6. J.I. Diaz (Madrid), "On the problem with an obstruction for the operator of minimal surfaces and the p-Laplacian".

1°. Introdution.

Let Ω be a bounded open regular subset of $\mathbb{R}^N$. We consider the problem

$$(1)\qquad \begin{cases} -\operatorname{div}[Q(|\nabla u|)\nabla u] \geqslant -\lambda, \quad u \geqslant 0, \ u\{\operatorname{div}[Q(|\nabla u|)\nabla u] - \lambda\} = 0 \ \text{ in } \Omega, \\ u = 1 \ \text{ on } \partial\Omega, \end{cases}$$

where λ is given and $Q(r)$ satisfies conditions which hold in particular cases: (a) $Q(r) = r^{p-2}$, $p > 1$ (p-Laplacian), (b) $Q(r) = (1+r^2)^{-1/2}$ (operator of minimal surfaces). Physical motivations can be found in [1]. We state results on the existence and uniqueness of the solution of the problem (1) and then describe certain properties of the coincidence set $\{x \in \Omega : u(x) = 0\}$, proved by the maximum principle and the principle of comparing masses.

2°. On the existence and uniqueness of the solution.

We suppose that $Q \in C^2((0, +\infty))$, $rQ(r) \to 0$ as $r \to 0$, and that $r^2Q(r)$ is a strictly increasing complex function. We denote by $W^{1,A}(\Omega)$ the Orlicz–Sobolev space corresponding to $A(r) \equiv \int_0^r sQ(s)\,ds$. The function $u \in W^{1,A}(\Omega)$ is called a *variational solution* of (1) if u realizes a minimum of the function $J(v) = \int_\Omega [A(|\nabla v|) + \lambda v]\,dx$ on the convex set

$$K = \{v \in W^{1,A}(\Omega);\ v \geqslant 0 \ \text{ in } \ \Omega,\ 1 - v \in W_0^{1,A}(\Omega)\}.$$

Theorem 1. *Let* $\lambda \in (W_0^{1,A}(\Omega))'$ *and suppose that the inequality*

$$\max\{|\Omega|^{1/N}, \|\lambda\|_{L^N(\Omega)}\} \leqslant N\omega_N^{1/N} \lim_{r\to\infty} rQ(r) \tag{2}$$

holds, where ω_N *is the volume of the unit ball in* $\mathbb{R}^N$. *Then there exists a unique variational solution of* (1).

Remark. If $Q(r) = r^{p-2}$, $p > 1$, then (2) holds automatically, but in the case $Q(r) = (1+r^2)^{-1/2}$ it is an "almost necessary" restriction (this follows from work by Miranda, Giusti, and others).

From now on we assume that λ is a positive constant. Then (1) can be rewritten as follows:

$$\begin{cases} -\operatorname{div}[Q(|\nabla u|\nabla u])+\lambda\beta(u) \ni 0 \text{ in } \Omega, \\ u = 1 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where $\beta(r)$ is the maximal monotone graph: $\beta(r) = \{0\}$ for $r < 0$, $\beta(0) = [0, 1]$, $\beta(r) = \{1\}$ for $r > 0$.

Proposition 1. *If* $u(x)$ *is a variational solution of* (1), *then there exists* $b_u \in L^1(\Omega)$ *such that* $b_u(x) \in \beta(u(x))$ *for almost all* $x \in \Omega$, *and the integral identity*

$$\int_\Omega Q(|\nabla u|)\nabla u\cdot\nabla v\,dx + \lambda\int_\Omega b_u v\,dx = 0$$

holds for any $v \in W_0^{1,A}(\Omega) \cap L^\infty(\Omega)$.

Under certain hypotheses with regard to $\partial\Omega$ a variational solution of (1) belongs to $W^{1,\infty}(\Omega)$.

3°. Estimates of the location of the coincidence set.

The results obtained in [1] for the case $Q(r) = r^{p-2}$ admit the following extension to the case of an operator of minimal surfaces.

Theorem 2. *Let* $R_N = \max\{1, N/\lambda\}$, $R_1 = \min\{1, 1/\lambda\}$. *Then the following inclusions hold for the coincidence set* $I \equiv \{x \in \Omega;\ u(x) = 0\}$:

$$\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geqslant R_N\} \subset I \subset \Omega \setminus \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \leqslant R_1\}.$$

Corollary 1. *Let* ρ_Ω *be the radius of the largest ball containing* Ω. *If* $\rho_\Omega > R_N$, *then* I *is non-empty; if, on the other hand,* $\rho_\Omega < R_1$, *then* I *is empty and* $u > 0$ *throughout* Ω.

In the proof of Theorem 2 the function $v(x, x_0) = R-(R^2-|x-x_0|^2)^{1/2}$ is used as a local supersolution, where x_0 is sufficiently far from $\partial\Omega$, and the function $v(x, x_0) = R-[R^2-(x_{0,1}-x_1)_+^2]^{1/2}$ is used as a subsolution, where $x_0 \in \partial\Omega$ and $x_{0,1}-x_1$ is the first component of the vector x_0-x.

