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# SPACE AND TIME LOCALIZATION IN THE FLOW OF TWO IMMISCIBLE FLUIDS THROUGH A POROUS MEDIUM: ENERGY METHODS APPLIED TO SYSTEMS

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## 1. INTRODUCTION

SINCE the beginnings of the 1980s some energy methods have been introduced as an alternative to comparison principles in order to prove space and time localization of solutions of suitable nonlinear parabolic or elliptic equations (see, for instance, the survey [4]). The study of nonhomogeneous equations (i.e. with prescribed right-hand terms) was considered by the authors in [5 and 7] proving new results on the retention of the free boundary (separating the region of the domain where the solution vanishes). In this work we wish to explain how to extend the results obtained in [5, 7] for scalar equations to the case of systems of equations (even of different types).

As an illustration of the applicability of the methods to the case of systems of equations we shall consider the model of the flow of two immiscible fluids through a porous medium. For the application of the methods to other nonlinear systems arising in continuum mechanics we refer the reader to [6 and 8].

The mathematical theory of the simultaneous motion of two immiscible incompressible liquids has been shown to be of a great interest in many different problems as, for instance, in oil reservoirs (a porous medium $\Omega$ of $R^N$, $N \leq 3$, whose pores contain some hydrocarbon component or "oil"). In the "secondary recovery" a second inexpansive fluid (water) is injected into the porous medium in order to push the oil towards the production wells. During the time in which the pressure is above the bubble pressure of the oil, the flow in the reservoir is of a two-phase immiscible type.

A formulation of the problem is given for the following system of equations for the unknowns $(s, p)$

$$\phi(x)\frac{\partial s}{\partial t} - \operatorname{div}(\mathbb{K}_0(x)a(s)\nabla s) = \operatorname{div}(\mathbb{K}_0(x)b(s)\nabla p) + f(x, t) \tag{1}$$

$$\operatorname{div}(\mathbb{K}_0(x)d(s)\nabla p) = 0. \tag{2}$$

Such a system is obtained from the basic equations

$$\mathbf{v}_i = -\mathbb{K}_0(x)\mu_i k_i(s)\nabla(p_i + \rho_i g h) \qquad \text{(Dárcy–Muskat law)} \tag{3}$$

$$\frac{\partial(\phi s_i \rho_i)}{\partial t} + \operatorname{div} \rho_i \mathbf{v}_i = f_i(x, t) \qquad \text{(continuity equation)} \tag{4}$$

$$p_2 - p_1 = p_c(x, s_1) \qquad \text{(Laplace law)} \tag{5}$$

where $\mathbf{v}_i$, $\rho_i$, $p_i$ and $s_i$ are the velocity, density, pressure and saturation of the phase $i$, $i = 1, 2$, $(s_1 + s_2 = 1)$, $\phi(x) > 0$ is the porosity of the medium $\mathbb{K}_0(x)$ is the matrix of absolute permeability, $\mu_i$ is the viscosity, $k_i(s_i) \geq 0$ ($k_i(0) = 0$, $k_i(s_i)$ increasing in $s_i$) is the relative permeability, $g$ is the acceleration due to gravity, $h$ is the distance to a fixed horizontal reference plane, $f_i$ is the injected mass of the phase $i$, and $p_c$ is the capillary pressure (increasing in $s_1$ and $p_c(x, 1) = 0$). Introducing $s = s_1$ and the "reduced pressure" $p$ (see [9, 12]) equations (3)–(5) lead to systems such as (1) and (2) where, for simplicity, we have assumed that the porous medium is homogeneous and planar (i.e. the unknowns do not depend on the vertical variable $z$). The qualitative assumptions on the structure of data in (1) and (2) are now the following: there exists some positive constants $C_j$, $j = 1, 2, 3$ and some exponents $\alpha$ and $\beta$ such that

$$C_1 \leq \phi(x) \leq \tilde{C}_1 \tag{6}$$

$$C_2^{-1}|\xi|^2 \leq (\mathbb{K}_0(x)\xi) \cdot \xi \leq C_2|\xi|^2, \qquad \forall\, \xi \in R^N - \{0\}, \qquad \text{a.e. } x \in \Omega. \tag{7}$$

$$C_3^{-1} s^\alpha (1 - s)^\beta \leq a(s) \leq C_3 s^\alpha (1 - s)^\beta, \qquad s \in (0, 1) \qquad \text{for some } \alpha, \beta \in (-1, \infty), \tag{8}$$

$b$ and $d$ are Lipschitz functions and

$$d(s) \geq C_4. \tag{9}$$

We remark that

$$a(s) = \bar{k}_1(s)\bar{k}_2(1 - s)\left|\frac{\partial p_c}{\partial s}\right| \Big/ (\bar{k}_1(s) + \bar{k}_2(s)), \qquad \bar{k}_i(s) = \mu_i k_i(s) \tag{10}$$

and so the exponents $\alpha$ and $\beta$ in (8) can take positive and negative values leading to very different types of behaviour for $s(x, t)$. In particular we shall show two different localization properties for $s(x, t)$:

(i) localization in time (or extinction in finite time) and

(ii) localization in space (or finite speed of propagation).

Property (i) occurs in the case of fast diffusion $\alpha \in (-1, 0)$ in contrast with property (ii) which is typical of a slow diffusion, $\alpha > 0$. We can state, more precisely, those properties in the terms below.

*Definition* 1. Let $\Omega$ be a bounded regular open set of $R^N$. We say that there is *localization in time* of $s(t, x)$ if given a datum $f(t, x)$ such that $f(t, \cdot) \equiv 0$ on $\Omega$ for $t \geq T_F$ there exists a $T_s > 0$ such that $s(t, \cdot) \equiv 0$ on $\Omega$ for $t \geq T_s$.

*Definition* 2. Given $x_0 \in \Omega$ and a ball $B_\rho(x_0) = \{x \in R^N : |x - x_0| < \rho\}$, we say that there is *localization in space* of $s(t, x)$ if there exists a $t_0 > 0$ and a function $\rho(t)$ such that $0 \leq \rho(t) \leq \rho_0$ and $s(t, x) = 0$ for any $t \in [0, t_0]$ and a.e. $x \in B_{\rho(t)}(x_0)$ assuming that $s(0, x) = 0$ for a.e. $x \in B_{\rho_0}(x_0)$ (here the source term $f(t, x)$ is assumed to be compatible with the conclusion on $s$).

