Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 177, No. 1, July 15, 1993

On Convexity and Starshapedness of Level Sets for Some Nonlinear Elliptic and Parabolic Problems on Convex Rings

J. I. DIAZ

Departamento de Matematica Aplicada, Universidad Compultense, E-28040 Madrid, Spain

AND

B. KAWOHL

SFB 123, Universität Heidelberg, Im Neuenheimer Feld 294, D-6900 Heidelberg, Germany

Submitted by Ivar Stakgold

Received February 15, 1992

We consider some degenerate parabolic problems on a convex (or starshaped) ring. We prove that if the initial data have convex (or starshaped) level sets, then the solution $u(t, \cdot)$ has the same property for any positive t. Similar results are shown for the corresponding stationary problems. Our results imply in particular the convexity (or starshapedness) of certain free boundaries. Other nonlinear parabolic problems are also discussed. © 1993 Academic Press, Inc.

1. INTRODUCTION

The main goal of this paper is to study some qualitative properties of the level sets of certain nonlinear parabolic problems. We show that some qualitative properties of the initial data $u_0(x)$, namely, convexity and starshapedness of the level sets $\{x \in \Omega \mid u_0(x) \geq c\}$, are preserved for all positive times. Both properties have been extensively studied in the last years for elliptic equations, and for the sake of completeness we give a result on those as well, but for parabolic problems only few results of this type seem to be mentioned in the literature.

Let Ω be a bounded, open, simply connected subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and let $G \subset \Omega$ be a compact, simply connected set with

0022-247X/93 $5.00

smooth boundary $\partial\Omega$. Consider the following nonlinear degenerate diffusion problem (P) in a ring domain $\Omega\setminus G$,

$$
(\mathrm{P})\begin{cases} u_t-\Delta_p\phi(u)+f(u)=0 & \text{in} \quad (0,T)\times(\Omega\setminus G) & (1.1)\\ \phi(u)\equiv 1 & \text{on} \quad (0,T)\times G & (1.2)\\ \phi(u)=0 & \text{on} \quad (0,T)\times\partial\Omega & (1.3)\\ u(0,x)=u_0(x) & \text{in} \quad \Omega, & (1.4)\end{cases}
$$

where

$$\Delta_p w=\operatorname{div}(|\nabla w|^{p-2}\nabla w) \qquad \text{and} \qquad p>1.$$

The functions ϕ and f are in general continuous and nondecreasing with $\phi(0)=f(0)=0$, but they can also be maximal monotone graphs in $\mathbb{R}^2$ containing the origin. The symbol ∇ denotes gradient with respect to x. The equation (1.1) becomes degenerate near the set where $|\nabla\phi(u)|=0$ if $p>2$ and near $\{u=0\}$ if $\phi'(0)=0$. We also remark that for $p=2$ the pseudo-Laplace operator Δ_p becomes the Laplace operator and that problem (P) is really a boundary value problem on $(0,T)\times(\Omega\setminus G)$ with constant boundary values on the interior boundary.

There are many physical problems which are formulated as special cases of (P). The presence of the pseudo-Laplace operator Δ_p is of interest in the study of non-Newtonian fluids (see, for instance, the references in Diaz and Herrero [17]). Nonlinear diffusion terms $\phi(u)$ occur in many applied problems: $\phi(u)=u^m$ with $m>1$, for instance, appears in the modelling of porous media flow (see references in Bertsch [6]) or the spread of some biological populations (Gurtin and MacCamy [25]). If $0<m<1$, this effect is called fast diffusion and it appears in plasma physics (see references in Bertsch [6]). Another relevant choice of ϕ is $\phi(s)=k(s-a)^+$ with k and a positive, which is related to the one phase Stefan problem (see, e.g., Friedman and Kinderlehrer [23]).

The term $f(u)$ describes absorption phenomena and is of particular interest in the study of chemical catalysts. Aris [2] gives $f(u)=u^q$ as a typical example for a reaction of order q. Existence, uniqueness and regularity results for problem (P) as well as for more general problems have been provided by many authors. We refer, for instance, to Alt and Luckhaus [1] and its references.

The main objective of this paper is to prove that under suitable assumptions for every positive t the level sets $\{x\in\Omega\,|\,u(t,x)\geqslant c\}$ are convex subsets of $\mathbb{R}^n$. As a preliminary step we first establish the starshapedness of level sets. This program is carried out for solutions u of (P) on the *ring* $\Omega\setminus G$, i.e., G is in general assumed to be nonempty. Nevertheless, some other results for the case $G=\varnothing$ are also mentioned.

For the sake of completeness, our paper contains also a study of the associated stationary problem

$$(\mathrm{SP})\begin{cases} -\Delta_p w + g(w) = 0 & \text{in } \Omega \setminus G \\ w = 1 & \text{on } G \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

The contents of this paper are organized in the following way. In Section 2 we study the starshapedness of level sets of solutions of (P). This property is derived for weak solutions of (P) under very general assumptions on ϕ, f and u_0. A stronger property, namely, $x \cdot \nabla\phi(u) < 0$, is needed in the proof of the convexity of level sets. We prove it under some additional assmptions on ϕ, f, and u_0 which removed in the proof of convexity by passing to the limit. The proof of this strict inequality is not difficult for $p = 2$. For $p \neq 2$, however, the arguments are quite involved and make use of suitable barrier functions which were inspired by the work of Lewis [37].

In Section 3 we study the convexity of level sets. We prove it for solutions of (P) under natural assumptions on the initial datum u_0 and under the additional assumption of *concavity on* ϕ but for any $p > 1$. Our approach has its origins in work of Gabriel [24] on the convexity of level sets of harmonic functions. It consists in proving that the *quasiconcavity* function

$$Q(z_1, z_2) := u((z_1 + z_2)/2) - \min\{u(z_1), u(z_2)\}$$

is nonnegative for any pair $z_1, z_2 = (t, x_1), (t, x_2)$ of points in $[0, \infty) \times \Omega$. References on related previous works will be given in Sections 3 and 4. We also include in Section 3 some bibliographical comments on the starshapedness and convexity of level sets of solutions of the *interior* problem ($G = \varnothing$) and the associated obstacle problem. Finally the stationary problem (SP) is investigated in Section 4.

We end this Introduction by noting that our results on the shape of level sets of u give useful information on free boundaries which are sometimes generated by u. It is well known that a free boundary $\mathscr{F}(t) := \partial\{x \in \Omega \mid u(t, x) > 0\} \cap (\Omega \setminus G)$ can occur if (1.1) is degenerate or if the absorption term $f(u)$ dominates the diffusion (see, e.g., Diaz [13]). Our results on the level sets imply, in particular, that such free boundaries are locally Lipschitz continuous in space (and time) or even convex, provided the data satisfy suitable assumptions.

2. Starshapedness of Level Sets

A set $D \subset \mathbb{R}^n$ is called *starshaped with respect to* $x^0 \in D$ iff for any $x \in D$ the line segment $\{y = \lambda x^0 + (1-\lambda)x,\ 0 \leqslant \lambda \leqslant 1\}$ is contained in D. For brevity of notation, we call a set *starshaped* iff it is starshaped with respect to the origin. Throughout this and the following section we assume that $G \neq \emptyset$.

Theorem 1. *Let $G \Subset \Omega$ and G and Ω be starshaped. Let $p > 1$, ϕ a nondecreasing function with $\phi(0) = 0$, and let f be the sum of a continuous nondecreasing and a Lipschitz-continuous function and suppose that $f(s) \geqslant 0$ for $s \geqslant 0$. Let $u_0 \in L^\infty(\Omega)$, $\phi(u_0) \in W^{1,p}(\Omega)$ be given and suppose*

$$0 \leqslant \phi(u_0) \leqslant 1 \quad in\ \Omega, \qquad \phi(u_0) \equiv 1 \quad on\ G, \tag{2.1}$$

$$\textit{the level sets of } u_0 \textit{ are starshaped,} \tag{2.2}$$

$$\Delta_p \phi(u_0) - f(u_0) \geqslant 0 \qquad in\ \mathscr{D}'(\Omega \setminus G). \tag{2.3}$$

If $u \in C([0, T] : L^1(\Omega))$, $\phi(u) \in L^\infty((0, T) : W^{1,p}(\Omega))$ is the solution of problem (P), *then we have*

$$x \cdot \nabla \phi(u(t, x)) \leqslant 0 \qquad for \quad a.e.\ t \in (0, T) \quad and \quad x \in \Omega. \tag{2.4}$$

Moreover, if $u \in C([0, T] \times \Omega)$, then the level sets $\{x \in \Omega \mid u(t, x) \geqslant c\}$ of $u(t, \cdot)$ are starshaped for every $t \in [0, T]$.

