

ISOPERIMETRIC INEQUALITIES IN THE PARABOLIC OBSTACLE PROBLEMS

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ABSTRACT. — In this paper, we are concerned with the parabolic obstacle problem ($u_t = \partial u / \partial t$)

$$(0.1) \quad \begin{cases} u_t + \mathcal{A}u + cu \geq f, & u \geq \psi, & (u_t + \mathcal{A}u + cu - f)(u - \psi) = 0 & \text{in } Q = (0, T) \times \Omega, \\ u = \psi & \text{on } \Sigma = (0, T) \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases}$$

(\mathcal{A} is a linear second order elliptic operator in divergence form or a nonlinear “pseudo-Laplacian”). We give an isoperimetric inequality for the concentration of $u - \psi$ around its maximum. Various consequences are given. In particular, it is proved that $u - \psi$ vanishes after a finite time, under a suitable assumption on $\psi_t + \mathcal{A}\psi + c\psi - f$. Other applications are also given.

These results are deduced from the study of the particular case $\psi = 0$. In this case, we prove that, among all linear second order elliptic operators \mathcal{A} , having ellipticity constant 1, all equimeasurable domains Ω , all equimeasurable functions f and u_0 , the choice giving the “most concentrated” solution around its maximum is: $\mathcal{A} = -\Delta$, Ω is a ball $\tilde{\Omega}$, f and u_0 are radially symmetric and decreasing along the radii of $\tilde{\Omega}$.

A crucial point in our proof is a pointwise comparison result for an auxiliary one-dimensional unilateral problem. This is carried out by showing that this new problem is well-posed in L^∞ in the sense of the accretive operators theory.

Key words: Parabolic obstacle problems, isoperimetric inequalities, comparison by rearrangement, accretive operators, extinction in finite time.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be a bounded regular domain, $Q = (0, T) \times \Omega$; a_{ij} being given functions in $L^\infty(Q)$, $A = (a_{ij}(t, x))$ is a $N \times N$ (not necessarily symmetric) matrix such that

$$(1.1) \quad A(\xi, \xi) = \sum_{i,j=1}^N a_{ij}(t, x) \xi_i \xi_j \geq |\xi|^2 \quad \text{a. e. } (t, x) \in Q, \quad \forall \xi \in \mathbb{R}^N.$$

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For u, v in $L^2(0, T; H_0^1(\Omega))$, denote

$$(1.2) \quad a(u, v) (= a(t, u, v)) = \int_{\Omega} \sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx;$$

$$(1.3) \quad \mathcal{A}u = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(a_{ij}(t, x) \frac{\partial u}{\partial x_i} \right),$$

where the derivatives are taken (at least) in the distribution sense. In the whole paper, c will be a given non-negative function in $L^\infty(Q)$. We denote $\underline{c} = \text{ess inf}_{\Omega} c$. ■

Our first concern will be the parabolic obstacle problem, with obstacle zero, under suitable regularity assumptions on the data. More general situations will be treated later (see Section 4).

With $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega)$, $u_0 \geq 0$, we suppose that u verifies

$$(1.4) \quad \left\{ \begin{array}{l} u \in L^2(0, T; H_0^1(\Omega)), \quad u \geq 0, \quad \frac{\partial u}{\partial t} \in L^2(Q), \quad u|_{t=0} = u_0, \\ \text{and, for almost every } t \text{ in } (0, T), \\ \int_{\Omega} \left(\frac{\partial u}{\partial t} + cu - f \right) (v - u) dx + a(u, v - u) \geq 0, \\ \forall v \in H_0^1(\Omega), \quad v \geq 0. \end{array} \right.$$

Of course, under stronger regularity assumptions, u verifies the complementarity system:

$$(1.4 \text{ bis}) \quad \left\{ \begin{array}{l} u_t + \mathcal{A}u + cu \geq f, \quad u \geq 0, \quad (u_t + \mathcal{A}u + cu - f)u = 0 \text{ in } Q, \\ u = 0 \text{ on } \Sigma = (0, T) \times \partial\Omega, \\ u|_{t=0} = u_0 \text{ in } \Omega. \quad \blacksquare \end{array} \right.$$

In this paper, we shall compare the solution u of (1.4) with the solution U of the "symmetrized problem"

$$(1.4) \quad \left\{ \begin{array}{l} U \in L^2(0, T; H_0^1(\tilde{\Omega})), \quad U \geq 0, \quad \frac{\partial U}{\partial t} \in L^2(\tilde{Q}), \quad U|_{t=0} = u_0, \\ \text{and, for almost every } t \text{ in } (0, T), \\ \int_{\tilde{\Omega}} \left(\frac{\partial U}{\partial t} + \underline{c}U - f \right) (v - U) dx + \int_{\tilde{\Omega}} \nabla U \cdot \nabla (v - U) dx \geq 0, \\ \forall v \in H_0^1(\tilde{\Omega}), \quad v \geq 0. \end{array} \right.$$

In (1.4), the data have been symmetrized in the following way: $\tilde{\Omega}$ is the ball of \mathbb{R}^N , centered in the origin (for example), which has the same measure as Ω , $\tilde{Q} = (0, T) \times \tilde{\Omega}$;

\underline{u}_0 [resp. $\underline{f}(t, \cdot)$] is the rearrangement of u_0 [resp. $f(t, \cdot)$] in $\tilde{\Omega}$, which decreases (non-increases) along the radii.

This rearrangement is defined as follows. If v is a real measurable ¹ function defined in Ω , the one-dimensional decreasing (non-increasing) rearrangement of v is defined in $\Omega^* = [0, |\Omega|]$ (here $\Omega^* = (0, |\Omega|)$) by

$$(1.5) \quad v_*(s) = \text{Inf} \{ \theta \in \mathbb{R}, |v > \theta| \leq s \},$$

where $|v > \theta| = \text{meas} \{ x \in \Omega, v(x) > \theta \}$ ². The rearrangement of v in $\tilde{\Omega}$, which decreases along the radii, is

$$(1.6) \quad \underline{v}(x) = v_*(\alpha_N |x|^N), \quad \text{for } x \in \tilde{\Omega},$$

where α_N is the measure of the unit ball of \mathbb{R}^N . If v is defined in Q , and is measurable with respect to the space variable x of Ω , we consider its rearrangements with respect to x :

$$(1.7) \quad v_*(t, s) = (v(t, \cdot))_*(s) = \text{Inf} \{ \theta \in \mathbb{R}, |v(t, \cdot) > \theta| \leq s \},$$

$$(1.8) \quad \underline{v}(t, x) = v_*(t, \alpha_N |x|^N).$$

Thus, in $(\widetilde{1.4})$, \underline{f} is defined as in (1.7), (1.8). The reader is referred to [2], [20], [24] for some well known properties of the rearrangement mapping (in particular continuity in L^p spaces and Hardy-Littlewood inequality). ■

Remark 1.1. — As $\underline{f} \in L^2(\tilde{Q})$, $\underline{u}_0 \in H_0^1(\tilde{\Omega})$ (by the classical properties of rearrangements—see for example G. Polya, C. Szego [24], C. Bandle [2], J. Mossino [20]), $U \in \mathcal{C}([0, T]; H_0^1(\tilde{\Omega})) \cap L^2(0, T; H^2(\tilde{\Omega}))$, $(\partial U / \partial t) \in L^2(\tilde{Q})$ (see H. Brézis [9]), and thus U verifies the complementarity system:

$$(\widetilde{1.4} \text{ bis}) \quad \left\{ \begin{array}{l} U_t - \Delta U + \underline{c} U \geq \underline{f}, \quad U \geq 0, \quad (U_t - \Delta U + \underline{c} U - \underline{f}) U = 0 \quad \text{a.e. in } \tilde{Q}, \\ U = 0 \quad \text{on } \tilde{\Sigma} = (0, T) \times \partial \tilde{\Omega}, \\ U|_{t=0} = \underline{u}_0 \quad \text{in } \tilde{\Omega}. \end{array} \right.$$

Of course, U is radially symmetric (by the uniqueness of the solution, and the symmetry of the problem). Moreover we shall prove that U decreases along the radii. ■

For the *elliptic obstacle problem*, a pointwise comparison $\underline{u}(x) \leq U(x)$ in $\tilde{\Omega}$ (or $u_*(s) \leq U_*(s)$ in Ω^*) was shown by C. Bandle and J. Mossino [3], who also gave an explicit optimal bound for the measure of the coincidence set $\{u=0\}$ (see also C. Maderna, S. Salsa [19] for a different obstacle problem, in which the coincidence set does not reach the boundary, and J. I. Diaz [13], Theorem 2.20, p. 152).

⁽¹⁾ In the whole paper, we consider the Lebesgue measure.

⁽²⁾ For any measurable set E , we denote by $|E|$ its measure.

In the case of *parabolic equations*, it is well known that pointwise comparisons $\underline{u}(t, x) \leq U(t, x)$ are not true in general, as it can be shown by suitable counterexamples. The “integral” comparison $\int_0^s u_*(t, \sigma) d\sigma \leq \int_0^s U_*(t, \sigma) d\sigma$ (or equivalently $\int_{B_r(0)} \underline{u}(t, x) dx \leq \int_{B_r(0)} U(t, x) dx$, where $B_r(0)$ is the ball of center 0 and radius $r \leq |\Omega|^{1/N} \alpha_N^{-1/N}$) was first proved by C. Bandle (see e.g. [2]), and then by J. L. Vazquez [26] (for the porous media equation) by using a semidiscretization in time, and J. Mossino, J. M. Rakotoson [21], by using the notion of directional derivative of the rearrangement mapping (or relative rearrangement) introduced by J. Mossino, R. Temam [22]. The method of [21] works under weaker hypotheses than in [2] (or [4]) and remains valid for non-autonomous equations (which was not clear in the approach of [26]). Other recent approaches to parabolic equations are due to A. Alvino, P. L. Lions, G. Trombetti [1] and P. Benilan, P. Aborjaily [7] (see also B. Gustafsson, J. Mossino [16]). ■

The main result, in the present paper, is

THEOREM 1. — *If u (resp. U) is the solution of (1.4) (resp. (1.4)), then*

$$(1.9) \quad \int_0^{s_c} u_*(t, \sigma) d\sigma \leq \int_0^s U_*(t, \sigma) d\sigma, \quad \forall t, \quad 0 \leq t \leq T, \quad \forall s, \quad 0 \leq s \leq |\Omega|.$$

It is worth noticing that this result is not obtained just by putting together arguments from [3] and [21]: the combination of these arguments just gives Lemma 2 below. The most difficult part in the proof of Theorem 1 consists in proving a maximum principle ($k \leq K$, with the notations of Lemma 2) which is non standard, in particular since the elliptic part of the operator degenerates, and the constraint is on the derivative ($(\partial k / \partial s) \geq 0$). This crucial part of the proof uses the accretive operators theory.

Of course, the comparison of the L^p (in space, $1 \leq p \leq \infty$) norms of u and U follows immediately from Theorem 1. Moreover, various consequences (for example extinction in finite time) can be derived from (1.9). We shall return on these in Section 3, Section 2 being devoted to a proof of Theorem 1. In Section 4, we shall extend our result to a more “general” obstacle problem in which the coincidence set reaches the boundary, the obstacle zero being replaced by a suitable function ψ [see (0.1)]. We shall also consider the case of “less regular” data, and the case of the operator \mathcal{A} being replaced by an analogous of the p -Laplacian.

Part of these results was announced in a Note aux *Comptes Rendus* [15].

Remark 1.2. — In [15], we got $\int_0^s u_* d\sigma \leq \int_0^s U'_* d\sigma$, where U' is the solution of (1.4) with \underline{c} replaced by 0; as $U \leq U'$, (1.9) is a slight improvement of the corresponding result in [15].