We remark that property (i) is global ($s(t, x)$ vanishes on the whole $\Omega$ if $t \geq T_s$) in contrast with (ii) which has a local character ($s(t, x)$ merely vanishes on $B_{\rho(t)}(x_0)$ if $t \leq t_0$). Then the region where $s(t, x)$ vanishes represents the region occupied (purely) by only one of the fluids. Similar studies can be carried out for the region where $s(t, x) \equiv 1$ which represents the region occupied (purely) only by the other fluid. We also remark that it is known experimentally that $s(t, x) \in [s_m, S_M]$ with $0 < s_m < S_M < 1$ ($s_m$, and $S_M$ are called the residual saturations). However, in this case assumption (8) must be replaced by

$$C_3^{-1}(s - s_m)^\alpha(S_M - s)^\beta \leq a(s) \leq C_3(s - s_m)^\alpha(S_M - s)^\beta, \qquad s \in (s_m, S_M)$$

and so our study is also valid for the region where $s = s_m$ (as well as for the region where $s = S_M$).

## 2. LOCALIZATION IN TIME

### 2.1. *General remarks on the method*

Before considering the special case of two immiscible fluids we shall make an "informal" outline of the application of the energy method to systems of equations. Consider the initial boundary-value problem given by a nonlinear system of p.d.e. equations

$$\mathbb{B}(t, x, \mathbf{w}, \mathbf{w}_t) = \mathbb{A}(t, x, \mathbf{w}, D\mathbf{w}, D^2\mathbf{w}) + \mathbf{f}(t, x) \qquad \text{on } Q_T \equiv (0, T)x\Omega \tag{11}$$

for the unknown $\mathbf{w}$ which we assume to be written in the form

$$\mathbf{w} = (\mathbf{w}^{(1)}, \mathbf{w}^{(2)}) = (w_1^{(1)}, \ldots, w_m^{(1)}, w_1^{(2)}, \ldots, w_q^{(2)}) \tag{12}$$

($\mathbf{w}^{(1)}$ are the unknown variables for which we want to prove localization in time), $\mathbb{B} = (\mathbb{B}^{(1)}, \mathbb{B}^{(2)})$ and $\mathbb{A} = (\mathbb{A}^{(1)}, \mathbb{A}^{(2)})$ are vector valued operators satisfying some assumptions that we shall state later, and $\mathbf{f} = (\mathbf{f}^{(1)}, \mathbf{f}^{(2)})$ is a given vector valued function satisfying

$$\exists\, T_f \in [0, T] \qquad \text{such that } \mathbf{f}^{(1)}(t, \cdot) \equiv 0 \qquad \text{on } \Omega \qquad \text{for } t \leq T_f. \tag{13}$$

We assume a boundary condition

$$\mathbb{C}(t, x, \mathbf{w}|_{\Sigma_\infty}, D\mathbf{w}|_{\Sigma_\infty} \cdot n) = 0 \qquad \text{on } \Sigma_\infty = (0, \infty)x\partial\Omega \tag{14}$$

(here $\mathbb{C}$ may represent a mixed type of boundary operator, i.e. Dirichlet type on a part $\Sigma_D$, Neumann type on other region $\Sigma_N$, etc.). Finally an initial condition is also known

$$\mathbf{w}(0, x) = \mathbf{w}_0(x) \qquad \text{on } \Omega. \tag{15}$$

The main steps in the application of energy methods are given below.

*First step*. Multiply the first $m$-components of (11) by $\mathbf{w}^{(1)}$ (or a monotone function of $\mathbf{w}^{(1)}$), and assume on $\mathbb{B}$ and $\mathbb{A}$ a nonlinear structure such that the global integration by parts (in space) leads to the inequalities

$$\int_\Omega \mathbb{B}^{(1)}(t, x, \mathbf{w}, \mathbf{w}_t) \cdot \mathbf{w}^{(1)} \geq C\frac{\mathrm{d}}{\mathrm{d}t}y(t) \tag{16}$$

$$\int_\Omega \mathbb{A}^{(1)}(t, x, \mathbf{w}, D\mathbf{w}, D^2\mathbf{w}) \cdot \mathbf{w}^{(1)} \leq -\Phi(\|\mathbf{w}^{(1)}\|^{p_2}_{L^{p_1}(\Omega)}) + F((T_f - t)_+). \tag{17}$$

Where $y(t)$ is defined by

$$y(t) = \int_\Omega \Lambda(w^{(1)}(t, x))\, \mathrm{d}x$$

for some suitable function $\Lambda$ leading to the inequality

$$y(t) \geq M \| w^{(1)}(t, \cdot) \|^{p_3}_{L^{p_1}(\Omega)}.$$

Here $p_1 \geq 1$ and $C, M, p_2$ and $p_3$ are positive numbers obtained from the structural assumption on $\mathbf{B}^{(1)}$, $\Phi$ is a real increasing function (also obtained from the hypotheses on the operators $\mathbf{A}$ and $\mathbf{C}$) and $F$ is a real continuous function determined by the assumptions on $\mathbf{f}^{(1)}$. We remark that to obtain (16) the rest of equations for $\mathbf{w}^{(2)}$ must be used as well as some interpolation inequalities.

*Second step.* The conclusion comes from the following result about ordinary differential inequalities (we take, for simplicity $C = 1$ in (16)).

LEMMA 1 [5, 7]. Let $y(t) \geq 0$ be such that

$$y'(t) + \Phi(y(t)) \leq F((T_f - t)_+) \qquad \text{a.e. on } (t_1, T) \tag{18}$$

where $\Phi$ is a nondecreasing function such that $\Phi(0) = 0$ and

$$\Phi(\cdot)^{-1} \in L^1(0, 1), \qquad \text{i.e. } \int_{0^+} \frac{\mathrm{d}s}{\Phi(s)} < \infty. \tag{19}$$

For any $\mu > 0$ and $\tau > 0$ we define

$$\Theta_\mu(\tau) = \int_0^\tau \frac{\mathrm{d}s}{\mu\Phi(s)} \tag{20}$$

and $\eta_\mu(s) = \Theta_\mu^{-1}(s)$. Assume

$$\exists\, \bar{\mu} < 1 \qquad \text{such that } F(s) \leq (1 - \bar{\mu})\Phi(\eta_{\bar{\mu}}(s)) \tag{21}$$

and

$$(T_f - t_1) \geq \Theta_{\bar{\mu}}(y(t_1)). \tag{22}$$

Then $y(t) \equiv 0$ for any $t \in [T_f, T]$.

We remark that if, in fact, $F \equiv 0$ on $(t_1, T)$ (for instance, if in assumption (13) we replace $T_f$ by a smaller time $t_1$) then (21) is trivially satisfied and then $T_f (> t_1)$ is determined by condition (22). This is merely the property of extinction in a finite time of $\mathbf{w}^{(1)}$. However, if $T_f$ is the first time in which $\mathbf{f}^{(1)}$ satisfies (13) and the decay assumption (21), $F$ given in (17), then the conclusion can be interpreted as the instantaneous extinction of the solution (the component $\mathbf{w}^{(1)}$ of the solution vanishes from the same time in which $\mathbf{f}^{(1)}$ vanishes identically on $\Omega$).