Proof. The existence and uniqueness of a weak solution (in $C[0, T] : L^1(\Omega))$, $\phi(u) \in L^\infty((0, T) : W^{1,p}(\Omega)))$ is a well known result (see, e.g., Alt and Luckhaus [1]) or [19]. More general results on regularity are given in Di Benedetto [20], Di Benedetto and Friedman [21], and Wiegner [46]. The maximum principle holds for (P), since for instance, the realization on $L^1(\Omega \setminus G)$ of the operator $Au = -\Delta_p \phi(u) + f(u)$ is T-accretive [14], and so we conclude that $0 \leqslant \phi(u) \leqslant 1$ in $(0, T) \times \Omega$. Without loss of generality, we may assume the initial data to be smooth. Indeed, otherwise we approach u_0 by a sequence u_{0n} satisfying (2.1), (2.2), and (2.3). If each of the associated solutions u_n has starshaped level sets, then

$$S_n(t, x, s) = u_n(t, sx) - u_n(t, x) \geqslant 0$$
$$\text{for} \quad t \in [0, T],\ s \in [0, 1] \text{ and a.e. } x \in \Omega. \tag{2.5}$$

Thus in the limit, using the continuity of u,

$$S(t, x, s) = u(t, sx) - u(t, x) \geqslant 0 \qquad \text{for} \quad t \in [0, T],\ s \in [0, 1].$$

As a consequence of (2.3) we have, for u_0 smooth, that

$$u_t \geqslant 0 \qquad \text{in} \quad \mathscr{D}'((0, T) \times \Omega).$$

This can be shown after approximating the data and applying the maximum principle to the approximate solutions, or by abstract monotonicity arguments; see, e.g., Damlamian [12].

To derive (2.4) we define $v(t, x) := u(t, sx)$ for a fixed $s \in [0, 1]$, and $(\Omega \setminus G)_s := \{x \in G \mid sx \in \Omega \setminus G\}$. Note that

$$v(0, x) \geqslant u(0, x) \qquad \text{in } (\Omega \setminus G)_s \tag{2.6}$$

and

$$\phi(v(t, x)) \geqslant \phi(u(t, x)) \qquad \text{on} \quad (0, T) \times \partial(\Omega \setminus G)_s \tag{2.7}$$

hold. On the other hand, we have

$$v_t - \Delta_p \phi(v) + f(v) = \tilde{g} \qquad \text{in} \quad \mathscr{D}'((0, T) \times (\Omega \setminus G)_s) \tag{2.8}$$

with

$$\tilde{g}(t, x) = u_t(t, sx) - s^p \Delta_p \phi(u(t, sx)) + f(u(t, sx)), \tag{2.9}$$

and so $\tilde{g} \geqslant 0$ on $(0, T) \times (\Omega \setminus G)_s$. Another application of the parabolic comparison principle yields $v(t, x) \geqslant u(t, x)$ for any $t \in [0, T]$ and a.e. $x \in \Omega$, as desired. ∎

Remark 1. In the proof of Theorem 1 we have derived $u_t \geqslant 0$ as well as $x \cdot \nabla u \leqslant 0$ in $(0, T) \times (\Omega \setminus G)$. Thus the level sets $\{(t, x) \in (0, T) \times \Omega \mid u(t, x) \geqslant c\}$ are starshaped with respect to the point $(T, 0) \in \mathbb{R} \times \mathbb{R}^n$ in space and time. It is not difficult to show the stronger conclusion

$$t \cdot \phi(u)_t + x \cdot \nabla \phi(u) \leqslant 0, \tag{2.10}$$

if u is smooth enough. In fact, by the comparison principle $u(st, sx) \geqslant u(t, x)$ for $s \in (0, 1)$ and $p \geqslant 2$, and $u(s^{(p-1)}t/(p-1), sx) \geqslant u(t, x)$ for $s \in (0, 1)$ and $p \in (1, 2)$. This proves (2.10). Incidentally, we use the word "smooth" to indicate that all the occuring derivatives are continuous functions in $(0, T) \times (\Omega \setminus G)$. Property (2.10) was derived by probabilistic methods in Borell [7] for $p = 2$, $\phi(s) = s$, $f \equiv 0$, and $u_0 = \chi_G$.

Remark 2. Several generalizations are possible: For instance, we can replace the interior boundary condition (1.2) by

$$\phi(u(t, x)) = w(t) \qquad \text{on} \quad (0, T) \times G,$$

with $w \in W^{1,1}(0, T)$; $w \geqslant 0$, $w' \geqslant 0$, $0 \leqslant u_0 \leqslant w(0)$ in Ω, and $u_0 \equiv w_0(0)$ on G. Then the conclusion of Theorem 1 remains true. Or we can let G shrink to a point $\{0\}$ and obtain starshapedness of level sets for solutions which are singular at the origin. It is also possible to apply the same kind of arguments in order to prove starshapedness of level sets for solutions to obstacle problems (now considered on the entire domain Ω), provided the level sets of the obstacle Ψ are also starshaped. For some related results see Kawohl [27] or Sakaguchi [42], [43].

In order to derive convexity of level sets for problem (P), we need the following stronger result $x \cdot \nabla\phi(u) < 0$ for smooth solutions. Due to the structure of the pseudo Laplace operator, it turns out that the proof of this inequality is quite delicate if $p \neq 2$. We start with a general result which gives the desired conclusion for $p = 2$.

LEMMA 1. *Let u be a nonnegative smooth function defined on an open set $D \subset (0, T) \times \mathbb{R}^n$ such that*

$$u_t - \Delta_p \phi(u) + f(u) = g(t, x) \qquad \text{in } D, \tag{2.11}$$

$$u_t \geqslant 0 \qquad \text{on } \bar{D} \tag{2.12}$$

and

$$x \cdot \nabla\phi(u) \leqslant 0 \qquad \text{on the parabolic boundary of } D. \tag{2.13}$$

Suppose that $\phi'(s) > \alpha$ for some $\alpha > 0$ and that

$$x \cdot \nabla g(t, x) + 2g(t, x) \leqslant 0 \qquad \text{in } D. \tag{2.14}$$

In addition, if $p > 2$ suppose that $|\nabla\phi(u)| > 0$ in D.

Then $x \cdot \nabla\phi(u(t, x)) \leqslant 0$ in D. Moreover, if $p = 2$ then $x \cdot \nabla\phi(u) < 0$ in D.

Proof. Let $v = \phi(u)$, then

$$\psi'(v)\, v_t - \Delta_p v + f(\psi(v)) = g(t, x),$$

where $\psi = \phi^{-1}$. Next we consider $w(t, x) = x \cdot |\nabla v(t, x)|^{p-2}\, \nabla v(t, x)$ and calculate

$$\begin{aligned}
\nabla w &= |\nabla v|^{p-2}\, \nabla v + x \cdot \Delta_p v \\
\Delta w &= x \cdot \nabla(\Delta_p v) + 2\Delta_p v \\
&= (x \cdot \nabla v)_t\, \psi'(v) + (x \cdot \nabla v)[\psi''(v)\, v_t + f'(\psi(v))\, \psi'(v)] \\
&\quad + 2\psi'(v)\, v_t + 2f(\psi(v)) - 2g - x \cdot \nabla g \\
&\geqslant a(t, x)(b(t, x)\, w)_t + c(t, x) w,
\end{aligned}$$

with

$$a(t,x)=\psi'(v(t,x))\geqslant\alpha,\qquad a\in L^\infty,$$

$$b(t,x)=|\nabla v(t,x)|^{p-2}\geqslant 0,\qquad b\in L^\infty,$$

and

$$c(t,x)=b(\psi''(v)\,v_t+f'(\psi(v)\,\psi'(v)),\qquad c\in L^\infty.$$

Since $w\leqslant 0$ on the parabolic boundary of D, the weak maximum principle implies that $w\leqslant 0$ in D. If $p=2$, then $b\equiv 1$ and $w<0$ in D by the strong maximum principle. ∎

