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On the initial growth of interfaces in reaction–diffusion equations with strong absorption

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Synopsis

We study the initial growth of the interfaces of non-negative local solutions of the equation $u_t = (u^m)_{xx} - \lambda u^q$ when $m \geqq 1$ and $0 < q < 1$. We show that if $u(x, 0) \geqq C(-x)_+^{2/(m-q)}$ with $C > C_0$, for some explicit $C_0 = C_0(\lambda, m, q)$, then the free boundary $\zeta(t) = \sup\{x: u(x, t) > 0\}$ is a "heating front". More precisely $\zeta(t) \geqq at^{(m-q)/2(1-q)}$ for any t small enough and for some $a > 0$. If on the contrary, $u(x, 0) \leqq C(-x)_+^{2/(m-q)}$ with $C < C_0$, then $\zeta(t)$ is a "cooling front" and in fact $\zeta(t) \leqq -at^{(m-q)/2(1-q)}$ for any t small enough and for some $a > 0$. Applications to solutions of the associated Cauchy and Dirichlet problems are also given.

1. Introduction

This paper deals with non-negative solutions of the scalar reaction-diffusion equation

$$u_t = (u^m)_{xx} - \lambda u^q \quad \text{in} \quad Q = (-L, L) \times (0, T) \tag{1.1}$$

under the assumptions $m \geqq 1$, $0 < q < 1$. Here, λ represents a positive number and $0 < T \leqq +\infty$, $0 < L \leqq +\infty$. Equation (1.1) arises in many physical situations such as thermal diffusion with absorption, chemical reactions, population dynamics etc. (see, for instance [**12**]). Due to the non-Lipschitz character of the absorption term λu^q, some interfaces may occur separating the definition set of u (a subset of $\mathbb{R} \times [0, \infty)$) in two different regions: a region where u vanishes and another where u is strictly positive. In this paper we are interested in the study of the initial development of such interfaces as they emerge from the boundary points of the support of $u(x, 0)$. Due to the invariance of the equation through the transformation $x \to -x$, it is enough to consider the case in which the support of $u(x, 0)$ is contained in $(-\infty, 0]$, i.e. we shall assume

$$\sup\{x: u(x, 0) > 0\} = 0. \tag{1.2}$$

If, to fix ideas, we assume $u(x, 0)$ to be a bounded function, it is possible to show (by using comparison techniques: [**25, 13, 14**], or by energy methods: [**31, 8, 5**])

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the existence of a free boundary

$$\zeta(t) = \sup\{x \in (-L, L): u(x, t) > 0\}$$

i.e. $|\zeta(t)| < +\infty$ for any $t \in [0, T^*)$, for some $T^* \in (0, T]$.

We remark that u is merely assumed to be a local non-negative solution of equation (1.1), i.e. such that $u \in C((-L, L) \times [0, T])$ and u satisfies (1.1) in the sense of distributions. So, in particular, the results of this paper apply for solutions of the Cauchy problem ($L = +\infty$) or any boundary value problem ($L < +\infty$) associated with equation (1.1).

In many applications it is important to know whether, for given initial data, the support of the solution $u(., t)$, expands or contracts with time ($\zeta(t)$ is a *heating* or *cooling* front). Our main result shows that the initial behaviour of $\zeta(t)$ depends on the "concentration" of the mass of $u(x, 0)$ near $x = 0$. We shall compare, locally, $u(x, 0)$ with the auxiliary function $u_\infty(x)$ given by

$$u_\infty(x) = C_0(-x)_+^{2/(m-q)}, \quad x \in (-\infty, +\infty),$$

where $(s)_+ = \max(s, 0)$

$$C_0 = \left[\frac{\lambda(m-q)^2}{2m(m+q)}\right]^{1/(m-q)}. \tag{1.3}$$

It is easy to see that u_∞ is the non-negative stationary solution of (1.1) in $(-\infty, +\infty)$ vanishing on $[0, +\infty)$.