4°. Estimates of the measure of the coincidence set: "principle of comparison of masses".

Let Ω^* be a ball with centre at the origin and $|\Omega^*| = |\Omega|$. Let U be a solution of (1) with Ω replaced by Ω^*. We denote by f^* the decreasing symmetrization of f.

Theorem 3. *Let* $b_u \in \beta(u)$ *and* $b_U \in \beta(U)$ *be functions of class* $L^1(\Omega)$ *in Proposition* 1. *Then*

$$\int_{|x|\leqslant r} (b_u)^* dx \geqslant \int_{|x|\leqslant r} (b_U)^* dx$$

for any $r > 0$.

Corollary 2. *If* $U > 0$ *in* Ω^*, *then* $u > 0$ *in* Ω. *In particular, if* $\operatorname{mes}\Omega \leqslant \omega_N R_N^N$, *then* $u > 0$ *in* Ω.

Estimates for $|I|$ yield the following result.

Theorem 4. *Let* $s \in (0, |\Omega|)$. *We set*

$$M_{\Omega,\lambda}(s) = \int_s^{|\Omega|} [1/\alpha(s)]\, B^{-1}(\lambda(\sigma-s)/\alpha(s))\,d\sigma,$$

where $B(r) = rQ(r)$, $\alpha(s) = N\omega_N^{1/N}s^{(N-1)/N}$. *Then either* $|I| = 0$ *or* $|I| \leqslant F_{\Omega,\lambda}$, *where* $s = F_{\Omega,\lambda}$ *is the unique solution of the equation* $M_{\Omega,\lambda}(s) = 1$.

Reference

[1] J.I. Diaz, Nonlinear partial differential equations and free boundaries. Vol. I, Elliptic equations, Pitman, Boston, MA—London 1985. MR 88d:35058.

7. A.I. Vol'pert and V.A. Vol'pert, "Wave solutions of parabolic systems".

8. M.V. Safonov, "On second-order non-linear parabolic equations".

9. A.E. Shishkov, "Behaviour of solutions of mixed problems for quasilinear parabolic equations in unbounded domains".

In the unbounded domain $Q = \Omega\times(0, T)$, $T < \infty$, $\Omega \subset \mathbb{R}^n$, we consider the problem

$$(1)\qquad Pu \equiv u_t + \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D_x^\alpha a_\alpha(x, t, u, \nabla_x u, \ldots, \nabla_x^m u) = F(x, t);$$

$$(2)\qquad u\,|_{t=0} = f,\quad D_x^\alpha u\,|_{\Gamma=\partial\Omega\times(0,T)} = 0 \quad \text{for all } \alpha\colon |\alpha| \leqslant m-1;$$

where the $a_\alpha(x, t, \xi)$ are Carathéodory functions satisfying the conditions

$$|a_\alpha(x, t, \xi)| \leqslant d_0|\xi^{(m)}|^{p-1} + d_1\sum_{i=0}^{m-1}|\xi^{(i)}|^{p_{|\alpha|,i}-1},\quad d_0<\infty,\ d_1<\infty,\ 2\leqslant p_{k,i}\leqslant p<\infty;$$

$$\sum_{|\alpha|=m} a_\alpha(x, t, \xi)\,\xi_\alpha^{(m)} \geqslant d_2|\xi^{(m)}|^p - d_3\sum_{i=0}^{m-1}|\xi^{(i)}|^{q_i},\quad d_2>0,\ d_3<\infty,\ q_i = \frac{(p_{m,i}-1)\,p}{p-1}.$$

We establish a general result of Phragmén–Lindelöf type on the behaviour of solutions of the problem (1), (2), which in the case of the structure equation of the "principal part" ($d_1 = 0$; $d_3 = 0$; $a_\alpha(x, t, \xi) \equiv 0$ for all α: $|\alpha| \leqslant m-1$) takes the form:

Theorem. Let $h(\tau)$ be an arbitrary non-decreasing positive function satisfying the following condition of Teklindovskii type:

$$\int_c^\infty (\tau h(\tau))^{-1}d\tau = \infty,\quad c<\infty.$$

Then for any non-trivial generalized solution of the homogeneous problem (1), (2) *there exists a sequence $\tau_i \to \infty$ such that*

$$\int_{Q_i\equiv Q\cap\{|x|<\tau_i\}} |u|^p dx\,dt)^{1/p}\tau_i^{-\gamma_1}(h(\tau_i))^{-\gamma_2} \to \infty \quad as \quad i\to\infty,$$

where $\gamma_1 = \dfrac{n}{p} + \dfrac{mp}{p-2}$, $\gamma_2 = \dfrac{2}{p(p-2)}$.

Under additional assumptions we establish similar growth estimates for the difference of solutions of the inhomogeneous problem (1), (2) possessing a certain additional smoothness. These estimates lead to new uniqueness classes of solutions of mixed problems for degenerate parabolic equations in unbounded domains.

Evening session, 31 May 1991, Section 3

1. A.A. Agrachev, "On pseudomanifolds of symmetric and Hermitian matrices with multiple eigenvalues".

2. V.I. Blagodatskikh, "The problem of optimal control for differential inclusions".