As mentioned above, the study of the strict inequality $x\cdot\nabla\phi(u)<0$ is delicate if $p\neq 2$. In order to prove it we shall make suitable assumptions on the initial datum u_0. We use the following barrier function $v(x)$ defined by

$$v(x):=\begin{cases}\alpha\,|x-z|^{(p-n)/(p-1)}+\beta, & \text{if}\quad p\neq n,\\ \alpha\log|x-z|+\beta, & \text{if}\quad p=n,\end{cases}\tag{2.15}$$

for$x\in B_\delta(z)\setminus\overline{B_{\delta/2}(z)}$, $v(x):=1$ on $\overline{B_{\delta/2}(z)}$, $v(x):=0$ on $\mathbb{R}^n\setminus B_\delta(z)$, where z will be chosen later, and

$$\alpha:=\begin{cases}(2^{(m-p)/(p-1)}-1)^{-1}\,\delta^{(n-p)/(p-1)}, & \text{if}\quad p\neq n\\ -(\log 2)^{-1}, & \text{if}\quad p=n.\end{cases}$$

and

$$\beta:=\begin{cases}[1-2^{(N-p)/(p-1)}]^{-1}, & \text{if}\quad p\neq n.\\ \log\delta/\log 2, & \text{if}\quad p=n.\end{cases}$$

It is a simple exercise to check that $-\Delta_p v=0$ in $B_\delta(z)\setminus\overline{B_{\delta/2}(z)}$, that v is continuous and $v\in W^{1,p}(\mathbb{R}^n)$. Moreover,

$$\nabla v(x)\,|\nabla v(x)|^{-1}\to(z-x_0)\,|z-x_0|^{-1}\tag{2.16}$$

as $x\to x_0\neq z$ in $B_\delta(z)$, and

$$|\nabla v(x)|\geqslant c>0\qquad\text{for}\quad x\in B_\delta(z)\setminus\overline{B_{\delta/2}(z)}.\tag{2.17}$$

THEOREM 2. *Let u be the solution to problem* (P) *and in addition to the assumptions of Theorem* 1 *suppose that the interior of G is not empty, and that G and Ω are convex and satisfy the following uniform sphere condition.*

$$\left.\begin{array}{l}\textit{There exists a } \delta>0 \textit{ such that for any } x_0\in\partial G\cup\partial\Omega\\ \textit{there is a } z \textit{ with } x_0\in B_\delta(z) \textit{ and } B_\delta(z)\subset\Omega\setminus G.\end{array}\right\}\tag{2.18}$$

Moreover, we assume that ϕ *and* f *are smooth, strictly increasing and satisfy*

$$\left.\begin{array}{l}\phi'(s) \leqslant M \text{ and } \inf_{s \in [0,1]}\{(d/ds)\, \phi'(\phi^{-1}(s))\} > -\infty \text{ and } \tilde{f} = f \circ \phi^{-1} \\ \text{is such that } \tilde{f} \in C^1([0,1]), \tilde{f}' \geqslant 0, \tilde{f}(0) = \tilde{f}'(0) = 0,\end{array}\right\} \tag{2.19}$$

$$\phi(u_0) \in C^1(\Omega \setminus G) \qquad \text{and} \qquad x \cdot \nabla\phi(u_0(x)) < 0 \quad \text{in } \Omega \setminus G, \tag{2.20}$$

and

$$\phi(u_0(x)) \geqslant \varepsilon v(x) \qquad \text{for some } \varepsilon > 0 \qquad \text{and for} \quad d(x, \partial\Omega) \leqslant \delta. \tag{2.21}$$

Let u *be the solution of problem* (P) *and assume that* ϕ *is convex or that* $u_t \in L^\infty((0, T) \times (\Omega \setminus G)$.

Then u *satisfies the nondegeneracy condition*

$$x \cdot \nabla u(t, x) < 0 \qquad \text{in} \quad (0, \infty) \times (\Omega \setminus G).$$

Proof. We adopt some ideas from Lewis [37], who considered the stationary problem with vanishing f. First we obtain some estimates on the growth of $\phi(u(t, sx))$ with respect to s, when $x \in \partial\Omega \cup \partial G$. Then we shall estimate from above the function $\phi(u(t, sx)) - \phi(u(t, x))$ when s decreases to one. This gives us the result.

Let $K = \{x \in \Omega \setminus G \mid d(x, \partial\Omega \cup \partial G) \geqslant \delta/2\}$. Because of the choice of u_0 $(0 < \phi(u_0) < 1$ on $\Omega \setminus G)$ and f, there is a constant $A > 0$ such that

$$\min\{1 - \phi(u(t, x)), \phi(u(t, x))\} \geqslant A \qquad \text{for} \quad x \in K, \quad t \in [0, T]. \tag{2.22}$$

Moreover, from the continuity of $\phi(u)$, A decreases to zero as δ goes to zero. Given $x_0 \in \partial\Omega \cup \partial G$ we choose z according to (2.18) and consider $v(x)$.

From the convexity of Ω and G and (2.17) we see that

$$|(z - x_0) \cdot x_0| \geqslant \tau\, |z - x_0|\, |x_0|,$$

where $\tau > 0$ is independent of x_0 and z. Then, if $x_0 \in \partial G$ we have that $sx_0 \in B(z, \delta) \setminus \overline{B(z, \delta/2)}$ for $1 < s \leqslant s_0$ and suitable s_0, and

$$v(sx_0) \geqslant 2^{-1}\tau(s - 1)\, c\, |x_0| \geqslant \mu\, |s - 1|. \tag{2.23}$$

Here v is the auxiliary function defined above Theorem 2. Similarly, if $x_0 \in \partial\Omega$, then $s^{-1}x_0 \in B(z, \delta) \setminus \overline{B(z, \delta/2)}$ and

$$v(x_0/s) \geqslant \mu\, |s - 1| \qquad \text{for} \quad 1 < s \leqslant s_0. \tag{2.24}$$

Note that s_0 and μ can be chosen independent of $x_0 \in \partial\Omega \cup \partial G$. If $x_0 \in \partial G$, it is clear from (2.22) that

$$\phi(u(t, x)) \leqslant 1 - Av(x), \qquad \text{for} \quad x \in \partial(B(z, \delta)\setminus\overline{B(z, \delta/2)}), \quad t \in (0, T],$$

while $-\Delta_p \phi(u) = -u_t - f(u) \leqslant 0$ in Q_T. Then by the comparison principle we have

$$\phi(u(t, x)) \leqslant 1 - Av(x) \qquad \text{for} \quad x \in B(z, \delta)\setminus B(z, \delta/2) \quad \text{and} \quad t \in (0, T] \tag{2.25}$$

If $x_0 \in \partial\Omega$ we consider the test function $\underline{v}(t, x) = \phi^{-1}(Av(x) - C_1 t)$ with $C_1 > 0$ to be chosen later. On $\partial(B(z, \delta)\setminus B(z, \delta/2)) \times (0, T)$ we have $\phi(u) \geqslant \underline{v}$. On the other hand, due to (2.21) we may suppose without loss of generality that $\phi(u_0(x)) \geqslant Av(x)$ on $B(z, \delta)\setminus B(z, \delta/2)$. Finally by assumption (2.19)

$$\begin{aligned}\underline{v}_t - \Delta_p \phi(\underline{v}) + f(\underline{v}) &\leqslant -MC_1 + f(\phi^{-1}(Av - C_1 t)) \\ &\leqslant C_1 + f(\phi^{-1}(A)) \leqslant 0,\end{aligned}$$

if we choose now $C_1 = \tilde{f}(A)$. Then again by the comparison principle

$$Av(x) - \tilde{f}(A)\, t \leqslant \phi(u(t, x)) \qquad \text{for} \quad x \in B(z, \delta)\setminus B(z, \delta/2) \quad \text{and} \quad t \in (0, T] \tag{2.26}$$

Now we may conclude from (2.24) and (2.23) that, if $x_0 \in \partial G$

$$1 - \phi(u(t, sx_0)) \geqslant A\mu(s-1) \qquad \text{for} \quad 1 < s \leqslant s_0 \tag{2.27}$$

and analogously, if $x_0 \in \partial\Omega$, from (2.26) and (2.24)

$$\phi(u(t, x_0/s)) \geqslant Av(x_0/s) - \tilde{f}(A)\, t \geqslant A\mu(s-1) - \tilde{f}(A)\, t \tag{2.28}$$

Since $\tilde{f}'(0) = 0$ we may assume A to be small enough so that $\tilde{f}(A)/A \leqslant \mu(s-1)/2T$ and then (2.28) leads to

$$\phi(u(t, x_0/s)) \geqslant \frac{A}{2}\, \mu(s-1) \qquad \text{for} \quad 1 < s \leqslant s_0. \tag{2.29}$$