2. Proof of Theorem 1

Our proof needs several preliminary lemmas. We have seen, in Remark 1.1, that U has radial symmetry and solves (1.4 bis). Moreover

LEMMA 1. — *The solution U of (1.4), (1.4 bis) decreases along the radii: $U = \underline{U}$.*

Proof of Lemma 1. — Let

$$g = \frac{\partial U}{\partial t} - \Delta U + \underline{c}U - \underline{f} = \frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial r^2} - \frac{N-1}{r} \frac{\partial U}{\partial r} + \underline{c}U - \underline{f}(r)$$

in terms of $r = |x|$ [$\underline{f}(r)$ is an abusive notation for $f_*(\alpha_N r^N)$]. By (1.4 bis), we have $gU = 0$ in \tilde{Q} . Under stronger regularity assumptions on \underline{f} and \underline{u}_0 , we can differentiate (with respect to r) $gU = 0$. We get, with $W = \partial U / \partial r$,

$$\left(\frac{\partial W}{\partial t} - \Delta W + \left(\underline{c} + \frac{N-1}{|x|^2} \right) W - \frac{\partial \underline{f}}{\partial r}(|x|) \right) U + gW = 0.$$

Now, on $\{U = 0\}$, $W = 0$ implies $(\partial W / \partial t) - \Delta W + (\underline{c} + (N-1)/|x|^2)W = 0$; on $\{U > 0\}$, $g = 0$ implies $(\partial W / \partial t) - \Delta W + (\underline{c} + (N-1)/|x|^2)W = (\partial \underline{f} / \partial r)(|x|) \leq 0$. Thus

$$\left\{ \begin{array}{l} \frac{\partial W}{\partial t} - \Delta W + \left(\underline{c} + \frac{N-1}{|x|^2} \right) W \leq 0 \quad \text{in } \tilde{Q}, \\ W|_{t=0} \leq 0, \\ W|_{\tilde{\Sigma}} \leq 0, \end{array} \right.$$

which gives $W \leq 0$, U decreases (non-increases) along the radii. Let D be the subset of functions in $L^2(\tilde{Q})$ which decrease (non-increase) along the radii, that is:

$$v \in D \quad \text{if and only if} \quad v(t, x) \leq v(t, y) \quad \text{a.e. } t \in (0, T), \quad \text{a.e. } x \in \tilde{\Omega}, \\ \text{a.e. } y \in \tilde{\Omega}, \quad \text{such that } |x| \geq |y|.$$

Clearly D is closed in $L^2(\tilde{Q})$. Let U_n be the solutions of (1.4), with \underline{f} , \underline{u}_0 replaced by f_n , u_{0n} , such that $f_n \rightarrow \underline{f}$, $u_{0n} \rightarrow \underline{u}_0$, as regular as we need in order to justify the previous computations, $f_n \rightarrow \underline{f}$ in $L^2(\tilde{Q})$, $u_{0n} \rightarrow \underline{u}_0$ in $L^2(\tilde{\Omega})$. Then U_n belongs to D by the previous arguments, $U_n \rightarrow U$ in $L^2(\tilde{Q})$, which implies that U belongs to D . ■

LEMMA 2. — *Let u (resp. U) be the solutions of (1.4) (resp. (1.4), (1.4 bis)). Define the functions*

$$(2.1) \quad k(t, s) = \int_0^s u_*(t, \sigma) d\sigma,$$

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$$(2.2) \quad F(t, s) = \int_0^s f_*(t, \sigma) d\sigma, \quad (F \in L^2(0, T; H^1(\Omega^*))),$$

$$(2.3) \quad k_0(s) = \int_0^s u_{0*}(\sigma) d\sigma, \quad (k_0 \in H^1(\Omega^*)),$$

$$(2.4) \quad d(s) = N^2 \alpha_N^{2/N} s^{2-(2/N)},$$

$$(2.5) \quad s(t) = |u(t, \cdot) > 0| = |u_*(t, \cdot) > 0|.$$

Then,

$$(2.6) \quad k \text{ and } K \text{ belong to } H^1(0, T; H^1(\Omega^*)) \cap L^2(0, T; H^2(\varepsilon, |\Omega|)),$$

$\varepsilon > 0$

k verifies

$$(2.7) \quad \left\{ \begin{array}{l} \frac{\partial k}{\partial t} - d \frac{\partial^2 k}{\partial s^2} + \underline{c}k \leq F \quad \text{and} \quad \frac{\partial k}{\partial s} > 0, \\ \text{a. e. } (t, s) \in Q_u^* = \{(t, s), t \in (0, T), s \in (0, s(t))\}, \\ \frac{\partial k}{\partial s} \geq 0, \quad \forall t \in [0, T], \quad \forall s \in \overline{\Omega^*}, \\ k(t, 0) = 0 = \frac{\partial k}{\partial s}(t, s(t)) \left(= \frac{\partial k}{\partial s}(t, |\Omega|) \right), \quad \forall t \in [0, T], \\ k(0, s) = k_0(s), \quad \forall s \in \overline{\Omega^*}; \end{array} \right.$$

K verifies

$$(2.8) \quad \left\{ \begin{array}{l} \frac{\partial K}{\partial t} - d \frac{\partial^2 K}{\partial s^2} + \underline{c}K \geq F \quad \text{and} \quad \frac{\partial K}{\partial s} \geq 0, \quad \text{a. e. } t, s \text{ in } Q^* = (0, T) \times \Omega^*, \\ \left(\frac{\partial K}{\partial t} - d \frac{\partial^2 K}{\partial s^2} + \underline{c}K - F \right) \frac{\partial K}{\partial s} = 0, \quad \text{a. e. } t, s \text{ in } Q^*, \\ K(t, 0) = 0 = \frac{\partial K}{\partial s}(t, |\Omega|), \quad \forall t \in [0, T], \\ K(0, s) = k_0(s), \quad \forall s \in \overline{\Omega^*}. \end{array} \right.$$

Proof of Lemma 2 : For fixed $t \in [0, T]$, we denote for convenience $u = u(t)$, $f = f(t), \dots$ We argue as in [3], for the elliptic obstacle problem, and in [21], for parabolic equations. We get, by taking $v = u \pm (u - \theta)_+$ ($\theta > 0$) in (1.4)

$$a(u, (u - \theta)_+) = \int_{\Omega} \left(f - cu - \frac{\partial u}{\partial t} \right) (u - \theta)_+ dx,$$

that is

$$\int_{u>\theta} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx = \int_{u>\theta} \left(f - cu - \frac{\partial u}{\partial t} \right) (u - \theta) dx,$$

and, differentiating with respect to θ (see for example [20], p. 9, 10),

$$(2.9) \quad -\frac{d}{d\theta} \int_{u>\theta} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx = \int_{u>\theta} \left(f - cu - \frac{\partial u}{\partial t} \right) dx.$$

By using the ellipticity of A [see (1.1)], and the Cauchy-Schwarz inequality (see for example [20], p. 25)

$$\left(-\frac{d}{d\theta} \int_{u>\theta} |\nabla u| dx \right)^2 \leq \mu'(\theta) \frac{d}{d\theta} \int_{u>\theta} \sum_{i,j} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx,$$

where $\mu(\theta) = |u > \theta|$, that is, with (2.9), the Fleming-Rishel formula, and the isoperimetric inequality for the perimeter in the sense of De Giorgi

$$(2.10) \quad \begin{cases} N^2 \alpha^{2/N} \mu(\theta)^{2-(2/N)} \leq -\mu'(\theta) \int_{u>\theta} \left(f - cu - \frac{\partial u}{\partial t} \right) dx, \\ \text{a. e. } \theta \in (0, m) \quad \text{where } m = \text{ess sup}_{\Omega} u. \end{cases}$$

But, for almost every θ (see [21])

$$\int_{u>\theta} \frac{\partial u}{\partial t} dx = w(t, \mu(\theta)),$$

where

$$w(t, s) = \int_0^s \frac{\partial w}{\partial s} d\sigma, \quad \frac{\partial w}{\partial s} = \left(\frac{\partial u}{\partial t} \right)_{*u} = \frac{\partial u_*}{\partial t},$$

v_{*u} denotes the relative rearrangement of v with respect to u , introduced in [22]. We have $w = \partial k / \partial t$, with k defined by (2.1). As, by the Hardy-Littlewood inequality (see [20])

$$\int_{u>\theta} (f - cu) dx \leq \int_0^{\mu(\theta)} (f_* - c^* u_*) d\sigma \leq \int_0^{\mu(\theta)} (f_* - \underline{c} u_*) d\sigma$$

(where c^* denotes the increasing rearrangement of c , defined by $c^*(s) = c_*(|\Omega| - s)$), we get from (2.10)

$$(2.11) \quad N^2 \alpha^{2/N} \mu(\theta)^{2-(2/N)} \leq -\mu'(\theta) G(\mu(\theta)), \quad \text{a. e. } \theta \in (0, m)$$

with $G = F - \underline{c}k - (\partial k / \partial t)$. For fixed t , G is continuous in Ω^* . Now, we prove that G is non-negative on $[0, s(t)]$. By (2.11), $G \circ \mu(\theta) \geq 0$, a. e. $\theta \in (0, m)$. As $G \circ \mu$ is right

continuous, $G \circ \mu \geq 0$ on $[0, m]$, but $G \circ \mu(m) = G(0) = 0$, and $G \circ \mu \geq 0$ on $[0, m]$. It follows $G \circ \bar{\mu} \geq 0$ on $]0, m]$, with $\bar{\mu}(\theta) = |u \geq \theta|$ (since $\mu(\theta - h) \rightarrow \bar{\mu}(\theta)$ when $h \searrow 0$). Let $s \in [0, s(t)[$, $\theta = u_*(s) > 0$. One has $s' = \mu(\theta) \leq s \leq s'' = \bar{\mu}(\theta)$, $G(s') = G \circ \mu(\theta) \geq 0$, $G(s'') = G \circ \bar{\mu}(\theta) \geq 0$, $\partial G / \partial s = f_* - \underline{c}u_* - (\partial u_* / \partial t)$ non increasing on (s', s'') (for f_* in non-increasing, $u_* = \theta$ and $\partial u_* / \partial t$ is a.e. constant on (s', s'') —see [21]). Thus G is concave on $[s', s'']$. We get $G(s) \geq 0$. By continuity it follows $G \geq 0$ on $[0, s(t)]$. By integrating (2.11), we get, for $0 \leq \theta \leq \theta' \leq m$,

$$(2.12) \quad \theta' - \theta \leq N^{-2} \alpha_N^{-2/N} \int_{\theta}^{\theta'} \mu(\tau)^{(2/N)-2} G(\mu(\tau)) (-\mu'(\tau)) d\tau$$

$$\leq N^{-2} \alpha_N^{-2/N} \int_{\mu(\theta')}^{\mu(\theta)} s^{(2/N)-2} G ds$$

and then

$$(2.13) \quad 0 \leq -\frac{\partial u_*}{\partial s} \leq N^{-2} \alpha_N^{-2/N} s^{(2/N)-2} G, \quad \text{a. e. } s \in (0, s(t)).$$

In fact, let $0 \leq s' < s \leq s(t)$ be such that $u_*(s) < u_*(s')$. Observe that, for any $\varepsilon > 0$, $|u_* > u_*(s)| \leq s$, $|u_* > u_*(s') - \varepsilon| \geq |u_* \geq u_*(s')| \geq s'$. Setting $\theta = u_*(s)$, $\theta' = u_*(s') - \varepsilon$ in (2.12), we get

$$u_*(s') - \varepsilon - u_*(s) \leq N^{-2} \alpha_N^{-2/N} \int_{s'}^s \sigma^{(2/N)-2} G d\sigma;$$

by letting ε tend to zero:

$$u_*(s') - u_*(s) \leq N^{-2} \alpha_N^{-2/N} \int_{s'}^s \sigma^{(2/N)-2} G d\sigma;$$

and (2.13) follows, as well as the first inequality in (2.7). Notice that u_* is continuous in $]0, |\Omega|]$, which implies $\partial k / \partial s(t, s(t)) = 0$.

As for the regularity of k , we have $u \in L^2(0, T; H_0^1(\Omega))$ (and $u \geq 0$), then $u_* = \partial k / \partial s \in L^2(0, T; L^2(\Omega^*)) \cap L^2(0, T; H^1(\varepsilon, |\Omega|))$ (see for example [20], remark $\varepsilon > 0$ page 17), which implies $k \in L^2(0, T; H^1(\Omega^*)) \cap L^2(0, T; H^2(\varepsilon, |\Omega|))$. Moreover $\varepsilon > 0$ $u \in H^1(0, T; L^2(\Omega))$ implies $u_* \in H^1(0, T; L^2(\Omega^*))$ (see [21]), and then $k \in H^1(0, T; H^1(\Omega^*))$.

Concerning U , we have, by (1.4 bis) and Lemma 1

$$(2.14) \quad \left\{ \begin{array}{l} \frac{\partial U_*}{\partial t} - \frac{\partial}{\partial s} \left(d \frac{\partial U_*}{\partial s} \right) + \underline{c} U_* \geq f_*, \quad U_* \geq 0, \quad \text{a.e. in } Q^*, \\ \left(\frac{\partial U_*}{\partial t} - \frac{\partial}{\partial s} \left(d \frac{\partial U_*}{\partial s} \right) + \underline{c} U_* - f_* \right) U_* = 0, \quad \text{a.e. in } Q^*, \\ U_*(t, |\Omega|) = 0, \quad \forall t \in [0, T], \\ U_{*|t=0} = u_{0*} \quad \text{in } \Omega^*, \end{array} \right.$$

(since $\partial/\partial s(d(\partial U_*/\partial s))$ is just the expression of ΔU in terms of s). And U_* (resp. K) has (at least) the same regularity as u_* (resp. k). Moreover, as $\Delta U \in L^2(\bar{\Omega})$, we get $\partial/\partial s(d(\partial U_*/\partial s)) \in L^2(\Omega^*)$. The same is true for $\partial U_*/\partial t$. By integrating the first inequality in (2.14) between 0 and s , we get the first inequality in (2.8) [we use $d(\partial U_*/\partial s) = 0(s)$ as in [21], p. 68]. Moreover, if $U_*(t, s) > 0$, we have (by Lemma 1) $U_*(t, \sigma) > 0$ for $\sigma \leq s$, $(\partial U_*/\partial t) - (\partial/\partial s)(d(\partial U_*/\partial s)) + \underline{c} U_* - f_* = 0$ on $(0, s)$, and, again by integrating between 0 and s , $((\partial K/\partial t) - d(\partial^2 K/\partial s^2) + \underline{c} K - F)(s) = 0$, which gives the second line in (2.8). ■

Now, Theorem 1 (to be proved) says that $k \leq K$. Setting $\chi = k - K$, we could check (as in [21], p. 69-70) that

$$\begin{aligned} \frac{\partial \chi}{\partial t} - d \frac{\partial^2 \chi}{\partial s^2} + \underline{c} \chi &\leq 0 \quad \text{in } Q_u^*, \\ \int_0^{s(t)} s^{(2/N)-2} \frac{\partial \chi}{\partial t} \chi_+ ds &\leq N^2 \alpha_N^{2/N} \int_0^{s(t)} \frac{\partial^2 \chi}{\partial s^2} \chi_+ ds \leq 0. \end{aligned}$$

Formally, as

$$\begin{aligned} \frac{\partial}{\partial t} \int_0^{s(t)} s^{(2/N)-2} \chi_+^2(s, t) ds &= s(t)^{(2/N)-2} \chi_+^2(s(t), t) s'(t) \\ &\quad + 2 \int_0^{s(t)} s^{(2/N)-2} \frac{\partial \chi}{\partial t} \chi_+ ds \leq s(t)^{(2/N)-2} \chi_+^2(s(t), t) s'(t), \end{aligned}$$

we get $\int_0^{s(t)} s^{(2/N)-2} \chi_+^2(s, t) ds$ non-increasing with respect to t , as soon as $t \mapsto s(t)$ is non-increasing. In this case

$$\int_0^{s(t)} s^{(2/N)-2} \chi_+^2(s, t) ds \leq \int_0^{s(0)} s^{(2/N)-2} \chi_+^2(s, 0) ds = 0$$

and then $\chi \leq 0$ in Q_u^* . As $\partial \chi / \partial s = u_* - U_* \leq 0$ outside Q_u^* , it follows $\chi \leq 0$ in Q^* .

