\partial E}{\partial\rho}\right)^{1/\alpha}$$

where $C = C(T, M, q, \delta, \varepsilon)$, $\omega = 8/(3q-1)$, $\alpha = 3q-2/4(q-1)$.

The joining of (1.27), (1.28), taking into account (1.22) and choosing $\varepsilon>0$ yields the final inequality

$$(1.29)\quad E \le E+\mathscr{B} \le a\cdot t^{\omega}\cdot\left(\frac{\partial E}{\partial\rho}\right)^{1/\alpha} + b(\rho-\rho_0)_+^{\frac{1}{1-\alpha}}$$

where a, b depend only on M, q, δ, T. According to results from [29], [30] for inequalities of the form (1.29), there exists $t_0>0$, such that

$$E(t,\rho_0) = 0\ ,\quad t\le t\le t_0\ .$$

This completes the proof of Theorem.

<u>Remark 4.</u> Thus, if non-Newtonian fluid (obeying the law (1.20)) is at rest ($w(0,x) = u(0,x) = 0$) in the domain B_{ρ_0} at $t=0$, then independent of the boundary conditions and "souces" beyond B_{ρ_0}, its motion is determined by relation

$$w(t,x) = -\int_0^t a(\tau)\,d\tau\ ,\quad 0\le t\le t_0\ ,\quad x\in B_{\rho_0}$$

In particular, the state of rest is retained ($w(t,x)=0$) at $t\in[0,t_0]$ if there is no pressure drop ($a=0$).

<u>Remark 5.</u> Analogously, the energy method can be used to study the problem (1.1) - (1.4) allowing for the medium temperature change $\theta(t,x)$, if the following equation is added to the system

$$\rho\frac{d\theta}{dt} \equiv \rho\left(\frac{\partial\theta}{\partial t} + (\vec{v}\nabla)\theta\right) = div\,\vec{A}(t,x,\theta,\nabla\theta) + L(t,x,\theta,\vec{v})$$

In this case we can take into account the dependence $F(\vartheta, \theta)$, as well as the nonlinear laws of thermal conductivity and volumetric thermal energy absorption in the following form [11] - [13]

$$\delta_1 |\theta|^{\alpha} \cdot |\nabla\theta|^{p} \leq \vec{A}\,\nabla\theta \leq \frac{1}{\delta_1} \cdot |\theta|^{\alpha} \cdot |\nabla\theta|^{p}, \quad -1<\theta, \quad 1<p,$$

$$L = -\gamma |\theta|^{\sigma-1} \cdot \theta + L_0(t, x, \vec{\vartheta}), \; 0 \leq \gamma, \quad 0<\sigma\leq 1$$

II. Joint Flows of Surface and Ground Waters.

In papers [11], [23] the mathematical models of joint flows of free surfface and ground waters were considered which were based on equations of plane filtration and hydraulics of open channels.

In the simplest case (constant-width rectangular cross-section channel, horizontal ground water bed and channel bottom, etc.) the correponding system of equations and the internal conjugation conditions have the form [23]

(2.1) $$\frac{\partial H}{\partial t} = div(H \nabla H) + f_{\Omega}(t, x), \quad x \in \Omega \setminus \Pi$$

(2.2) $$\frac{\partial u}{\partial t} = \frac{\partial}{\partial s}\left(\psi(s,u) \cdot |u_s|^{-1/2} \frac{\partial u}{\partial s}\right) - \left[H \frac{\partial H}{\partial n}\right]_{\Pi} + f_{\Pi}(t,x), \quad x \in \Pi$$

(2.3) $$H \frac{\partial H}{\partial n}\Big|_{\Pi_{\pm}} = \sigma_{\pm}(u - H_{\pm}), \quad x \in \Pi, \quad 0 < \sigma_{\pm} = const.$$

Here $H(t,x)$ is the ground-water level in the domain $\Omega \subset R^2$; $u(t,s)$- is the water level in the channel, to which the curve Π in Ω corresponds; s is the arc length along Π ; $\vec{n}$ is the vector of a normal to Π ; $H_{\pm}$ is the value of H at approaching to Π from different sides; respectively, $[H]_{\Pi} = H_+ - H_-$; $f_{\Omega}(t,x)$, $f_{\Pi}(t,x)$ are the prescribed external water inflows - the "sources".

At $f_{\Pi} = f_{\Omega} = 0$ in [11], [23] the energy method was used to prove the finite disturbance-propagation velocity for $H(t,x)$, $u(t,x)$

from zero initial data. In this case the presence of metastable localization is proved for solutions (2.1) - (2.3).

