

6. J.I. Diaz (Madrid), "On the problem with an obstruction for the operator of minimal surfaces and the p -Laplacian".

1°. Introduction.

Let Ω be a bounded open regular subset of \mathbb{R}^N . We consider the problem.

$$(1) \quad \begin{cases} -\operatorname{div} [Q(|\nabla u|)\nabla u] \geq -\lambda, & u \geq 0, \quad u \{\operatorname{div} [Q(|\nabla u|)\nabla u] - \lambda\} = 0 \quad \text{in } \Omega, \\ u = 1 \quad \text{on } \partial\Omega, \end{cases}$$

where λ is given and $Q(r)$ satisfies conditions which hold in particular cases: (a) $Q(r) = r^{p-2}$, $p > 1$ (p -Laplacian), (b) $Q(r) = (1+r^2)^{-1/2}$ (operator of minimal surfaces). Physical motivations can be found in [1]. We state results on the existence and uniqueness of the solution of the problem (1) and then describe certain properties of the coincidence set $\{x \in \Omega : u(x) = 0\}$, proved by the maximum principle and the principle of comparing masses.

2°. On the existence and uniqueness of the solution.

We suppose that $Q \in C^2((0, +\infty))$, $rQ(r) \rightarrow 0$ as $r \rightarrow 0$, and that $r^2Q(r)$ is a strictly increasing complex function. We denote by $W^{1,A}(\Omega)$ the Orlicz-Sobolev space corresponding to $A(r) \equiv \int_0^r Q(s) ds$. The function $u \in W^{1,A}(\Omega)$ is called a *variational solution* of (1) if u realizes a minimum of the function $J(v) = \int_{\Omega} [A(|\nabla v|) + \lambda v] dx$ on the convex set

$$K = \{v \in W^{1,A}(\Omega); v \geq 0 \text{ in } \Omega, 1 - v \in W_0^{1,A}(\Omega)\}.$$

Theorem 1. Let $\lambda \in (W_0^{1,p}(\Omega))'$ and suppose that the inequality

$$(2) \quad \max \{ |\Omega|^{1/N}, \|\lambda\|_{L^N(\Omega)} \} \leq N \omega_N^{1/N} \lim_{r \rightarrow \infty} r Q(r)$$

holds, where ω_N is the volume of the unit ball in \mathbb{R}^N . Then there exists a unique variational solution of (1).

Remark. If $Q(r) = r^{p-2}$, $p > 1$, then (2) holds automatically, but in the case $Q(r) = (1+r^2)^{-1/2}$ it is an "almost necessary" restriction (this follows from work by Miranda, Giusti, and others).

From now on we assume that λ is a positive constant. Then (1) can be rewritten as follows:

$$(3) \quad \begin{cases} -\operatorname{div} [Q(|\nabla u|) \nabla u] + \lambda \beta(u) = 0 & \text{in } \Omega, \\ u = 1 & \text{on } \partial\Omega, \end{cases}$$

where $\beta(r)$ is the maximal monotone graph: $\beta(r) = \{0\}$ for $r < 0$, $\beta(0) = [0, 1]$, $\beta(r) = \{1\}$ for $r > 0$.

Proposition 1. If $u(x)$ is a variational solution of (1), then there exists $b_u \in L^1(\Omega)$ such that $b_u(x) \in \beta(u(x))$ for almost all $x \in \Omega$, and the integral identity

$$\int_{\Omega} Q(|\nabla u|) \nabla u \cdot \nabla v \, dx + \lambda \int_{\Omega} b_u v \, dx = 0$$

holds for any $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Under certain hypotheses with regard to $\partial\Omega$ a variational solution of (1) belongs to $W^{1,\infty}(\Omega)$.

3°. Estimates of the location of the coincidence set.

The results obtained in [1] for the case $Q(r) = r^{p-2}$ admit the following extension to the case of an operator of minimal surfaces.

Theorem 2. Let $R_N = \max\{1, N/\lambda\}$, $R_1 = \min\{1, 1/\lambda\}$. Then: the following inclusions hold for the coincidence set $I \equiv \{x \in \Omega; u(x) = 0\}$:

$$\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) \geq R_N\} \subset I \subset \Omega \setminus \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) \leq R_1\}.$$

Corollary 1. Let ρ_Ω be the radius of the largest ball containing Ω . If $\rho_\Omega > R_N$, then I is non-empty; if, on the other hand, $\rho_\Omega < R_1$, then I is empty and $u > 0$ throughout Ω .

In the proof of Theorem 2 the function $v(x, x_0) = R - (R^2 - |x - x_0|^2)^{1/2}$ is used as a local supersolution, where x_0 is sufficiently far from $\partial\Omega$, and the function $v(x, x_0) = R - [R^2 - (x_{0,1} - x_1)^2]^{1/2}$ is used as a subsolution, where $x_0 \in \partial\Omega$ and $x_{0,1} - x_1$ is the first component of the vector $x_0 - x$.