We are going to develop a rigorous proof of Theorem 1, which does not use any monotonicity (or regularity) assumption on $t \mapsto s(t)$. Our proof uses the theory of maximal monotone graphs (see for example [10]) and accretive operators [5]. ■

Let β be the multivalued operator, mapping its domain $D(\beta) = \mathbb{R}^+$ into $P(\mathbb{R})$ (the set of all subsets of \mathbb{R}), defined by

$$(2.15) \quad \beta(t) = \begin{cases} (-\infty, 0] & \text{if } t=0, \\ 0 & \text{if } t>0. \end{cases}$$

Let the Banach space $X = L^\infty(\Omega^*)$, and let the multivalued operator $M : D(M) \rightarrow P(X)$ be defined by

$$(2.16) \quad \left\{ \begin{array}{l} D(M) = \{ w \in \mathcal{C}^0(\overline{\Omega^*}) \cap \underset{\varepsilon > 0}{H^2(\varepsilon, |\Omega|)}; \quad w(0) = w'(|\Omega|) = 0; \\ w'(s) \geq 0, \quad \forall s \in]0, |\Omega|]; \\ \exists b \in L^2(\Omega^*), \quad b(s) \in \beta(w'(s)) \text{ a. e. } s \in \Omega^*, \quad -dw'' + \underline{c}w + b \in L^\infty(\Omega^*) \}, \\ \text{(Note } D(M) \subset \mathcal{C}^0(\overline{\Omega^*}) \cap \mathcal{C}^1(]0, |\Omega|]), \\ Mw = \{ -dw'' + \underline{c}w + b \in L^\infty(\Omega^*) \text{ such that } b \in L^2(\Omega^*), \\ b(s) \in \beta(w'(s)) \text{ a. e. } s \in \Omega^* \}. \end{array} \right.$$

LEMMA 3. — *The functions k and K defined in Lemma 2 are strong solutions of two Cauchy problems associated with M , i. e.*

$$k, K \in \mathcal{C}^0([0, T]; X); \quad \frac{dk}{dt} \text{ and } \frac{dK}{dt} \in L^1(0, T; X);$$

and for almost every t in $(0, T)$:

$$(2.17) \quad k(t) \text{ and } K(t) \in D(M) \text{ and } \left. \begin{array}{l} \frac{dk}{dt} + Mk \ni g \\ \frac{dK}{dt} + MK \ni G \end{array} \right\} \text{ in } X$$

(and $k(0) = K(0) = k_0 \in X$), where

$$(2.18) \quad g = \text{Min} \left(\frac{\partial k}{\partial t} - d \frac{\partial^2 k}{\partial s^2} + \underline{c}k, F \right) = \begin{cases} \frac{\partial k}{\partial t} - d \frac{\partial^2 k}{\partial s^2} + \underline{c}k & \text{in } Q_u^*, \\ \text{Min} \left(\frac{\partial k}{\partial t} + \underline{c}k, F \right) & \text{in } Q^* \setminus Q_u^*, \end{cases}$$

$$(2.19) \quad G = F(\geq g); \quad g \text{ and } G \in L^1(0, T; X).$$

Proof of Lemma 3. — It is an easy consequence of the regularity and properties in Lemma 2. We have $g \in L^1(0, T; X)$ as k, F and $(\partial k/\partial t) \in L^2(0, T; H^1(\Omega^*)) \subset L^2(0, T; X)$,

$$\begin{aligned} \frac{\partial k}{\partial t} &\leq \frac{\partial k}{\partial t} - d \frac{\partial^2 k}{\partial s^2} + \underline{c}k \left\{ \begin{array}{l} \leq F \text{ in } Q_u^* \\ = \frac{\partial k}{\partial t} + \underline{c}k \text{ in } Q \setminus Q_u^* \end{array} \right\}, \\ &\leq \text{Max} \left\{ F, \frac{\partial k}{\partial t} + \underline{c}k \right\}. \end{aligned}$$

Moreover $k(t)$ [resp. $K(t)$] belongs to $D(M)$, since e. g.

$$-d \frac{\partial^2 k}{\partial s^2}(t, \cdot) + \underline{c}k(t, \cdot) \in L^\infty(\Omega^*) \quad [\text{take } b=0 \text{ in (2.16)}].$$

In fact, by the first inequality in (2.7), we get, using Cauchy-Schwarz inequality as in [21], p. 68,

$$0 \leq -d \frac{\partial^2 k}{\partial s^2} + \underline{c}k \left\{ \begin{array}{l} \leq s^{1/2} \left[\|f(t)\|_{L^2(\Omega)} + \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^2(\Omega)} \right] \text{ in } Q_u^*, \\ = \underline{c}k \text{ in } Q^* \setminus Q_u^*. \end{array} \right.$$

It follows that also $d(\partial^2 k/\partial s^2)$ belongs to $L^\infty(\Omega^*)$ and the second assertion in (2.17) easily follows from the definitions of g and G and from Lemma 2. ■

LEMMA 4. — *The operator M , defined in (2.16), is m -T accretive in $X = L^\infty(\Omega^*)$, that is*

$$(2.20) \quad R(I+M) = X \quad (X \text{ is exactly the image of } I+M),$$

i. e. M is m accretive, and

$$(2.21) \quad \|[(I+M)^{-1}g - (I+M)^{-1}\hat{g}]_+\|_X \leq \|(g - \hat{g})_+\|_X.$$

i. e. M is T accretive.

The proof of Lemma 4 is technical and is given in the Appendix. ■

Now, we can apply the abstract theory (see P. Benilan [5], C. Picard [23], P. Benilan, M. G. Crandall, A. Pazy [6]) to obtain our main result. Let $\Phi: X \rightarrow \mathbb{R}$ be a continuous convex function on some Banach space X . We define the product $[\cdot, \cdot]_\Phi: X \times X \rightarrow \mathbb{R}$, by

$$(2.22) \quad [x, y]_\Phi = \lim_{\lambda \searrow 0} \frac{\Phi(x + \lambda y) - \Phi(x)}{\lambda}$$

We also recall a well known result [generalization of a lemma due to Kato (see e.g. [5], [25]): If $u: [0, T] \rightarrow X$ is absolutely continuous on $[0, T]$, then $(d/dt) \Phi(u(t)) = [u(t), (du/dt)(t)]_{\Phi}$, a.e. $t \in (0, T)$. Now if $u(t)$ and $v(t)$ are strong solutions of an abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} + A u(t) \ni f(t), \\ u(0) = u_0, \end{cases} \quad \begin{cases} \frac{dv}{dt} + A v(t) \ni g(t), \\ v(0) = v_0, \end{cases}$$

they are absolutely continuous (on every compact set in $]0, T[$, see [10], p. 64), and

$$\begin{aligned} \frac{d}{dt} \Phi(u(t) - v(t)) &= \left[u(t) - v(t), \frac{du}{dt} - \frac{dv}{dt} \right]_{\Phi} \\ &\leq [u(t) - v(t), -A u(t) + A v(t)]_{\Phi} + [u(t) - v(t), f(t) - g(t)]_{\Phi}, \end{aligned}$$

(in general, the equality is not true, see e.g. [5], Chap. I). So if A is a Φ accretive operator, *i. e.* if

$$[x - y, -A x + A y]_{\Phi} \leq 0, \quad \forall x, y \in D(A),$$

then, by integrating, we obtain

$$(2.23) \quad \Phi(u(t) - v(t)) \leq \Phi(u(0) - v(0)) + \int_0^t [u(\tau) - v(\tau), f(\tau) - g(\tau)]_{\Phi} d\tau.$$

It is very easy to see that

$$A \text{ is a } \Phi\text{-accretive operator} \Leftrightarrow \begin{cases} (I + \lambda A)^{-1} \text{ is a } \Phi\text{-contraction, } i. e. \\ \Phi(x - y) \leq \Phi((x + \lambda A x) - (y + \lambda A y)). \end{cases}$$

In the special case when X is a Banach lattice, the Φ -accretive operators for $\Phi(x) = \|x_+\|$ are called T -accretive operators. With $X = L^\infty(\Omega^*)$, we have checked in Lemma 4 (see the Appendix) that our operator M is T -accretive in X . Moreover, by the convexity of Φ , $[x, y]_{\Phi} \leq \Phi(y)$, and so, from (2.23), we conclude that (see Lemma 3)

$$(2.24) \quad \begin{aligned} \|(k(t) - K(t))_+\|_{L^\infty(\Omega^*)} &\leq \|(k(0) - K(0))_+\|_{L^\infty(\Omega^*)} \\ &\quad + \int_0^t \|(g(\tau) - G(\tau))_+\|_{L^\infty(\Omega^*)} d\tau = 0. \end{aligned}$$

3. Some consequences

3 a. L^r -ESTIMATES. — As usual, we deduce from Theorem 1:

COROLLARY 1. — *If u (resp. U) is the solution of (1.4) [resp. $(\widetilde{1.4})$], then*

$$(3.1) \quad \left\{ \begin{array}{l} \forall t \in [0, T], \quad \forall r \in [1, \infty], \\ \|u(t, \cdot)\|_{L^r(\Omega)} \leq \|U(t, \cdot)\|_{L^r(\widetilde{\Omega})} = \|U_*(t, \cdot)\|_{L^r(\Omega^*)}. \end{array} \right.$$

Corollary 1 is a simple application of a lemma in [2] (p. 174) for all r in $[1, \infty[$, and then for $r = +\infty$. More generally, by this lemma in [2],

$$(3.2) \quad \left\{ \begin{array}{l} \forall t \in [0, T], \\ \int_{\Omega} \Phi(u(t, x)) dx \leq \int_{\widetilde{\Omega}} \Phi(U(t, x)) dx = \int_{\Omega^*} \Phi(U_*(t, s)) ds, \end{array} \right.$$

for every $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}$, continuously differentiable, non-decreasing convex function.

3 b. EXTINCTION IN FINITE TIME. — The following corollary shows that if $f(t)$ becomes negative after a finite time, then the solution must coincide with the obstacle ($u \equiv 0$) after a time large enough.

COROLLARY 2. — *Assume there exists $t_0 \in [0, T]$ such that $U(t_0) \in L^\infty(\Omega)$,*

$$(3.5) \quad \varepsilon(t) = -\operatorname{ess\,sup}_{x \in \Omega} f(x, t) \geq 0, \quad \text{a. e. } t \in (t_0, T),$$

$$(3.6) \quad \|U(t_0)\|_{L^\infty(\Omega)} \leq \frac{1}{2} \int_{t_0}^T \varepsilon(\tau) d\tau.$$

Then the solutions u (resp. U) of (1.4) [resp. $(\widetilde{1.4})$] verify

$$(3.7) \quad u(t, \cdot) \equiv U(t, \cdot) \equiv 0, \quad \forall t \in [T_0, T],$$

where $T_0 \geq t_0$ is the smallest time such that

$$(3.8) \quad \|U(t_0)\|_{L^\infty(\Omega)} = \frac{1}{2} \int_{t_0}^{T_0} \varepsilon(\tau) d\tau.$$

Proof. — The definition of T_0 is possible, thanks to (3.6). Now, define

$$\varphi(t) = \frac{1}{|\Omega|} \left[\|U(t_0)\|_{L^\infty(\Omega)} - \frac{1}{2} \int_{t_0}^t \varepsilon(\tau) d\tau \right]_+.$$

We have $\varphi > 0$ on $[t_0, T_0[$, $\varphi = 0$ on $[T_0, T]$. As in Lemma 3, it is easy to check that $\tilde{K}(t, s) = \varphi(t)s(2|\Omega| - s)$ is (as K with second member F) a strong solution of the Cauchy

problem:

$$\hat{K} \in \mathcal{C}^0([t_0, T]; X), \quad \frac{d\hat{K}}{dt} \in L^1(t_0, T; X);$$

and for almost every t in (t_0, T) ,

$$\hat{K}(t) \in D(M) \quad \text{and} \quad \frac{d\hat{K}}{dt} + M\hat{K} \ni \hat{F} \quad \text{in } X,$$

with

$$\hat{F}(t, s) = \varphi(t) [2d + \underline{c}s(2|\Omega| - s)] - \varepsilon(t)s \left(1 - \frac{s}{2|\Omega|}\right); \quad \hat{F} \in L^1(t_0, T; X);$$

$$\hat{F} \geq F \quad (\text{as } F(t, s) = \int_0^s f_*(t, \sigma) d\sigma \leq -\varepsilon(t)s \text{ by (3.11)});$$

$$\begin{aligned} \hat{K}(t_0) &\geq K(t_0) \left(\text{as } K(t_0, s) = \int_0^s u_*(t_0) d\sigma \leq s \|u_*(t_0)\|_{L^\infty(\Omega^*)} = s \|u(t_0)\|_{L^\infty(\Omega)} \right. \\ &\leq s \|U(t_0)\|_{L^\infty(\Omega)} \text{ (by Corollary 1)} \leq \|U(t_0)\|_{L^\infty(\Omega)} s \left(2 - \frac{s}{|\Omega|}\right) = \hat{K}(t_0, s) \Big). \end{aligned}$$

By T-accretivity of M [see (2.24)],

$$\|(K(t) - \hat{K}(t))_+\|_{L^\infty(\Omega^*)} \leq \|(K(t_0) - \hat{K}(t_0))_+\|_{L^\infty(\Omega^*)} + \int_0^t \|(F(\tau) - \hat{F}(\tau))_+\|_{L^\infty(\Omega^*)} d\tau = 0,$$

and (using Theorem 1)

$$0 \leq k(t, s) \leq K(t, s) \leq \hat{K}(t, s), \quad \forall t \in [t_0, T], \quad \forall s \in [0, |\Omega|].$$

As $\hat{K}(t, \cdot) \equiv 0$ on $[T_0, T]$, so are $\partial K/\partial s = U_*$ and $\partial k/\partial s = u_*$, and then (by equimeasurability) u and U .

Remark 3.1. — One could also take (as in [15])

$$\hat{F} = -\varepsilon(t)s, \quad \hat{K} = \varphi(t)s,$$

with

$$\varphi(t) = \left(\|U(t_0)\|_{L^\infty(\Omega)} - \int_{t_0}^t \varepsilon(\tau) d\tau \right)_+$$

[resp. $\varphi(t) = \left(\|U\|_{L^\infty(Q)} - \int_{t_0}^t \varepsilon(\tau) d\tau \right)_+$, if U is in $L^\infty(Q)$]. Condition (3.6) is then replaced by $\|U(t_0)\|_{L^\infty(\Omega)}$ (resp. $\|U\|_{L^\infty(Q)} \leq \int_{t_0}^{T_0} \varepsilon(\tau) d\tau$), and (3.8) is replaced by

$\|U(t_0)\|_{L^\infty(\Omega)}$ (resp. $\|U\|_{L^\infty(\Omega)}$) = $\int_{t_0}^{T_0} \varepsilon(\tau) d\tau$. One could check that \hat{K} is a supersolution of the system verified by K . Then, as $\partial\hat{K}/\partial s(t, |\Omega|) = \varphi(t)$ is (non negative, but) not identically zero, the inequality $K \leq \hat{K}$ does not result directly (as in the previous proof) from T -accreitivity of M . For sake of brevity, we do not develop this argument here.

Remark 3.2. — If (3.5), (3.6) are verified with t_0 finite, $T = +\infty$, then Corollary 2 proves that both solutions of (1.4), $(\widetilde{1.4})$ vanish after a finite time $\bar{T}_0 \leq T_0$, where T_0 is the smallest time such that (3.8) is verified. This extinction in finite time can be compared with the results in [8], [11], [12], [14]. Our formulation here includes, as particular cases, those given in the mentioned references.

Remark 3.3. — In [15], we assumed $T = +\infty$, U in $L^\infty(Q)$, and obtained T_0 by $\int_{t_0}^{T_0} \varepsilon(\tau) d\tau = \|U\|_{L^\infty(Q)}$. This bound corresponds to the choices of \hat{F} , \hat{K} , φ given in Remark 3.1.

4. Some generalizations

4a. EXTENSION TO WEAKER FORMULATIONS, WITH OBSTACLE ZERO. — Let us consider an obstacle problem, with obstacle zero, where the data are less regular than above. Then inequality (1.9) will be still valid, as soon as the original and the symmetrized problems can be approximated by problems $(1.4)_n$, $(\widetilde{1.4})_n$ as before, whose solutions u_n and U_n satisfy (1.9) (by Theorem 1), and (the best case), for all t in $[0, T]$,

$$u_n(t) \rightarrow u(t) \quad \text{in } L^1(\Omega), \quad U_n(t) \rightarrow U(t) \quad \text{in } L^1(\bar{\Omega}).$$

In fact, we then have $u_{n^*}(t)$ [resp. $U_{n^*}(t)$] $\rightarrow u_*(t)$ [resp. $U_*(t)$] in $L^1(\Omega^*)$.

More generally (see e. g. [13], [16]),

PROPOSITION. — Assume $u_n(t)$ [resp. $U_n(t)$] in $L^1(\Omega)$ [resp. $L^1(\bar{\Omega})$] verify

$$(1.9)_n \quad \int_0^s u_{n^*}(t, \sigma) d\sigma \leq \int_0^s U_{n^*}(t, \sigma) d\sigma,$$

together with

$$u_n(t) \rightarrow u(t) \quad \text{weakly in } L^1(\Omega),$$

and either $U_n(t) \rightarrow U(t)$ strongly in $L^1(\bar{\Omega})$ or

$$\left\{ \begin{array}{l} U_n(t) \rightarrow U(t) \quad \text{weakly in } L^1(\bar{\Omega}) \\ \text{and } U_n(t) \text{ has radial symmetry and non-increases along the radii.} \end{array} \right.$$

Then u and U satisfy (1.9).

In fact, one has (by Hardy-Littlewood inequality)

$$\int_0^s u_{n*}(t, \sigma) d\sigma \geq \int_{E(t)} u_n(t, x) dx,$$

as soon as $|E(t)|=s$. The last term tends to $\int_{E(t)} u(t, x) dx$ which is $\int_0^s u_*(t, \sigma) d\sigma$ for a good choice of $E(t)$. Thus

$$\liminf_{n \rightarrow \infty} \int_0^s u_{n*}(t, \sigma) d\sigma \geq \int_0^s u_*(t, \sigma) d\sigma.$$

If $U_n(t) \rightarrow U(t)$ strongly in $L^1(\bar{\Omega})$, then $U_n(t)_* \rightarrow U(t)_*$ strongly in $L^1(\Omega^*)$ and the proof is complete. Otherwise $U(t)$ [as $U_n(t)$] has radial symmetry and non-increases along the radii,

$$\int_0^s U_{n*}(t, \sigma) d\sigma = \int_{\alpha_N |x|^N < s} U_n(t, x) dx \rightarrow \int_{\alpha_N |x|^N < s} U(t, x) dx = \int_0^s U_*(t, \sigma) d\sigma.$$

4 b. A MORE GENERAL OBSTACLE PROBLEM. — Let [see (1.3)]

$$D(\mathcal{A}) = \{ \psi \in L^2(Q); \mathcal{A}\psi \in L^2(Q) \}.$$

Consider ψ in $H^1(Q) \cap D(\mathcal{A})$, $\psi_0 = \psi|_{t=0} \in H^1(\Omega)$. As previously, $a_{ij} \in L^\infty(Q)$, $c \in L^\infty(Q)$, $\underline{c} = \text{ess inf}_\Omega c \geq 0$, $f \in L^2(Q)$. Given $u_0 \in H^1(\Omega)$, $u_0 \geq \psi_0$ a. e. in Ω , $u_0 = \psi_0$ on $\partial\Omega$, we consider the obstacle problem

$$\left\{ \begin{array}{l} u \in L^2(0, T; H^1(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2(Q), \quad u \geq \psi \text{ a. e. in } Q, \\ u = \psi \text{ a. e. on } \Sigma = (0, T) \times \partial\Omega, \quad u|_{t=0} = u_0, \\ \text{and for almost every } t \text{ in } (0, T), \\ \int_\Omega \left(\frac{\partial u}{\partial t} + cu - f \right) (v - u) dx + a(u, v - u) \geq 0, \quad \forall v \in H^1(\Omega), \\ v \geq \psi(t) \text{ a. e. in } \Omega, \quad v = \psi(t) \text{ a. e. on } \partial\Omega. \end{array} \right.$$

Let $\bar{u} = u - \psi$, $\bar{v} = v - \psi(t)$, $\bar{u}_0 = u_0 - \psi_0$, $\bar{f} = f - (\partial\psi/\partial t) - \mathcal{A}\psi - c\psi$. We have $\bar{f} \in L^2(Q)$, $\bar{u}_0 \in H_0^1(\Omega)$, $\bar{u}_0 \geq 0$, $\bar{u} \in L^2(0, T; H_0^1(\Omega))$, $\bar{u} \geq 0$, $\partial\bar{u}/\partial t = (\partial u/\partial t) - (\partial\psi/\partial t) \in L^2(Q)$, $\bar{u}|_{t=0} = \bar{u}_0$, and for almost every t in $(0, T)$

$$\int_\Omega \left(\frac{\partial \bar{u}}{\partial t} + c\bar{u} - \bar{f} \right) (\bar{v} - \bar{u}) dx + a(\bar{u}, \bar{v} - \bar{u}) \geq 0, \quad \forall \bar{v} \in H_0^1(\Omega), \quad \bar{v} \geq 0.$$

Let \bar{U} be the solution of the symmetrized problem (corresponding to \bar{f}, \bar{u}_0 , where \bar{f} (resp. \bar{u}_0) is the rearrangement of f (resp. u_0), that is,

$$\left\{ \begin{array}{l} \bar{U}_t - \Delta \bar{U} + c \bar{U} \geq \bar{f}, \quad \bar{U} \geq 0, \quad \text{a. e. in } \bar{Q}, \\ (\bar{U}_t - \Delta \bar{U} + c \bar{U} - \bar{f}) \bar{U} = 0, \quad \text{a. e. in } \bar{Q}, \\ \bar{U} = 0 \quad \text{on } \bar{\Sigma} = (0, T) \times \partial \bar{\Omega}, \\ \bar{U}|_{t=0} = \bar{u}_0 \quad \text{in } \bar{\Omega}. \end{array} \right.$$

By Theorem 1, we have

$$\int_0^s (u - \psi)_*(t, \sigma) d\sigma = \int_0^s \bar{u}_*(t, \sigma) d\sigma \leq \int_0^s \bar{U}_*(t, \sigma) d\sigma$$

and the consequences given in Section 3 are valid for \bar{u}, \bar{U} .

Remark 4.1. — In particular, let us assume $\psi(t) \in L^r(\Omega)$, $1 \leq r \leq \infty$. Then

$$\begin{aligned} \|u(t)\|_{L^r(\Omega)} &\leq \| (u - \psi)(t) \|_{L^r(\Omega)} + \| \psi(t) \|_{L^r(\Omega)} \\ &= \| (u - \psi)_*(t) \|_{L^r(\Omega^*)} + \| \psi(t) \|_{L^r(\Omega)} \leq \| \bar{U}_*(t) \|_{L^r(\Omega^*)} + \| \psi(t) \|_{L^r(\Omega)}. \end{aligned}$$

But we have more. In fact, by Lemma 2.1 in [3]

$$\|u(t)\|_{L^r(\Omega)} \leq \| (u - \psi)_*(t) + \psi_*(t) \|_{L^r(\Omega^*)} \leq \| \bar{U}_*(t) + \psi_*(t) \|_{L^r(\Omega^*)}.$$

For $r < +\infty$, the last inequality is obtained from

$$\int_0^s [(u - \psi)_*(t) + \psi_*(t)] d\sigma \leq \int_0^s [\bar{U}_*(t) + \psi_*(t)] d\sigma,$$

which gives (see again the lemma in [2], p. 174)

$$\int_0^{|\Omega|} \Phi [(u - \psi)_*(t) + \psi_*(t)] d\sigma \leq \int_0^{|\Omega|} \Phi [\bar{U}_*(t) + \psi_*(t)] d\sigma,$$

for every Φ continuously differentiable, convex, non-decreasing (e. g. $\Phi(\theta) = \theta^r$, $r \geq 1$). For $r = +\infty$, it is obtained by passing to the limit from $r' < +\infty$, $r' \rightarrow +\infty$.

4c. THE CASE OF A NON-LINEAR OPERATOR OF "PSEUDO-LAPLACE TYPE". — Let $\Omega, Q, A = (a_{ij})$, as before [in particular A verifies (1.1)]. For $p \in]2, \infty)$, p integer (or at least verifying the technical condition $p > 5/2(1 + (1/N))$, see (A.22) below), define, for u, v in $L^2(0, T; W_0^{1,p}(\Omega))$,

$$(4.1) \quad a(u, v) = \int_{\Omega} A(\nabla u, \nabla u)^{(p/2)-1} A(\nabla u, \nabla v) dx;$$

$$(4.2) \quad \mathcal{A} u = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left(A(\nabla u, \nabla u)^{(p/2)-1} a_{ij}(t, x) \frac{\partial u}{\partial x_i} \right).$$

We recall that $W_0^{1,p}(\Omega)$ is the completion of $\mathcal{D}(\Omega) = \mathcal{C}_0^\infty(\Omega)$, with respect to the metric

$$\|u\| = \left\{ \int_{\Omega} \sum_{|\alpha| \leq 1} |D^\alpha u|^p dx \right\}^{1/p}.$$

Observe that, for $A=I$, $a(u, v) = \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx$ is associated with the “pseudo-Laplace” operator

$$\mathcal{A}u = - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) = -\Delta_p u.$$

We consider the obstacle problem, with obstacle zero.