The application of this method to other systems can be found in [6 or 8]. For instance, this is the case of the following formulation arising in the theory of incompressible nonhomogeneous non-Newtonian fluids

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = 0, \qquad \mathbf{v} = (v_1, \ldots, v_N),$$

$$\operatorname{div} \mathbf{v} = 0$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right) = \operatorname{div} \mathbb{P} + \rho \mathbf{f}$$

$$\mathbb{P} = -p\mathbb{I} + \mathbb{F}(\mathbb{D}), \qquad \mathbb{D} = \left\{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\}$$

$$c|\mathbb{D}|^q \leq \mathbb{F}(\mathbb{D})\mathbb{D}, \qquad c > 0, \qquad 1 < q < 2.$$

Extinction properties for $\mathbf{v}$ can be found in [8].

### 2.2. *Localization in time for two-immiscible fluids in a porous medium*

We now consider the system (1) and (2) on a bounded regular open set $\Omega \subset R^N$ representing the porous medium ($N \leq 3$ in the applications). One way to take into account the influence of the injection and production wells is to consider a partition of $\partial\Omega$ as $\partial\Omega = \Gamma_1 \cup \Gamma_e \cup \Gamma_s$ corresponding to the impermeable boundary, the injection and production boundaries. As boundary conditions we shall assume the following:

*In the impermeable boundary* $(0, T)x\Gamma_1$

$$\mathbb{K}_0(x)a(s)\nabla s \cdot \mathbf{n} = 0 \tag{23}$$

$$d(s)\nabla p \cdot \mathbf{n} = 0 \tag{24}$$

*In the injection wells* $(0, T)x\Gamma_e$

$$s = 0 \tag{25}$$

$$\mathbb{K}_0(x)\mathrm{d}(s)\nabla p \cdot \mathbf{n} = -g(t, x) \tag{26}$$

with $g \in C^0([0, T] : L^\infty(\Gamma_e))$, $g \geq 0$.

*In the production wells* $(0, T)x\Gamma_s$

$$\left.\begin{aligned} s \geq 0, \qquad \mathbb{K}_0(x)a(s)\nabla s \cdot \mathbf{n} + \mathbb{K}_0(x)b(s)\nabla p \cdot \mathbf{n} \geq 0 \\ s(\mathbb{K}_0(x)a(s)\nabla s \cdot \mathbf{n} + \mathbb{K}_0(x)b(s)\nabla p \cdot \mathbf{n}) = 0 \end{aligned}\right\} \tag{27}$$

$$\mathbb{K}_0(x)d(s)\nabla p \cdot \mathbf{n} = \mu(p - h(t, x)) \tag{28}$$

with $h \in C^0([0, T] : L^2(\Gamma_s))$. Boundary condition (27) is an unilateral condition corresponding to the fact that the water flow can only be directed outward from $\Omega$ on $\Gamma_1$ (see [12, Section III.V] and [14]).

It is well known that, in general, there is no classical solution of problem (1), (2) and (23)–(28) and so suitable notions of weak solutions must be introduced. In order to make explicit the notion of solution that we shall use in the following, some auxiliary functions are introduced

$$A(r) = \int_0^r a(\tau)\,\mathrm{d}\tau \tag{29}$$

$$\Psi(r) = A^{-1}(r) \tag{30}$$

$$\tilde{b}(r) = b(A^{-1}(r)), \qquad \tilde{d}(r) = d(A^{-1}(r)). \tag{31}$$

Defining

$$u(t, x) = A(s(t, x)) \tag{32}$$

equation (1) and boundary conditions (23), (25) and (27) are formulated in a strong way by the following terms

$$\phi(x)\frac{\partial \Psi(u)}{\partial t} - \operatorname{div}(\mathbb{K}_0(x)\nabla u) = \operatorname{div}(\mathbb{K}_0(x)\tilde{b}(u)\nabla p) + f(x, t) \tag{33}$$

$$\mathbb{K}_0(x)\nabla u \cdot \mathbb{n} \qquad \text{in } (0, T)x\Gamma_1 \tag{34}$$

$$u = 0 \qquad \text{in } (0, T)x\Gamma_e \tag{35}$$

$$\left.\begin{aligned} u \geq 0, \qquad \mathbb{K}_0\nabla u \cdot \mathbb{n} + \mathbb{K}_0\tilde{b}(u)\nabla p \cdot \mathbb{n} \geq 0 \\ u(\mathbb{K}_0\nabla u \cdot \mathbb{n} + \mathbb{K}_0\tilde{b}(u)\nabla p \cdot \mathbb{n}) = 0 \end{aligned}\right\} \quad \text{in } (0, T)x\Gamma_s. \tag{36}$$

The variational formulation is the following: let

$$V = \{v \in H^1(\Omega);\ v|_{\Gamma_e} = 0\}$$

$$K = \{v \in V : v|_{\Gamma_s} \geq 0\}$$

then $(u, p)$ is a weak solution of (33), (2), (34), (24), (35), (26), (36), (28) and the initial condition

$$u(0, x) = u_0(x), \qquad (0 \leq u_0(x) \leq A(1)) \qquad \text{a.e. on } \Omega \tag{37}$$

if $u \in L^2(0, T : V)$, $p \in L^2(0, T : H^1(\Omega))$, $\Psi(u) \in L^\infty(0, T : L^1(\Omega))$, $\Psi(u)_t \in L^2(0, T : V')$ ($V'$ dual of $V$) and the following inequalities are satisfied a.e. $t \in (0, T)$

$$\langle \phi\Psi(u)_t, v - u\rangle_{V'V} + \int_\Omega \mathbb{K}_0(x)\nabla u \cdot \nabla(v - u)\,\mathrm{d}x$$

$$\geq -\int_\Omega \mathbb{K}_0(x)\tilde{b}(u)\nabla p \cdot \nabla(v - u) + \langle f, v - u\rangle_{V'V}$$

for any $v \in L^2(0, T : V)$, $v(t) \in K$, a.e. $t \in (0, T)$, and

$$\int_\Omega \tilde{d}(u)\nabla p . \nabla z + \int_{\Gamma_s} \lambda p z\,\mathrm{d}\Gamma = \int_{\Gamma_e} g z\,\mathrm{d}\Gamma + \int_{\Gamma_s} \mu(p - h)\,\mathrm{d}\Gamma$$

for any $z \in H^1(\Omega)$. The existence of weak solutions has been stabilised under different kinds of assumptions (see, for instance, [15, 9, 12, 14, 1]).

We shall prove first the localization in time of $s(t, x)$. To do that we shall use some extra regularity on $p$ which has been proved in [9] under suitable extra conditions. The well-known additional information $0 \leq u \leq A(1)$ will be used too.

THEOREM 1. Assume that

$$C^{-1}r^m \le \Psi(r) \le Cr^m \qquad \text{for some } C \ge 1, m > 1 \qquad \text{and any } r \in [0, A(1)), \tag{38}$$

$$|\tilde{b}(r)| \le Cr^{\omega} \qquad \text{for some } \omega > 0 \qquad \text{and any } r \in [0, A(1)). \tag{39}$$

Let $(u, p)$ be a weak solution such that

$$\|\nabla p(t, \cdot)\|_{L^q(\Omega)} \le M \qquad \text{for some } q > 2 \qquad \text{and for any } t \ge 0. \tag{40}$$

Let $m > 1$ and let $u_0 \in L^{m+k}(\Omega)$ with $k = \max\{1, N/2(m-1)\}$ if $m < (2N/\mathrm{N} - 2) - 1$ and $k = 1$ otherwise, $u_0$ satisfying (37). Also assume that

$$q \ge \frac{2(m+k)}{m-1} \qquad \text{and} \qquad \omega \ge \frac{m+k}{2}\left(\frac{q-2}{q}\right).$$

Let $f \in L^2(0, \infty : H^{-1}(\Omega))$ such that $f(t, \cdot) = 0$ for $t \ge T_f$. Then there exists $T_u \ge T_f$ such that $u(t, \cdot) \equiv 0$ on $\Omega$ for any $t \ge T_u$, assuming that $M$ is small enough.

*Remark*. It is not difficult to show that condition (38) is implied by (8) with $m = 1/(\alpha + 1)$. So the crucial assumption $m > 1$ corresponds to $\alpha < 0$.

*Proof*. We follow the general method explained in Section 2.1. The equivalent idea to multiply the equation by $u$ (or more generally by $u^k$ with $k$ integer to be suitably choosen) is to take $v = 0$ (or $v = u - A(1)^{1-k}u^k$). Then $v \ge 0$ and so is a test function (if $k \ne 1$, we argue by regularizing and passing to the limit). Formula (16) is obtained in the following way

$$-\langle \phi\Psi(u)_t, v - u\rangle_{V'V} = \langle \phi\Psi(u)_t, u\rangle = \phi \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Lambda(u(t, x))\,\mathrm{d}x$$

where

$$\Lambda(r) = \int_0^r (\Psi(r) - \Psi(\tau))\,\mathrm{d}\tau \qquad \left(= \int_0^r \Psi'(\tau)\tau\,\mathrm{d}\tau \qquad \text{if } \Psi' \text{ exists}\right).$$

This integration by parts for weak solutions can be made rigorous (see lemma 1.5 of [2] or lemma 2.1 of [10]). Now we define

$$y(t) = \int_\Omega \Lambda(u(t, x))\,\mathrm{d}x.$$

From assumption (38) we deduce that

$$y(t) \le C\|u(t, \cdot)\|^{m+1}_{L^{m+1}(\Omega)} \tag{41}$$

for some suitable constant $C > 0$ (if $k \ne 1$ we must replace $m + 1$ by $m + k$ in (41)). Obtaining formula (17) is the crucial point of the proof. From the condition of weak solution and the choice $v = 0$ we see that the expression of the right-hand side of (17) is now equivalent to the terms

$$-\int_0^t\int_\Omega (\mathbb{K}_0(x)\nabla u)\nabla u + \int_0^t\int_\Omega \mathbb{K}_0(x)\tilde{b}(u)\nabla p\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}\tau = I_2 + I_2.$$

Using the assumption (7) and Sobolev's inequality we deduce that if

$$m + 1 \leq \frac{2N}{N-2}, \qquad \text{if } N > 2 \tag{42}$$

then

$$I_1 \leq -C^* \|u(t, \cdot)\|^2_{L^{m+1}(\Omega)} \tag{43}$$

for some constant $C^* > 0$. On the other hand using (7), Holder and Young inequalities, we deduce that

$$|I_2| \leq \left(\int_\Omega C\tilde{b}(u)^2 |\nabla p|^2\right)^{1/2} \left(\int_\Omega |\nabla u|^2\right)^{1/2} \leq C\left(\frac{1}{2\varepsilon}\int_\Omega \tilde{b}(u)^2|\nabla p|^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2\right) = I_3 + I_4.$$

Moreover, using assumptions (39), (40) and (41) we deduce, by Holder, that

$$I_4 \leq \frac{C}{2\varepsilon}\left(\int_\Omega |\nabla p|^q\right)^{2/q}\left(\int_\Omega \tilde{b}(u)^{2q/(q-2)}\right)^{(q-2)/q} \leq \frac{C}{2\varepsilon} M^{2/q} \|u(t, \cdot)\|^{(m+1)(q-2)/q}_{L^{m+1}(\Omega)}$$

$$\leq \frac{C^{**}}{\varepsilon} \|u(t, \cdot)\|^2_{L^{m+1}(\Omega)}.$$

In conclusion we have

$$I_1 + I_2 \leq \left(-C^* + \frac{C^{**}}{\varepsilon} + \frac{C\varepsilon}{2}\right)\|u(t, \cdot)\|^2_{L^{m+1}(\Omega)}$$

and then we have the inequality (17) if $C^{**}/\varepsilon + C\varepsilon/2 < C^*$ (which is easily seen implied if we optimizate $\varepsilon > 0$ and choose $M$ small enough) for a function $\Phi$ given by $\Phi(r) = Cr^{2/(m+1)}$ with C a constant. Condition (19) is satisfied and the conclusion follows.

