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Sufficient and Necessary Initial Mass Conditions for the Existence of a Waiting Time in Nonlinear-Convection Processes

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We study the initial behavior of the fronts (for interfaces) generated by the solutions of the equation $u_t = (u^m)_{xx} + b(u^\lambda)_x$, where m, b, $\lambda > 0$ are real numbers. We prove a mass comparison principle that allows us to give necessary and sufficient conditions in order to have waiting time at the fronts. Different regions in the (λ, m) parameter space must be introduced leading to answer of very different nature.

1. INTRODUCTION

In this paper we study the diffusion-convection equation

$$u_t = (u^m)_{xx} + b(u^\lambda)_x \qquad \text{in} \quad Q = \mathbb{R} \times \mathbb{R}^+ \tag{1.1}$$

$$u(x, 0) = u_0(x) \qquad \text{in} \quad \mathbb{R}, \tag{1.2}$$

where m, b, $\lambda > 0$ and $u_0(x)$ is a continuous non-negative function on $\mathbb{R}$ with compact support.

This equation arises as a model for a number of different physical phenomena. For instance, when u denotes unsaturated soil-moisture content, the equation describes the infiltration of water in a homogeneous

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porous medium and some natural assumption in the context are $m > 1$ and $\lambda > 0$ (see [2]). The equation also appears in the study of the flow of a thin viscous film over an inclined bed for the specific exponents $m = 3$ and $\lambda = 4$ (see [3]). By analogy with the classical equations from statistical mechanics (see [4]), Eq. (1.1) is often referred to as the nonlinear Fokker–Plank equation. Equation (1.1) is also used in connection with transport of thermal energy in plasma (then $0 < m < 1$ and $\lambda = 1$ (see [15]). Finally, the equation has additional interest as a generalization of the well-known equation of Burgers approximating the associated hyperbolic conservation law equation.

It is well known that nonnegative solutions u of (1.1) may give rise to interfaces (or free boundaries) separating regions where $u > 0$ from ones where $u = 0$:

$$\zeta_-(t) = \inf\{x : u(x, t) > 0\} \tag{1.3}$$

$$\zeta_+(t) = \sup\{x : u(x, t) > 0\}. \tag{1.4}$$

These fronts are relevant in the physical problems modeled and their occurrence is essentially due to slow diffusion ($m > 1$) or to convective phenomena dominating over diffusion ($\lambda < m$) (see, e.g., [9, 10, 7]). Sometimes $\zeta_-(t)$ or $\zeta_+(t)$ can remain static in relation to the boundary of initial data $u_0(x)$, i.e., there exists $t^* > 0$, called waiting time, such that $\zeta_-(t) = \zeta_-(0)$ or $\zeta_+(t) = \zeta_+(0)$ if $0 < t < t^*$. The main goal of this work lies in the study of this "waiting time."

We continue the research initiate in [1] by generalizing the conditions on the initial data in order to have such a phenomenon.

In the non-convective case ($b = 0$, $m > 1$), a necessary and sufficient condition was given in [16]: there exists waiting time for $\zeta_+(t)$ if and only if $u_0(x)$ satisfies

$$\operatorname*{Lim\,sup}_{x \to \zeta_+(0)^-} |x - \zeta_+(0)|^{(m+1)/(1-m)} \int_x^{+\infty} u_0(s)\, ds < +\infty. \tag{1.5}$$

Let us remark that in this case, the behavior of $\zeta_-(t)$ is the same as that of $\zeta_+(t)$ due to the symmetry of Eq. (1.1) when $b = 0$.

In the case $b > 0$, a separate study of $\zeta_-(t)$ and $\zeta_+(t)$ is needed because the convective term introduces an inherent asymmetry into the problem. In [1], we showed, that the initial growth of the interfaces is different in each one of the regions of (λ, m) parameter space shown in Fig. 1.

In order to describe our results we remark that the behavior of the interface $\zeta_\pm(t)$ depends only on the values m and λ as well as on the local behavior of the initial data $u_0(x)$ near $\zeta_\pm(0)$. In [1], we assume a "pointwise" growth of $u_0(x)$ away from $\zeta_\pm(0)$, i.e., $u_0(x)$ is bounded from

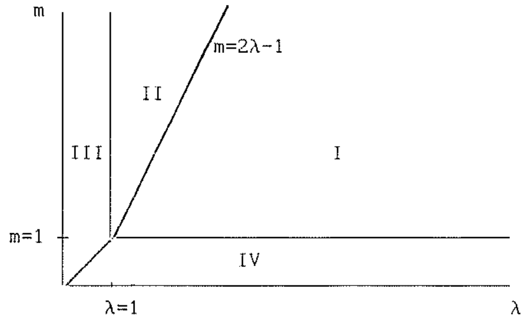

FIGURE 1

below or from above by $C\,|x-\zeta_\pm(0)|^\gamma$ for x near $\zeta_\pm(0)$, with C, $\gamma>0$, suitable constants.

In this paper we change this "pointwise" growth assumption under $u_0(x)$ for a "mass" growth condition (in a similar sense to (1.5)), in order to improve the characterization of waiting time existence.