Next fix $1 < s \leqslant s_0$ and let $\Omega_s = \{y \in \Omega \mid sy \in \Omega\}$. Set $w(t, x) := u(t, sx)$. Then

$$\begin{aligned} w_t - \Delta_p \phi(w) + f(w) &= 0 && \text{in} \quad (0, T) \times (\Omega_s \setminus G), \\ \phi(w) &\leqslant 1 && \text{on} \quad (0, T) \times G, \\ \phi(w) &= 0 && \text{on} \quad (0, T) \times \partial\Omega_s, \\ w(0, x) &= u_0(sx) && \text{on} \quad \Omega_s. \end{aligned}$$

We want to estimate $\phi(u(t, sx)) - \phi(u(t, x))$ from above, i.e.,, we want to show that there is a positive function ψ depending on t (and s) such that $\underline{w}(t, x) = \phi^{-1}(\phi(w(t, x)) + \psi(t)) \leqslant u(t, x)$ on $(0, T) \times (\Omega_s \backslash G)$. The following considerations are aimed at the construction of ψ. On $(0, T) \times \partial G$ and $(0, T) \times \partial\Omega_s$ we may use (2.27) and (2.29) to conclude that $\underline{w} \leqslant u$ on $(0, T) \times \partial(\Omega_s \backslash G)$, provided

$$\psi(t) \leqslant \frac{A}{2}\mu(s-1) \qquad \text{for any} \quad t \in [0, T]. \tag{2.30}$$

Since $\underline{w} \leqslant u$ should hold on the parabolic boundary of $(0, T) \times (\Omega_s \backslash G)$ we set $\omega = \min\{|x \cdot \nabla\phi(u_0(x))| \mid x \in \overline{\Omega \backslash G}\}$, then $\phi^{-1}(\phi(u_0(sx)) - \omega(s-1)) \leqslant u_0(x)$ and $\underline{w}(0, x) \leqslant u_0(x)$ in $\Omega_s \backslash G$ is implied by the condition

$$\psi(0) \leqslant \omega(s-1). \tag{2.31}$$

On the other hand

$$\begin{aligned} &\underline{w}_t - \Delta_p \phi(\underline{w}) + f(\underline{w}) \\ &\quad = \frac{1}{\phi'(\phi^{-1}(\phi(w) + \psi))} [\psi' + w_t(\phi'(w) - \phi'(\phi^{-1}(\phi(w) + \psi))] \\ &\qquad + \phi'(\phi^{-1}(\phi(w) + \psi))(\tilde{f}(\phi(w) + \psi(t)) - \tilde{f}(\phi(w)) \\ &\quad \leqslant \frac{1}{\phi'(\phi^{-1}(\phi(w) + \psi))} [\psi' - H_1\psi + H_2\psi], \end{aligned}$$

where

$$H_1 = \min_{(t, x) \in Q_T} w_t(t, x) \cdot \min_{s \in [0, 1]} \frac{d}{ds} \phi'(\phi^{-1}(s))$$

(or $H_1 \equiv 0$ if ϕ is convex), and

$$H_2 = M \cdot \min_{s \in [0, 1]} \tilde{f}(s).$$

It is clear that it is always possible to choose $\psi(t) \geqslant 0$ satisfying (2.30), (2.31), and

$$\psi' + (H_2 - H_1)\psi \leqslant 0. \tag{2.32}$$

In fact we can choose $\psi(t) = C(s-1)\exp(H_1 - H_2)t$ with C given by $C = \min\{\omega, A\mu/2\}$ if $H_2 \geqslant H_1$, and $C = \min\{\omega, (A\mu/2)\exp(H_2 - H_1)T\}$ if

$H_2 < H_1$. Then by the comparison principles we conclude $\underline{w} \leqslant u$ on $(0, T) \times (\Omega_s \setminus G)$, i.e.,

$$\phi(u(t, sx)) - \phi(u(t, x)) \leqslant -C(s-1) \exp(H_1 - H_2) t,$$

and consequently

$$x \cdot \nabla\phi(u, t, x)) \leqslant \lim_{s \to 1} \frac{\phi(u(t, sx)) - \phi(u(t, x))}{(s-1)} \leqslant -C \exp(H_1 - H_2)\, t \leqslant 0$$

as desired. This completes the proof of Theorem 2. ∎

Problem (P) can give rise to free boundaries such as

$$\mathscr{F}(t) := \partial\{x \in \Omega \mid u(t, x) > 0\} \cap (\Omega \setminus G).$$

The occurrence of such a free boundary can be caused by the degeneracy of the equation, i.e., if ϕ satisfies

$$\int_0^1 \left(\frac{\phi'(s)}{s}\right)^{1/(p-1)} ds < \infty,$$

or by sufficiently strong absorption such as $f(s) = |s|^{q-1} s$ for $\phi(s) = |s|^{m-1} s$ and $(p-1)\, m > q \geqslant 0$. Results on the existence of such free boundaries have been obtained by many different authors. We refer to the expository paper [13] and the lecture notes [14] and its references.

Note that solutions of problem (P) can (for $p > 2$) have the property that they may be constant and nonzero on sets of positive measure. The boundaries of such sets can be again considered as free boundaries. For an elliptic version of this phenomenon, see Diaz [14, p. 41].

In any case it is of interest to study the regularity of the free boundary. According to the above theorem those free boundaries are starshaped with respect to the origin. From this property we may deduce that $\mathscr{F}(t)$ is Lipschitz continuous in space and that $\mathscr{F} = \bigcup_{t \in [0, T)} \mathscr{F}(t)$ is locally Lipschitz continuous in space and time.

COROLLARY 1. *Assume the hypotheses of Theorem* 1 *as well as condition* (2.33):

$$\left.\begin{array}{l}\textit{The interior of } G \textit{ contains the origin, and } \Omega \textit{ and } G \textit{ are starshaped}\\ \textit{with respect to an open neighborhood of the origin.}\end{array}\right\} \qquad (2.33)$$

Then for each $t \in [0, T]$ *the free boundary* $\mathscr{F}(t)$ *is Lipschitz continuous in* x, *and* $\mathscr{F} = \bigcup_{t \in [0, T)} \mathscr{F}(t)$ *is locally Lipschitz continuous in* (t, x).

Proof. By Theorem 1 any ray originating in (t, x^0) with $x^0 \in B_\delta(0) \subset \mathbb{R}^n$ and $0 < \delta$ intersects $\mathscr{F}(t)$ at most once, provided δ is sufficiently small. If one varies x^0 in $B_\delta(0)$ one finds that $\mathscr{F}(t)$ cannot lie inside certain cones with vertex in $\mathscr{F}(t)$, and hence $\mathscr{F}(t)$ is Lipschitz continuous. To prove the analogous result for $\mathscr{F}$ one has to recall Remark 1 and apply the cone type argument in $(0, T-\delta)\times\mathbb{R}^n$. ∎

Corollary 1 recovers the result of Friedman and Kinderlehrer [23] on the Stefan problem. Recent results for the porous medium equation ($p=2$ and $\phi(u)=u^m$ with $m>1$) on $(0, T)\times\mathbb{R}^n$ are due to Caffarelli *et al.* [11] and to Bardi [4]. They prove the starshapedness of the support of $u(t,\cdot)$ as a preliminary step to other qualitative results.

3. Convexity of Level Sets

Now we shall prove the convexity (in space) of level sets for solutions of problem (P). It is an easy exercise to show that this is equivalent to establishing the nonnegativity of the quasiconcavity function

$$Q(t, x_1, x_2) := \phi(u(t, (x_1+x_2)/2)) - \min\{(\phi(u(t, x_1)), \phi(u(t, x_2))\}$$

for any fixed t in $(0, T)$ and any pair x_1, x_2 of points in $\Omega\times\Omega$. In this case we call $u(t,\cdot)$ *quasiconcave on* Ω.