With $f \in L^2(0, T; L^q(\Omega))$ ($(1/p) + (1/q) = 1$), $u_0 \in W_0^{1,p}(\Omega)$, $u_0 \geq 0$, u verifies

$$(4.3) \quad \left\{ \begin{array}{l} u \in L^2(0, T; W_0^{1,p}(\Omega)), \quad u \geq 0, \\ \frac{\partial u}{\partial t} \in L^2(0, T; L^q(\Omega)), \quad u|_{t=0} = u_0, \\ \text{and, for almost every } t \text{ in } (0, T), \\ \int_{\Omega} \left(\frac{\partial u}{\partial t} - f \right) (v - u) dx + a(u, v - u) \geq 0, \quad \forall v \in W_0^{1,p}(\Omega), \quad v \geq 0. \end{array} \right.$$

The reference “symmetrized problem”, which we consider, is then

$$(4.3) \quad \left\{ \begin{array}{l} U \in L^2(0, T; W_0^{1,p}(\tilde{\Omega})), \quad U \geq 0, \\ \frac{\partial U}{\partial t} \in L^2(0, T; L^q(\tilde{\Omega})), \quad U|_{t=0} = u_0, \\ \text{and, for almost every } t \text{ in } (0, T), \\ \int_{\tilde{\Omega}} \left(\frac{\partial U}{\partial t} - f \right) (v - U) dx + \int_{\tilde{\Omega}} |\nabla U|^{p-2} \nabla U \cdot \nabla (v - U) dx \geq 0, \\ \forall v \in W_0^{1,p}(\tilde{\Omega}), \quad v \geq 0, \end{array} \right.$$

and we get the same comparison:

THEOREM 1' — *If u (resp. U) is the solution of (4.3), [resp. (4.3)], then*

$$(1.9)' \quad \int_0^s u_*(t, \sigma) d\sigma \leq \int_0^s U_*(t, \sigma) d\sigma, \quad \forall t, \quad 0 \leq t \leq T, \quad \forall s, \quad 0 \leq s \leq |\Omega|.$$

Remark 4.2. — It seems more difficult to include, in the inequalities in (4.3) (4.3), the terms $\int_{\Omega} cu^{p-1}(v-u) dx$ and $\int_{\tilde{\Omega}} \underline{c}U^{p-1}(v-U) dx$ respectively (see in the proof below).

Proof of Theorem 1': It is easy to prove that (4.3) can be written

$$(4.3 \text{ bis}) \quad \left\{ \begin{array}{l} U_t - \Delta_p U \geq f, \quad U \geq 0, \quad (U_t - \Delta_p U - f) U = 0 \quad \text{in } \tilde{Q}, \\ U = 0 \quad \text{on } \tilde{\Sigma}, \\ U|_{t=0} = u_0 \end{array} \right.$$

and we have again

LEMMA 1' - U decreases along the radii: $U = \tilde{U}$.

Proof of Lemma 1': Let $g = (\partial U / \partial t) - \Delta_p U - f$. By (4.3 bis), $gU = 0$ in \tilde{Q} . Under stronger regularity assumptions on f and u_0 , we get, with $W = \partial U / \partial r$: on $\{U = 0\}$, $W_t - (\partial / \partial r)(\Delta_p U) = 0$; on $\{U > 0\}$, $g = 0$ implies $\partial g / \partial r = 0$, or $W_t - (\partial / \partial r)(\Delta_p U) = \partial f / \partial r \leq 0$. In any case $W_t - (\partial / \partial r)(\Delta_p U) \leq 0$ in \tilde{Q} . Now, an easy computation gives

$$\begin{aligned} \Delta_p U &= \left| \frac{\partial U}{\partial r} \right|^{p-2} \left[(p-1) \frac{\partial^2 U}{\partial r^2} + \frac{N-1}{r} \frac{\partial U}{\partial r} \right] = |W|^{p-2} \left[(p-1) \frac{\partial W}{\partial r} + \frac{N-1}{r} W \right], \\ \frac{\partial}{\partial r} (\Delta_p U) &= |W|^{p-2} \\ &\times \left[(p-1) \frac{\partial^2 W}{\partial r^2} + \frac{(p-1)(p-2)}{W} \left(\frac{\partial W}{\partial r} \right)^2 + \frac{(p-1)(N-1)}{r} \frac{\partial W}{\partial r} - \frac{N-1}{r^2} W \right] \\ &= |W|^{p-2} \left[(p-1) \Delta W - \frac{N-1}{r^2} W + \frac{(p-1)(p-2)}{W} \left(\frac{\partial W}{\partial r} \right)^2 \right]. \end{aligned}$$

We get

$$\begin{aligned} W_t &\leq |W|^{p-2} \left[(p-1) \Delta W - \frac{N-1}{r^2} W + \frac{(p-1)(p-2)}{W} |\nabla W|^2 \right] \quad \text{in } \tilde{Q}, \\ W|_{t=0} &\leq 0, \\ W|_{\tilde{\Sigma}} &\leq 0. \end{aligned}$$

Multiplying by W_+ and integrating over $\tilde{\Omega}$ gives

$$\begin{aligned} \int_{\tilde{\Omega}} W_t W_+ dx &\leq (p-1) \int_{\tilde{\Omega}} W_+^{p-1} \Delta W dx + (p-1)(p-2) \int_{\tilde{\Omega}} W_+^{p-2} |\nabla W|^2 dx \\ &= -(p-1) \int_{\tilde{\Omega}} W_+^{p-2} |\nabla W|^2 dx, \end{aligned}$$

(by integrating by parts, and using $W_+|_{\tilde{\Sigma}} = 0$). Thus

$$\frac{\partial}{\partial t} \int_{\tilde{\Omega}} W_+^2 dx \leq 0, \quad \int_{\tilde{\Omega}} W_+(t)^2 dx \leq \int_{\tilde{\Omega}} W_+(0)^2 dx = 0, \quad W_+ \equiv 0,$$

or $W \leq 0$ in \tilde{Q} , U decreases along the radii. The general case (f, u_0 less regular) is solved by passing to the limit as in Lemma 1. ■

Lemma 2 becomes

LEMMA 2' - Let u (resp. U) be the solutions of (4.3) [resp. (4.3), (4.3 bis)]. Define k, K, F, k_0, s as before (see (2.1), (2.1), (2.2), (2.3), (2.5)). Now (2.4) is replaced by

$$(4.4) \quad d(s) = N^p \alpha_N^{p/N} s^{p-(p/N)}.$$

Then $F \in L^2(0, T; W^{1,q}(\Omega^*))$,

$$(4.5) \quad k \text{ and } K \text{ belong to } H^1(0, T; W^{1,q}(\Omega^*)) \cap L^2(0, T; W^{2,p}(\epsilon, |\Omega|)),$$

$\epsilon > 0$

k verifies

$$(4.6) \quad \left\{ \begin{array}{l} \frac{\partial k}{\partial t} + d \left(-\frac{\partial^2 k}{\partial s^2} \right)^{p-1} \leq F \quad \text{and} \quad \frac{\partial k}{\partial s} > 0, \\ \text{a. e. } (t, s) \in Q_{ii}^* = \{ (t, s), t \in (0, T), s \in (0, s(t)) \}, \\ \frac{\partial k}{\partial s} \geq 0, \quad \forall t \in [0, T], \quad \forall s \in \overline{\Omega^*}, \\ k(t, 0) = 0 = \frac{\partial k}{\partial s}(t, s(t)) \left(= \frac{\partial k}{\partial s}(t, |\Omega|) \right), \quad \forall t \in [0, T], \\ k(0, s) = k_0(s), \quad \forall s \in \overline{\Omega^*}; \end{array} \right.$$

K verifies

$$(4.7) \quad \left\{ \begin{array}{l} \frac{\partial K}{\partial t} + d \left(-\frac{\partial^2 K}{\partial s^2} \right)^{p-1} \geq F \quad \text{and} \quad \frac{\partial K}{\partial s} \geq 0, \quad \text{a. e. } t, s \text{ in } Q^*, \\ \left[\frac{\partial K}{\partial t} + d \left(-\frac{\partial^2 K}{\partial s^2} \right)^{p-1} - F \right] \frac{\partial K}{\partial s} = 0, \quad \text{a. e. } t, s \text{ in } Q^*, \\ K(t, 0) = 0 = \frac{\partial K}{\partial s}(t, |\Omega|), \quad \forall t \in [0, T], \\ K(0, s) = k_0(s), \quad \forall s \in \overline{\Omega^*}. \end{array} \right.$$

Proof of Lemma 2'. - With $\alpha(N, p) = N^p \alpha_N^{p/N}$, (2.10) is replaced by (see [20], p. 40-43).

$$(2.10)' \quad \left\{ \begin{array}{l} \alpha(N, p) \mu(\theta)^{p-(p/N)} \leq (-\mu'(\theta))^{p-1} \int_{u>0} \left(f - \frac{\partial u}{\partial t} \right) dx, \\ \text{a. e. } \theta \in (0, m), \quad m = \text{ess sup}_\Omega u(t) \end{array} \right.$$

and (2.11) is replaced by

$$(2.11)' \quad \alpha(N, q) \mu(\theta)^{q-(q/N)} \leq -\mu'(\theta) [G(\mu(\theta))]^{q/p},$$

with $G = F - (\partial k / \partial t) \geq 0$ on $[0, s(t)]$ (same proof as before); (2.13) is replaced by

$$(2.13)' \quad 0 \leq -\alpha(N, q) \frac{\partial u_*}{\partial s} \leq s^{(q/N)-q} G^{q/p}, \quad \text{a. e. } s \in (0, s(t)),$$

or $(\partial k / \partial t) + d(-\partial u_* / \partial s)^{p/q} \leq F$, a. e. $s \in (0, s(t))$. That is (2.7) becomes (4.6); (4.5) is easy to show [as we did for (2.6) before].

As for U , we have by (4.3 bis) and Lemma 1' before

$$(4.8) \quad \left\{ \begin{array}{l} \frac{\partial U_*}{\partial t} + \frac{\partial}{\partial s} \left(d \left(-\frac{\partial U_*}{\partial s} \right)^{p-1} \right) \geq f_*, \quad U_* \geq 0 \quad \text{a. e. in } Q^*, \\ \left(\frac{\partial U_*}{\partial t} + \frac{\partial}{\partial s} \left(d \left(-\frac{\partial U_*}{\partial s} \right)^{p-1} \right) - f_* \right) U_* = 0 \quad \text{a. e. in } Q^*, \\ U_*(t, |\Omega|) = 0, \quad \forall t \in [0, T], \\ U_{*|t=0} = u_{0*} \quad \text{in } \Omega^*. \end{array} \right.$$

(since $(\partial / \partial s)(d|\partial U_* / \partial s|^{p-2}(\partial U_* / \partial s)) = -(\partial / \partial s)(d(-\partial U_* / \partial s)^{p-1})$ is just the expression of $\Delta_p U$ in terms of s); $\partial U_* / \partial t, \Delta_p U \in L^q(\Omega^*)$. By integrating the first inequality in (4.8) between 0 and s , we get the first inequality in (4.7). The remaining in (4.7) is obtained as before.

Remark 4.3. - In (4.6), (4.7), $d(-(\partial^2 k / \partial s^2))^{p-1}$ can be written as

$$-d \left| \frac{\partial^2 k}{\partial s^2} \right|^{p-2} \frac{\partial^2 k}{\partial s^2} \quad (\text{for } \frac{\partial^2 k}{\partial s^2} = \frac{\partial u_*}{\partial s} \leq 0).$$

The same for K . ■

With β defined in (2.15), $X = L^\infty(\Omega^*)$, we consider the multivalued operator $M : D(M) \rightarrow P(X)$ given by

$$(2.16)' \quad \left\{ \begin{array}{l} D(M) = \{ w \in \mathcal{C}^0(\overline{\Omega^*}) \cap \bigcap_{\varepsilon > 0} W^{2,p}(\varepsilon, |\Omega|); w(0) = 0 = w'(|\Omega|); \\ \quad w'(s) \geq 0, \forall s \in]0, |\Omega|]; \\ \exists b \in L^q(\Omega^*), b(s) \in \beta(w'(s)) \text{ a. e. } s \in \Omega^*, -d|w''|^{p-2}w'' + b \in L^\infty(\Omega^*) \} \\ (\text{again } D(M) \subset \mathcal{C}^0(\overline{\Omega^*}) \cap \mathcal{C}^1(]0, |\Omega|]), \\ M w = \{ -d|w''|^{p-2}w'' + b \in L^\infty(\Omega^*) \text{ such that } b \in L^q(\Omega^*), \\ \quad b(s) \in \beta(w'(s)) \text{ a. e. } s \in \Omega^* \}. \end{array} \right.$$

As in Section 3, the inequality $k \leq K$ (Theorem 1' to be proved) follows directly from Lemma 4', the proof of which is very technical and is given in the Appendix.