Here the local mass of u_0 near the point $\zeta_\pm(0)$ is given by the term

$$\int_{\zeta_-(0)}^{x} u_0(s)\,ds=\int_{-\infty}^{x} u_0(s)\,ds \qquad \text{or} \qquad \int_{x}^{\zeta_+(0)} u_0(s)\,ds=\int_{x}^{+\infty} u_0(s)\,ds.$$

The paper is divided into several sections according to the following plan: In Section 2 we introduce a comparison principle, based on the evaluation of masses which is used systematically in the rest of the paper. A first result in this direction was given in [17] for the case without convection (i.e., $b=0$) by using different technics. As a corollary of this masses comparison principle we derive the comparison of interfaces associated to the cases $b>0$ and $b=0$.

In Section 3 we give necessary and sufficient conditions on u_0 for a positive waiting time in the Region I $=\{(\lambda, m): 1<m\leqslant 2\lambda-1\}$. In Section 4 we study the waiting time in the Region II, defined by $\{(\lambda, m): 2<2\lambda<m+1\}$. Finally, Section 5 is devoted to the Region III, given by $\{(\lambda, m): \lambda<m \text{ and } \lambda\leqslant 1\}$.

Let us remark that Region IV, defined by $\{(\lambda, m): m\leqslant 1 \text{ and } \lambda\geqslant m\}$, corresponds to a fast or linear diffusion with a weak convection. In this case, none of the fronts exist [6, 10]. Hence this region is not of interest to us.

2. Comparison of Masses

In order to improve the pointwise criteria given in [1] we start by proving a comparison principal which does not have a local character: we shall compare the masses from $-\infty$ to a given point x. The proof will be carried out by using an approximation argument since it is well known that we cannot expect, in general, to have classical solutions. Approximation arguments are very common in order to obtain the existence, uniqueness, and regularity of weak solutions of (1.1), (1.2) (see [5, 11] and their references).

LEMMA 2.1. *Let* $u \in C(Q)$ *be weak solution of* (1.1)–(1.2) *and let* $\mathbf{u} \in C(Q)$ *be the solution of* $\mathbf{u}_t = (\mathbf{u}^m)_{xx} + \mathbf{b}(\mathbf{u}^\lambda)_x$ *on* Q *with initial value* $\mathbf{u}_0(x)$, $\mathbf{u}_0 \in L^1(\mathbb{R})$, $\mathbf{u}_0 \geqslant 0$. *Assume* $b \geqslant \mathbf{b}$, *and the following initial mass comparison*:

$$\int_{-\infty}^{x} u_0(s)\,ds \geqslant \int_{-\infty}^{x} \mathbf{u}_0(s)\,ds \qquad \forall x \in \mathbb{R}. \tag{2.2}$$

Then for any $t > 0$ *we have*

$$\int_{-\infty}^{x} u(s,t)\,ds \geqslant \int_{-\infty}^{x} \mathbf{u}(s,t)\,ds \qquad \forall x \in \mathbb{R}. \tag{2.3}$$

Proof. We start by assuming that $u(x,t)$ and $\mathbf{u}(x,t)$ are strictly positive classical solutions (for instance, because u_0 and $\mathbf{u}_0$ are strictly positive functions). We define the functions $w(x,t)$ and $\mathbf{w}(x,t)$ by

$$w(x,t) = e^{-\varepsilon t} \int_{-\infty}^{x} u(s,t)\,ds; \qquad \mathbf{w}(x,t) = e^{-\varepsilon t} \int_{-\infty}^{x} \mathbf{u}(s,t)\,ds.$$

It is known (see [9]), that any solution of the Eq. (1.1) verifies the time-invariance of the mass, i.e., for every $t > 0$,

$$\int_{-\infty}^{+\infty} u(s,t)\,ds = \int_{-\infty}^{+\infty} u_0(s)\,ds. \tag{2.4}$$

Hence from (2.2) we have, for any $t > 0$, that

$$\operatorname{Lim}_{x \to +\infty} w(x,t) - \mathbf{w}(x,t) \geqslant 0. \tag{2.5}$$

Suppose now that there exists $(x_0, t_0) \in (-\infty, \infty) \times (0, \infty)$ such that $w(x_0, t_0) < \mathbf{w}(x_0, t_0)$. Let $P = \mathbb{R} \times [0, t_0]$. By the continuity, (2.2) and (2.5), we obtain that the function $w(x,t) - \mathbf{w}(x,t)$ has a minimum in some interior point (x_1, t_1) of P. Hence in (x_1, t_1) we have

(i) $w_x(x_1, t_1) = \mathbf{w}_x(x_1, t_1) > 0$

(ii) $(w - \mathbf{w})_t\,(x_1, t_1) \leqslant 0$.

(iii) $((w_x)^m - (\mathbf{w}_x)^m)_x\,(x_1, t_1) \geqslant 0$.