The precise statement of our main result is the following:

THEOREM 1.1. *Let u be any local non-negative solution of* (1.1) *satisfying* (1.2).

(i) *Assume that there exist $x_0 \in (-L, 0)$ and $C < C_0$ such that*

$$u(x, 0) \leqq C(-x)_+^{2/(m-q)} \quad \textit{for} \quad x \in [x_0, 0].$$

Then

$$\zeta(t) \leqq -at^{(m-q)/2(1-q)} \quad \textit{for any} \quad t \in [0, t_0],$$

for some $a > 0$ and $t_0 \in (0, T]$.

(ii) *Assume that there exist $x_0 \in (-L, 0)$ and $C > C_0$ such that*

$$u(x, 0) \geqq C(-x)_+^{2/(m-q)} \quad \textit{for} \quad x \in [x_0, 0].$$

Then

$$\zeta(t) \geqq at^{(m-q)/2(1-q)} \quad \textit{for any} \quad t \in [0, t_0],$$

for some $a > 0$ and $t_0 \in (0, T]$.

To prove the above theorem, we shall use the following programme. First we shall obtain the conclusions for the solution u of the Cauchy problem

$$\begin{cases} u_t = (u^m)_{xx} - \lambda u^q, & \text{in} \quad (-\infty, +\infty) \times (0, +\infty), \\ u(x, 0) = u_0(x), & \text{for} \quad x \in (-\infty, +\infty), \end{cases} \tag{1.4}$$

when u_0 is given by

$$u_0(x) = C(-x)_+^{2/(m-q)}. \tag{1.5}$$

To do that, we shall prove that under those conditions the solution has a self-similar structure

$$u(x, t) = t^{1/(1-q)} f(xt^{-(m-q)/2(1-q)}) \tag{1.6}$$

for some real function f. The first step ends by comparing the solution with a family of travelling waves in a similar way to the technique introduced in [**4**]. The second step makes the above conclusions local by using the comparison principle again.

The organisation of the paper is as follows: In Section 2 we prove the main conclusions for the self-similar solution (1.6). Section 3 contains some existence, uniqueness and regularity results for solutions of the Cauchy problem (1.4) corresponding to general unbounded initial data u_0 satisfying

$$0 \leqq u_0(x) \leqq h_m(|x|) \quad \text{for} \quad x \in (-\infty, +\infty), \tag{1.7}$$

where

$$h_m(|x|) = (K + |x|)^\alpha, \quad 0 \leqq \alpha < 2/(m-1), \quad K > 0, \tag{1.8}$$

if $m > 1$ and, for all $\alpha > 0$,

$$h_1(x) \leqq Ke^{\alpha|x|^2} \quad \text{for} \quad x \in (-\infty, +\infty). \tag{1.9}$$

The complete proof of Theorem 1.1 is then given in Section 4.

A previous version of the results of this paper was presented in the Ph.D dissertation of the first author at the University Complutense of Madrid in September 1988. While preparing the present article, the authors became aware of the paper [**18**] in which Grundy and Peletier obtain, by asymptotic methods, some related estimates on the interface $\zeta(t)$ of the solution of the Cauchy problem associated with (1.1) when $m = 1$. We point out the difference between the methods of proof used in both works, and the generality of our assumptions and conclusions.

2. The interface for self-similar solutions

This section is devoted to the study of qualitative properties of solutions of the Cauchy problem associated with (1.1) with the special initial data

$$u_0(x) = C(-x)_+^{2/(m-q)}, \quad \text{for some} \quad C > 0. \tag{2.1}$$

In the next section we shall show the existence and uniqueness of such a solution as a consequence of more general results related to initial data satisfying the growth condition (1.7). We remark that such a condition is trivially satisfied when u_0 is given by (1.9) (since $0 < q < 1$) and that, of course, the proofs of the basic theory (existence, uniqueness etc.) are completely independent of the results in this section.

We start by showing that the solutions are self-similar.