If $m$ is big and (42) is not satisfied then we take $v = u - A(1)^{1-k}u^k$ in the definition of weak solution and modify the definition of $y(t)$ in an obvious way. In this case the equivalent terms to the right-hand side of (17) are (up to the constant $kA(1)^{1-k}$)

$$-\int_0^t (\mathbb{K}_0(x)\nabla u) \cdot \nabla u u^{k-1} + \int_0^t \int_\Omega \mathbb{K}_0(x)\tilde{b}(u)\nabla p \cdot \nabla u u^{k-1} \, dx \, dt = \tilde{I}_1 + \tilde{I}_2.$$

Applying Sobolev inequality to $w = u^{(k+1)/2}$ we deduce (using (7)) that

$$\tilde{I}_1 \leq -\tilde{C}^* \|u(t, \cdot)\|^{k+1}_{L^\lambda(\Omega)}$$

for any $\lambda \geq 1$ such that $\lambda \leq N(k+1)/N - 2$ if $N > 2$ and $\lambda$ arbitrary if $N \leq 2$. Then using the Holder inequality we conclude that there exists $\tilde{C} = \tilde{C}(|\Omega|, m, k)$ such that

$$\tilde{I}_1 \leq -\tilde{C}^*\tilde{C}\|u(t, \cdot)\|^{k+1}_{L^{m+k}(\Omega)}$$

if $\lambda \geq m + k$ and so if $k \geq N/2(m-1)$. Arguing as before we deduce that

$$\tilde{I}_1 + \tilde{I}_2 \leq \left(-\tilde{C}^*\tilde{C} + \frac{\tilde{C}\varepsilon}{\varepsilon} + \frac{\tilde{C}\varepsilon}{2}\right)\|u(t, \cdot)\|^{k+1}_{L^{m+k}(\Omega)}$$

and again we have the inequality (17), if $M$ is small enough, for a function $\Phi(r) = Cr^{(k+1)/(m+k)}$ where C is a constant and $F \equiv 0$. ∎

Concerning the instantaneous extinction of the solution we have the following theorem.

THEOREM 2. Assume the same hypothesis on $\Psi$, $\tilde{b}$, $p$ and $u_0$. Let $f \in L^2(0, \infty : L^{(m+k)/m}(\Omega))$ such that $f$ vanishes on $\Omega$ after a finite time $T_f$ and that there exists $t_1 \in [0, T_f)$ such that

$$\|f(\cdot, t)\|^{k+1}_{L^{(m+k)/k}(\Omega)} \leq C^1(T_f - t)_+^{\alpha/(1-\alpha)}, \qquad \text{for a.e. } t \in (t_1, +\infty) \tag{44}$$

where $h_+ = \max(h, 0)$, $\alpha = (k+1)/(m+k)$, and $C^1$ is a small constant. Then there exists a constant $C^2$ (depending on $C^1$, $\|u_0\|_{m+k}$ and $\alpha$) such that if

$$\int_\Omega |u(t_1, x)|^{m+k}\, dx \leq C^2(T_f - t_1)_+^\alpha \tag{45}$$

then $u(t, \cdot)$ vanishes on $\Omega$ for any $t \geq T_f$.

*Proof.* It comes from lemma 1 once we make explicit the function $F$ in inequality (17) (and so in (18)). Following the proof of theorem 1 we take $v = u - A(1)^{1-k}u^k$ in the condition for $u$ to be a weak solution. Then by the Holder and Young inequalities

$$\begin{aligned}
\langle f(t, \cdot), v - u\rangle_{V'V} &= -A(1)^{1-k}\int_\Omega f(t, x)u^k(t, x)\, dx \\
&\geq -A(1)^{1-k}\left(\int_\Omega f(t, x)^{(m+k)/m}\, dx\right)^{m/(m+k)}\left(\int_\Omega u^{m+k}(t, x)\, dx\right)^{k/(m+k)} \\
&\geq -A(1)^{1-k}\left\{\frac{1}{\mu'}\left(\frac{1}{\varepsilon}\|f(t, \cdot)\|_{(m+k)/L^m(\Omega)}\right)^{\mu'} + \frac{1}{\mu}\left(\varepsilon\int_\Omega u^{m+k}(t, x)\, dx\right)^{\mu k/(m+k)}\right\}.
\end{aligned}$$

Taking $\mu = (k+1)/k$ then $\mu' = k+1$ and we conclude that

$$\langle f(t, \cdot), v - u\rangle_{V'V} \geq -A(1)^{1-k}\left\{\left(\frac{1}{\varepsilon(k+1)}\|f(t, \cdot)\|^{k+1}_{(m+k)/L^m(\Omega)}\right) + \frac{\varepsilon k}{k+1}\|u(t, \cdot)\|^{k+1}_{L^{m+k}(\Omega)}\right\}.$$

Then choosing $\varepsilon$ small enough we add the last term to the expression $\tilde{I}_l + \tilde{I}_2$ of the proof of theorem 1. Using (44) it is easy to see that condition (21) is verified for $\Phi(r) = Cr^\alpha$, $\alpha = (k+1)/(m+k)$ and $C$ a suitable positive constant. Finally condition (22) is implied by assumption (45) and lemma 1 can be applied. ∎

*Remark*. Theorems 1 and 2 generalize previous results given in [9] and [6], respectively.

## 3. SPACE LOCALIZATION

### 3.1. *General remarks on the method*

As in the case of time localization we start by giving an "informal" outline of the method for a general system (11), in which now we pay special attention to the principal part that we assume in divergence form

$$\mathbb{B}(t, x, \mathbf{w}, \mathbf{w}_t) = \operatorname{div} \mathbf{A}(t, x, \mathbf{w}, \nabla\mathbf{w}) + \mathbb{D}(t, x, \mathbf{w}, \nabla\mathbf{w}) + \mathbf{f}(t, x). \tag{46}$$

Assume that the unknown $\mathbf{w}$ is written in the form $\mathbf{w} = (\mathbf{w}^{(1)}, \mathbf{w}^{(2)})$, where $\mathbf{w}^{(1)}$ collects the $m$ unknown variables for which we want to prove space localization. As has already been remarked in the Introduction, we do not need to know any boundary conditions because the required property has a local character.

As in Section 2.1 the idea is to obtain an ordinary differential inequality from which we obtain the conclusion as a last step. However, the preliminary work is more elaborate than in the case of time localization.

*First step.* Multiply the first $m$-components of (11) by $\mathbf{w}^{(1)}$ and integrate by parts on the domain $(0, t)xB_\rho(x_0)$, where $B_\rho(x_0) = \{x \in \Omega : |x - x_0| < \rho\}$ (of course, since $\mathbf{w}^{(1)}$ is not a good test function on that domain suitable truncation and regularizing processes are needed to justify the result for the case in which $\mathbf{w}$ is merely a weak solution). Now we assume $B$ and $A$ have a nonlinear structure such that the following inequality holds

$$\int_0^t \int_{B_\rho} \mathbb{B}^{(1)}(t, x, \mathbf{w}, \mathbf{w}_t)\mathbf{w}^{(1)}\, dx\, d\tau \geq C\left\{\int_{B_\rho} |\mathbf{w}^{(1)}(t, x)|^{p_1}\, dx - \int_{B_\rho} |\mathbf{w}^{(1)}(0, x)|^{p_1}\, dx\right\} \quad (47)$$