THEOREM 3. *Let Ω and G be convex domains with smooth boundaries satisfying the uniform interior sphere condition* (2.18). *Let ϕ and f be continuous and nondecreasing functions with $\phi(0)=0$, $f(0)\geqslant 0$ and assume that ϕ is concave. Let $u_0\in C(\bar{\Omega})$ with $\phi(u_0)\in W_0^{1,p}(\Omega)$ satisfy* (2.1), (2.3), *and*

$$u_0 \qquad \text{is quasiconcave.} \tag{3.1}$$

Then the solution $u\in C([0,\infty)\times\Omega)$ of problem (P) *is quasiconcave on Ω for any fixed $t\in[0,\infty)$.*

Proof. Without loss of generality, we may assume that u_0, ϕ, and f are sufficiently regular and satisfy the assumptions of Lemma 1, Theorem 2, and Theorem 4, and that

$$\phi(u_0(x_1+x_2)/2) \leqslant -\min\{(\phi(u_0(x_1)), \phi(u_0(x_2))\}. \tag{3.2}$$

If this is not the case we can approximate them by regular functions $u_{0,n}$, ϕ_n and f_n in such a way that $u_{0,n}\to u_0$ in $L^2(\Omega)$, $f_n(s)\to f(s)$ and $\phi_n(s)\to\phi(s)$ uniformly on compact sets, and such that $u_{0,n}$, ϕ_n and f_n satisfy the required assumptions. The corresponding solutions of the regularized problems are denoted by u_n. Arguing as in Damlamian [12, Thm. 2.3], we conclude that $u_n \rightharpoonup u$ weakly in $W^{1,2}([0,T]:W^{-1,p'}(\Omega\setminus G))$

and $\phi_n(u_n(t,\cdot)) \rightharpoonup \phi(u(t,\cdot))$ weakly in $L^2(\Omega\setminus G)$ and pointwise a.e. in x for every $t\in[0,T]$. Once we manage to prove the theorem for regular data, we are done, because the quasiconcavity functions

$$Q_n(t,x_1,x_2) := \phi_n(u_n(t,x_1+x_2)/2)) - \min\{\phi_n(u_n(t,x_1)), \phi_n(u_n(t,x_2))\}$$

are all nonnegative and converge pointwise for every $t\in(0,T]$ and a.e. $(x_1,x_2)\in\Omega\times\Omega$ to the quasiconcavity function Q of $\phi(u)$. Moreover, we can send T to ∞. Now we know from Theorems 1 and 2 that $|\nabla u(t,x)|$ is nonzero in $(0,T)\times(\Omega\setminus G)$ and that the solution to our problem is a classical one and satisfies

$$x\cdot\nabla\phi(u(t,x))<0, \quad u>0 \quad \text{and} \quad u_t\geqslant 0 \quad \text{in} \quad (0,T)\times(\Omega\setminus G), \tag{3.3}$$

as well as

$$u_t,\ \phi(u)_{x_i},\ \phi(u)_{x_ix_j},\ \phi(u)_{x_ix_jx_j}\in C((0,T)\times(\Omega\setminus G)),$$

see, e.g., Ladyzhenskaya *et al.* [35].

We proceed by contradiction. Assume that

$$Q(t,x_1,x_2) := \phi(u(t,(x_1+x_2)/2)) - \min\{(\phi(u(t,x_1)), \phi(u(t,x_2))\}\geqslant 0 \tag{3.4}$$

is not true. Then Q has a negative infimum on $[0,\infty)\times\bar\Omega\times\bar\Omega$, and by continuity of u, the infimum is either attained for some finite t_0 and $(y_1,y_2)\in\bar\Omega\times\bar\Omega$, or as $t\to\infty$. To deal with the last possibility, recall that $u(t,\cdot)\to u_\infty$ as $t\to\infty$ at least in $L^2(\Omega)$, where u_∞ is the solution of the associated stationary problem (see, e.g., [19, 22, 36]). But $u_\infty(x)$ is quasiconcave (see Section 4), and, as $u_t\geqslant 0$, Q cannot become infimal as $t\to\infty$. Clearly, by assumption (3.1) (or, equivalently (3.2)) $Q(0,x_1,x_2)\geqslant 0$, so if Q fails to satisfy (3.5), then it attains its negative minimum in $(t_0,y_1,y_2)\in(0,\infty)\times\Omega\times\Omega$.

The following information is easily derived:

$$y_1, y_2 \quad \text{and} \quad (y_1+y_2)/2\in\Omega\setminus G \tag{3.5}$$

and

$$u(t_0,y_1)=u(t_0,y_2)>u(t_0,(y_1+y_2)/2). \tag{3.6}$$

Property (3.5) follows from the starshapedness of level sets of u. To prove (3.6) by contradiction, suppose that $u(t_0,y_1)<u(t_0,y_2)$. Then, locally near (y_1,y_2) the function Q would have the representation

$$Q(t,x_1,x_2) := \phi(u(t,(x_1+x_2)/2)) - \phi(u(t,x_1)),$$

and therefore the spatial gradient of Q would have to vanish at (t_0, y_1, y_2), i.e., $\nabla v(t_0, (y+y_2)/2 = 0 = \nabla v(t_0, y_1)$. But this would contradict Theorem 2.

We can now vary the points y_1 and y_2 and deduce that the vectors $\nabla v(t_0, (y_1+y_2)/2)$, $\nabla v(t_0, y_1)$ and $\nabla v(t_0, y_2)$ are parallel and equally directed. Indeed, if $\nabla v(t_0, y_1)$ and $\nabla v(t_0, y_2)$ are not parallel and equally directed, then there exists a unit vector $\xi \in \mathbb{R}^n$ such that $v_\xi(t_0, y_1) < 0$ and $v_\xi(t_0, y_2) > 0$. This contradicts the minimality of Q, since one could diminish Q by moving y_1 in direction $-\xi$ and y_2 in direction ξ. In a similar fashion we get the conclusion that $\nabla v(t_0, (y_1+y_2)/2)$ and $\nabla v(t_0, y_1)$ are parallel and equally directed.

The crucial ingredient of the rest of the proof is the following auxiliary result.

LEMMA 2. *Let t_0, y_1 and y_2 be as above. We introduce the notation*

$$(t_0, (y_1+y_2)/2) = z_{12}, \qquad (t_0, y_1) = z_1, \qquad \text{and} \qquad (t_0, y_2) = z_2,$$

and

$$a = |\nabla v(z_{12})|, \qquad b = |\nabla v(z_1)|, \qquad c = |\nabla v(z_2)|, \qquad \text{and} \qquad \mu = c/(b+c).$$

Then the following relations hold:

$$\frac{1}{a} = \frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right) \qquad (i.e., a = \mu b + (-\mu)c) \tag{3.7}$$

$$v_t(z_{12}) = \mu v_t(z_1) + (1-\mu) v_t(z_2) \tag{3.8}$$

$$\frac{1}{a^p} \Delta_p \phi(u(z_{12})) \geqslant \frac{\mu}{b^p} \Delta_p \phi(u(z_1)) + \frac{1-\mu}{c^p} \Delta_p \phi(u(z_2)). \tag{3.9}$$

Proof. The proof uses ideas of a method which was presented in [28, 29, 37] for the treatment of stationary problems. We define $\tilde{n} := (1/b) \nabla\phi(u(z_1))$ and fix a unit vector $h \in \mathbb{R}^n$ with the property

$$h \cdot \tilde{n} \neq 0. \tag{3.10}$$

Since $|\nabla\phi(u(t, x))| > 0$ in $(0, T) \times (\Omega \setminus G)$, for any δ small enough there exists a C^2-function $r_\delta(s)$ such that for every small real number s the relation

$$\phi\left(u\left(t_0 + s\delta,\, y_1 + \frac{s}{b} h\right)\right) = \phi\left(u\left(t_0 + s\delta,\, y_2 + \frac{r_\delta(s)}{c} h\right)\right) \tag{3.11}$$

holds. This follows from the implicit function theorem and property (3.6). Now let us consider the auxiliary function

$$Q^*(s;\delta,h):=Q\left(t_0+s\delta;\ y_1+\frac{s}{b}h,\ y_2+\frac{r_\delta(s)}{c}h\right).$$

By (3.5) and (3.6), for any δ and h the function Q^* attains a negative minimum at $s=0$. Therefore $\partial Q^*/\partial s$ has to vanish and $\partial^2 Q^*/\partial s^2 \geqslant 0$ at $s=0$. Before we calculate those two expressions, we note that by the choice of $r_\delta(s)$ we have the representation

$$\begin{aligned} Q^*(s;\delta,h) :=\ & \phi\left(u\left(t_0+s\delta,\frac{y_1+y_2}{2}+\frac{h}{2}\left(\frac{s}{b}+\frac{r_\delta(s)}{c}\right)\right)\right) \\ & -\phi\left(u\left(t_0+s\delta,\ y_1+\frac{s}{b}h\right)\right). \end{aligned} \tag{3.12}$$