4°. Estimates of the measure of the coincidence set: "principle of comparison of masses".

Let Ω^* be a ball with centre at the origin and $|\Omega^*| = |\Omega|$. Let U be a solution of (1) with Ω replaced by Ω^* . We denote by f^* the decreasing symmetrization of f .

Theorem 3. Let $b_u \in \beta(u)$ and $b_U \in \beta(U)$ be functions of class $L^1(\Omega)$ in Proposition 1. Then

$$\int_{|x| \leq r} (b_u)^* dx \geq \int_{|x| \leq r} (b_U)^* dx$$

for any $r > 0$.

Corollary 2. If $U > 0$ in Ω^* , then $u > 0$ in Ω . In particular, if $\operatorname{mes} \Omega \leq \omega_N R_N^N$, then $u > 0$ in Ω .

Estimates for $|I|$ yield the following result.

Theorem 4. Let $s \in (0, |\Omega|)$. We set

$$M_{\Omega, \lambda}(s) = \int_s^{|\Omega|} [1/\alpha(s)] B^{-1}(\lambda(\sigma - s)/\alpha(s)) d\sigma,$$

where $B(r) = rQ(r)$, $\alpha(s) = N\omega_N^{1/N} s^{(N-1)/N}$. Then either $|I| = 0$ or $|I| \leq F_{\Omega, \lambda}$, where $s = F_{\Omega, \lambda}$ is the unique solution of the equation $M_{\Omega, \lambda}(s) = 1$.

Reference

- [1] J.I. Díaz, Nonlinear partial differential equations and free boundaries. Vol. I, Elliptic equations, Pitman, Boston, MA—London 1985. MR 88d:35058.
- 7. A.I. Vol'pert and V.A. Vol'pert, "Wave solutions of parabolic systems".
- 8. M.V. Safonov, "On second-order non-linear parabolic equations".
- 9. A.E. Shishkov, "Behaviour of solutions of mixed problems for quasilinear parabolic equations in unbounded domains".

In the unbounded domain $Q = \Omega \times (0, T)$, $T < \infty$, $\Omega \subset \mathbb{R}^n$, we consider the problem

$$(1) \quad Pu \equiv u_t + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_x^\alpha a_\alpha(x, t, u, \nabla_x u, \dots, \nabla_x^m u) = F(x, t);$$

$$(2) \quad u|_{t=0} = f, \quad D_x^\alpha u|_{\Gamma = \partial\Omega \times (0, T)} = 0 \quad \text{for all } \alpha: |\alpha| \leq m-1;$$

where the $a_\alpha(x, t, \xi)$ are Carathéodory functions satisfying the conditions

$$|a_\alpha(x, t, \xi)| \leq d_0 |\xi|^{(m)} |x|^{p-1} + d_1 \sum_{i=0}^{m-1} |\xi^{(i)}|^{p|\alpha|, i-1}, \quad d_0 < \infty, \quad d_1 < \infty, \quad 2 \leq p_k, i \leq p < \infty;$$

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi) \xi_\alpha^{(m)} \geq d_2 |\xi|^{(m)} |x|^p - d_3 \sum_{i=0}^{m-1} |\xi^{(i)}|^{q_i}, \quad d_2 > 0, \quad d_3 < \infty, \quad q_i = \frac{(p_{m, i-1}) p}{p-1}.$$

We establish a general result of Phragmén—Lindelöf type on the behaviour of solutions of the problem (1), (2), which in the case of the structure equation of the "principal part" ($d_1 = 0$; $d_3 = 0$; $a_\alpha(x, t, \xi) \equiv 0$ for all $\alpha: |\alpha| \leq m-1$) takes the form:

Theorem. Let $h(\tau)$ be an arbitrary non-decreasing positive function satisfying the following condition of Teklindovskii type:

$$\int_c^\infty (\tau h(\tau))^{-1} d\tau = \infty, \quad c < \infty.$$

Then for any non-trivial generalized solution of the homogeneous problem (1), (2) there exists a sequence $\tau_i \rightarrow \infty$ such that

$$\int_{Q_i \equiv Q \cap \{t \leq \tau_i\}} |u|^p dx dt)^{1/p} \tau_i^{-\gamma_1} (h(\tau_i))^{-\gamma_2} \rightarrow \infty \quad \text{as } i \rightarrow \infty,$$

where $\gamma_1 = \frac{n}{p} + \frac{mp}{p-2}$, $\gamma_2 = \frac{2}{p(p-2)}$.

Under additional assumptions we establish similar growth estimates for the difference of solutions of the inhomogeneous problem (1), (2) possessing a certain additional smoothness. These estimates lead to new uniqueness classes of solutions of mixed problems for degenerate parabolic equations in unbounded domains.

Evening session, 31 May 1991, Section 3

- 1. A.A. Agrachev, "On pseudomanifolds of symmetric and Hermitian matrices with multiple eigenvalues".
- 2. V.I. Blagodatskikh, "The problem of optimal control for differential inclusions".