Proposition 2.1. *Let u be the solution of the Cauchy problem* (1.1) *corresponding to u_0 given by* (2.1). *Then there exists a function $f : \mathbb{R} \to [0, \infty)$ such that*:

$$u(x, t) = t^{1/(1-q)} f(xt^{-(m-q)/2(1-q)}). \tag{2.2}$$

Moreover $\operatorname{supp}(f) = (-\infty, a]$ with $a < +\infty$, i.e.

$$f(\eta) = 0 \quad \text{for any} \quad \eta \geqq a. \tag{2.3}$$

Proof. Given $k > 0$, we define the function

$$v(x, t; k) = ku(k^{-(m-q)/2}x, k^{(q-1)}t)$$

where u is the solution of the Cauchy problem (1.4). Since

$$v_t - (v^m)_{xx} + \lambda v^q = k^q(u_t - (u^m)_{xx} + \lambda u^q)$$

and $v(x, 0; k) = u_0(x)$ (due to (2.1)), we conclude (by the uniqueness of the solution of the Cauchy problem) that $v(x, t; k) = u(x, t)$, for any $k > 0$. Now, given $t_0 \in (0, T]$, we choose $k = t_0^{1/(1-q)}$ and so we deduce that

$$u(x, t_0) = t_0^{1/(1-q)} u(x t_0^{-(m-q)/2(1-q)}, 1) \quad \text{for any} \quad t_0 \in (0, T] \quad \text{and} \quad x \in \mathbb{R}.$$

Finally, given $\eta \in \mathbb{R}$, we define $f(\eta) = u(\eta, 1)$. Making $t = t_0$, we obtain (2.2) (t_0 is arbitrary). In order to prove (2.3) we first consider the case $m > 1$. Let $\bar{u}$ be the solution of the equation without absorption:

$$\bar{u}_t - (\bar{u}^m)_{xx} = 0$$

and with the same initial data u_0. Then, since $\lambda u^q \geqq 0$, we conclude that $0 \leqq u(x, t) \leqq \bar{u}(x, t)$ for any $(x, t) \in \mathbb{R} \times [0, T]$. By the result of [**33**] we know that for any $t \geqq 0$ we have that $\sup\{x : \bar{u}(x, t) > 0\} < +\infty$ and hence $\sup\{x : u(x, t) > 0\}$ is also finite. Choosing $t = 1$, we obtain (2.3). If $m = 1$, the existence of the free boundary

$$\zeta(t) = \sup\{x : u(x, t) > 0\} \tag{2.4}$$

has already been shown in [**23**] for initial data satisfying (1.7). □

Now we shall prove the conclusions of Theorem 1.1 when $L = +\infty$ and $u(x, 0) = u_0(x)$, with u_0 given by (2.1). We remark that from Proposition 2.1 we deduce that if $\zeta(t)$ is given by (2.4) then $\zeta(t) = at^{(m-q)/2(1-q)}$ and so the behaviour of $\zeta(t)$ is determined by the sign of a.

THEOREM 2.2. *Let u_0 and f be given by* (2.1) *and* (2.2), *respectively. Let $a \in \mathbb{R}$ defined by $a = \sup\{\eta : f(\eta) > 0\}$. Then we have*:

(i) $C = C_0$ *implies* $a = 0$;
(ii) $C < C_0$ *implies* $a < 0$;
(iii) $C > C_0$ *implies* $a > 0$.

Proof. If $C = C_0$, we deduce from the uniqueness of solutions of the Cauchy problem (1.4) that $u(x, t) = u_\infty(x)$ and so (i) follows. To complete the proof, we shall adapt the arguments of [**4**] which consist in comparing $u(x, t)$ with different families of travelling wave solutions. The structure of such special solutions is different according to the values of q and m:

Region 1: ($m + q = 2$). In that special case it is not difficult to see that the function f can be made explicit and so

$$u(x, t) = C\left[\left(\left(\frac{m}{m-1} C^{m-1} - \lambda(m-1)C^{1-m}\right)t - x\right)_+\right]^{1/(m-1)}.$$

The assertions (ii) and (iii) follow easily in this way.