We calculate

$$\begin{aligned} \frac{\partial Q^*}{\partial s}(s;\delta,h) =\ & \delta\frac{\partial}{\partial t}\phi\left(u\left(t_0+s\delta,\frac{y_1+y_2}{2}+\frac{h}{2}\left(\frac{s}{b}+\frac{r_\delta(s)}{c}\right)\right)\right) \\ & +\frac{1}{2}\left(\frac{1}{b}+\frac{r'_\delta(s)}{c}\right)D_h\phi\left(u\left(t_0+\delta s,\frac{y_1+y_2}{2}+\frac{h}{2}\left(\frac{s}{b}+\frac{r_\delta(s)}{c}\right)\right)\right) \\ & -\delta\frac{\partial}{\partial t}\phi\left(u\left(t_0+\delta s,\ y_1+\frac{s}{b}h\right)\right) \\ & -\frac{1}{b}D_h\phi\left(u\left(t_0+\delta s,\ y_1+\frac{s}{b}h\right)\right) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} 0 &= \frac{\partial Q^*}{\partial s}(0;\delta,h) \\ &= \delta\frac{\partial}{\partial t}\phi(u(z_{12}))+\frac{1}{2}\left(\frac{1}{b}+\frac{r'_\delta(0)}{c}\right)D_h\phi(u(z_{12})) \\ &\quad -\delta\frac{\partial}{\partial t}\phi(u(z_1))-\frac{1}{b}D_h\phi(u(z_1)). \end{aligned} \tag{3.14}$$

But from (3.11) we deduce, taking $h=\tilde{n}$ and using (3.8) that

$$r'_\delta(0)=\delta\left(\frac{\partial}{\partial t}\phi(u(z_1))-\frac{\partial}{\partial t}\phi(u(z_2))\right)+1. \tag{3.15}$$

So the first property (3.7) of Lemma 2 follows from (3.14) and (3.15) after the choice $\delta = 0$ and $h = \tilde{n}$. Analogously one obtains (3.8) for $\delta \neq 0$, because then (3.14) becomes

$$0 = \delta v_t(z_{12}) + \frac{\delta a}{2c}(v_t(z_1) - v_t(z_2)) - \delta v_t(z_1), \tag{3.16}$$

and $a/2c = (1-\mu)$. To prove (3.9) we have to consider second derivatives. We note

$$\begin{aligned} 0 \leqslant \frac{\partial^2 Q^*}{\partial s^2}(0; 0, h) = \left[\frac{1}{2}\left(\frac{1}{b} + \frac{1}{c}\right)\right]^2 D_{hh}\phi(u(z_{12})) \\ + \frac{1}{2}\frac{r_0''(0)}{c}a(n \cdot h) - \frac{1}{b^2}D_{hh}\phi(u(z_1)), \end{aligned} \tag{3.17}$$

and

$$r_0''(0)(n \cdot h) = \frac{1}{b^2}D_{hh}\phi(u(z_1)) - \frac{1}{c^2}D_{hh}u(z_2)). \tag{3.18}$$

A combination of (3.18) and (3.7) yields

$$\frac{1}{a^2}D_{hh}\phi(u(z_{12})) \geqslant \frac{\mu}{b^2}D_{hh}\phi(u(z_1)) + \frac{(1-\mu)}{c^2}D_{hh}\phi(u(z_2)). \tag{3.19}$$

In particular we have that

$$\begin{aligned} \left[\frac{\mu}{b^2}\frac{\partial^2}{\partial x_i\,\partial x_j}\phi(u(z_1)) + \frac{(1-\mu)}{c^2}\frac{\partial^2}{\partial x_i\,\partial x_j}\phi(u(z_2))\right. \\ \left. - \frac{1}{a^2}\frac{\partial^2}{\partial x_i\,\partial x_j}\phi(u(z_{12}))\right]\xi_i\xi_j \leqslant 0 \end{aligned} \tag{3.20}$$

for any unit vector $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$. Here the summation convention is used. Finally we note that

$$\Delta_p w(x) = |\nabla w(x)|^{p-2}\left(\Delta w(x) + \sum_{i,j} w_{ij}(x)\frac{w_i(x)}{|\nabla w(x)|}\frac{w_j(x)}{|\nabla w(x)|}(p-2)\right).$$

Then

$$\begin{aligned} \frac{1}{|\nabla w|^2}\left[\Delta w(x) + (p-2)\sum_{i,j} w_{ij}(x)\frac{w_i(x)}{|\nabla w(x)|}\frac{w_j(x)}{|\nabla w(x)|}\right] \\ = \frac{1}{|\nabla w|^p}\Delta_p w. \end{aligned} \tag{3.21}$$

So we deduce (3.9) from (3.20) and (3.21) by means of a suitable choice of ξ.

This completes the proof of Lemma 2 and it remains to complete the proof of Theorem 3. ∎

Using the differential equation and (3.9) we have that

$$\frac{u_t(z_{12})+f(u(z_{12}))}{a^p} \geqslant \frac{\mu u_t(z_1)}{b^p}+\frac{(1-\mu)\, u_t(z_2)}{c^p}$$
$$+f(u(z_1))\left[\frac{\mu}{b^p}+\frac{1-\mu}{c^p}\right]. \tag{3.22}$$

Because of (3.6), the strict monotonicity and positivity of f, and the convexity of the mapping $d \mapsto d^p$, the last term in (3.22) is strictly bigger than $(1/a^p)\, f(u(z_{12}))$, so that

$$\frac{1}{a^p}\, u_t(z_{12}) > \frac{\mu}{b^p}\, u_t(z_1)+\frac{1-\mu}{c^p}\, u_t(z_2). \tag{3.23}$$

Without loss of generality we may assume that $u_t(z_{12})>0$. Then (3.8) implies

$$1=\frac{\phi'(u(z_1))}{\phi'(u(z_{12}))}\left[\frac{\mu u_t(z_1)}{u_t(z_{12})}+\frac{(1-\mu)u_t(z_2)}{u_t(z_{12})}\right] := \alpha+(1-\alpha),$$

where $\alpha \in (0, 1)$. But now (3.23) can be rewritten as

$$\frac{1}{a^p} > \frac{\phi'(u(z_{12}))}{\phi'(u(z_1))}\left[\frac{\alpha}{b^p}+\frac{1-\alpha}{c^p}\right].$$

Using the concavity of ϕ, i.e., $\phi'(u(z_{12})) \geqslant \phi'(u(z_1))$ we may conclude that

$$\frac{1}{a^p} > \left[\frac{\alpha}{b^p}+\frac{1-\alpha}{c^p}\right]$$

for some $\alpha \in (0, 1)$, a contradiction to the convexity of the mapping $d \mapsto d^p$. This completes the proof of Theorem 3. ∎

The key point in the proof of Theorem 3 is Lemma 2, which has its origin in work of Gabriel [24] on the convexity of level sets of harmonic functions. Some variants of it have been used in the stationary setting by several authors; we shall comment on those results in Section 4.

As for parabolic problems on "convex rings" we are aware only of the results of Borell [7]. He studied the special case $p=2$, $f \equiv 0$ and $u_0 \equiv 0$ in

$\Omega \setminus G$. He interpreted level sets in time and space as sets in $\mathbb{R}^{1+n}$ and proved quasiconcavity in time and space. His results were generalized in [18].

Remark 3. As usual in parabolic differential equations one can sometimes remove monotonicity assumptions on $f(u)$. To be specific, for $p=2$ and $\phi(s)=s$ the function f may be the sum of a maximal monotone and a Lipschitz-continuous map. In that case, the transformation $v(t,x)=e^{-\lambda t}u(t,x)$ leads to a parabolic equation with monotone nonlinearity (for $\lambda > -f'$) and with time dependent boundary condition on G. As long as one can prove $v_t \geqslant 0$, Theorem 3 remains valid in this situation, see also Remark 2. The same applies to the case $p \neq 2$, and $\phi(s)=s^m$, $0<m\leqslant 1$, except that now one has to use the substitution

$$v(t,x)=e^{-\lambda t}u(\tau(t),x)$$

with

$$\tau(t)=(e^{\lambda[(m-1)+(p-2)]t}-1)/(\lambda[(m-1)+(p-2)]).$$

Problems of this nature were introduced in Gurtin and McCamy [25] to model spatial spread of some biological populations.

Remark 4. In Theorem 3 the function f does not have to be continuous. If for instance or $\phi(s)\equiv s$ and if f is the maximal monotone mapping defined by

$$f(s)=\begin{cases}\varnothing & \text{if} \quad s<0,\\ (-\infty,1] & \text{if} \quad s=0,\\ \{1\} & \text{if} \quad s>0,\end{cases}$$

then we can approximate f by continuous monotone functions and show that the correspondig solution is quasiconcave. Problems of this type arise in the modelling of galvanization processes (see [45] for the stationary case), and one can show (see [3, 17]), that for sufficiently large Ω the solution will have compact support in Ω.