The proof of (iii) is an easy consequence of (i) and that $w_{xx} \geqslant \mathbf{w}_{xx}$ at (x_1, t_1). Moreover, $w(x, t)$ and $\mathbf{w}(x, t)$ satisfy

$$\varepsilon e^{\varepsilon t}(w-\mathbf{w}) = e^{\varepsilon m t}((w_x)^m - (\mathbf{w}_x)^m)_x + e^{\varepsilon \lambda t}(b(w_x)^\lambda - b(\mathbf{w}_x)^\lambda) - e^{\varepsilon t}(w-\mathbf{w})_t.$$

Using (i)–(iii) we have that the left hand of the latter equality is less than zero at the point (x_1, t_1) but the right hand is larger than or equal to zero, and so we arrive to a contradiction.

Finally, in the general case we approximate u and $\mathbf{u}$ by sequences of classical solutions u_n, $\mathbf{u}_n$ with convergence at least in $C([0, T] : L^1(\mathbb{R}))$. Moreover, it is possible to take $u_{0,n}$ satisfying the inequality (2.2) and so the conclusion comes passing to the limit in the mass comparison of u_n and $\mathbf{u}_n$ (i.e., in the associated inequality (2.3)).

Remark 2.1. The above lemma remains true for solutions of the boundary value problem associate to introduce a lateral boundary condition in the set $(-\infty, \delta] \times [0, T]$ $(\delta \in \mathbb{R})$, assuming that $u(x, t)$, $\mathbf{u}(x, t)$ satisfy

$$\int_{-\infty}^{\delta} u(s, t)\, ds \geqslant \int_{-\infty}^{\delta} \mathbf{u}(s, t)\, ds \qquad \text{for any } t \in [0, T]. \tag{2.6}$$

Indeed, $w(x, t) - \mathbf{w}(x, t) \geqslant 0$ at the boundary of $(-\infty, \delta] \times [0, T]$, and we can use the same argument as in Lemma 2.1 in order to prove (2.3) for any $x \in (-\infty, \delta]$.

We also remark that Lemma 2.1 remains true, if we change $u(x, t)$ by supersolutions $(u_t - (u^m)_{xx} - b(u^\lambda)_x \geqslant 0)$ and $\mathbf{u}(x, t)$ by subsolutions $(\mathbf{u}_t - (\mathbf{u}^m)_{xx} - \bar{b}(\mathbf{u}^\lambda)_x \leqslant 0)$.

The following result give a first result about how the presence of a convection term modifies the behavior of the free boundary.

COROLLARY 2.1. *Let $u(x, t)$ be a solution of* (1.1) *and* $\mathbf{u}(x, t)$ *solution of the porous medium equation* $(b = 0)$, *for the same initial data* $u_0 \in L^1(\mathbb{R})$, $u_0 \geqslant 0$. *Then*

(i) $\zeta_-(t) \leqslant \boldsymbol{\zeta}_-(t)$ *for all* $t > 0$ (2.7)

(ii) $\zeta_+(t) \leqslant \boldsymbol{\zeta}_+(t)$ *for all* $t > 0$. (2.8)

Proof. Let $\mathbf{b} = 0$, by Lemma 2.1 we have that for any $x \in \mathbb{R}$

$$\int_{-\infty}^{x} \mathbf{u}(s, t)\, ds \leqslant \int_{-\infty}^{x} u(s, t)\, ds. \tag{2.9}$$

Choosing $x=\zeta_-(t)$ in (2.9), we obtain

$$\int_{-\infty}^{\zeta_-(t)} \mathbf{u}(s, t)\, ds \leqslant 0.$$

Therefore (2.7) is proved. In order to show (2.8), we choose $x=\zeta_+(t)$ in (2.9), and using (2.4), we obtain

$$\int_{-\infty}^{+\infty} u_0(s)\, ds \leqslant \int_{-\infty}^{\zeta_+(t)} u(s, t)\, ds$$

and then (2.8) follows. ∎

The mass comparison also admits a statement in terms of the reverse inequalities.

COROLLARY 2.3. *Let* $u(x, t)$ *be the solution of* (1.1)–(1.2) *and* $\mathbf{u}(x, t)$ *solution of* $\mathbf{u}_t=(\mathbf{u}^m)_{xx}+\mathbf{b}(\mathbf{u}^\lambda)_x$ *with the initial value* $\mathbf{u}_0 \in L^1(\mathbb{R})$, $\mathbf{u}_0 \geqslant 0$. *Let* $\mathbf{b} \geqslant b$, *and assume* $u_0(x)$ *and* $\tilde{u}_0(x)$ *such that*

$$\int_{x}^{+\infty} \mathbf{u}_0(s)\, ds \leqslant \int_{x}^{+\infty} u_0(s)\, ds \qquad \textit{for any} \quad x \in \mathbb{R}. \tag{2.10}$$

Then for any $t>0$ *and any* $x \in \mathbb{R}$ *we have*

$$\int_{x}^{+\infty} \mathbf{u}(s, t)\, ds \leqslant \int_{x}^{+\infty} u(s, t)\, ds \qquad \textit{for any} \quad x \in \mathbb{R}. \tag{2.11}$$

Proof. Let $u_1(x, t)=u(-x, t)$ and $\mathbf{u}_1(x, t)=\mathbf{u}(-x, t)$. These functions are solutions of the associate equation (1.1), where b and $\mathbf{b}$ are changed by $-b$ and $-\mathbf{b}$. Moreover, we have the integral equality

$$\int_{x}^{+\infty} u(s, t)\, ds = \int_{-\infty}^{-x} u_1(s, t)\, ds$$

and so the corollary follows from Lemma 2.1. ∎

3. WAITING TIME CONDITIONS FOR (λ, m) IN REGION I

We shall assume in this section that $1 < m \leqslant 2\lambda - 1$. Under this condition it turns out that the initial behavior of the interfaces is the same as the porous medium equation ($b=0$). In order to describe ours results we separate the interfaces $\zeta_-(t)$ and $\zeta_+(t)$.

We start by considering $\zeta_-(t)$. The following result gives a necessary and sufficient condition for the existence of a positive waiting time.

THEOREM 3.1. *Let $u(x, t)$ be the solution of (1.1)–(1.2). There exists $t^*>0$ such that $u(\zeta_-(0), t)=0$ for all $t \in [0, t^*]$ if and only if*

$$\operatorname*{Lim\ sup}_{x \to \zeta_-(0)^+} |x-\zeta_-(0)|^{(m+1)/(1-m)} \int_{-\infty}^{x} u_0(s)\,ds < \infty. \tag{3.1}$$

Proof. Assume by contrary that this limit is equal to ∞. By Corollary 2.1, $\zeta_-(t) \leqslant \bar{\zeta}_-(t)$ (where $\bar{\zeta}_-(t)$ corresponds to the case $b=0$). But by the result of [16], $\bar{\zeta}_-(t)$ has no waiting time and so the same occurs with the $\zeta_-(t)$.

Assume, now that the limit in (3.1) is finite. We define the following separable function

$$\bar{u}(x, t)=\begin{cases}(Ax^2)^{1/(m-1)}\left(\dfrac{T}{T-t}\right)^{\alpha} & \text{in} \quad [0, \delta] \times [0, T] \\ 0 & \text{in} \quad (-\infty, 0] \times [0, T].\end{cases} \tag{3.2}$$

Where A, δ, α, $T>0$ will be chosen later. Without lost of generality we may assume that $\zeta_-(0)=0$. We introduce also the notations

$$M_0=\int_{-\infty}^{+\infty} u_0(s)\,ds$$

$$M_1=\left(\operatorname*{Lim\ sup}_{x \to 0^+} |x|^{(m+1)/(1-m)} \int_{-\infty}^{x} u_0(s)\,ds\right)+1<\infty$$

$$M_2=\int_{-\infty}^{+\infty} u_0(s)\,ds<\infty.$$

By (3.1) there exists $1>\delta>0$ such that

$$\int_{-\infty}^{x} u_0(s)\,ds \leqslant M_1\,|x|^{(m+1)/(m-1)} \qquad \text{if} \quad x \in (-\infty, \delta]. \tag{3.3}$$

We define α, A, and T given by

$$\begin{aligned}A&=(m-1)^{1-m}\max\{(M_1(m+1))^{m-1},\\&\qquad (M_0(m-1)\,\delta^{-1})^{m-1}, (M_2(m+1))^{m-1}\}\\ \alpha&=\min\{(m-1)^{-1}, (\lambda-1)^{-1}\}\\ T&=\tfrac{1}{2}\alpha(m-1)^2\,[m(m+1)A+\lambda(m-1)\,A^{1/2}(A\delta^2)^{\lambda}]^{-1}.\end{aligned}$$

With this choice of constants we obtain that $\bar{u}(x, t)$ satisfies

$$\text{(a)} \qquad \bar{u}_t-(\bar{u}^m)_{xx}-b(\bar{u}^{\lambda})_x \geqslant 0 \qquad \text{in} \quad [-\delta, \infty) \times [0 \times T]$$

(b) $$\int_{-\infty}^{x} \bar{u}(s, 0)\, ds \geqslant A^{1/(m-1)} \frac{m-1}{m+1} |x|^{(m+1)/(m-1)}$$

$$\geqslant M_1 |x|^{(m+1)/(m-1)} \geqslant \int_{-\infty}^{x} u_0(s)\, ds$$

(c) $$\int_{-\infty}^{\delta} \bar{u}(s, t)\, ds \geqslant A^{1/(m-1)} \frac{m-1}{m+1} |\delta|^{(m+1)/(m-1)} \geqslant M_0$$

$$= \int_{-\infty}^{+\infty} u_0(s)\, ds \geqslant \int_{-\infty}^{\delta} u(s, t)\, ds.$$

Hence, by Lemma 2.1 (see Remark 2.1) we have that

$$\int_{-\infty}^{x} u(s, t) \leqslant \int_{-\infty}^{x} \bar{u}(s, t)\, ds \qquad \text{in} \quad (-\infty, \delta] \times [0, T].$$

Therefore $u(x, t) = 0$ for all $x \in (-\infty, 0] \times [0, T]$. ∎

We consider now the other interface $\zeta_+(t)$:

THEOREM 3.2. *Let $u(x, t)$ be the solution of* (1.1)–(1.2). *We assume that the following limit exists*

$$L_0 = \operatorname*{Lim}_{x \to \zeta_+(0)^-} |x - \zeta_+(0)|^{(m+1)/(1-m)} \int_x^{+\infty} u_0(s)\, ds \qquad (L_0 \leqslant +\infty). \quad (3.4)$$

Then we have the characterization:

"$\exists t^* > 0$ *such that* $u(t, \zeta_+(0)) = 0$ $\forall t \in [0, t^*]$ *if and only if* $L_0 < \infty$."