Region 2: ($m + q > 2$). Given $k \in \mathbb{R}$ and $\eta \in \mathbb{R}$, the existence of travelling wave solutions to equation (1.1) of the form

$$w(x, t; k, \eta) = \phi_k((kt - x + \eta)_+) \tag{2.5}$$

was obtained by Herrero and Vazquez in **[22]**, where they also showed that for any $k \in \mathbb{R}$, ϕ_k is a suitable continuous non-negative function such that $\phi_k(0)=0$ and

$$\lim_{\xi \to +\infty} \phi_k(\xi)\xi^{-2/(m-q)} = C_0 \quad (C_0 \text{ given by (1.3)}). \tag{2.6}$$

Now let $C < C_0$. By (2.6) there exists $M > 0$ such that

$$\phi_k(\xi) > C\xi^{2/(m-q)} \quad \text{for} \quad \xi > M. \tag{2.7}$$

Let $\eta = M$. Then $w(x, t; k, M)$ is a solution of (1.4) with initial datum $\phi_k((-x+M))$. Besides, from (2.7), we deduce that

$$\phi_k((-x+M)_+) \geqq C(-x)_+^{2/(m-q)} \quad \text{for any} \quad x \in \mathbb{R}.$$

Then by the comparison result of Section 3 we have that

$$w(x, t; k, M) \geqq u(x, t) \quad \text{for any} \quad x \in \mathbb{R}, \quad t \geqq 0.$$

Finally, choosing $k<0$, we see that $w(x, t; k, M)$ is a cooling wave and so we deduce (ii). If $C > C_0$, we choose $k > 0$. By (2.6) we have that

$$\phi_k(\xi) < C\xi^{2/(m-q)} \quad \text{for} \quad \xi > M,$$

for some $M>0$. Let $N = \max\{\phi_k(\xi): 0 \leqq \xi \leqq M\}$ and $\eta = -\max\{M, (N/C)^{(m-q)/2}\}$. Hence $w(x, t; k, \eta)$ is a solution to (1.4) with initial datum $\phi_k((-x+\eta))$. Moreover

$$\phi_k((-x+\eta)_+) \leqq C(-x)_+^{2/(m-q)} \quad \text{for any} \quad x \in \mathbb{R}.$$

Then by the comparison result $w(x, t; k, \eta) \leqq u(x, t)$ for any $x \in \mathbb{R}$, $t \geqq 0$. Since $k>0$, we have that $w(x, t; k, \eta)$ is a heating wave and (iii) follows.

Region 3: $(m+q<2)$. Again by the result of **[22]** there exists a family of travelling wave solutions to equation (1.1) of the form

$$w(x, t; k) = \phi_k((kt - x)), \tag{2.8}$$

for arbitrary $k \in \mathbb{R}$. Moreover, ϕ_k is a continuous non-negative function satisfying $\phi_k(0) = 0$ and

$$\lim_{\xi \to 0} \phi_k(\xi)\xi^{-2/(m-q)} = C_0 \quad (C_0 \text{ given by (1.3)}). \tag{2.9}$$

Now let $C < C_0$ and take $k<0$. From (2.9) there exists $M>0$ such that

$$\phi_k(\xi) > C\xi^{2/(m-q)} \quad \text{for} \quad 0 < \xi < M.$$

On the other hand, by the continuity of ϕ_k and u, there exists $\tau > 0$ such that

$$w(-M, t; k) \geqq u(-M, t) \quad \text{for any} \quad t \in [0, \tau].$$

Then we can compare $w(x, t; k)$ and $u(x, t)$ in the region $[-M, M] \times [0, \tau]$ and so

$$w(x, t; k) \geqq u(x, t) \quad \text{in} \quad [-M, M] \times [0, \tau].$$

Thus, since $k<0$, conclusion (ii) follows. Finally, if $C > C_0$, we choose $k>0$ and by a similar argument we obtain (iii). □