LEMMA 4'. — *The operator M , defined in (2.16)', is m -T accretive in $X=L^\infty(\Omega^*)$ [see (2.20), (2.21)].* ■

All the consequences and generalizations given in Sections 3 and 4 *a, b*, are valid (with easy modifications) for this pseudo-Laplace operator. ■

APPENDIX

A.1. PROOF OF LEMMA 4. — We shall use β_λ , the Yosida approximation of β (so β_λ is a Lipschitz non decreasing real function). By the classical theory (see e. g. Ladyzenskaya and Ural'tseva [18], Chap. 10), we know that, for any $h \in L^\infty(\Omega^*)$, there exists a unique $w_\lambda \in \mathcal{C}^1(\Omega^*)$, solution of

$$(A.1) \quad \begin{cases} -d_\lambda w_\lambda'' + \beta_\lambda(w_\lambda') + (\underline{c} + 1)w_\lambda = h, \\ w_\lambda(0) = w_\lambda'(|\Omega|) = 0, \end{cases}$$

with $d_\lambda = d + \lambda$ (However the introduction of d_λ in place of d is not absolutely necessary, since the degeneracy of d takes place only at $s = 0$, and does not introduce any important difficulty — with d in place of d_λ in (A.1), w_λ should be in $\mathcal{C}^0(\bar{\Omega}^*) \cap \mathcal{C}^1(]0, |\Omega|])$, and the following should be true with minor changes —). Now we will show that, when $\lambda \searrow 0$, $w_\lambda \rightarrow w$ (in a suitable sense), with $w \in D(M)$ satisfying

$$(A.2) \quad dw'' - (\underline{c} + 1)w + h \in \beta(w') \quad \text{a. e. in } \Omega^*.$$

In order to prove this, we will use the following lemma

LEMMA A.1. — *Let w_λ and \hat{w}_λ be two solutions of (A.1) corresponding to h and \hat{h} respectively. Then*

$$(A.3) \quad \|(w_\lambda - \hat{w}_\lambda)_+\|_\infty \leq \|(h - \hat{h})_+\|_\infty.$$

Proof of Lemma A.1. — Assume that $\|(w_\lambda - \hat{w}_\lambda)_+\|_\infty = (w_\lambda - \hat{w}_\lambda)(s_0) > \|(h - \hat{h})_+\|_\infty$, for some $s_0 \in]0, |\Omega|]$ (it is clear that $s_0 > 0$). If $s_0 < |\Omega|$, then the function $z = w_\lambda - \hat{w}_\lambda$ verifies $z \in W^{2,\infty}(s_0 - \varepsilon, s_0 + \varepsilon)$ and

$$(A.4) \quad -d_\lambda z'' < 0 \quad \text{a. e. in } (s_0 - \varepsilon, s_0 + \varepsilon).$$

Here, we have used that

$$\beta_\lambda(w_\lambda'(s)) - \beta_\lambda(\hat{w}_\lambda'(s)) = z'(s) c_\lambda(s),$$

with

$$c_\lambda(s) = \int_0^1 \beta_\lambda'(\tau w_\lambda'(s) + (1 - \tau)\hat{w}_\lambda'(s)) d\tau, \\ -d_\lambda z'' = -c_\lambda z' + h - \hat{h} - (\underline{c} + 1)z \leq -c_\lambda z' + \|(h - \hat{h})_+\|_\infty - z,$$

a continuous function, < 0 on $(s_0 - \varepsilon, s_0 + \varepsilon)$ (ε small) as it is at the point s_0 . Now, (A.4) contradicts that z achieves its maximum at the interior point s_0 . If $s_0 = |\Omega|$, (A.4) holds on $(|\Omega| - \varepsilon, |\Omega|)$, implying $z'(|\Omega|) > 0$, which contradicts the boundary conditions.

Proof of Lemma 4 (continued). — In order to justify the convergence $w_\lambda \rightarrow w$, we first note that

$$(A.5) \quad \|w_\lambda\|_\infty \leq \|h\|_\infty.$$

This is a consequence of (A.3). By taking $\hat{h} = 0$ ($\hat{w}_\lambda = 0$), we get $\|w_{\lambda,+}\|_\infty \leq \|h_+\|_\infty$. Similarly (by taking $h = 0$ in (A.3)), $\|w_{\lambda,-}\|_\infty \leq \|h_-\|_\infty$, and the addition of both gives (A.5). By (A.5), we also conclude that

$$(A.6) \quad \|-d_\lambda w''_\lambda + \beta_\lambda(w'_\lambda)\|_\infty \leq C,$$

(independent of λ). Now, we shall prove an uniform estimate for $d_\lambda w''_\lambda, dw''_\lambda$:

$$(A.7) \quad \|dw''_\lambda\|_{L^2(\Omega^*)}^2 \leq \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2 \leq C,$$

for another constant C , independent of λ . From the equation (A.1), and using Young inequality

$$-\int_{\Omega^*} ((\varepsilon + 1)w_\lambda - d_\lambda w''_\lambda + \beta_\lambda(w'_\lambda)) d_\lambda w''_\lambda ds \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2.$$

But

$$\int_{\Omega^*} -(\varepsilon + 1)w_\lambda d_\lambda w''_\lambda ds \geq -(\varepsilon + 1) \|w_\lambda\|_\infty \|d_\lambda w''_\lambda\|_{L^1(\Omega^*)} \geq -C \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)},$$

[by (A.5), where C is independent of λ]

$$\geq -\frac{\alpha}{2} C^2 - \frac{1}{2\alpha} \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2,$$

for any $\alpha > 0$ (α will be chosen later on), and

$$-\int_{\Omega^*} d_\lambda \beta_\lambda(w'_\lambda) w''_\lambda ds = [-d_\lambda j_\lambda(w'_\lambda)]_0^{\Omega^*} + \int_{\Omega^*} d'_\lambda j_\lambda(w'_\lambda) ds,$$

(where $j_\lambda(\theta) = \int_0^\theta \beta_\lambda(\tau) d\tau \geq 0$) which is non-negative. We get

$$\begin{aligned} & -\frac{\alpha}{2} C^2 - \frac{1}{2\alpha} \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2 + \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2 \\ & \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \|d_\lambda w''_\lambda\|_{L^2(\Omega^*)}^2. \end{aligned}$$

Taking α large enough gives (A.7). Now, by (A.1),

$$\begin{aligned} \int_{\Omega^*} (\underline{c} + 1) w_\lambda - d_\lambda w_\lambda'' + \beta_\lambda(w_\lambda') w_\lambda' ds &= \int_{\Omega^*} h w_\lambda' ds, \\ \int_{\Omega^*} \beta_\lambda(w_\lambda') w_\lambda' ds &\geq 0, \quad \int_{\Omega^*} (\underline{c} + 1) w_\lambda w_\lambda' ds = \frac{\underline{c} + 1}{2} w_\lambda (|\Omega|)^2 \geq 0, \\ - \int_{\Omega^*} d_\lambda w_\lambda'' w_\lambda' ds &= - \int_{\Omega^*} \frac{d_\lambda}{2} \frac{d}{ds} (w_\lambda')^2 ds = \int_{\Omega^*} \frac{d_\lambda'}{2} (w_\lambda')^2 ds + \frac{d_\lambda(0)}{2} w_\lambda'(0)^2 \geq \int_{\Omega^*} \frac{d_\lambda'}{2} w_\lambda'^2 ds, \\ (A.8) \quad \frac{1}{2} \int_{\Omega^*} d_\lambda' w_\lambda'^2 ds &\leq \int_{\Omega^*} h w_\lambda' ds \leq \int_{\Omega^*} \left(\frac{d_\lambda'}{4} w_\lambda'^2 + \frac{1}{d_\lambda'} h^2 \right) ds, \\ &\int_{\Omega^*} \frac{1}{d_\lambda'} h^2 ds \leq \|h\|_\infty^2 \left\| \frac{1}{d'} \right\|_{L^1(\Omega^*)} = C \quad (\text{independent of } \lambda), \end{aligned}$$

and then by (A.8)

$$\begin{aligned} \int_{\Omega^*} d' w_\lambda'^2 ds &\leq C \text{ independent of } \lambda, \\ (A.9) \quad \|\sqrt{d'} w_\lambda'\|_{L^2(\Omega^*)} &\leq C \text{ independent of } \lambda. \end{aligned}$$

Remark A.1. — As $(dw_\lambda')' = d' w_\lambda' + dw_\lambda''$,

$$\int_{\Omega^*} d'^2 w_\lambda'^2 ds \leq \|d'\|_\infty \int_{\Omega^*} d' w_\lambda'^2 ds,$$

we deduce, from (A.7), (A.9), that $(dw_\lambda')'$ is bounded independently of λ in $L^2(\Omega^*)$, and dw_λ' is bounded independently of λ in $H^1(\Omega^*)$. Now, as

$$(dw_\lambda)' = d' w_\lambda + dw_\lambda',$$

dw_λ' , $d' w_\lambda$ bounded (independently of λ) in $L^\infty(\Omega^*)$, dw_λ is bounded independently of λ in $W^{1, \infty}(\Omega^*)$. ■

Now we will show that w_λ' is bounded in $L^\alpha(\Omega^*)$ for any $\alpha < N/N - 1$, if $N > 1$, and any $\alpha < 2$ if $N = 1$. In fact, for any α ,

$$\int_{\Omega^*} |w_\lambda'|^\alpha ds = \int_{\Omega^*} \left(\frac{1}{d'} \right)^{\alpha/2} d'^{\alpha/2} |w_\lambda'|^\alpha ds \leq \left(\int_{\Omega^*} \left(\frac{1}{d'} \right)^{\alpha p/2} ds \right)^{1/p} \left(\int_{\Omega^*} (d'^{1/2} |w_\lambda'|)^{\alpha q} ds \right)^{1/q},$$

$((1/p) + (1/q) = 1)$. Let $q = 2/\alpha (> 1$ as soon as $\alpha < 2)$,

$$\begin{aligned} p = \frac{q}{q-1} = \frac{2}{2-\alpha}, \quad \int_{\Omega^*} |w_\lambda'|^\alpha ds &\leq \left(\int_{\Omega^*} \left(\frac{1}{d'} \right)^{\alpha/(2-\alpha)} ds \right)^{(2-\alpha)/2} \left(\int_{\Omega^*} d' w_\lambda'^2 ds \right)^{\alpha/2}, \\ \left(\frac{1}{d'} \right)^{\alpha/(2-\alpha)} \in L^1(\Omega^*) &\Leftrightarrow \left(\frac{2}{N} - 1 \right) \frac{\alpha}{2-\alpha} > -1 \Leftrightarrow \alpha < \frac{N}{N-1}, \end{aligned}$$

if $N > 1$, and, if $N = 1$, we have the only condition $\alpha < 2$. We conclude from (A.9)

$$(A.10) \quad \|w'_\lambda\|_{L^\alpha(\Omega^*)} \leq C_\alpha \text{ independent of } \lambda,$$

and [by (A.5)] w_λ is bounded independently of λ in any $W^{1,\alpha}(\Omega^*)$ ($\alpha < 2$ if $N = 1$, $\alpha < N/(N-1)$ if $N > 1$). ■

For convenience, we shall also denote by w_λ a subsequence of the whole sequence w_λ . By (A.10) [and (A.5)],

$$\begin{aligned} w_\lambda &\rightarrow w \quad \text{in } \mathcal{C}^0(\overline{\Omega^*}) \quad (\text{in particular } w(0) = 0), \\ w'_\lambda &\rightharpoonup w' \quad \text{in } L^\alpha(\Omega^*) \text{ weakly.} \end{aligned}$$

By (A.7)

$$d_\lambda w''_\lambda \rightharpoonup dw'' \quad \text{in } L^2(\Omega^*) \text{ weakly,}$$

($d_\lambda w''_\lambda \rightharpoonup l$ in $L^2(\Omega^*)$ weakly, and for $\Omega_\varepsilon^* = (\varepsilon, |\Omega|)$, φ in $L^2(\Omega^*)$,

$$\int_{\Omega_\varepsilon^*} w''_\lambda \varphi \, ds = \int_{\Omega_\varepsilon^*} d_\lambda w''_\lambda \frac{\varphi}{d_\lambda} \, ds \rightarrow \int_{\Omega_\varepsilon^*} l \frac{\varphi}{d} \, ds$$

as $\varphi/d_\lambda \rightarrow \varphi/d$ strongly in $L^2(\Omega_\varepsilon^*)$, $d_\lambda w''_\lambda \rightharpoonup l$ weakly in $L^2(\Omega_\varepsilon^*)$; $w''_\lambda \rightharpoonup l/d$ in $L^2(\Omega_\varepsilon^*)$ weakly, $w'_\lambda \rightharpoonup w'$ in $L^\alpha(\Omega_\varepsilon^*)$ weakly, $dw'' = l$ a.e. in Ω_ε^* , for any $\varepsilon > 0$, $dw'' = l$ a.e. in Ω^*). By Remark A.1,

$$dw'_\lambda \rightarrow dw' \quad \text{in } \mathcal{C}^0(\overline{\Omega^*})$$

and this implies $w'(|\Omega|) = 0$. By (A.1), (A.6), (A.7), and the previous convergences

$$\beta_\lambda(w'_\lambda) = h - (\underline{c} + 1)w_\lambda + d_\lambda w''_\lambda \rightharpoonup b = h - (\underline{c} + 1)w + dw''$$

[weakly in $L^2(\Omega^*)$], with $b \in \beta(w')$ classically. In other words, we know that $w \in D(M)$ and $Mw + w \ni h$ in $L^\infty(\Omega^*)$, that is (2.20) is proved.

In order to prove (2.21), we only need to check that there is a unique solution $w \in D(M)$ of the equation $Mw + w \ni h$. After that, (2.21) is a trivial consequence of (A.3), and lemma 4 will be completely proved. ■

LEMMA A.2. — *There is a unique $w \in D(M)$ solution of (A.2).*

Proof of Lemma A.2. — Let w and \hat{w} , in $D(M)$, be two solutions of (A.2), with b and \hat{b} belonging to $\beta(w')$ and $\beta(\hat{w}')$ respectively, such that

$$\begin{aligned} -dw'' + b + (\underline{c} + 1)w &= h, & -d\hat{w}'' + \hat{b} + (\underline{c} + 1)\hat{w} &= h \quad \text{in } \Omega^*, \\ w(0) = \hat{w}(0) = w'(|\Omega|) &= \hat{w}'(|\Omega|) = 0. \end{aligned}$$

Let $z = w - \hat{w} \in \mathcal{C}^0(\bar{\Omega}^*) \cap H^2(\varepsilon, |\Omega|)$. By multiplying the difference of the two equations in Ω^* , by z' , and by using that $(b - \hat{b})z' \geq 0$ a. e., we have

$$(A.11) \quad (-dz'' + (c+1)z)z' \leq 0 \text{ a. e. on } \Omega^*.$$

Now we introduce the change of variable $r = (s/\alpha_N)^{1/N}$, and define $\bar{z}(r) = z(s)$. Then

$$(A.12) \quad \left\{ \begin{array}{l} \frac{dz}{ds} = \frac{d\bar{z}}{dr} \frac{dr}{ds} = \frac{d\bar{z}}{dr} r^{1-N}, \\ \frac{d^2z}{ds^2} = \frac{d^2\bar{z}}{dr^2} \left(\frac{dr}{ds}\right)^2 + \frac{d\bar{z}}{dr} \frac{d^2r}{ds^2}, \\ \frac{d^2z}{ds^2} = \frac{d^2\bar{z}}{dr^2} - \frac{N-1}{r} \frac{d\bar{z}}{dr}. \end{array} \right.$$

From (A.11), (A.12), we deduce

$$\left((c+1)\bar{z} - \frac{d^2\bar{z}}{dr^2} + \frac{N-1}{r} \frac{d\bar{z}}{dr} \right) \frac{d\bar{z}}{dr} \leq 0, \quad \text{a. e. } r \in (0, R),$$

with $R = (|\Omega|/\alpha_N)^{1/N}$. In particular

$$(A.13) \quad ((c+1)\bar{z} - \bar{z}')\bar{z}' = \frac{1}{2}((c+1)\bar{z}^2 - \bar{z}'^2)' \leq 0, \quad \text{a. e. } r \in (0, R).$$

From the boundary conditions for z , we have $\bar{z}(0) = \bar{z}'(R) = 0$. We do not know *a priori* if $\bar{z}'(0)$ is defined, but, integrating (A.13) between $r(0 < r \leq R)$ and R , we get

$$(A.14) \quad (c+1)(\bar{z}^2(R) - \bar{z}^2(r)) + \bar{z}'^2(r) \leq 0 \quad (\text{for } 0 < r \leq R)$$

and then [since $\bar{z}(r) \rightarrow 0$ when $r \rightarrow 0$]

$$(c+1)\bar{z}^2(R) + \overline{\lim}_{r \rightarrow 0} [\bar{z}'(r)]^2 \leq 0.$$

It follows $\bar{z}(R) = 0 = \overline{\lim}_{r \rightarrow 0} [\bar{z}'(r)]^2 = \lim_{r \rightarrow 0} [\bar{z}'(r)]^2$, $\bar{z}'(r) \rightarrow 0$ when $r \rightarrow 0$. Integrating (A.13)

between ε and $r(0 < \varepsilon < r \leq R)$, we get

$$(c+1)(\bar{z}^2(r) - \bar{z}^2(\varepsilon)) - \bar{z}'^2(r) + \bar{z}'^2(\varepsilon) \leq 0,$$

and (with $\varepsilon \rightarrow 0$)

$$(c+1)\bar{z}^2(r) - \bar{z}'^2(r) \leq 0 \quad (0 < r \leq R).$$

From (A.14) and $\bar{z}(R) = 0$, we conclude

$$(A.15) \quad (c+1)\bar{z}^2 - \bar{z}'^2 = 0,$$

which implies

$$(A.16) \quad \bar{z}=0 \quad \text{or} \quad \frac{\bar{z}'}{\bar{z}} = \pm \sqrt{\underline{c}+1}.$$

(Notice that $z \in \mathcal{C}^1(]0, |\Omega|)$ implies $\bar{z} \in \mathcal{C}^1(]0, \mathbb{R}[)$). Let $\mathcal{O} = \{r \in]0, \mathbb{R}[, \bar{z}(r) \neq 0\} = \{\bar{z} \neq 0\}$. This is an open set, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ (\mathcal{O}_i open) with

$$\begin{aligned} \mathcal{O}_1 &= \{r \in]0, \mathbb{R}[, \bar{z}(r) \neq 0, \bar{z}'(r) = \sqrt{\underline{c}+1} \bar{z}(r)\}, \\ \mathcal{O}_2 &= \{r \in]0, \mathbb{R}[, \bar{z}(r) \neq 0, \bar{z}'(r) = -\sqrt{\underline{c}+1} \bar{z}(r)\}. \end{aligned}$$

As \bar{z} is \mathcal{C}^1 , $\bar{z}=0$ on $\partial\mathcal{O}_i$ ($i=1, 2$). If we assume $\mathcal{O}_1 \neq \emptyset$, then,

$$\frac{\bar{z}'}{\bar{z}} = \sqrt{\underline{c}+1} \quad \text{on } \mathcal{O}_1,$$

$$\text{Log}|\bar{z}| = \sqrt{\underline{c}+1}r + C_j \text{ i. e. } |\bar{z}| = \lambda_j e^{\sqrt{\underline{c}+1}r} \quad (\lambda_j \text{ constant}),$$

on each connected component \mathcal{O}_1^j of \mathcal{O}_1 . But $\bar{z}=0$ on $\partial\mathcal{O}_1^j$ implies that $\lambda_j=0$, which is a contradiction. It follows $\mathcal{O}_1 = \emptyset$, and similarly $\mathcal{O}_2 = \emptyset$, that is $\bar{z} \equiv 0$. A different (even simpler) proof will be given in Lemma (A.2)' below. ■

A.2. Proof of Lemma 4'. — First, we have to prove that, for any $h \in L^\infty(\Omega^*)$, there exists a unique $w_\lambda \in \mathcal{C}^1(\Omega^*)$, solution of

$$(A.1)' \quad \begin{cases} -d_\lambda |w_\lambda''|^{p-2} w_\lambda'' + \beta_\lambda(w_\lambda) + w_\lambda = h, \\ w_\lambda(0) = w_\lambda'(|\Omega|) = 0. \end{cases}$$

This can be shown by different methods. For instance, it is possible to apply the method of sub and supersolutions (sometimes also called monotone iterative method). Indeed, it is enough to check that $\underline{w} = -\|h_-\|_\infty$ and $\bar{w} = \|h_+\|_\infty$ are sub- and supersolutions of (A.1)'. On the other hand, the differential equation can be written as $-w_\lambda'' = f(x, w_\lambda, w_\lambda')$, with

$$f(x, u, p) = \alpha \left(\frac{1}{d_\lambda} (h - u - \beta_\lambda(p)) \right), \quad \alpha(s) = |s|^{1/(p-1)} \cdot \text{sign } s,$$

and so f satisfies a Nagumo condition with respect to p (see e.g. [17], Theorem 2.1.5). We remark that the method of sub and supersolutions must be applied in the space $W^{2, \infty}(\Omega^*)$, and not in $\mathcal{C}^2(\Omega^*)$ as usual, but in any case, this is a trivial modification of the theory. The uniqueness comes from Lemma A.1' below. ■

The proof of the first part of Lemma 4' (2.20) consists in showing that, when $\lambda \searrow 0$, $w_\lambda \rightarrow w$ (in a suitable sense), with $w \in D(M)$ satisfying

$$(A.2)' \quad d|w''|^{p-2} w'' - w + h \in \beta(w') \quad \text{a. e. in } \Omega^*.$$

We use again

LEMMA A.1'. — Let w_λ and \hat{w}_λ be two solutions of (A.1)' corresponding to h and \hat{h} respectively. Then

$$(A.3)' \quad \|(w_\lambda - \hat{w}_\lambda)_+\|_\infty \leq \|(h - \hat{h})_+\|_\infty.$$

The proof is very similar to that of Lemma A.1.

Proof of Lemma 4' (continued). — The convergence $w_\lambda \rightarrow w$ is obtained by *a priori* estimates. Again, by (A.3)',

$$(A.5)' \quad \|w_\lambda\|_\infty \leq \|h\|_\infty,$$

which gives

$$(A.6)' \quad \|-d_\lambda |w'_\lambda|^{p-2} w'_\lambda + \beta_\lambda(w'_\lambda)\|_\infty \leq C.$$

Now, we shall prove

$$(A.7)' \quad \|d_\lambda^{2/p} w'_\lambda\|_{L^p(\Omega^*)} \leq \|d_\lambda^{2/p} w'_\lambda\|_{L^p(\Omega^*)} \leq C,$$

for another constant — as usually, in the following, C will denote several constants, independent of λ —. From (A.1)'

$$-\int_{\Omega^*} (w_\lambda - d_\lambda |w'_\lambda|^{p-2} w'_\lambda + \beta_\lambda(w'_\lambda)) d_\lambda w'_\lambda ds \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \|d_\lambda w'_\lambda\|_{L^2(\Omega^*)}^2.$$

As previously,

$$\int_{\Omega^*} -w_\lambda d_\lambda w'_\lambda ds \geq -\frac{\alpha}{2} C^2 - \frac{1}{2\alpha} \|d_\lambda w'_\lambda\|_{L^2(\Omega^*)}^2, \quad -\int_{\Omega^*} d_\lambda \beta_\lambda(w'_\lambda) w'_\lambda ds \geq 0.$$

Thus, we get, with

$$X_\lambda = \int_{\Omega^*} d_\lambda^2 |w'_\lambda|^p ds = \|d_\lambda^{2/p} w'_\lambda\|_{L^p(\Omega^*)}^p,$$

$$X_\lambda \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{\alpha}{2} C^2 + \frac{1}{2} \left(1 + \frac{1}{\alpha}\right) \|d_\lambda w'_\lambda\|_{L^2(\Omega^*)}^2.$$

By Hölder inequality,

$$\int_{\Omega^*} d_\lambda^2 w'_\lambda ds = \int_{\Omega^*} d_\lambda^{2-(4/p)} d_\lambda^{4/p} w'_\lambda ds \leq \left(\int_{\Omega^*} d_\lambda^2 ds \right)^{1-(2/p)} X_\lambda^{2/p},$$

that is

$$\|d_\lambda w'_\lambda\|_{L^2(\Omega^*)}^2 \leq C X_\lambda^{2/p}.$$

It comes

$$X_\lambda \leq C X_\lambda^{2/p} + C',$$

(with $p > 2$), which implies that X_λ is bounded independently of λ (easily proved by contradiction), that is we have (A. 7)'. It follows

$$(A. 7)'' \quad \|d |w_\lambda''|^{p-2} w_\lambda''\|_{L^q(\Omega^*)} \leq \|d_\lambda |w_\lambda''|^{p-2} w_\lambda''\|_{L^q(\Omega^*)} \leq C.$$

In fact,

$$\int_{\Omega^*} d_\lambda^q |w_\lambda''|^p ds = \int_{\Omega^*} d_\lambda^{q-2} d_\lambda^2 |w_\lambda''|^p ds \leq \|d_\lambda\|_{\infty}^{q-2} X_\lambda \leq C.$$

Now, we are going to prove that w_λ' is bounded. When $p=2$, we proved [see (A. 10)] that w_λ' is bounded in $L^\alpha(\Omega^*)$, for any $\alpha < 2$ if $N=1$, or $\alpha < N/(N-1)$ if $N > 1$. The argument which was used [*i. e.* multiply (A. 1) by w_λ' and integrate] seems to be no more valid when $p > 2$. On the contrary, the following one (boundedness of $s^i w_\lambda''$ for some suitable power i) would not be valid (for every N) for $p=2$. By (A. 1)', with $\gamma > 0$ to be chosen,

$$\begin{aligned} & - \int_{\Omega^*} (w_\lambda - d_\lambda |w_\lambda''|^{p-2} w_\lambda'' + \beta_\lambda(w_\lambda')) s^\gamma w_\lambda'' ds \\ & = - \int_{\Omega^*} h s^\gamma w_\lambda'' ds \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \|s^\gamma w_\lambda''\|_{L^2(\Omega^*)}^2; \\ & - \int_{\Omega^*} w_\lambda s^\gamma w_\lambda'' ds \geq -C \|s^\gamma w_\lambda''\|_{L^2(\Omega^*)} \geq -\frac{\alpha}{2} C^2 - \frac{1}{2\alpha} \|s^\gamma w_\lambda''\|_{L^2(\Omega^*)}^2; \\ & - \int_{\Omega^*} s^\gamma \beta_\lambda(w_\lambda') w_\lambda'' ds = \int_{\Omega^*} \gamma s^{\gamma-1} j_\lambda(w_\lambda') ds \geq 0. \end{aligned}$$