for some $p_1 \geq 1$ and $C > 0$. We also assume that $\mathbb{A}$ is such that

$$C^{-1}\int_0^t \int_{B_\rho} |\nabla \mathbf{w}^{(1)}|^{p_2}\, dx\, d\tau \leq \int_0^t \int_{B_\rho} \mathbb{A}^{(1)}(\tau, x, \mathbf{w}, \nabla \mathbf{w}) \cdot \nabla \mathbf{w}^{(1)}\, dx\, d\tau \leq C\int_0^t \int_{B_\rho} |\nabla \mathbf{w}^{(1)}|^{p_2}\, dx\, d\tau \quad (48)$$

for some $p_2 \geq 1$ and $C > 0$. We introduce the "energy functions"

$$b(t, \rho) = \operatorname*{sup\,ess}_{0 \leq \tau \leq t} \int_{B_\rho(x_0)} |\mathbf{w}^{(1)}(\tau, x)|^{p_1}\, dx \quad (49)$$

and

$$E(t, s) = \int_0^t \int_{B_\rho(x_0)} \mathbb{A}^{(1)}(\tau, x, \mathbf{w}, \nabla \mathbf{w}) . \nabla \mathbf{w}^{(1)}\, dx\, d\tau. \quad (50)$$

The main idea, in this first step, is to obtain an inequality of the type

$$\Phi(E + b) \leq Ct^{\omega}\left(\frac{\partial E}{\partial \rho}\right) + G((\rho - \rho_0)_+) \quad (51)$$

where $\omega \geq 0$, $\Phi$ is an increasing function and $G$ is nonnegative continuous and $G(0) = 0$. The presence of the term $\partial E/\partial \rho$ comes from the contribution of boundary term in the integrations by parts formula. Notice that

$$\frac{\partial E}{\partial \rho}(t, s) = \int_0^t \int_{\partial B_\rho(x_0)} \mathbb{A}^{(1)}(\tau, x, \mathbf{w}, \nabla \mathbf{w}) \cdot \nabla \mathbf{w}^{(1)}\, dx\, d\tau. \quad (52)$$

The obtain (51) requires some suitable assumption on $w^{(1)}(0, x)$ on $B_{\rho_0}(x_0)$ and the application of different inequalities (Holder–Young inequalities of interpolation and traces, etc.). This constitutes the core of the method.

*Second step.* The conclusion comes now from the treatment of inequality (51).

LEMMA 2 [5, 7]. Let $y \in C^0([0, t_1]x[0, \rho_0 + \delta])$, $y \geq 0$ and such that

$$\Phi(y(t, s)) \leq Ct^{\omega} \frac{\partial y}{\partial \rho}(t, s) + G((\rho - \rho_0)_+) \tag{53}$$

for a.e. $\rho \in (0, \rho_0 + \delta)$ and for any $t \in [0, t_1]$, where $\omega \geq 0$ and $\Phi$ is a continuous nondecreasing function such that $\Phi(0) = 0$ and

$$\Phi(\cdot)^{-1} \in L^1(0, R), R > 0 \qquad \left(\text{i.e.} \int_{0^+} \frac{\mathrm{d}s}{\Phi(s)} < +\infty\right). \tag{54}$$

Assume $G \equiv 0$. Then there exists $t_0 \in (0, t_1)$ and a function $\rho(t)$ with $0 \leq \rho(t) \leq \rho_0 + \delta$ such that $y(t, \rho) = 0$ for any $t \in [0, t_0]$ and any $\rho \in [0, \rho(t)]$. Moreover, define $\Theta_\mu(s)$ and $\eta_\mu(s)$ as in lemma 1. Assume that $G \not\equiv 0$ satisfies

$$\exists\, \bar{\mu} > 0 \quad \text{and} \quad \varepsilon < 1 \quad \text{such that } G(s) \leq \varepsilon\Phi(\eta_{\bar{\mu}}(s)), \quad \text{a.e. } s \in (0, \delta). \tag{55}$$

Then there exists $t^* \leq t_1$ such that $y(t, \rho) = 0$ for any $0 \leq \rho \leq \rho_0$ and $t \in [0, t^*]$.

The first conclusion of lemma 2 leads to the *space localization property* of $\mathbf{w}^{(1)}$. The second corresponds to the *waiting time* or *nondiffusion of the support properties*.

This method can be easily adapted to the consideration of special systems. So, for the treatment of the system arising in non-Newtonian fluids (see formulation at the end of Section 2.1, but now with $q > 2$) a suitable new unknown must be introduced [8]. In other cases, as for instance the system of simultaneous flow of surface and ground waters, the components of $\mathbf{w}^{(1)}$ are defined on different spatial domains, but the method applies under natural modifications [4, 8].

### 3.2. *Space localization for two immiscible fluids in a porous medium*

Consider $(u, p)$ satisfying (weakly) the following equations

$$\phi(x) \frac{\partial \Psi(u)}{\partial t} - \operatorname{div}(\mathbb{K}_0(x)\nabla u) = \operatorname{div}(\mathbb{K}_0(x)\tilde{b}(u)\nabla p) + f(x, t)$$

$$\operatorname{div}(\mathbb{K}_0(x)\tilde{d}(u)\nabla p) = 0.$$

As before we shall make use only of information on $u$ and $p$ on a local domain of the form $(0, t_1) \times B_{\rho_1}(x_0)$ and so no information on $\partial\Omega$ is needed.

THEOREM 3. Let $(u, p)$ be a local weak solution on $(0, T)xB_{\rho_1}(x_0)$ and assume that $p \in L^\infty(0, T\colon W^{1,q}(B_{\rho_1}(x_0)))$ for some $q > 2$. Let $\Psi$ satisfying (38) with $m < 1$. Let $\tilde{b}$ and $\tilde{d}$ such that

$$\left| \tilde{d}(u) \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\tilde{b}(u)}{\tilde{d}(u)}\right) \right| \leq Cu^\mu, \qquad \text{for any } u \in (0, 1) \tag{56}$$

and with $\mu > 0$ such that

$$\mu \geq (m(q - 2) - q - 2)/2q. \tag{57}$$

Let $u_0$ and $f$ such that $u_0(\cdot) = f(t, \cdot) = 0$ on $B_{\rho_1}(x_0)$ and for $t \in (0, T)$. Then there exists a $t_0 \in (0, T)$ and a function $\rho(t)$, $0 \leq \rho(t) \leq \rho_1$ such that $u(t, \cdot) = 0$ for any $t \in [0, t_0]$ on $B_{\rho(t)}(x_0)$.

*Proof.* We follow the ideas explained in Section 3.1. Here $\mathbf{w} = (u, p)$ and $\mathbf{w}^{(1)} = u$. We multiply the first equation by $u$ and integrate by parts on $(0, t_0)xB_\rho(x_0)$. Using inequality (47) which now holds with $p_1 = m + 1$ (this is a local version of (41)) we conclude that

$$\begin{aligned} C\int_{B_\rho} |u(t,x)|^{m+1}\,\mathrm{d}x &+ \int_0^t\int_{B_\rho} \mathbb{K}_0(x)\nabla u\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^t\int_{\partial B_\rho} u\mathbb{K}_0(x)\nabla u\cdot\mathbb{n} + \int_0^t\int_{B_\rho} u\,\mathrm{div}(\mathbb{K}_0(x)\tilde{b}(u)\nabla p) \\ &\quad + C\int_{B_\rho}|u(0,x)|^{m+1}\,\mathrm{d}x + \int_0^t\int_{B_\rho} f(\tau,x)u(\tau,x)\,\mathrm{d}\tau\,\mathrm{d}x. \end{aligned} \tag{58}$$

We introduce the energy functions $b$ and $E$. In particular $E$ is given by

$$E(t,\rho) = \int_0^t\int_{B_\rho(x_0)} \mathbb{K}_0(x)\nabla u\cdot\nabla u\,\mathrm{d}x\,\mathrm{d}\tau$$

and so, by assumption (7), inequality (48) holds with $p_2 = 2$. In order to prove the crucial estimate (51) we start by considering the contribution at the boundary. Applying the Holder inequality, using (52) and the assumption (7) we obtain

$$\begin{aligned} \int_0^t\int_{\partial B_\rho} u\mathbb{K}_0(x)\nabla u\cdot\mathbb{n} &\le C\int_0^t\int_{\partial B_\rho} |u|\,|\nabla u|\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &\le C\left(\int_0^t\int_{\partial_\rho}|u|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^t\int_{\partial_\rho}|\nabla u|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2} \\ &\le \bar{C}\left(\int_0^t\int_{\partial B_\rho}|u|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2}\left(\frac{\partial E}{\partial p}(\tau,\rho)\right)^{1/2}. \end{aligned}$$

We use now the following interpolation-trace inequality (see [13, formula (2.13) and (2.13)]) which is true under the general condition $m \le 1$

$$\begin{aligned} \left(\int_0^t\int_{\partial B_\rho}|u|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right) &\le C\int_0^t \{(\|\nabla u(\tau,\cdot)\|_{L^2(B_\rho)} + \|u(\tau,\cdot)\|_{L^{m+1}(B_\rho)})^{2\Theta}(\|u(\tau,\cdot)\|_{L^{m+1}(B_\rho)})^{2(1-\Theta)}\}\,\mathrm{d}\tau \\ &\le C\left(\int_0^t\int_{B_\rho}|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\|u(\tau,\cdot)\|^2_{L^{m+1}(B_\rho)}\right)^{\Theta}\left(\int_0^t\|u(\tau,\cdot)\|^2_{L^{m+1}(B_\rho)}\right)^{1-\Theta} \end{aligned}$$

which implies

$$\begin{aligned} \left(\int_0^t\int_{\partial B_\rho}|u|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)^{1/2} &\le C(E(t,s) + tb(t,s)^{2/(m+1)})^{\Theta/2}(tb(t,s)^{2/(m+1)})^{(1-\Theta)/2} \\ &\le Ct^{(1-\Theta)/2}(E^{1/2} + t^{1/2}b^{1/(m+1)})^{\Theta}b^{(1-\Theta)/(m+1)} \\ &\le Ct^{(1-\Theta)/2}K(T)(E+b)^{\Theta/2+(1-\Theta)/(m+1)} \end{aligned}$$

where $K(T) = \max(1, T^{\Theta/2})\max(1, b(T,\rho_1)^{\Theta(1-m)/2(m+1)})$ and

$$\Theta = \frac{N(1-m)+m+1}{N(1-m)+2m+2}.$$

Then introducing the exponent $\mathcal{K} = \Theta/2 + (1 - \Theta)/(m + 1)$ it is clear that $\mathcal{K} \in (0, 1]$, since $m > 0$, and then using Young's inequality we deduce that

$$\int_0^t \int_{\partial B_\rho} u \mathbb{K}_0(x) \nabla u \cdot \mathbf{n} \le CK(T) t^{1-\Theta/2} (E + b)^{\mathcal{K}} \left(\frac{\partial E}{\partial \rho}\right)^{1/2}$$

$$\le \varepsilon(E + b) + C_\varepsilon (CK(T) t^{1-\Theta/2})^{1/(1-\mathcal{K})} \left(\frac{\partial E}{\partial \rho}\right)^{1/\beta} \tag{59}$$

where $\beta = 2(1 - \mathcal{K})$. Later we shall use the crucial fact that (lying $\Theta$ in $(0, 1)$) $\beta < 1$ if (and only if) $m < 1$. On the other hand, using the second equation we have

$$\begin{aligned}
&\int_0^t \int_{B_\rho} u \operatorname{div}(\mathbb{K}_0(x) \tilde{b}(u) \nabla p) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^t \int_{B_\rho} u \operatorname{div}\left(\mathbb{K}_0(x) \tilde{d}(u) \nabla p \frac{\tilde{b}(u)}{\tilde{d}(u)}\right) \mathrm{d}x \, \mathrm{d}t = \int_0^t \int_{B_\rho} u \tilde{d}(u) \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\tilde{b}(u)}{\tilde{d}(u)}\right) \nabla u . \mathbb{K}_0(x) \nabla p \, \mathrm{d}x \, \mathrm{d}t \\
&\le C \int_0^t \int_{B_\rho} |\nabla u| \left| u \tilde{d}(u) \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\tilde{b}(u)}{\tilde{d}(u)}\right) \right| |\nabla p| \\
&\le C \left( \int_0^t \int_{B_\rho} |\nabla u|^2 \right)^{1/2} \left( \int_0^t \int_{B_\rho} B(u)^2 |\nabla p|^2 \, \mathrm{d}x \, \mathrm{d}t \right)^{1/2} \\
&\le CE(t, \rho)^{1/2} \left( \int_0^t \left( \int_{B_\rho} B(u)^{q/(q-2)} \, \mathrm{d}x \right)^{(q-2)/q} \left( \int_{B_\rho} |\nabla p|^q \, \mathrm{d}x \right)^{2/q} \mathrm{d}t \right)^{1/2} \\
&\le CMt^{(q-2)/q} E(t, s)^{1/2} \left( \int_0^t \int_{B_\rho} B(u)^{2q/(q-2)} \, \mathrm{d}x \, \mathrm{d}t \right)^{(q-2)/2q} \equiv I
\end{aligned}$$

where

$$B(u) = u \tilde{d}(u) \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\tilde{b}(u)}{\tilde{d}(u)}\right).$$

We now use the assumptions (56) and (57). Then by Young's inequality

$$\begin{aligned}
I &\le CMt^{3(q-2)/2q} E(t, \rho)^{1/2} b(t, \rho)^{(q-2)/2q} \le CMt^{3(q-2)/2q} (E + b)^{1/2 + (q-2)/2q} \\
&\le CMt^{3(q-2)/2q} (E + b).
\end{aligned} \tag{60}$$

The final conclusion comes from lemma 2 once inequality (53) is satisfied on $(0, t_0) \times (0, \rho_1)$, with $t_0$ small enough, $\Phi(r) = r^\beta$, $\beta < 1$, $\omega = \beta(1 - \Theta/2)/(1 - \mathcal{K})$ and $G \equiv 0$. ∎

Some extra information on $\rho(t)$ can be obtained under additional information on $u(0, \cdot)$ and $f$.

THEOREM 4. Assume $\Psi$, $\tilde{b}$, $\tilde{d}$ and $(u, p)$ as in theorem 3. Let $u_0 \in L^{m+1}(B_{\rho_1}(x_0))$ and $f \in L^2(0, T : L^r(B_{\rho_1}(x_0)))$, for some $r \in [m + 1, 2N/(N - 2)]$ such that there exists $t_f > 0$, $\rho_0 \in (0, \rho_1)$ and $\varepsilon > 0$ ($\varepsilon$ small enough) such that

$$\int_{B_\rho(x_0)} |u(0, x)|^{m+1} \, \mathrm{d}x + \left( \int_0^{t_f} \int_{B_\rho(x_0)} |f(\tau, x)|^{r/(r-1)} \, \mathrm{d}\tau \, \mathrm{d}x \right)^{(r-1)/(1-\nu)r} \le \varepsilon(\rho - \rho_0)_+^{1/(1-\beta)} \tag{61}$$

for a.e. $\rho \in (0, \rho_1)$, where

$$\beta = 2(1 - (\Theta/2 + (1 - \Theta)/(m + 1))), \qquad \nu = (\lambda/2) + (1 - \lambda)/(m + 1) \tag{62}$$

$$\lambda = (1/(m + 1) - 1/r)(1/(m + 1) - (N - 2)/2N). \tag{63}$$

Then there exists $t^* > 0$, $t^* \leq t_f$, such that $\rho(t) = \rho_0$ for any $t \in [0, t^*]$, i.e. $u(t, x) = 0$ a.e. $x \in B_{\rho_0}(x_0)$ and $t \in [0, t^*]$.

*Proof.* We argue as in the proof of theorem 3, but taking into account the contributions of the independent term $f$ and the initial datum $u(0, \cdot)$. We shall use the following interpolation inequality

$$\|u(t, \cdot)\|_{L^r(B_\rho)} \leq C(\|\nabla u(t, \cdot)\|_{L^2(B_\rho)} + \|u(t, \cdot)\|_{L^{m+1}(B_\rho)})^\lambda \|u(t, \cdot)\|_{L^{m+1}(B_\rho)}^{(1-\lambda)}$$

for any $r \in [m + 1, 2N/(N - 2)]$ and where $\lambda = (1/(m + 1) - 1/r)(1/(m + 1) - (N - 2)/2N)$ (see [16]). Then

$$\begin{aligned}
&\int_0^t \int_{B_\rho} |u(\tau, x)|^r \,\mathrm{d}x \,\mathrm{d}\tau \\
&\leq C\left(\int_0^t \|\nabla u(\tau, \cdot)\|_{L^2(B_\rho)}^r \,\mathrm{d}\tau + \int_0^t \|u(\tau, \cdot)\|_{L^{m+1}(B_\rho)}^r \,\mathrm{d}\tau\right)^\lambda \left(\int_0^t \|u(\tau, \cdot)\|_{L^{m+1}(B_\rho)}^r\right)^{1-\lambda} \\
&\leq C(t^{(2-r)/2} E(t, s)^{r/2} + t b(t, s)^{r/(m+1)})^\lambda t^{(1-\lambda)} b(t, s)^{(1-\lambda)r/(m+1)}
\end{aligned}$$

which implies

$$\begin{aligned}
\left(\int_0^t \int_{B_\rho} |u(\tau, x)|^r \,\mathrm{d}x \,\mathrm{d}\tau\right)^{1/r} &\leq C t^{(2-\lambda r)/2r} (E^{1/2} + t^{1/2} b^{1/(m+1)})^\lambda b^{(1-\lambda)/(m+1)} \\
&\leq C t^{(2-\lambda r)/2r} K^*(T)(E + b)^{(\lambda r/2) + (1-\lambda)r/(m+1)}
\end{aligned}$$

where $K^*(T) = \max(1, T^{1/2}) \max(1, b(T, \rho_1)^{\lambda/(m+1)})$. We have

$$\left(\int_0^t \int_{B_\rho} f(\tau, x) u(\tau, x) \,\mathrm{d}\tau \,\mathrm{d}x \leq \int_0^t \int_{B_\rho} |u(\tau, x)|^r \,\mathrm{d}x \,\mathrm{d}\tau\right)^{1/r} \left(\int_0^t \int_{B_\rho} |f(\tau, x)|^{r/(r-1)} \,\mathrm{d}x \,\mathrm{d}\tau\right)^{(r-1)/r} \equiv I_2.$$

Applying Young's inequality and calling $\nu = \lambda/2 + (1 - \lambda)/(m + 1)$ we conclude

$$I_2 \leq C t^{(2-\lambda r)/2r\nu} K^*(T)^{1/\nu}(E + b) + \left(\int_0^t \int_{B_\rho} |f(\tau, x)|^{r/(r-1)}\right)^{(r-1)/(1-\nu)r}. \tag{64}$$

Finally, choosing $t^*$ small enough, inequality (53) with $\Phi(s) = s^\beta$, $\beta < 1$, $\omega$ as in the proof of theorem 1 and $G(r) = \varepsilon r^{1/(1-\beta)}$, holds on $(0, t^*) \times (0, \rho_1)$ as we deduce from (59), (60), (64) and assumption (61). ■

*Remark.* Theorem 3 improves a previous result given in [3]. Theorem 4 was announced in [6], but only a sketch of the proof was indicated.

As mentioned in the Introduction, similar results to theorems 3 and 4 can be obtained on the set where $\Psi(u) = 1$. On the other hand global statements giving location estimates on the sets $u = 0$ or $\Psi(u) = 1$ can be easily obtained once that boundary conditions are given and some information on the location of the sets where $u_0 = 0, f = 0$ or $\Psi(u_0) = 1$ is known. It is enough to obtain *a priori* estimates on the energy functions $b$ and $E$ on the global domain $(0, T) \times \Omega$. We leave the details to the reader.

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