Proof. Assume that the limit is finite. By Corollary 2.1, $\zeta_+(t) \leqslant \zeta_+(t)$ (where $\zeta_-(t)$ corresponds to case $b = 0$). Moreover by (1.5) and the result of [16], $\zeta_+(t)$ has waiting time. Therefore $\zeta_+(t)$ has waiting time too.

In order to complete the proof we assume that $\zeta_+(0) = 0$ and we define a family of auxiliary functions depending of two parameters k and $\bar{x}$ in the following way

$$\mathbf{u}(x, t; k, \bar{x}) = [k^2 t - k M_1 (1 - M_2 t)(x - \bar{x})]_+^{1/(m-1)}, \qquad (3.5)$$

where $[u]_+ = \max\{u, 0\}$, k and $\bar{x}$ are arbitrary constants. In [1], we show that if M_1, M_2 are suitable constants, then $\mathbf{u}(x, t; k, \bar{x})$ are subsolutions of (1.1) in the set $[-\delta, \infty) \times [0, T]$ ($T > 0$ arbitrary).

Given $\bar{x} \in (-\delta, 0)$ and $C > 0$, by (3.4) there exists $\delta = \delta(C) > 0$ such that

$$C |x|^{(m+1)/(m-1)} \leqslant \int_x^{+\infty} u_0(s)\, ds \qquad \text{if} \quad x \in [-\delta, 0].$$

Moreover, $\mathbf{u}(x, t; k, \bar{x})$ satisfies

$$\int_x^{+\infty} \mathbf{u}(s, 0; k, \bar{x})\, ds = \frac{m-1}{m} (M_1 k)^{1/(m-1)} (\bar{x} - x)_+^{m/(m-1)}.$$

Next we choose $k = k(\bar{x}, C)$ such that

$$\frac{m-1}{m} (M_1 k)^{1/(m-1)} (\bar{x} - x)_+^{m/(m-1)} \leqslant C\, |x|^{(m+1)/(m-1)} \qquad \text{if} \quad x \in [-\delta, 0].$$

By a convexity argument it is easy to see that the last inequality holds if we choose

$$k = \frac{m-1}{m} \left(\frac{m+1}{m-1}\right)^m M_1^{-1} C^{m-1}\, |\bar{x}|. \tag{3.6}$$

Moreover, if $\bar{x}$ is enough small,

$$\int_{-\delta}^{+\infty} \mathbf{u}(s, t; k, \bar{x})\, ds \leqslant \int_{-\delta}^{+\infty} u(s, t)\, ds \qquad \text{if} \quad t \in [0, 1].$$

Hence, by Lemma 2.1 we have that

$$\int_x^{+\infty} \mathbf{u}(s, t; k, \bar{x})\, ds \leqslant \int_x^{+\infty} u(s, t)\, ds \qquad \text{in} \quad [-\delta, +\infty) \times [0, 1].$$

Therefore,

$$\zeta_+(t, \bar{x}) \leqslant \zeta_+(t) \qquad \text{if} \quad t \in [0, 1],$$

where $\zeta_+(t, \bar{x}) = \sup\{x : \mathbf{u}(x, t; k, \bar{x}) > 0\}$. To be explicit

$$\zeta_+(t; \bar{x}) = \left(\frac{m-1}{m} \left(\frac{m+1}{m-1}\right)^m M_1^{-2} \frac{C^{m-1}}{(1 - M_2 t)}\, t - 1\right) |\bar{x}|.$$

But if $t > 0$ small, we can choose $C = C(t)$ such that $\zeta_+(t; \bar{x}) > 0$ and hence $\zeta_+(t) > 0$. ∎

4. Waiting Time Conditions for (λ, m) in Region II

We shall assume, in this section that $1 < \lambda < (m+1)/2$. As we shall see the influence of convection on the behavior of each front has a different nature. We start by studying the left interface $\zeta_-(t)$ and we give a necessary and sufficient condition for the existence of a positive waiting time.

THEOREM 4.1. *Let $u(x, t)$ be the solution of (1.1)–(1.2). We assume that there exists the limit*

$$L_0 = \operatorname*{Lim}_{x \to \zeta_-(0)^+} |x - \zeta_-(0)|^{\lambda/(1-\lambda)} \int_{-\infty}^{x} u_0(s)\, ds \qquad (L_0 \leqslant +\infty). \tag{4.1}$$

Then we obtain the following characterization:

"$\exists t^* > 0$ *such that* $u(t, \zeta_-(0)) = 0$ $\forall t \in [0, t^*]$ *if and only if* $L_0 < \infty$."