Remark 2.3. *Regularity of the interface of self-similar solutions.* The initial growth of the interface $\zeta(t) = at^{(m-q)/2(1-q)}$ associated with the self-similar solutions is quite different according to the values of m and q. Thus in Region 1 $(m+q=2)$, we have $(m-q)/2(1-q)=1$ and then $\zeta(t)=at$. In Region 2 $(m+q>2)$, $(m-q)/2(1-q)>1$ and thus $\zeta(t)$ is a Lipschitz function for $t \in [0, M)$. Finally, in Region 3 $(m+q<2)$, $(m-q)/2(1-q)<1$ and thus $\zeta(t)$ is not Lipschitz at $t=0$.

Remark 2.4. The technique of comparing the solution with a family of travelling waves was introduced in **[4]** (see also **[3]**) for the study of the equation $u_t = (u^m)_{xx} + (u^\lambda)_x$. When the exponent λ is in $(0, 1)$, there is a stationary solution $u_\infty(x)$ of this equation and the initial behaviour of the front depends on how concentrated is the mass of the initial datum $u_0(x)$ with respect to that of $u_\infty(x)$.

3. Existence, uniqueness and regularity of the solutions of the Cauchy problem

In this section, we are going to show some existence, uniqueness and regularity results of non-negative solutions of the Cauchy problem (1.4) corresponding to unbounded initial data u_0 satisfying the growth assumption (1.7).

3.1. Existence

We define a generalised solution of (1.4) as a continuous non-negative function $u(x, t)$ in $\mathbb{R} \times [0, \infty)$ such that $u(x, 0) = u_0(x)$ and, for all $0 \leqq t_0 < t_1$ and $x_0 < x_1$, we have:

$$I(u, \phi, P) = \int_{x_0}^{x_1}\int_{t_0}^{t_1} (u^m\phi_{xx} + u\phi_t - \lambda u^q\phi)\,dx\,dt - \left.\int_{x_0}^{x_1} u\phi\,dx\right]_{t_0}^{t_1} - \left.\int_{t_0}^{t_1} u^m\phi_x\,dt\right]_{x_0}^{x_1} = 0,$$

where $\phi(x, t) \in C^{2,1}_{x,t}$ is any arbitrary function satisfying $\phi(x, t) = 0$ in $[t_0, t_1] \times \{x_0\} \cup [t_0, t_1] \times \{x_1\}$ and $P = [t_0, t_1] \times [x_0, x_1]$. Given $m > 1$, we introduce the functional space E_m defined by:

$$E_m = \left\{u \in C(\mathbb{R} \times [0, \infty)) : \forall T > 0\ \exists K > 0 \text{ and } \alpha \in \left(0, \frac{2}{m-1}\right) : 0 \leqq u(x, t) \leqq (K + |x|)^\alpha\ \forall x \in \mathbb{R} \quad 0 < t < T\right\}.$$

If $m = 1$, we define

$$E_1 = \{u \in C(\mathbb{R} \times [0, \infty)) : \forall T, \exists K > 0 \text{ and } \alpha > 0 : 0 \leqq u(x, t) \leqq Ke^{\alpha x^2}\ \forall x \in \mathbb{R}, 0 < t < T\}.$$

We have

PROPOSITION 3.1. *Let* $m \geqq 1$. *If* $u_0(x)$ *satisfies* (1.7), *then there exists a generalised solution* $u(x, t)$ *of* (1.1) *and* $u \in E_m$.

In order to prove the above result, we start by showing some preliminary results, using some techniques introduced in **[27]** and **[28]**.