Let us now study the local properties of the solution $w=(H(t,x), u(t,x))$ of the system (2.1) - (2.3) within the circle

$$B_{\rho_1}(x_0)=\{x : x\in\Omega,\ |x-x_0|<\rho\},\quad x_0\in\Pi$$

assuming without restriction of generality, that $x_0=0$,

$$\Pi_\rho=\{x : x\in\Omega,\ x=0,\ |x_1|<\rho\},\quad s=x_1,\quad B_\rho^{\pm}=\{x : x\in B_\rho,\ 0\leq x_2\}.$$

For the system (2.1) - (2.3) in [23] it was proved for the principal initial boundary-value problems that there exists the generalized solution $w=(H,u)\in V$, where

$$V=\{(H,u):\ 0\leq(H,u)\leq M,\quad \sqrt{H}\,\nabla H\in L^2(0,T;\ L^2(B_{\rho_1})),$$
$$\psi^{2/3}\cdot|u_s|\in L^{3/2}(0,T;\ L^{3/2}(\Pi_{\rho_1}))\},\quad (|\ln(\psi u^{-5/3})|\leq M).$$

Theorem 3. (Metastable localization). Let $w=(H,u)\in V$ be the generalized solution of the system (2.1) - (2.3) in $B_{\rho_1}\times(0,T)$ and

$$(2.4)\quad (\|H(0;\cdot)\|^2_{2,B^{\pm}_\rho}+\|u(0,\cdot)\|^2_{2,\Pi_\rho}+\int_0^T(\|f_\Omega\|^2_{2,B^{\pm}_\rho}+\|f_\Pi\|^2_{2,\Pi_\rho})\,d\tau)\leq C(\rho-\rho_0)_+^{1/1-\alpha}$$

$$\rho\in(0,\rho_1),\quad 0<\rho_0<\rho_1,\quad \alpha=5/6$$

Then there exists $t_0=t_0(M,C,\rho_1,T)$, such that

$$w=(H,u)=0,\qquad x\in B_{\rho_0},\quad 0\leq t\leq t_0$$

Proof. Introduce the notations

$$\Pi(t,\rho)=(\|H(t,\cdot)\|^2_{2,B_\rho}+\|u(t,\cdot)\|^2_{2,\Pi_\rho}),\quad b(t,\rho)=\sup_{0\leq\tau\leq t}\Pi(\tau,\rho)$$

$$E(t,\rho)=\int_0^t((H\nabla H,\nabla H)_{B_\rho}+(\psi,|u_s|^{3/2})_{\Pi_{\rho_t}})\,d\tau$$

$$\mathcal{D}^2=\int_0^t\sum_{\pm}(\sigma_\pm,(u-H_\pm)^2)_{\Pi_\rho}\,d\tau\ ;\quad F=\int_0^t((f_\Omega,H)_{B_\rho}+(f_\Pi,u)_{\Pi_\rho})\,d\tau.$$

Then the energy equality, corresponding to system (2.1) - (2.3), can be written as follows

$$(2.5)\quad \frac{1}{2}\left(\Pi(t,\rho)-\Pi(0,\rho)\right)+E(t,\rho)+\mathcal{D}^2=$$

$$=F+\int_0^t\left(\left(H,\frac{\partial H}{\partial n},H\right)_{\partial B_\rho}+\psi\,|u_s|^{-1/2}\cdot u_s\cdot u\Big|_{x_1=-\rho}^{x_1=\rho}\right)d\tau\ .$$

Obtaining the estimates analogous to (1.28), we are led to the inequality of the form (1.29), the analysis of which completes the proof of the Theorem 3.

Remark 6. Theorem 3 is interpreted in the physical sense as follows. The domain B_{ρ_0} not filled in with water at the initial time moment $t=0$, remains empty at $t\le t_0$ independently of the boundary conditions and the sources beyond B_{ρ_0}.

III. Two-Phase Filtration of Non-mixing Incompressible Fluids.

Nonstationary filtration of two immissible incompressible fluids in nonhomogeneous anisotropic porous medium is described by the following system of mixed-type equations [22]

$$(3.1)\quad m(x)\frac{\partial s}{\partial t}=\operatorname{div}\left(K_0(x)\,a(s)\nabla s-b(s)\vec{v}+\vec{F}(x,s)\right),$$

$$(3.2)\quad \operatorname{div}\vec{v}=0,\quad \vec{v}=-\left(K(x,s)\nabla p+\vec{f}(x,s)\right),\quad x\in\Omega\subset R^n.$$