Remark 5. The proof of Theorem 3 uses in a fundamental way, that the solutions of (P) can be approximated by functions which satisfy $u_t \geqslant 0$ and $x\cdot\nabla u<0$. The same arguments can be applied to show the convexity of level sets for the system of ODEs

$$u_t(t,x)+f(u(t,x))=0 \qquad \text{in } (0,\infty)\times\Omega$$

$$u(0,x)=u_0(x) \qquad \text{in } \Omega,$$

where $x \in \Omega$ is considered to be a system-parameter. Then by using Kato-Trotter's formula (see, e.g., Brezis [9]) the problem of showing quasiconcavity of solutions to the quasilinear equation (1.1), however on all of $(0, \infty) \times \Omega$ (i.e., with $G = \emptyset$), and with boundary condition (1.3) is reduced to a corresponding proof for the nonperturbed case $f \equiv 0$.

Some references on the convexity and related properties of level sets for various special cases of this interior problem are Brascamp and Lieb [8], Lions [38], Korevaar [34], Kawohl [29], Keady [31], Keady and Stakgold [32], Kennington [33], Matano [39], Nickel [40], Polya [41], and Benilan and Vazquez [5].

4. The Stationary Problem

In this section we consider the elliptic problem

$$
\text{(SP)} \begin{cases} -\Delta_p w + g(w) = 0 & \text{in } \Omega \setminus G \qquad (4.1) \\ w = 1 & \text{on } G \qquad (4.2) \\ w = 0 & \text{on } \partial\Omega, \qquad (4.3) \end{cases}
$$

where Ω and G are as in the previous section. We remark that (SP) can be considered as the stationary case associated to the parabolic problem (P). Indeed, since $u(t, \cdot) \to u_\infty(x)$ as $t \to \infty$ (see [19]), the function $w = \phi(u_\infty)$ satisfies (SP) with $g(r) = f(\phi^{-1}(r))$, provided we assume $\phi(1) = 1$ without loss of generality.

Some results on the convexity of level sets for solutions to (SP) are already documented in the literature. We mention the deep work of Lewis [37] for the case $g \equiv 0$, the results of Caffarelli and Spruck [10] and Kawohl [27] for the semilinear case ($p = 2$), and the result of Kawohl [28] for $p \geqslant 2$ and g strictly monotone, but under the nondegeneracy assumption $|\nabla w| > 0$. The motivation to reinvestigate the stationary case comes from the parabolic problem. In the proof of Theorem 3 we have used the fact that $Q(\infty, x_1, x_2) \geqslant 0$, and this is exactly the quasiconcavity of u_∞. The main goal of this section is therefore to extend the results of Kawohl [28] to a more general setting.

Theorem 4. *Let Ω and G be convex domains with smooth boundaries satisfying the uniform interior sphere condition* (2.18). *Let g be a continuous and nondecreasing function with $g(0) \geqslant 0$ (or, more generally, a maximal monotone graph in $\mathbb{R}^2$ with $0 \in g(0)$).*

Then the solution w of (SP) *is quasiconcave on Ω.*

The proof of Theorem 4 resembles the one of Theorem 3 to a certain extent. As in the proof of Theorem 3 it suffices to prove the conclusion for a sequence of functions w_n, which converges to w in $L^1(\Omega)$, say, and which satisfies (4.1) with a smooth, strictly increasing function g. Since conclusions (3.5) and (3.6) and the stationary part of Lemma 2 ((3.7) and (3.9)) are proved in the same way as in Theorem 3 by ignoring the role of the time variable t, the only remaining step is to prove the nondegeneracy statement $|\nabla w_n| > 0$. Observe that the starshapedness of level sets can be shown exactly as in Theorem 1, but the nondegeneracy requires more delicate arguments. Therefore the following theorem completes the proof of Theorem 4.

THEOREM 5. *Let $w(x)$ be a solution of*

$$\left.\begin{aligned} -\Delta_p w + g(w) &= 0 \qquad \text{in } \Omega \setminus G \\ w &= 1 \qquad \text{on } G \\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{4.4}$$

where $G \Subset \Omega$, G and Ω are convex and satisfy the uniform sphere condition (2.18). *Moreover we assume that $p > 1$ and*

$$g \in C^0([0,1]), \qquad g \text{ nondecreasing}, \qquad g(0) = 0, \qquad \text{and} \tag{4.5}$$

$$\int_{0^+} \frac{ds}{G(s)^{1/p}} = \infty, \qquad \text{where} \quad G(t) = \int_0^t g(s)\, ds. \tag{4.6}$$

Then the solution u satisfies the nondegeneracy condition

$$x \cdot \nabla u < 0 \qquad \text{in } \Omega \setminus G.$$

Note that (4.6) holds in particular for Lipschitz continuous f. In order to prove this theorem, we shall use the auxiliary function $v(x)$ defined in (2.15) as well as the function $W(r)$, which satisfies

$$\left.\begin{aligned} -(|W'|^{p-2} W' + \theta\, |W'|^{p-2} W' + g(AW)\, A^{1-p} &= 0 \qquad \text{in } (0, \delta/2), \\ w(0) &= 0, \\ w(\delta/2) &= 1, \end{aligned}\right\} \tag{4.7}$$

where $A > 0$ and $\theta > 0$ are suitably fixed and will be chosen later.

LEMMA 3. *Under assumption* (4.6) *there exists a constant $c^* > 0$ such that*

$$W'(r) > c^* \qquad \text{for} \quad r \in [0, \delta/2]. \tag{4.8}$$

This result goes back to Vazquez [44], who indicated that it could be generalized to $p \in (1, \infty)$, see details in Diaz [14, p. 55]. In the proof of Lemma 1.21 in Diaz [14] it is shown

(i) that $e^{-\theta r} |W'(r)|^{p-2} W'(r)$ is nondecreasing, and

(ii) that $W'(0) > 0$.

Now (4.8) follows from (i) and (ii), and this proves Lemma 3.

Proof of Theorem 5. This is very similar to the proof of Theorem 2. As in the parabolic case (2.22), we deduce from (4.5) and (4.6) (see Diaz [14]), that there exists a positive constant A such that

$$\min\{1-w(x), w(x)\} \geqslant A \qquad \text{for} \quad x \in K = \{x \in \Omega \setminus G \mid d(x, \partial\Omega \cup \partial G \geqslant \delta/2\}.$$

Then, similar to the evolution case, we prove that if $x_0 \in \partial G$,

$$w(x) \leqslant 1 - Av(x) \qquad \text{for} \quad x \in B(z, \delta) \setminus B(z, \delta/2), \tag{4.9}$$

where $v(x)$ is defined in (2.15), and z, δ are given by (2.18). Now let $x_0 \in \partial\Omega$ and consider the test function

$$\underline{v}(x) = A\,W(\delta - |x-z|).$$

Then $\underline{v} \leqslant w$ on $\partial(B(z, \delta) \setminus B(z, \delta/2))$, and if we choose $\theta \geqslant 2(n-1)/\delta$

$$\Delta_p \underline{v} + g(\underline{v}) \leqslant 0.$$

Thus, by the comparison principle

$$A\,W(\delta - |x-z|) \leqslant w(x) \qquad \text{for} \quad x \in B(z, \delta) \setminus B(z, \delta/2). \tag{4.10}$$

From (4.9) and (2.19) we conclude that if $x_0 \in \partial G$

$$1 - w(sx_0) \geqslant A\mu(s-1) \qquad \text{for} \quad 1 < s \leqslant s_0. \tag{4.11}$$

Analogously, if $x_0 \in \partial\Omega$, we deduce from (4.10) and (4.8) that there exists a positive constant μ^* (independent of x_0) such that

$$W\left(\delta - \left|\frac{x_0}{s} - z\right|\right) \geqslant \mu^* \, |s-1| \qquad \text{for} \quad 1 < s \leqslant s_0$$

and

$$w(x) \geqslant A W\left(\delta - \left|\frac{x_0}{s} - z\right|\right) \geqslant A\mu^* \, |s-1|.$$

Now the rest of the proof is the same as for Theorem 2, after choosing $\mu^{**} = \min\{\mu, \mu^*\}$. ∎

Acknowledgments

This research was begun at the MSRI in Berkeley. The authors gratefully acknowledge financial support by the DGICYT of Spain under Project PB 90/0620 and by the Deutsche Forschungsgemeinschaft (DFG). An earlier version of the results was presented at the Irsee Conference on Free Boundary Problems in June 1987.