LEMMA 3.2. *Let $m \geqq 1$ and $T > 0$. If $u_0(x)$ satisfies* (1.7), *then there exists a function $\bar{u} \in E_m$ such that $\bar{u}(x, 0) > u_0(x) + 1$ and*

$$\bar{u}_t \geqq (\bar{u}^m)_{xx} - \lambda \bar{u}^q + \lambda \quad in \quad \mathbb{R} \times (0, T)$$

Proof. (i) *Case $m > 1$.* Let $T > 0$, $\alpha < 2/(m-1)$ and $\bar{u}(x, t) = e^t(K + |x|)^\alpha$. We have:

$$\bar{u}_t - (\bar{u}^m)_{xx} + \lambda \bar{u}^q \geqq e^t[(K + |x|)^\alpha - e^{T(m-1)}\alpha m(\alpha m - 1)(K + |x|)^{\alpha m - 2}].$$

Since $\alpha > \alpha m - 2$, the right-hand side of the above inequality tends to infinity as $K \to +\infty$, and the lemma follows by choosing K large enough.

(ii) *Case $m = 1$.* Let $\bar{u}(x, t) = \int_{-\infty}^{+\infty} (u_0(s) + 1)G(x - s, t)\, ds + \lambda \int_0^t \int_{-\infty}^{+\infty} G(x - s, t - \tau)\, ds\, d\tau$, where $G(x, t)$ is the heat kernel defined by

$$G(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \quad \text{in} \quad Q = \mathbb{R} \times \mathbb{R}^+.$$

It is well known that $\bar{u}(x, t)$ is the solution of the linear equation $\bar{u}_t = \bar{u}_{xx} + \lambda$ in Q for the initial datum $\bar{u}(x, 0) = u_0(x) + 1$. Moreover, since $u_0(x)$ verifies (1.7), we deduce that $\bar{u} \in E_1$. □

LEMMA 3.3 [**28**]. *Let $m \geqq 1$ and $u_0(x)$ satisfying* (1.7). *Let $T > 0$ and $Q_n = (-n, n) \times (0, T)$. Then the Cauchy problem $\mathfrak{L}(v, \varepsilon) \equiv v_t - (v^m)_{xx} + \lambda v^q - \lambda \varepsilon^q = 0$ in Q_n with the auxiliary conditions*

$$\left.\begin{aligned} v(x, 0) &= v_0(x; \varepsilon, n), \quad for \quad x \in (-n, n), \\ v(\pm n, t) &= \bar{u}(\pm n, t), \quad for \quad t \in (0, T), \end{aligned}\right\}$$

has a unique classical solution such that

$$0 < \varepsilon \leqq v(x, t) \leqq \bar{u}(x, t) \quad in \ Q_n, \tag{3.1}$$

where $v_0(x; \varepsilon, n)$ is a sequence of regular functions satisfying

(i) $\varepsilon < v_0(x; \varepsilon, n) \leqq \bar{u}(x, t)$ *for all $x \in \mathbb{R}$;*
(ii) $v_0(\pm n; \varepsilon, n) = \bar{u}(\pm n, 0)$;
(iii) $v_0(x; \varepsilon, n)$ *decreases as $n \to +\infty$ and converges to $u_0(x) + \varepsilon$ as $n \to +\infty$.*

Proof. This is analogous to the one given in [**28**]. Inequality (3.1) holds by comparing $v(x, t)$ with the subsolution $\underline{v} = \varepsilon$ and the supersolution $\bar{u}(x, t)$ defined in Lemma 3.2. □

Let $\varepsilon = 1/n$ and $u_n(x, t)$ be the function obtained in Lemma 3.3. The next result shows that u_n decreases with respect to n.

LEMMA 3.4. *Let $p, n \in \mathbb{N}$ with $p > n$; then*

$$u_p(x, t) \leqq u_n(x, t) \quad in \quad Q_n = (-n, n) \times (0, T).$$

Proof. By Lemma 3.3(iii) we obtain $u_p(x, 0) \leqq u_n(x, 0)$. Moreover, $u_p(\pm n, t) \leqq \bar{u}(\pm n, t) = u_n(\pm n, t)$ and in Q_n we have:

$$\mathfrak{L}(u_p, n^{-1}) = \mathfrak{L}(u_n, p^{-1}) + \lambda n^{-q} - \lambda p^{-q} > 0.$$