Here the unknown functions are saturation $s(t,x)$, ($0\le s\le 1$), reduced pressure $p(t,x)$, mixture velocity $\vec{v}$. The coefficients of system (3.1), (3.2) are defined by formulas

$$(3.3)\quad a(s)=K_{01}\cdot K_{02}\left|\frac{\partial p_K}{\partial s}\right|/(K_{01}+K_{02}),\quad b=K_{01}/K$$

$$\vec{F}=-\frac{K_{01}\cdot K_{02}}{K}\cdot K_0\left(\nabla_x P_K+(\rho_2-\rho_1)\vec{g}\right),\quad K=K_{01}+K_{02}\quad ,$$

where p_K is the capillary pressure; $\overline{K}_{0i}=\mu_i K_{0i}$ is the relative phase penetration; μ_i, ρ_i are viscosity and density of fluids, respectively; $\vec{g}$ is the gravitational acceleration; K_0 is the symmetrical filtration tensor for the homogeneous fluid; m is the porosity. Depending on the form of the functional model parameters K_{0i}, P_K, the coefficient $a(s)$ in (3.1) can either vanish or tend to infinity at $s = 0, 1$, thereby determining the different character of disturbance propagation of saturation $s(t,x)$. In [22] for system (3.1), (3.2) the existence theorem was proved for the generalized solution $w = (s, p) \in V$,

$$V = \{(s,p): 0 \leq s \leq 1, \ \sqrt{a}\,\nabla s \in L^2(0,T; L^2(\Omega)), \ \nabla p \in L^\infty(0,T; L^q(\Omega))\}_{n<q\leq\infty} .$$

In [10], [11] there were established the finite time of localization $s(t,x)$ ($a(0)=\infty$, $s=0$) for the boundary-value problem and the finite velocity of initial data disturbance propagation ($s(0,x)=0$, or $s(0,x)=1$, $a(0)=a(1)=0$). It is shown in the sequel that under an additional restriction on the initial data, $s(0,x) = s_0(x)$, the solution under consideration possesses also the property of metastable localization (localization with inertia) at $a(0)=0$. Consider now the system (3.1), (3.2) in the domain $B_{\rho_1} \times (0,T)$, assuming the following conditions to hold:

$$(3.4) \quad M^{-1} \leq \left(m; \ \frac{(K_0\vec{\xi}, \vec{\xi})}{(\vec{\xi}, \vec{\xi})}; \ K, \ \left|\frac{\partial P_K}{\partial s}\right| \right) \leq M, \quad 0 \leq s \leq 1$$

$$(3.5) \left(|\ln(a s^{-\alpha})|; \ |\vec{F}_s'|\cdot s^{1-\alpha}; \ |div_x \vec{F}|\cdot s^{\alpha}; \ |b_s'|\cdot s^{-\frac{\alpha+\nu}{2}} \right) \leq M$$

$$(3.6) \quad \|s(0,x)\|^2_{2,B_\rho} \leq C\cdot(\rho-\rho_0)_+^{\frac{1}{1-\alpha}}, \quad 2+\frac{\alpha n}{q} \leq \nu + 2 \leq \alpha .$$

<u>Theorem 4.</u> (Metastable localization). Let $w = (s, p) \in V$ be the generalized solution of system (3.1), (3.2) and assume conditions (3.4), (3.5) to be staisfied. There exists $t_0 = t_0(M, C, \alpha, q, \rho_1) > 0$, such

that

$$s(t,x)=0\,,\quad x\in B_{\rho_0}\,,\quad 0\le t\le t_0\quad .$$

Proof. Introduce the energy functions

$$\Pi(t,\rho)=(m\,s(t,\cdot),\,s(t,\cdot))_{B_\rho}\ ;\quad E(t,\rho)=\int_0^t(K_0\,a\,\nabla s,\nabla s)_{B_\rho}\,d\tau$$

and utilize equation (3.1) in the following form

$$(3.7)\quad m\,\frac{\partial s}{\partial t}=div\,(K_0\,a\,\nabla s)-(\theta_s'\cdot\vec v+\vec F_s'\,)\,\nabla s+div_x\vec F\ .$$

The energy equality

$$\frac12\,(\Pi(t,\rho)-\Pi(0,\rho))+E(t,\rho)=\int_0^t((\theta_s'\cdot\vec v+\vec F_s')\nabla s,\,s\,)_{B_\rho}+$$

$$+\ (div_x\,\vec F,\,s\,)_{B_\rho}\ +\ (K_0\,a\,\nabla s\,\vec n,\,s\,)_{\partial B_\rho}\ )\,d\tau$$

that corresponds to (3.7), leads, as in the prooves of Theorems 2,3, to the inequality of the form (1.29), the study of which completes the proof of Theorem.

Remark 7. An analogous assertion is also valid for the function $\bar s\,(t,x)=1-s(t,x)$.

Remark 8. In the domain $B_{\rho_0}\times(0,t_0)$ the reduced pressure satisfies the elliptic equation

$$div\,(K(x,0)\,\nabla p\ +\vec f\,(x,0))=0\,.$$

Remark 9. Theorem 4 admits the following physical interpretation. Let the domain B_{ρ_0} be occupied completely with one of the fluids at time $t=0$ ($s(0,x)=0$ or $s(0,x)=1$). Whatever the influences outside B_{ρ_0} may be, the displacement of this fluid out of B_{ρ_0} would not start before time $t_0>0$.

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