Proof. We shall assume that $\zeta_-(0) = 0$. Assume that $L_0 < \infty$., Then there exists $\delta > 0$ such that

$$\int_{-\infty}^{x} u_0(s)\, ds \leqslant (L_0 + 1)\, |x|^{\lambda/(\lambda-1)} \qquad \text{if} \quad x \in [0, \delta].$$

We define the separable function,

$$\bar{u}(x, t) = \begin{cases} \left(\dfrac{1}{k_1 - k_2 t}\right)^{1/(m-1)} |x|^{1/(\lambda-1)} & \text{in} \quad [0, \delta] \times [0, T_1] \\ 0 & \text{in} \quad (-\infty, 0] \times [0, T_1], \end{cases}$$

where

$$k_1 = \max\left\{1, \left(\frac{(\lambda-1)}{\lambda(L_0+1)}\right)^{m-1}, \left(\frac{(\lambda-1)\delta^{-\lambda/(\lambda-1)}}{\lambda M}\right)^{m-1}\right\},$$

$$k_2 = (m-1)\left[\frac{m(m-\lambda+1)}{(\lambda-1)(\lambda-1)} + b\,\frac{\lambda}{\lambda-1}\right],$$

$$T_1 = k_1/k_2,$$

and $M = \int_{-\infty}^{+\infty} u_0(s)\, ds$. For this choice of k_1, k_2, and M, $\bar{u}(x, t)$ satisfies

(a) $\quad \bar{u}_t - (\bar{u}^m)_{xx} - b(\bar{u}^\lambda)_x \geqslant 0 \qquad \text{in} \quad (-\infty, \delta] \times [0, T_1]$

(b) $$\int_{-\infty}^{x} u_0(s)\, ds \leqslant (L_0 + 1)\, |x|^{\lambda/(\lambda-1)} \leqslant \int_{-\infty}^{x} \bar{u}(s, 0)\, ds \qquad \text{in} \quad [0, \delta]$$

(c) $$\int_{-\infty}^{\delta} u(s, t)\, ds \leqslant M \leqslant \int_{-\infty}^{\delta} u(s, t)\, ds \qquad \text{if} \quad t \in [0, T_1].$$

Hence, by Lemma 2.1 we have that

$$\int_{-\infty}^{x} u(s,t)\,ds \leqslant M \leqslant \int_{-\infty}^{x} \bar{u}(s,t)\,ds \qquad \text{in} \quad (-\infty, \delta] \times [0, T_1]$$

and then $\zeta_-(t) = 0$ for all $t \in [0, T_1]$.

Now, let us assume that $L_0 = \infty$. Then for any $C > 0$ there exists $\delta > 0$ such that

$$C\,|x|^{\lambda/(\lambda-1)} \leqslant \int_{-\infty}^{x} u_0(s)\,ds \qquad \text{if} \quad x \in [0, \delta].$$

Next, we introduce the family of travelling wave solutions $v(x, t; k, \bar{x})$,

$$v(x, t; k, \bar{x}) = \mu_k([x + kt - \bar{x}]_+) \qquad \text{in} \quad Q = \mathbb{R} \times \mathbb{R}^+,$$

where $k > 0$, $\bar{x} \in \mathbb{R}$, and μ_k is the function defined implicitly by the condition

$$\xi = m \int_0^{\mu_k} \frac{s^{m-2}}{k - bs^{\lambda-1}}\,ds.$$

Notice that $v(x, t; k, \bar{x})$ satisfies

$$\|v(x, 0; k, \bar{x})\|_{L^\infty(\mathbb{R})} = bk^{1/(\lambda-1)}$$

and that

$$\int_{-\infty}^{x} v(s, 0; k, \bar{x})\,ds \leqslant bk^{1/(\lambda-1)}(x - \bar{x})_+ \qquad \text{if} \quad x \in \mathbb{R}.$$

Next, we choose $k = k(\bar{x}, C)$ by

$$k = \left(\frac{\lambda C}{(\lambda-1)b}\right)^{\lambda-1} |\bar{x}|.$$

Then, by a convexity argument we conclude that

$$bk^{1/(\lambda-1)}\,|x - \bar{x}| \leqslant C\,|x|^{\lambda/(\lambda-1)} \qquad \text{if} \quad x \in [0, \delta].$$

Moreover, if $\bar{x}$ is enough small

$$\int_{-\infty}^{\delta} v(s, t; k, \bar{x})\,ds \leqslant \int_{-\infty}^{\delta} u(s,t)\,ds \qquad \text{if} \quad t \in [0, 1].$$

Then by Lemma 2.1 we have

$$\int_{-\infty}^{x} v(s, t; k, \bar{x})\, ds \leqslant \int_{-\infty}^{x} u(s, t)\, ds \qquad \text{in} \quad (-\infty, \delta] \times [0, 1].$$

Therefore,

$$\zeta_-(t; \bar{x}) \geqslant \zeta_-(t) \qquad \text{if} \quad t \in [0, 1],$$

where $\zeta_-(t; \bar{x})$ is defined by

$$\zeta_-(t; \bar{x}) = \left[1 - \left(\frac{\lambda C}{(\lambda - 1) b}\right)^{\lambda - 1} t\right] |\bar{x}|.$$

Then for $t > 0$ small, we can choose $C > 0$ such that $\zeta_-(t; \bar{x}) < 0$ and hence $\zeta_-(t) < 0$. ∎

We shall study now the initial behavior of the right front $\zeta_+(t)$.