References

1. H. W. Alt and S. Luckhaus, Quasilinear elliptic–parabolic differential equations, *Math. Z.* **183** (1983), 311–341.
2. R. Aris, The Mathematical Theory of Diffusion and Reaction in Permeable Catalysts," Vol. I and II, Clarendon, Oxford, 1975.
3. C. Bandle and I. Stakgold, The formation of the dead core in parabolic reaction diffusion problems, *Trans. Amer. Math. Soc.* **286** (1984), 275–293.
4. M. Bardi, Asymptotic spherical symmetry of the free boundary in degenerate diffusion equations, *Ann. Math. Pura Appl. (4)* **148** (1987), 117–130.
5. P. Benilan, and J. L. Vazquez, Concavity of solutions of the porous medium equation, *Trans. Amer. Math. Soc.* **299** (1987), 81–93.
6. M. Bertsch, "Nonlinear Diffusion Problems: The Large Time Behaviour," Thesis, Leiden, 1983.
7. C. Borell, Brownian motion in a convex ring and quasiconcavity, "*Comm. Math. Phys.* **86** (1982), 143–147.
8. H. J. Brascamp and E. H. Lieb, On extensions of the Brunn–Minkowski and Prekopa–Leindler theorems, including inequalities for log-concave functions, and with applications to the diffusion equation, *J. Funct. Anal.* **22** (1976), 366–389.
9. H. Brezis, "Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam, 1973.
10. L. A. Caffarelli and J. Spruck, Convexity properties of solutions of some classical variational problems, *Comm. Partial Differential Equations* **7** (1982), 1337–1379.
11. L. A. Caffarelli, J. L. Vazquez, and N. I. Wolanski, Lipschitz continuity of solutions and interfaces of the N-dimensional porous medium equation, *Indiana Univ. Math. J.* **36** (1987), 373–401.
12. A. Damlamian, Some results on the multi-phase Stefan problem, *Comm. Partial Differential Equations* **2** (1977), 1017–1044.
13. J. I. Diaz, Elliptic and parabolic quasilinear equations giving rise to a free boundary: The boundary of the support of the solutions, *Proc. Symp. Pure Math.* **45**, Part 1 (1986), 381–393.

14. J. I. DIAZ, "Nonlinear Partial Differential Equations and Free Boundaries. Vol. I. Elliptic Equations," Pitman Research Notes in Math., Vol. 106, London, Pitman, 1985.
15. J. I. DIAZ AND J. HERNANDEZ, Some results on the existence of free boundaries for parabolic reaction diffusion systems, *in* "Trends in Theory and Practice of Nonlinear Differential Equations" (V. Lakshmikantham, Ed.), pp. 149–156, Dekker, New York, 1984.
16. J. I. DIAZ AND J. HERNANDEZ, Qualitative properties of free boundaries for some nonlinear degenerate parabolic equations, *in* Nonlinear Parabolic Equations: Qualitative Properties of Solutions" (L. Boccardo and A. Tesei, Eds.), pp. 85–93, Longman, London, 1987.
17. J. I. DIAZ AND M. A. HERRERO, Estimates on the support of the solution of some nonlinear elliptic and parabolic problems, *Proc. Roy. Soc. Edinburgh* **89** (1981), 249–258.
18. J. I. DIAZ AND B. KAWOHL, Convexity and starshapedness of level sets for some nonlinear parabolic problems, *in* "Free Boundary Problems, Theory and Applications" (K. H. Hoffman and J. Sprekels, Eds.), Pitman Res. Notes in Math., Vol. 186, pp. 883–887, Pitman, London, 2990.
19. J. I. DIAZ AND F. DE THELIN, On a nonlinear parabolic problem arising in some models related to turbulent flows, to appear in *SIAM Math. Anal.*
20. E. A. DI BENEDETTO, A boundary modulus of continuity for a class of singular parabolic equations, *J. Differential Equations* **63** (1986), 418–447.
21. E. DI BENEDETTO AND A. FRIEDMAN, Regularity of solutions of non-linear degenerate parabolic systems, *J. Reine Angew. Math.* **349** (1984), 83–128.
22. A. EL HACHIMI AND F. DE THELIN, Supersolutins and stabilization of the solutions of the equation $u_t - \Delta_p u = f(x, u)$, *Nonlinear Anal. TMA* **12** (1988), 1385–1398.
23. A. FRIEDMAN AND D. KINDERLEHRER, A one phase Stefan problem, *Indiana Univ. Math. J.* **24** (1975), 1005–1035.
24. R. GABRIEL, A result concerning convex level surfaces of 3-dimensional harmonic functions, *J. London Math. Soc.* **32** (1957), 286–294.
25. M. E. GURTIN AND R. C. MCCAMY, On the diffusion of biological populations, *Math. Biosci.* **33** (1977), 35–49.
26. J. HULSHOF AND N. WOLANSKY, Monotone flows in N-dimensional partially saturated porous media, *Arch. Rational Mech. Anal.* **102** (1988), 287–305.
27. B. KAWOHL, Starshapedness of level sets for the obstacle problem and for the capacitory potential problem, *Proc. Amer. Math. Soc.* **89** (1983), 637–640.
28. B. KAWOHL, Geometrical properties of level sets of solutions to elliptic problems, *Proc. Symp. Pure Math.* **45**, Part 2 (1986), 25–36.
29. B. KAWOHL, "Rearrangements and Convexity of Level Sets in PDE," Springer Lecture Notes in Math., Vol. 1150, Springer, Heidelberg, 1985.
30. B. KAWOHL, Qualitative properties of solutions to semilinear heat equations, *Exposition. Math.* 4 (1986), 257–270.
31. G. KEADY, The persistence of log-concavity for positive solutions of the one-dimensional hear equation, *J. Austral. Math. Soc. Ser. A* **48** (1990), 1–16.
32. G. KEADY AND I. STAKGOLD, Some geometric properties of solids in combustion, *in* "Geometry of Solutions to Partial Differential Equations" (G. Talenti, Ed.), Symposia Mathematica, Vol. XXX, pp. 137–151, Academic Press, London, 1989.
33. A. U. KENNINGTON, Convexity of level curves for an initial value problem, *J. Math. Anal. Appl.* **133** (1988), 324–330.
34. N. KOREVAAR, Convex solutions to nonlinear elliptic and parabolic boundary problems, *Indiana Univ. Math. J.* **32** (1983), 603–614.
35. O. A. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URAL'TSEVA, "Linear and Quasilinear Equations of Parabolic Type," Transl. of Math. Monographs, Vol. 23, Amer. Math. Soc., Providence, 1968.

36. M. LANGLAIS, AND D. PHILLIPS, Stabilization of solutions of non-linear and degenerate evolution equations, *Nonlinear Anal.* **9** (1985), 321–333.
37. J. L. LEWIS, Capacitary functions in convex rings, *Arch. Rational Mech. Anal.* **66** (1977), 201–224.
38. P. L. LIONS, Two geometrical properties of solutions of semi-linear problems, *Appl. Anal.* **12** (1981), 267–272.
39. H. MATANO, Nonincrease of the lap number of a solutions for a one-dimensional semi-linear parabolic equation, *J. Fac. Sci. Univ. Tokyo, Sect. IA Math.* **29** (1982), 401–441.
40. K. NICKEL, Gestaltaussagen über Lösungen parabolischer Differentialgleichungen, *J. Reine Angew. Math.* **211** (1962), 78–94.
41. G. POLYA, Qualitatives über den Wärmeausgleich, *Z. Angew. Math. Mech.* **13** (1933), 125–128.
42. S. SAKAGUCHI, Starshaped coincidence sets in the obstacle problem, *Ann. Scuola Norm. Sup. Pisa (4)* **11** (1984), 123–128.
43. S. SAKAGUCHI, Coincidence sets in the obstacle problem for the p-harmonic operator, *Proc. Amer. Math. Soc.* **95** (1985), 382–386.
44. J. L. VAZQUEZ, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191–202.
45. T. I. VOGEL, A free boundary problem arising from a galvanizing process, *SIAM J. Math. Anal.* **16** (1985), 970–979.
46. M. WIEGNER, On C^{α}-regularity of the gradient of solutions of degenerate parabolic systems, *Ann. Mat. Pura Appl. (4)* **145** (1986), 385–405.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium