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Uniqueness and continuum of foliated solutions for a quasilinear elliptic equation with a non lipschitz nonlinearity

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1 Introducción.

We are interested in this article in quasilinear elliptic equations of the form

$$-\Delta u + \mathrm{H}(x, \nabla u) + \lambda u = 0 \quad \text{in } \Omega \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda \geq 0$ and H is a given continuous function.

It is well-known that (1) satisfies the classical Maximum Principle in $C^2(\Omega) \cap C(\overline{\Omega})$, provided that $\lambda > 0$. This result remains true when $\lambda = 0$ if $\mathrm{H}(x,p)$ is assumed to be locally Lipschitz continuous in p, by using Hopf's Maximum Principle (*c.f.* [4] or [5]). The aim of this paper is to investigate

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the case when $\lambda = 0$ and when we drop the locally Lipschitz assumption on p. To simplify the presentation, we will consider only the case when

$$\mathrm{H}(x,p) = |\, p\,|^m - f(x), \ (x,p) \in \overline{\Omega} \times \mathbf{R}^N, \tag{2}$$

where $0 < m < 1$ and $f \in C^{0,\alpha}(\overline{\Omega})$.

We will be interested first in existence results. With $\lambda = 0$ and H being given by (2), the equation (1) becomes

$$-\Delta u + |\, \nabla u\,|^m = f \quad \text{in } \Omega, \tag{3}$$

and we complement it with some Dirichlet boundary condition

$$u = \varphi \quad \text{on } \partial\Omega \tag{4}$$

provided $\varphi \in C(\partial\Omega)$.

Concerning the existence of a solution we have

Theorem 1 *Assume that Ω is a $C^{2,\beta}$ domain for some $0 < \beta \leq 1$, then the problem* (3) *and* (4) *has a minimal* u *and a maximal solution* $\mathrm{U} \in C^{2,\delta}(\Omega) \cap C(\overline{\Omega})$ *for some δ depending on α, β and m.*

Of course, u and U can be different in general, as we will show it by several examples. But uniqueness still holds under some additional assumptions. So, a first result is the following

Theorem 2 *Assume that $f(x) \neq 0$, $\forall x \in \Omega$. Then, the Maximum Principle holds for* (3) *in $C^2(\Omega) \cap C(\overline{\Omega})$. In particular, the solution of* (3)-(4) *is unique.*

Then, the natural question is to know whether we can relax or not the assumption on f in Theorem 2. It is a curious fact that the answer will depend on the sign of f. Indeed, we will give an example of function f, $f > 0$ in $\Omega \backslash \{x_0\}$ and $f(x_0) = 0$ for some $x_0 \in \Omega$ for which (3)-(4) exhibits three solutions. By the contrary, we shall show that uniqueness still remains true if $f \leq 0$ and the set where f vanishes is small enough. As a matter of fact, our answer for this last case only concerns with radially symmetric solutions of (3)-(4) on a ball. Finally, we shall prove that there exists a continuum of solutions for the Dirichlet problem, whenever uniqueness fails. In fact, this set generates, near a point where f vanishes, a foliation of dimension N and codimension 1.

The paper is organized as follows: Section 2 is devoted to the proof of Theorem 1 and also contains a first example of non uniqueness for the radial case. Theorem 2 will be proved in Section 3. Finally, in Section 4 we study the case when f may vanish and we give uniqueness and non uniqueness results according the sign of f. We end the paper by studying the structure of the set of solutions in absence of uniqueness.

2 Remarks on the existence results.

We first give the proof of Theorem 1. We start by considering the case when $\varphi \in C^{2,\gamma}(\partial\Omega)$ for some $\gamma > 0$. We approximate (3) and (4) by the problems $(\underline{P}_\epsilon)$:

$$\begin{aligned} -\Delta \mathrm{u}_\epsilon + (\epsilon^2 + \mid \nabla \mathrm{u}_\epsilon \mid^2)^{\frac{m}{2}} &= f \quad \text{in } \Omega, \\ \mathrm{u}_\epsilon &= \varphi \quad \text{on } \partial\Omega, \end{aligned}$$

and $(\overline{P}_\epsilon)$:

$$\begin{aligned} -\Delta \mathrm{U}_\epsilon + (\epsilon^2 + \mid \nabla \mathrm{U}_\epsilon \mid^2)^{\frac{m}{2}} - \epsilon^2 &= f \quad \text{in } \Omega, \\ \mathrm{U}_\epsilon &= \varphi \quad \text{on } \partial\Omega. \end{aligned}$$

By classical results ($c.f.$ [4]), $(\underline{P}_\epsilon)$ and $(\overline{P}_\epsilon)$ have unique solutions since, now, the nonlinearities are locally Lipschitz continuous. Then, due to $0 < m < 1$, using Schauder estimates, one proves easily that

$$\| \mathrm{u}_\epsilon \|_{2,\delta}, \ \| \mathrm{U}_\epsilon \|_{2,\delta} \leq C \quad (\text{ independent on } \epsilon)$$

for some δ depending on α, β, γ and m ($c.f.$ [4]) and by Ascoli Theorem

$$\mathrm{u}_\epsilon \to \mathrm{u} \quad \text{in } C^2(\overline{\Omega}),$$

$$\mathrm{U}_\epsilon \to \mathrm{U} \quad \text{in } C^2(\overline{\Omega}),$$

and u, U are both $C^{2,\delta}(\overline{\Omega})$ solutions of (3)-(4).

Moreover, if we point out the dependence of u and U in the boundary datun φ by denoting them respectively $\mathrm{u}(\varphi), \mathrm{U}(\varphi)$, one has

$$\| v(\varphi) - v(\psi) \|_{L^\infty(\Omega)} \leq \| \varphi - \psi \|_{L^\infty(\partial\Omega)} \tag{5}$$

for $v = \mathrm{u}, \mathrm{U}$ and $\varphi, \psi \in C^{2,\gamma}(\partial\Omega)$. Indeed, (5) is true for $v = \mathrm{u}_\epsilon$ (or $v = \mathrm{U}_\epsilon$) and we pass to the limit in the inequality. In order to solve (3)-(4) for $\varphi \in C(\partial\Omega)$, we introduce two sequences $\{\underline{\varphi}_\epsilon\}, \{\overline{\varphi}_\epsilon\}$ such that:

1. $\underline{\varphi}_\epsilon \leq \varphi \leq \overline{\varphi}_\epsilon$
2. $\underline{\varphi}_\epsilon, \overline{\varphi}_\epsilon \in C^{2,\gamma}(\partial\Omega)$ for some $\gamma \in]0,1[$,
3. $\underline{\varphi}_\epsilon, \overline{\varphi}_\epsilon \to \varphi$ in $C(\partial\Omega)$ when $\epsilon \to 0$.

Using (5), it is clear enough that $\{\mathrm{U}(\overline{\varphi}_\epsilon)\}$ and $\{\mathrm{u}(\underline{\varphi}_\epsilon)\}$ are Cauchy sequences in $C(\overline{\Omega})$ and therefore

$$\mathrm{u}(\underline{\varphi}_\epsilon) \to \mathrm{u} \quad \text{in } C(\overline{\Omega})$$

$$\mathrm{U}(\overline{\varphi}_\epsilon) \to \mathrm{U} \quad \text{in } C(\overline{\Omega}).$$

Moreover, $\mathrm{u}(\underline{\varphi}_\epsilon)$ and $\mathrm{U}(\overline{\varphi}_\epsilon)$ satisfy uniform interior estimates in $C^{2,\delta}(\overline{\Omega})$ (for some δ depending on α, β and m) and therefore u and U solve (3).

Finally, let w be a solution of (3)-(4) $w \in C^2(\Omega) \cap C(\overline{\Omega})$. Then using the above notations and the Maximum Principle for $(\underline{P}_\epsilon)$ and $(\overline{P}_\epsilon)$, one has

$$\mathrm{u}(\underline{\varphi}_\epsilon) \leq w \leq \mathrm{U}(\overline{\varphi}_\epsilon) \quad \text{on } \overline{\Omega}.$$

And letting $\epsilon \to 0$, we get

$$\mathrm{u} \leq w \leq \mathrm{U} \quad \text{on } \overline{\Omega}.$$

Therefore, $\mathbf{u}$ is the minimal solution of (3)-(4) and $\mathbf{U}$ is the maximal solution of (3)-(4) and the proof is complete.$_\#$

Now, we turn to a first example of non-uniqueness in the radial case.

Example 1.

We consider the boundary problem

$$\begin{aligned} -\Delta u + \mid \nabla u \mid^m &= 0 \quad \text{in } B(0,R), \\ u &= c \quad \text{on } \partial B(0,\mathrm{R}) \end{aligned} \tag{6}$$

where $B(0,\mathrm{R})$ is the open ball in $\mathbb{R}^N$ of radius R centered at the origin and $c \in \mathbb{R}$. Of course, $u \equiv c$ is a solution of (6). Easy computations show that the function

$$\mathrm{w}(x) = C_m(R^k - \mid x \mid^k) + c, \ x \in B(0,\mathrm{R}) \tag{7}$$

for

$$k = \frac{2-m}{1-m}$$

and

$$C_m = k(k+N-2)^{\frac{1}{m-1}}$$

is also a solution of (6).$_\#$

Function $\mathbf{w}$ given by (7), or a modification of it, is very useful in order to show the existence and localization of a free boundary for the problem

$$\begin{aligned} -\Delta u + \mid \nabla u \mid^m + \lambda u &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{8}$$

Clearly, the first order nonlinearity becomes, near the set $\{x : \mid \nabla u \mid = 0\}$, the dominant term of the PDE. In fact, this set is not empty provided suitable conditions on φ, f and Ω. Indeed, assume, for instance $f \in C^{0,\alpha}(\Omega)$, $\varphi \in C(\partial\Omega)$ and $\lambda > 0$. Standard arguments lead to the existence of a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (8).

Theorem 3 *Let us suppose that there exists $c \in \mathbb{R}$ such that $f \geq c$ in Ω and that*

$$\mathcal{N}_c(f) = \{x \in \Omega : \ f(x) = c\} \quad \textit{has a not empty interior.}$$

Assume also that

$$\varphi \geq \frac{c}{\lambda} \quad on\ \partial\Omega.$$

and let u be the solution of (8). *Then*

$$u(x_0) = \frac{c}{\lambda}$$

for any $x_0 \in \overline{\mathcal{N}_c(f)}$ such that

$$dist\ (x_0,\ supp\ \ (\varphi - (\frac{c}{\lambda})) \cup \partial\mathcal{N}_c(f)) \geq \mathrm{R} \tag{9}$$

with

$$\mathrm{R} = \left[\frac{\lambda \parallel u \parallel_{L^\infty(\Omega)} - c}{\lambda C_m}\right]^{\frac{1}{k}}$$

for k and C_m as in Example 1.

Proof.

The Maximum Principle and assumptions imply

$$u(x) \geq \frac{c}{\lambda},\ x \in \overline{\Omega}.$$

Let $x_0 \in \overline{\mathcal{N}_c(f)}$ satisfying (9). Then the function

$$u_c(x) = \frac{c}{\lambda} + C_m \mid x - x_0 \mid^k$$

is a supersolution of the PDE on $\tilde{\Omega} = B(x_0, \mathrm{R}) \cap \Omega$ (notice the inclusion $\tilde{\Omega} \subset \mathcal{N}_c(f)$).

Since

$$u_c \geq \parallel u \parallel_{L^\infty(\Omega)} \geq u \quad \text{on } \partial\tilde{\Omega},$$

we obtain

$$u_c \geq u \quad \text{in } \tilde{\Omega}$$

and consequently

$$\frac{c}{\lambda} = u_c(x_0) \geq u(x_0) \geq \frac{c}{\lambda}.\#$$

The boundary of the set $\{x:\ u = \frac{c}{\lambda}\}$ is a free boundary whose study can be carryed out following the techniques of [3]. The above result shows that the behaviour of the solutions of (8) is very close to the one of solutions of elliptic equations degenerating on the set $\{\nabla u = 0\}$ such as, for instance,

$$-\ \mathrm{div}\ \ (\mid \nabla u \mid^{p-2} \nabla u) + \lambda u = f,\ p > 2 \tag{10}$$

(see [3, Theorem 1.14]).

Finally, we pointed out that when $c \neq 0$ all those free boundaries disappear if $\lambda \to 0$. Moreover, if $\lambda = 0,\ f \leq 0$ the Strong Maximum Principle for nonnegative subharmonic functions shows that any solution u of (3) verifies $u < 0$ in Ω, unless $u \equiv 0$.

3 Uniqueness for non vanishing right side terms.

We give in this part the proof of Theorem 2. Since f is continuous in Ω it is enough to prove the uniqueness for $f < 0$ or $f > 0$. We first consider the case when $f < 0$ in Ω. The idea is to make some suitable change of variable $u = \Phi(v)$, where Φ is a strictly monotone smooth function. After this change, (3) becomes

$$-\Delta v - \frac{\Phi''(v)}{\Phi'(v)} \mid \nabla v \mid^2 + (\Phi'(v))^{m-1} \mid \nabla v \mid^m - (\Phi'(v))^{-1} f = 0 \quad \text{in } \Omega. \quad (11)$$

Since we deal with smooth solutions, the only property to be checked is that the nonlinearity is strictly increasing with respect to v, provided we make a suitable choice of Φ. We choose

$$\Phi(t) = -\exp\{-t\}, \; t \in \mathbb{R}.$$

Then (11) may be rewritten as

$$-\Delta v + \mid \nabla v \mid^2 + \exp\{(1-m)v\} \mid \nabla v \mid^m - \exp\{v\} f = 0 \quad \text{in } \Omega, \quad (12)$$

and the desired property is obviously satisfied since $1 - m > 0$ and $f < 0$. Now, [illegible] turn to the case when $f > 0$ in Ω. We are going to give two proofs in this case. The first one follows exactly the idea explained just above: unfortunately, the choice of Φ is not obvious anymore. So, we turn to the computations made in [2] which say that v has to be defined through the relation

$$\int_0^{u(x)} \frac{ds}{\mathrm{M} - \exp\{-\Lambda s\}} = v(x), \; x \in \overline{\Omega},$$

where M, Λ are large constants chosen later. Constant M has, in particular, to be chosen large enough in order to ensure that the above integral makes sense. This means that the function Φ satisfies the ODE

$$\Phi'(v) = \mathrm{M} - \exp\{-\Lambda\Phi(v)\}.$$

We denote by $\mathrm{H}(x,t,p)$ the nonlinearity of the equation (11), *i.e.*

$$\mathrm{H}(x,t,p) = -\frac{\Phi''(t)}{\Phi'(t)} \mid p \mid^2 + (\Phi'(t))^{m-1} \mid p \mid^m - (\Phi'(t))^{-1} f(x),$$

we obtain

$$\frac{\partial \mathrm{H}(x,t,p)}{\partial t} = -\Big(\frac{\Phi''(t)}{\Phi'(t)}\Big)' \mid p \mid^2 + (m-1)\Phi''(t)(\Phi'(t))^{m-2} \mid p \mid^m + \frac{\Phi''(t)}{\Phi'(t)} f(x).$$

But

$$\Phi''(v) = \Lambda \exp\{-\Lambda\Phi(v)\}\Phi'(v)$$

and therefore

$$\Big(\frac{\Phi''(v)}{\Phi'(v)}\Big)' = -\Lambda^2 \exp\{-\Lambda\Phi(v)\}\Phi'(v) = -\Lambda\Phi''(v).$$

We obtain

$$\frac{\partial \mathrm{H}(x,v,p)}{\partial t} = \Phi''(v)\Big[\Lambda \mid p \mid^2 + (m-1)(\Phi'(v))^{m-2} \mid p \mid^m + (\Phi'(v))^{-1} f(x)\Big].$$

Since $\Phi'' > 0$, we are just interested in the sign of the bracket. Using Young inequality, we have

$$(1-m)(\Phi'(v))^{m-2} \mid p \mid^m$$
$$\leq \frac{m}{2}\Lambda \mid p \mid^2 + \frac{2-m}{2}\Lambda^{\frac{-m}{2-m}}(1-m)^{\frac{2}{2-m}}(\Phi'(v))^{-2}$$

and the bracket is estimed by

$$\Lambda(1-\frac{m}{2}) \mid p \mid^2 + (\Phi'(v))^2\Big[f(x) - \frac{2-m}{2}\Lambda^{\frac{-m}{2-m}}(1-m)^{\frac{2}{2-m}}\Big].$$

To conclude, it is suffices to choose Λ large in order to have the last term strictly positive and then M is chosen verifying

$$\mathrm{M} \ > \ \exp\{\Lambda \parallel u \parallel_{L^\infty(\Omega)}\}$$

and the proof is completed.

We give now another proof of the above result. We consider a subsolution w and a supersolution W of (3); we assume $w, W \in C^2(\Omega) \cap C(\overline{\Omega})$ and $w \leq W$ on $\partial\Omega$. Our aim is to show that $w \leq W$ on $\overline{\Omega}$. To do so, we argue by contradiction assuming that max $(w - W) > 0$. This implies, in particular, that no maximum point of $w - W$ lies on $\partial\Omega$. Moreover, if $x_0 \in \Omega$ is a maximum point of $w - W$, then

$$\nabla w(x_0) = \nabla W(x_0) = 0.$$

Indeed, if $\nabla w(x_0) \neq 0$, the contradiction follows from the fact that the Hopf Principle applies, since in a neighborhood of x_0, functions w and W are solutions of a quasilinear elliptic equation with a Lipschitz nonlinearity. So that, we consider the function

$$x \mapsto \mu w(x) - W(x) \quad \text{for } \mu < 1, \text{ close to } 1.$$

This function has at least a maximum point x_μ and (taking if necessary a subsequence) we may assume that

$$x_\mu \to x_0 \quad \text{when } \mu \to 1,$$

where x_0 is a maximum point of $w - W$. Properties of the interior maxima yield

$$\mu\Delta w(x_\mu) \le \Delta W(x_\mu),$$
$$\mu\nabla w(x_\mu) = \nabla W(x_\mu).$$

But w and W are respectively sub and supersolutions of (3); hence

$$-\Delta w(x_\mu) + \mid \nabla w(x_\mu) \mid^m \le f(x_\mu),$$
$$-\Delta W(x_\mu) + \mid \nabla W(x_\mu) \mid^m \ge f(x_\mu).$$

Then, we multiply the first inequality by μ, we subtract the second one and using the properties above we are led to

$$(\mu^{1-m} - 1) \mid \nabla W(x_\mu) \mid^m \le (\mu - 1) f(x_\mu).$$

We know divide by $\mu - 1$ and we let $\mu \to 1$. We obtain

$$(1 - m) \mid \nabla W(x_0) \mid^m \ge f(x_0)$$

which is the desired contradiction since $\nabla W(x_0) = 0$ and $f(x_0) > 0$.$_\#$

Remark 1.
A slight modification of the above arguments gives also another proof for the "$f < 0$" case.$_\#$

4 The case of f vanishing at one point.

It is natural to wonder whether Theorem 2 is sharp or if one can relax the assumptions by allowing, for example, f to vanish at a point or on some "small" subset of $\overline{\Omega}$. We first build an example showing that the "$f > 0$" result is sharp.

Example 2.
For $f(x) = \nu \mid x \mid^{\frac{m}{1-m}}$, $\nu > 0$, $c \in \mathbb{R}$, we consider the boundary problem

$$\begin{aligned} -\Delta u + \mid \nabla u \mid^m &= f \quad \text{in } B(0, \mathrm{R}), \\ u &= c \quad \text{on } \partial B(0, \mathrm{R}) \end{aligned} \tag{13}$$

We are looking for solutions of the type

$$u(x) = C(\mathrm{R}^k - |\, x \,|^k) + c, \ x \in B(0,\mathrm{R}), \ C, k \in \mathbf{R}.$$

Let the function

$$g(C) = \left[\frac{2-m}{(1-m)^2} + N - 1\right] C + \left(\frac{2-m}{1-m}\right)^m \cdot |\, C \,|^m, \ C \in \mathbf{R}.$$

Easy computations show that u solves (13) if and only if and only if

$$k = \frac{2-m}{1-m}$$

and C satisfies

$$g(C) = \nu.$$

But if ν is small enough, this algebraic equation has two negative roots and one positive which lead to three different radially symmetric solutions for (13). We emphasize that $f > 0$ in $B(0,\mathrm{R})\backslash\{0\}$ and $f(0) = 0$. We also remark that, in fact, two of those solutions are negative solutions although $f \geq 0$.#

Our next result shows that the situation changes when f is nonpositive and the set where f vanishes is small enough. We only consider the case of radially symmetric solutions

Theorem 4 *Let $\varphi \in \mathbf{R}$ and $f(x) = \mathrm{f}(|\, x \,|)$ such that $\mathrm{f} \in C([0,\mathrm{R}])$ and*

$$\mathrm{f}(r) < 0 \text{ a.e. } r \in]0,\mathrm{R}[. \tag{14}$$

Then, for $\Omega = B(0,\mathrm{R})$ the problem (3)-(4) has at most one radially symmetric solution.

Proof.

Let $v(x) = \mathrm{v}(|\, x \,|)$ be any radially symmetric solution of (3)-(4) on $\Omega = B(0,\mathrm{R})$ and define

$$\mathrm{w}(r) = \frac{d\mathrm{v}}{dr}(r) r^{N-1}, \quad 0 < r < \mathrm{R}.$$

Then,

$$-\mathrm{w}'(r) = r^{N-1}\mathrm{f}(r) - r^{(N-1)(1-m)} \,|\, \mathrm{w}(r) \,|^m, \quad 0 < r < \mathrm{R}. \tag{15}$$

So, $\mathrm{w}'(r) > 0$ *a.e.* $r \in]0,\mathrm{R}[$. Moreover, as $\mathrm{w}(0) = 0$, we have that $\mathrm{w}(r) > 0$ for any $r \in]0,\mathrm{R}[$ (notice that $\mathrm{w} \in C^1([0,\mathrm{R}[))$. Then, we can define the inverse function $r = \mathrm{a}(t)$ (*i.e.* $\mathrm{w}(\mathrm{a}(t)) = t$). From (15) we deduce that

$$-\frac{1}{\mathrm{a}'(t)} = \mathrm{a}(t)^{N-1}\mathrm{f}(\mathrm{a}(t)) - \mathrm{a}(t)^{(N-1)(1-m)}t^m, \quad 0 < t < \mathrm{T}(=\| \mathrm{w} \|_{L^\infty(\Omega)})$$

or

$$-1 = \mathrm{a}'(t)\mathrm{a}(t)^{N-1}\mathrm{f}(\mathrm{a}(t)) - \mathrm{a}'(t)\mathrm{a}(t)^{(N-1)(1-m)}t^m.$$

Now, let

$$\mathrm{b}(t) = \frac{1}{q+1}\mathrm{a}(t)^{q+1}, \quad q = (N-1)(1-m).$$

(We note that $q+1>0$ and $\mathrm{w}(0)=0$ implies $\mathrm{b}(0)=0$). Then, we have

$$-1 = [\mathcal{F}(\mathrm{b}(t))]' - \mathrm{b}'(t)t^m \tag{16}$$

with $\mathcal{F}(0) = 0$ and

$$\mathcal{F}'(\eta) = [(q+1)\eta]^{\frac{m(N-1)}{q+1}}\mathrm{f}(((q+1)\eta)^{\frac{1}{q+1}}).$$

Notice that $\mathcal{F}'(\eta) < 0$ *a.e.* $\eta > 0$.

At this point, assume that (3)-(4) has two different radially symmetric solutions $\mathbf{u}_1$ and $\mathbf{u}_2$. They generate two solutions $\mathrm{b}_1(t)$ and $\mathrm{b}_2(t)$ of (14) with $\mathrm{b}_1 \neq \mathrm{b}_2$. Since $\mathrm{b}_1(0) = \mathrm{b}_2(0)$, one has

$$\mathrm{b}_1(t) = \mathrm{b}_2(t), \text{ if } 0 \le t \le \mathrm{t} \quad \text{and} \quad \mathrm{b}_1(t) \neq \mathrm{b}_2(t), \text{ if } \mathrm{t} < t \le \mathrm{T}$$

for some $0 \le \mathrm{t} < \mathrm{T}$. So, there exists $\mathrm{s} \in]\mathrm{t}, \mathrm{T}]$ such that

$$\| \mathrm{b}_1 - \mathrm{b}_2 \|_{L^\infty(\mathrm{t},\mathrm{T})} = (\mathrm{b}_1 - \mathrm{b}_2)(\mathrm{s}) > 0.$$

Integrating by parts in (16) in $[\mathrm{t}, \mathrm{s}]$, properties $\mathcal{F}(0) = 0$ and $(\mathrm{b}_1 - \mathrm{b}_2)(\mathrm{t}) = 0$ lead to

$$-\mathcal{F}(\mathrm{b}_1(\mathrm{s})) + \mathcal{F}(\mathrm{b}_2(\mathrm{s})) + (\mathrm{b}_1(\mathrm{s}) - (\mathrm{b}_2(\mathrm{s}))\mathrm{s}^m = m\int_{\mathrm{t}}^{\mathrm{s}} (\mathrm{b}_1(s) - (\mathrm{b}_2(s))s^{m-1}ds,$$

that is,

$$-\mathcal{F}(\mathrm{b}_1(\mathrm{s})) + \mathcal{F}(\mathrm{b}_2(\mathrm{s})) + \mathrm{s}^m \| \mathrm{b}_1 - \mathrm{b}_2 \|_{L^\infty(\mathrm{t},\mathrm{T})} \le \mathrm{s}^m \| \mathrm{b}_1 - \mathrm{b}_2 \|_{L^\infty(\mathrm{s},\mathrm{T})}$$

which contradicts that $\mathcal{F}$ is strictly decreasing. Then, $\mathrm{b}_1 \equiv \mathrm{b}_2$ and so $\mathrm{w}_1 \equiv \mathrm{w}_2$ in $]0, \mathrm{R}[$. In particular, $\mathrm{u}_1'(r) = \mathrm{u}_2'(r)$ in $]0, \mathrm{R}[$ and as

$$\mathrm{u}_i(r) = \varphi - \int_r^{\mathrm{R}} \mathrm{u}_i'(s)ds, \quad 0 < r < \mathrm{R} \qquad (i = 1, 2)$$

the result follows.$_\#$

Remark 2.

We do not know, update, if the above result can be extended to the general (non radially symmetric) case for functions f satisfying $f < 0$ in $\Omega\backslash\{x_0\}$, $f(x_0) = 0$, for some $x_0 \in \Omega$.#

We conclude this paper by studying the structure of the set of solutions provided that the following condition holds:

$$\text{there exists } x_0 \in \Omega \text{ such that } f(x_0) = 0 \text{ and } f(x) \neq 0 \quad \forall x \in \Omega\backslash\{x_0\}. \tag{17}$$

Theorem 5 *Assume* (17). *Then, for any $y \in \Omega$ and any $t \in [\mathbf{u}(y), \mathbf{U}(y)]$ there exists a solution u of* (3)-(4) *in $C^2(\Omega)\cap C(\overline{\Omega})$ satisfying $u(y) = t$, where* $\mathbf{u}$ *and* $\mathbf{U}$ *denote respectively the minimal and the maximal solution of* (3)-(4).

Proof.

First step.

We start by proving the result for the special case $y = x_0$. Let $t_1, t_2 > 0$ be defined by

$$t = \mathbf{u}(x_0) + t_1 = \mathbf{U}(x_0) - t_2.$$

We denote by $u_1(.)$ the function $\mathbf{u}(.) + t_1$ and by $u_2(.)$ the function $\mathbf{U}(.) - t_2$. We will proceed in the following way: we will first prove that there exists a unique solution $u_t \in C^{2,\gamma}(\Omega\backslash\{x_0\}) \cap C(\overline{\Omega})$ of

$$\begin{aligned} -\Delta u + |\nabla u|^m &= f \quad \text{in } \Omega\backslash\{x_0\}, \\ u(x_0) &= t, \\ u &= \varphi \quad \text{on } \partial\Omega, \end{aligned} \tag{18}$$

and then we will show that $u_t \in C^{2,\gamma}(\Omega)$ and that the equation holds at x_0. To do so, we introduce the approximated problem

$$\begin{aligned} -\Delta u_\epsilon + |\nabla u_\epsilon|^m &= f \quad \text{in } \Omega\backslash B(x_0,\epsilon), \\ u_\epsilon &= u_2 \quad \text{on } \partial B(x_0,\epsilon), \\ u_\epsilon &= \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{19}$$

where ϵ is small enough. By Theorem 1 and 2, there exists a unique solution u_ϵ of (19) in $C^{2,\gamma}(\Omega\backslash B(x_0,\epsilon)) \cap C(\overline{\Omega\backslash B(x_0,\epsilon)})$. The aim is to pass to the limit in (19). The first ingredient is, of course, the uniform interior estimates for u_ϵ in $C^{2,\gamma}(\Omega\backslash\{x_0\})$ for some $0 < \gamma < 1$. Then we have to work in a neighborhood of the boundaries. We first claim that

$$u_1 \geq u_2 \quad \text{in } \overline{\Omega}; \tag{20}$$

indeed, u_1 and u_2 are solutions of (3), $u_1(x_0) = u_2(x_0)$ and $u_1 \geq u_2$ on $\partial\Omega$; hence (20) is a consequence of Theorem 2 since f does not vanishes in $\Omega\backslash\{x_0\}$. By the same argument, one has also

$$u_2 \le u_\epsilon \le u_1 \quad \text{in } \overline{\Omega \backslash B(x_0, \epsilon)}. \tag{21}$$

Finally, since $t_1, t_2 > 0$, for ϵ small enough, one has

$$\mathrm{u} \le u_\epsilon \le \mathrm{U} \quad \text{in } \overline{\Omega \backslash B(x_0, \epsilon)}. \tag{22}$$

By the uniform interior estimates, we can consider a sequence $\{u_{\epsilon'}\}_{\epsilon'}$ converging in $C^2(\Omega\backslash\{x_0\})$ to some function u_t satisfying

$$\begin{array}{rl} -\Delta u_t + \mid \nabla u_t \mid^m = f & \text{in } \Omega\backslash\{x_0\}, \\ u_2 \le u_t \le u_1 & \text{in } \Omega\backslash\{x_0\}, \\ \mathrm{u} \le u_t \le \mathrm{U} & \text{on } \overline{\Omega}. \end{array} \tag{23}$$

Therefore $u_t \in C(\overline{\Omega})$, $u_t(x_0) = t$, $u_t = \varphi$ on $\partial\Omega$. We still have to obtain the equation at x_0. Recalling the arguments of the proof of Theorem 2, we know that if $\mathrm{u} \ne \mathrm{U}$, x_0 is the only maximum point of $\mathrm{U} - \mathrm{u}$ on $\overline{\Omega}$, moreover $\nabla \mathrm{u}(x_0) = \nabla \mathrm{U}(x_0) = 0$. Since

$$D^2\mathrm{U}(x_0) \le D^2\mathrm{u}(x_0)$$

and, by the equation,

$$\Delta\mathrm{U}(x_0) = \Delta\mathrm{u}(x_0) = 0,$$

we have

$$D^2\mathrm{U}(x_0) = D^2\mathrm{u}(x_0);$$

(indeed a nonpositive symmetric matrix with a zero trace is clearly the zero matrix). Therefore, (23) implies that u_t is twice differentiable at x_0 and

$$D^2\mathrm{U}(x_0) = D^2 u_t(x_0) = D^2\mathrm{u}(x_0).$$

And finally this implies that the equation holds at x_0, that $u_t \in C^1(\Omega)$ and classical regularity results show that $u_t \in C^{2,\gamma}(\Omega)$. Uniqueness of u_t follows from Theorem 2.