THEOREM 4.2. *Let $u(x, t)$ be the solution of* (1.1)–(1.2). *We assume that there exists the limit*

$$L_0 = \operatorname*{Lim}_{x \to \zeta_+(0)^-} |x - \zeta_+(0)|^{(m - \lambda + 1)/(m - \lambda)} \int_x^{+\infty} u(s)\, ds \qquad (L_0 \leqslant +\infty).$$

Then

(i) *if* $L_0 < ((m - \lambda)/(m - \lambda + 1))\, C_0$ *there exists* $t^* > 0$ *such that* $u(\zeta_+(0), t) = 0$ *for all* $t \in [0, t^*]$

(ii) *if* $L_0 > ((m - \lambda)/(m - \lambda + 1))\, C_0$ *there exist* C_1 *and* $t_1 > 0$ *such that*

$$\zeta_+(t) \geqslant \zeta_+(0) + C_1 t^{(m - \lambda)/(m + 1 - 2\lambda)} \qquad \textit{if} \quad t \in [0, t_1],$$

where $C_0 = (b(m - \lambda)/m)^{1/(m - \lambda)}$.

Proof. We first remark that the function $z(x, t) = C_0(-x)_+^{1/(m - \lambda)}$ is a stationary solution of Eq. (1.1). If $L_0 < ((m - \lambda)/(m - \lambda + 1))\, C_0$, from the continuity of u we deduce that there exists $t^* > 0$ such that for any $t \in [0, t^*]$ we have

$$\int_x^{+\infty} u(s, t)\, ds \leqslant \int_x^{+\infty} z(s, t)\, ds \qquad \text{for} \quad x \text{ near } \zeta_-(0).$$

Therefore, by Lemma 2.1 we have that $\int_{\zeta_-(0)}^{+\infty} u(s, t)\, ds = 0$ for all $t \in [0, t^*]$, and (i) follows.

In order to proof (ii), we choose $C>0$ such that $C_0<C<((m-\lambda+1)/(m-\lambda))\,L_0$, and we define $v_0(x)=C(-x)_+^{1/(m-\lambda)}$. Then there exists $\delta>0$ such that

$$\int_x^{\infty} u_0(s)\,ds \leqslant \int_x^{\infty} v_0(s)\,ds \qquad \text{if} \quad x\in[-\delta, 0].$$

Hence we can use the Lemma 2.1 for comparing $\zeta_+(t)$ and the interface of $v(x, t)$ (solution corresponding to the initial data $v_0(x)$). Moreover, since $C>C_0$, we conclude (ii) by using the Theorem 6 of [1]. ∎

5. Waiting Time Conditions for (λ, m) in Region III

In this last section we shall assume that $\lambda\leqslant 1$ and $\lambda<m$. The first important difference compared to the above cases appears already for $\lambda=1$. Indeed, in this case it is well known that $m>1$ implies the existence of the interfaces $\zeta_-(t)$ and $\zeta_+(t)$ (see [6]), nevertheless, as we showed in [1], in this case $\zeta_-(t)$ can never exhibit a waiting time ($\zeta_-(t)\leqslant\zeta_-(0)-bt$ for all $t>0$).

When $\lambda<1$ it turns out that the convection dominates diffusion in such a strong way that, in fact, the left interface $\zeta_-(t)$ does not exist, i.e., $\inf\{x : u(x, t)>0\}=-\infty$ for any $t>0$. This result was first proved in [6] for $m\geqslant 1$, and later, for $m>\lambda$, and $m>\lambda$, and $m>0$ arbitrary in [10].

The behavior of $\zeta_+(t)$ is completely different. This front exists for any value (λ, m) in that region (see [6] for $\lambda<1\leqslant m$ and [10] for the general case $\lambda<m$). The following result shows that, under a suitable assumption on $u_0(x)$, $\zeta_+(t)$ is in fact a reversing front near $t=0$.

Theorem 5.1. *Let $u(x, t)$ be the solution of* (1.1)–(1.2). *Assume that there exists the limit*

$$L_0=\operatorname*{Lim}_{x\to\zeta_+(0)^-} |x-\zeta_+(0)|^{(m-\lambda+1)/(m-\lambda)} \int_x^{+\infty} u_0(s)\,ds \qquad (L_0\leqslant +\infty).$$

Then

(i) *if $L_0<((m-\lambda)/(m-\lambda+1))\,C_0$ there exist C_1 and $t_1>0$ such that*

$$\zeta_+(t)\leqslant\zeta_+(0)-C_1 t^{(m-\lambda)/(m+1-2\lambda)} \qquad \text{if} \quad t\in[0, t_1]$$

(ii) *if* $L_0 > ((m-\lambda)/(m-\lambda+1))\, C_0$ *there exist* C_1 *and* $t_1 > 0$ *such that*

$$\zeta_+(t) \geq \zeta_+(0) + C_1 t^{(m-\lambda)/(m+1-2\lambda)} \qquad \text{if} \quad t \in [0, t_1],$$

where $C_0 = (b(m-\lambda)/m)^{1/(m-\lambda)}$.

Proof. (i) If $L_0 < (m-\lambda)/(m-\lambda+1)\, C_0$, we choose $C > 0$ such that $C_0 > C > ((m-\lambda+1)/(m-\lambda))\, L_0$, and we define $v_0(x) = C(-x)_+^{(m-\lambda)}$. Then there eixsts $\delta > 0$ such that

$$\int_x^\infty u_0(s)\, ds \leq \int_x^\infty v_0(s)\, ds \qquad \text{if} \quad x \in [-\delta, 0].$$

Hence we use the Lemma 2.1 to compare $\zeta_+(t)$ and the interface of $v(x, t)$ (solution with initial data $v_0(x)$). Moreover, since $C < C_0$, we conclude (i) by using the Theorem 8 of [1].

(ii) If $L_0 > ((m-\lambda)/(m-\lambda+1)\, C_0$, we choose $C > 0$ such that $C_0 < C < ((mm-\lambda+1)/(m-\lambda))\, L_0$, and we define $v_0(x) = C(-x)_+^{1/(m-\lambda)}$, there exists $\delta > 0$ such that

$$\int_x^\infty u_0(s)\, ds \leq \int_x^\infty v_0(s)\, ds \qquad \text{if} \quad x \in [-\delta, 0].$$

Hence, again by Lemma 2.1 we compare $\zeta_+(t)$ and the interface of $v(x, t)$ (solution with initial data $v_0(x)$). Moreover, since $C > C_0$, we conclude (ii) by using the Theorem 9 of [1]. ∎

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References

1. L. Alvarez, J. I. Diaz, and R. Kersner, On the initial growth of the interfaces in nonlinear diffusion-convection processes, *in* "Nonlinear Diffusion Equations and Their Equilibrium States I" (Ni, Peletier, and Serrin, Eds.), pp. 1–20, Springer-Verlag, New York/Berlin, 1987.
2. J. Bear, "Dynamics of Fluids in Porous Media," American Elsevier, New York, 1972.
3. J. Buckmaster, Viscous sheets advancing over dry bed, *J. Fluid. Mech.* **81** (1977), 735–756.
4. S. Chandrasekhar, Stochastic problems in physics and astronomy, *Rev. Modern Phys.* **15** (1943), 1–89.

5. J. I. DIAZ AND R. KERSNER, On a nonlinear degenerate parabolic equation in infiltration or evaporation, *J. Differential Equations* **69** (1987), 368–403.
6. J. I. DIAZ AND R. KERSNER, Non existence d'une des frontieres libres dans une equation dégénérée en theorie de la filtration, *C. R. Acad. Sci. Paris* **296** (1983), 505–508.
7. J. I. DIAZ AND R. KERSNER, On the behavior and cases of nonexistence of the free boundary in a semibounded porous medium, *J. Math. Anal. Appl.* **132** (1988), 281–289.
8. A. FRIEDMAN, "Partial Differential Equations of the Parabolic Type," Prentice-Hall, Englewood Cliffs, NJ, 1969.
9. B. H. GILDING, Properties of solutions of an equation in the theory of infiltration, *Arch. Rational Mech. Anal.* **65** (1977), 203–225.
10. B. H. GILDING, "The Occurrence of Interfaces in Nonlinear Diffusion-Advection Processes," Memorandum 595, Department of Applied Mathematics, Twente University of Technology, 1986.
11. B. H. GILDING, "Improved Theory for a Nonlinear Degenerate Parabolic Equation," Memorandum 587, Department of Applied Mathematics, Twente University of Technology, 1986.
12. B. F. KNERR, The porous medium equation in one dimension, *Trans. Amer. Math. Soc.* **234** (1977), 381–415.
13. O. LADYZHENSKAYA, V. SOLONNIKOV, AND URAL'CEVA, "Linear and Quasilinear Equations of Parabolic Type," translation of Math. Monograph, Vol. 23, Amer. Math. Soc., Providence, RI, 1968.
14. O. OLEINIK, A. KALASHNIKOV, AND YUI-LIN, The Cauchy problem and boundary value problems for equations of the type of nonstationary filtration, *Izv. Akad. Nauk. SSSR. Ser. Mat.* **22** (1958), 667–704.
15. P. ROSENAU AND S. KAMIN, Thermal waves in an absorbing and convective medium, *Physica D* **8** (1963), 273–283.
16. J. L. VAZQUEZ, The interface of one-dimensional flows in porous media, *Trans. Amer. Math. Soc.* **286** (1984), 787–802.
17. J. L. VAZQUEZ, Asymptotic behaviour and propagation properties of the one-dimensional flow of gas in porous medium, *Trans. Amer. Math. Soc.* **277** (1983), 507–527.

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