Thus

$$(A. 17) \quad -\frac{\alpha}{2} C^2 + \int_{\Omega^*} d_\lambda s^\gamma |w_\lambda''|^p ds \leq \frac{1}{2} \|h\|_{L^2(\Omega^*)}^2 + \frac{1}{2} \left(1 + \frac{1}{\alpha}\right) \|s^\gamma w_\lambda''\|_{L^2(\Omega^*)}^2.$$

Now, by Hölder inequality,

$$\begin{aligned} \int_{\Omega^*} s^{2\gamma} w_\lambda''^2 ds &= \int_{\Omega^*} d_\lambda^{-2/p} s^{2\gamma/q} d_\lambda^{2/p} s^{2\gamma/p} |w_\lambda''|^2 ds \\ &\leq \left(\int_{\Omega^*} d_\lambda s^\gamma |w_\lambda''|^p ds \right)^{2/p} \left(\int_{\Omega^*} d_\lambda^{-2/(p-2)} s^{2\gamma/(2-q)} ds \right)^{1-(2/p)}. \end{aligned}$$

For convenient γ , the last integral is bounded, as $d^{-2/(p-2)} s^{2\gamma/(2-q)}$ is in $L^1(\Omega^*)$, as soon as

$$(A.18) \quad \gamma > \text{Max}(0, \gamma_1) \quad \text{with} \quad \gamma_1 = \frac{p(1-(2/N))+2}{2(p-1)} = q\left(\frac{3}{2} - \frac{1}{N}\right) - 1.$$

From (A.17), we then obtain, with $X_\lambda = \int_{\Omega^*} d_\lambda s^\gamma |w'_\lambda|^p ds$, $X_\lambda \leq C + C' X_\lambda^{2/p}$, that is X_λ is bounded, when $\lambda \rightarrow 0$. Now

$$\begin{aligned} |w'_\lambda(s)| &\leq \int_s^{|\Omega|} |w''_\lambda(\sigma)| d\sigma = \int_s^{|\Omega|} d_\lambda^{-1/p} \sigma^{-\gamma/p} d_\lambda^{1/p} \sigma^{\gamma/p} |w''_\lambda| d\sigma \\ &\leq X_\lambda^{1/p} \left(\int_s^{|\Omega|} d_\lambda^{-q/p} \sigma^{-\gamma q/p} d\sigma \right)^{1/q} \quad (\text{by Hölder inequality}) \\ &\leq C \left(\int_s^{|\Omega|} \sigma^{q[(1/N)-1-(\gamma/p)]} d\sigma \right)^{1/q}. \end{aligned}$$

Assume $\gamma > \gamma_2 = p/N - 1$. Then, (A.18) is replaced by

$$(A.18)' \quad \gamma > \text{Max}(0, \gamma_1, \gamma_2),$$

and $q((1/N) - 1 - (\gamma/p)) < -1$. We have

$$|w'_\lambda(s)| \leq C s^{(1/N) - (\gamma+1)/p},$$

and w'_λ is bounded in $L^\alpha(\Omega^*)$, as soon as $s^{(1/N) - (\gamma+1)/p}$ is in $L^\alpha(\Omega^*)$, that is

$$(A.19) \quad \gamma < p\left(\frac{1}{N} + \frac{1}{\alpha}\right) - 1 = p\left(1 + \frac{1}{N}\right) - 1 - \varepsilon$$

if we set $p/\alpha = p - \varepsilon$. With (A.18)', (A.19), we get the condition

$$(A.20) \quad \text{Max}(0, \gamma_1, \gamma_2) < \gamma < p\left(1 + \frac{1}{N}\right) - 1 - \varepsilon,$$

that is the choice of γ giving that w'_λ is bounded in $L^\alpha(\Omega^*)$,

$$\left(\alpha = \frac{p}{p-\varepsilon}, \varepsilon < p\left(1 + \frac{1}{N}\right) - 1 - \text{Max}(0, \gamma_1, \gamma_2) \right),$$

is possible as soon as

$$\text{Max}(0, \gamma_1, \gamma_2) < p\left(1 + \frac{1}{N}\right) - 1.$$

It is easily seen that $p(1 + (1/N)) - 1 > \text{Max}(0, \gamma_2)$, and the previous condition is reduced to

$$(A.21) \quad \gamma_1 < p \left(1 + \frac{1}{N} \right) - 1.$$

But, (A.21) is fulfilled for example for p integer > 2 , since it is equivalent to

$$(A.22) \quad p > \frac{5}{2(1 + (1/N))}.$$

Otherwise, we need the restriction (A.22). Seemingly, this is just a technical restriction. ■

Using these *a priori* estimates, we can pass to the limit ($\lambda \searrow 0$) in (A.1)'. Denoting again by w_λ a subsequence of the whole sequence, we have

$$\begin{aligned} w_\lambda &\rightarrow w \quad \text{in } \mathcal{C}^0(\overline{\Omega^*}), \quad \text{which implies } w(0) = 0, \\ w'_\lambda &\rightharpoonup w' \quad \text{in } L^q(\Omega^*) \text{ weakly.} \end{aligned}$$

By (A.7)',

$$(A.23) \quad d_\lambda^{2/p} w'_\lambda \rightharpoonup d^{2/p} w'' \quad \text{in } L^p(\Omega^*) \text{ weakly,}$$

(same proof as for $p=2$). We note that

$$(d_\lambda^{2/p} w'_\lambda)' = d_\lambda^{2/p} w'_\lambda + \frac{2}{p} d_\lambda^{(2/p)-1} d'_\lambda w'_\lambda.$$

The first term in the second member is bounded in $L^p(\Omega^*)$, and the second one is bounded in $L^q(\Omega^*)$. Thus $d_\lambda^{2/p} w'_\lambda$ is bounded in $W^{1,q}(\Omega^*)$,

$$d_\lambda^{2/p} w'_\lambda \rightarrow d^{2/p} w' \quad \text{in } \mathcal{C}^0(\overline{\Omega^*}),$$

which implies $w'(|\Omega|) = 0$, together with

$$w'_\lambda \rightarrow w' \quad \text{in } \mathcal{C}^0([\varepsilon, |\Omega|]), \quad \forall \varepsilon > 0.$$

By (A.6)', (A.7)'', $\beta_\lambda(w'_\lambda)$ is bounded in $L^q(\Omega^*)$, thus

$$\beta_\lambda(w'_\lambda) \rightharpoonup b \quad \text{in } L^q(\Omega^*) \text{ weakly.}$$

Classically, $w' \geq 0$ a. e., $b \in \beta(w')$. On each compact set in $\{w' > 0\} \setminus \{0\}$, $w'_\lambda > 0$ for λ big enough, $\beta_\lambda(w'_\lambda) = 0 = b$. From (A.1)',

$$\begin{aligned} \varphi(w'_\lambda) &= -|w'_\lambda|^{p-2} w'_\lambda = \frac{h-w_\lambda}{d_\lambda}, \\ d_\lambda^{2/p} w'_\lambda &= d_\lambda^{2/p} \varphi^{-1} \left(\frac{h-w_\lambda}{d_\lambda} \right) \rightarrow d^{2/p} \varphi^{-1} \left(\frac{h-w}{d} \right) \quad \text{a. e.} \end{aligned}$$

By (A.23), on $\{w' > 0\}$, $w'' = \varphi^{-1}(h - w/d)$ a. e., or

$$-d|w''|^{p-2}w'' + b + w = h.$$

Now, we want to get the same equation on $\{w' = 0\}$. Multiplying (A.1)' by $w'_\lambda \varphi$, $\varphi \in \mathcal{D}(\Omega^*)$ with support in $\{w' = 0\}$, and integrating, we get

$$\int_{\Omega^*} -d_\lambda |w''_\lambda|^p \varphi ds + \int_{\Omega^*} \beta_\lambda(w'_\lambda) w''_\lambda \varphi ds + \int_{\Omega^*} w_\lambda w''_\lambda \varphi ds = \int_{\Omega^*} h w''_\lambda \varphi ds.$$

As

$$\int_{\Omega^*} w_\lambda w''_\lambda \varphi ds \rightarrow \int_{\Omega^*} w w'' \varphi ds, \quad \int_{\Omega^*} h w''_\lambda \varphi ds \rightarrow \int_{\Omega^*} h w'' \varphi ds$$

and

$$\begin{aligned} \int_{\Omega^*} \beta_\lambda(w'_\lambda) w''_\lambda \varphi ds &= \int_{\Omega^*} \frac{d}{ds} (j_\lambda(w'_\lambda)) \varphi ds = - \int_{\Omega^*} j_\lambda(w'_\lambda) \varphi' ds \\ &= - \int_{\Omega^*} \frac{1}{2\lambda} w_{\lambda-}^{\prime 2} \varphi' ds = \frac{1}{2} \int_{\Omega^*} \beta_\lambda(w'_\lambda) w'_{\lambda-} \varphi' ds, \end{aligned}$$

$w'_{\lambda-} \varphi' \rightarrow w'_- \varphi' = 0$ uniformly, $\beta_\lambda(w'_\lambda) \rightarrow b$ in $L^q(\Omega^*)$ weakly,

$$\int_{\Omega^*} \beta_\lambda(w'_\lambda) w''_\lambda \varphi ds \rightarrow 0,$$

we get

$$\int_{\Omega^*} d_\lambda |w''_\lambda|^p \varphi ds \rightarrow \int_{\Omega^*} (w - h) w'' \varphi ds.$$

Taking for φ any positive function with support in $\{w' = 0\}$, where $w'' = 0$, we get $\int_{\Omega^*} d_\lambda |w''_\lambda|^p \varphi ds \rightarrow 0$, then $d_\lambda |w''_\lambda|^p \rightarrow 0$ almost everywhere on $\{w' = 0\}$. Thus $-d|w''|^{p-2}w''$ converges to zero, almost everywhere on $\{w' = 0\}$, and weakly converges to $h - w - b$ in $L^q(\Omega^*)$, that is

$$0 = h - w - b = -d|w''|^{p-2}w'' \quad \text{a. e. on } \{w' = 0\},$$

as desired, and we obtain (A.2)'. ■

The proof of the second part of Lemma 4', i. e. (2.21), is reduced to

LEMMA (A.2)' - There is a unique $w \in D(M)$ solution of (A.2)'.

Proof of Lemma (A.2)'. - The proof of Lemma (A.2)' which we give here is different and simpler than the proof of Lemma (A.2) [and it is also valid to prove Lemma

(A.2)]. Let w and \hat{w} be two solutions of (A.2)′.

$$\begin{cases} -d|w''|^{p-2}w'' + b + w = h, & b \in \beta(w'), & w(0) = w'(|\Omega|) = 0, \\ -d|\hat{w}''|^{p-2}\hat{w}'' + \hat{b} + \hat{w} = h, & \hat{b} \in \beta(\hat{w}'), & \hat{w}(0) = \hat{w}'(|\Omega|) = 0. \end{cases}$$

We multiply the difference of these equations by z' [$z = w - \hat{w} \in C^0(\overline{\Omega^*}) \cap W^{2,p}(\varepsilon, |\Omega|)$],

and use $(b - \hat{b})z' \geq 0$ a.e. We get (with ξ between w'' and \hat{w}'')

$$z'(z - (p-1)d|\xi|^{p-2}z') \leq 0 \quad \text{a.e. on } \Omega^*.$$

Let \mathcal{O} be the open set $\Omega^* \cap \{z' > 0\}$, and let $\mathcal{O}_i = (a_i, b_i)$ be the connected components of \mathcal{O} . On each of \mathcal{O}_i , we have

$$z - (p-1)d|\xi|^{p-2}z'' \leq 0 \quad \text{a.e. on } \mathcal{O}_i,$$

and

$$\text{either } z' = 0, \quad \text{or } z = 0 \quad \text{on } \partial\mathcal{O}_i.$$

(for $z(a_i) = 0$ if $a_i = 0$, $z'(b_i) = 0$ if $b_i = |\Omega|$, $z' = 0$ on $\partial\mathcal{O}_i$ if $a_i \neq 0$, $b_i \neq |\Omega|$). It follows, in \mathcal{O}_i : $z'' > 0$ if $z > 0$,

$$\begin{aligned} 0 &\leq \int_{\mathcal{O}_i} z'' z_+ ds = - \int_{\mathcal{O}_i} (z_+)'{}^2 ds, \\ z_+ &= \text{const.} \quad \text{on } \mathcal{O}_i. \end{aligned}$$

This constant is necessarily 0, since $z = \text{const.} > 0$ should imply $z' = 0$, in contradiction with the definition of \mathcal{O} . Thus $z \leq 0$ on \mathcal{O}_i , and we have proved that $z' > 0$ implies $z \leq 0$. Exchanging w and \hat{w} , we get $z' < 0$ implies $z \geq 0$. Hence $zz' \leq 0$ on Ω^* ,

$$\frac{1}{2}z^2(s) = \int_0^s zz' d\sigma \leq 0, \quad z \equiv 0 \quad \text{on } \Omega^*.$$

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