<u>Second step</u>.

We still denote by u_t the solution of (3)-(4) built in the above step such that $u_t(x_0) = t$, for $t \in [\mathrm{u}(x_0), \mathrm{U}(x_0)]$. We claim that for any $y \in \Omega$ the function

$$t \mapsto u_t(y)$$

is continuous; consequently, the set $\{u_t(y) : t \in [\mathrm{u}(x_0), \mathrm{U}(x_0)]\}$ is a nonempty subinterval which is necessarily equals to $[\mathrm{u}(y), \mathrm{U}(y)]$, whence the result follows. Indeed, $\{u_t\}$ is relatively compact in $C^{2,\gamma}(\Omega)$ and when $t \to \tau$, u_t converges necessarily to u_τ, because if the subsequence $u_{t'} \to v$ in $C^2(\Omega)$, v is a solution of the equation in $\Omega\backslash\{x_0\}$ and $v(x_0) = \tau$; therefore $v = u_\tau$ by uniqueness in $\Omega\backslash\{x_0\}$. In particular,

$$u_t(y) \to u_\tau(y)$$

and the claim is proved.$_\#$

Remark 3.

From the above proof it is easy to see that for $t, s \in [\mathrm{u}(x_0), \mathrm{U}(x_0)]$ with $t < s$ implies

$$\mathrm{u}(x) \leq u_t(x) \leq u_s(x) \leq \mathrm{U}(x), \text{ for all } x \in \overline{\Omega}$$

and

$$u_t(\mathrm{x}) < u_s(\mathrm{x}),$$

provided $|\nabla u_t(\mathrm{x})| + |\nabla u_s(\mathrm{x})| > 0$ or x close enough x_0.$_\#$

Theorem 5 shows that the set of solutions generates a compact $N+1$ dimensional manifold

$$\mathcal{N} = \{(x, z) \in \mathbf{R}^{N+1} : \ x \in \overline{\Omega}, \ z \in [\mathrm{u}(x_0), \mathrm{U}(x_0)]\}.$$

Roughly speaking, $\mathcal{N}$ can be seen as a "nice stacked" family of submanifolds

$$\mathcal{N}_t = \{(x, u_t(x)) : \ x \in \overline{\Omega}, \ u_t \text{ is the solution of (18)}\}, \quad t \in [\mathrm{u}(x_0), \mathrm{U}(x_0)].$$

In fact, we have

Theorem 6 *There exists a positive constant σ such that $\mathcal{N} \cap (B(x_0, \sigma) \times \mathbf{R})$ is a foliation of dimension N and codimension 1.*

Proof.

Theorem 5 implies the relation

$$\mathcal{N} = \bigcup \{\mathcal{N}_t : \ t \in [\mathrm{u}(x_0), \mathrm{U}(x_0)]\}.$$

Moreover, from the compactness of the interval $[\mathrm{u}(x_0), \mathrm{U}(x_0)]$ there exists $\sigma > 0$ such that the sets $\mathcal{N}_t$ are connected and disjoints when they intersect the cylinder $B(x_0, \sigma) \times \mathbf{R}$. As $\mathcal{N}_t$ are N-dimensional manifolds, it is easy to check that for any $(y_0, t_0) \in \mathcal{N} \cap (B(x_0, \sigma) \times \mathbf{R})$ there exist two "small" balls $B(y_0, \delta) \subset B(x_0, \sigma)$ and $B(z_0, \gamma)$, a positive constant τ and a map

$$\vartheta_t : \ B(y_0, \delta) \times]t_0 - \tau, t_0 + \tau[\to \mathbf{R}^{N+1}$$

such that

$$\vartheta_t(B(y_0, \delta) \times]t_0 - \tau, t_0 + \tau[) \cap \mathcal{N}_t \subset B(z_0, \gamma) \times \{t\}.$$

So, according of [1, Definition 4.4.4] the set $\mathcal{N} \cap (B(x_0, \sigma) \times \mathbf{R})$ is a foliation of dimension N and codimension 1.$_\#$

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