Then the conclusion follows by comparison. □

Proof of Proposition 3.1. Since $u_n(x, t)$ is a decreasing sequence and $u_n \geqq 0$, we deduce that, for any $(x, t) \in (-\infty, +\infty) \times [0, T]$, $u_n(x, t)$ converges to a function $u(x, t)$ as $n \to +\infty$. Moreover, by Lemma 3.3, $u(x, t)$ satisfies $u(x, t) \leqq \bar{u}(x, t)$ and we have

$$I(u_n, \phi, P) = -\lambda n^{-q} \int_{x_0}^{x_1} \int_{t_0}^{t_1} \phi(x, t)\, dx\, dt.$$

Therefore the conclusion follows by application of the Lebesgue convergence theorem to the sequence $u_n(x, t)$. □

We postpone for a while the proof of the continuity of the function u obtained above.

3.2. Comparison and uniqueness

We begin by proving a comparison principle.

PROPOSITION 3.5. *Let* $u_0, v_0 \in C(\mathbb{R})$ *satisfying* (1.7) *and such that* $0 \leqq u_0(x) \leqq v_0(x)\ \forall x \in \mathbb{R}$. *Let* $u, v \in E_m$ *be solutions of the Cauchy problem* (1.4) *with initial data* u_0 *and* v_0, *respectively. Then we have*

$$u(x, t) \leqq v(x, t) \quad \text{in} \quad Q = \mathbb{R} \times \mathbb{R}^+.$$

Proof. We can suppose that one of the functions $u(x, t)$, $v(x, t)$ is the solution obtained in the proof of Proposition 2.1. We shall prove the following inequality

$$\int_{-\infty}^{+\infty} [u(x, t_1) - v(x, t_1)] w(x)\, dx \leqq 0 \tag{3.2}$$

for all $t_1 > 0$ and for any $w \in C^\infty$, $w(x) > 0$, w with compact support in $\mathbb{R}$. Proposition 3.5 follows in an obvious way from (3.2). Assume that $u(x, t)$ is the solution obtained in Proposition 3.1 (the same argument remains true if $v(x, t)$ is such a function). Let $t_1 \in (0, T]$ and $w(x)$ such that $w(x) = 0$ if $x \notin (-r, r)$. Let $n > r$ and $P = (-r, r) \times (0, T)$. Assume that $m > 1$ and define the functions

$$A_n(x, t) = \int_0^1 m(\theta u_n + (1 - \theta) v)^{m-1}\, d\theta$$

and

$$C_n(x, t) = \int_0^1 \lambda q(\theta u_n + (1 - \theta) v)^{q-1}\, d\theta.$$

Let $A_{nkr}(x, t)$ and $C_{nkr}(x, t)$ be two sequences of regular positive functions such that:

$\{A_{nkr}\}$ is decreasing and converges uniformly to A_n as $k \to +\infty$ and

$A_{nkr}(x, t) \leqq (K + |x|)^2$ in $Q_n = (-n, n) \times (0, T)$ for any n, k and r.

$\{C_{nkr}\}$ is decreasing and converges uniformly to C_n as $k \to +\infty$.

Moreover, we can also assume that $A_n(x, t)$ and $C_n(x, t)$ satisfy

$$A_{nkr}(x, t) \geqq A_n(x, t) \geqq n^{1-m} \quad \text{in} \quad (-r, r) \times [0, T]$$

and

$$C_{nkr}(x,t) \leqq C_n(x,t) \leqq \lambda n^{1-q} \quad \text{in} \quad (-r,r)\times[0,T].$$

Then we can write $I(u-v,\phi,P)$ as

$$\int_{-r}^{+r}[u_n(x,t_1)-v(x,t_1)]\phi(x,t_1)\,dx = \int_{-r}^{+r}[u_n(x,0)-v(x,0)]\phi(x,0)\,dx$$
$$-\int_0^{t_1}(u_n^m-v^m)\phi_x\Big]_{-r}^{+r} + \int_{-r}^{+r}\int_0^{t_1}(A_n-A_{nkr})(u_n-v)\phi_{xx}\,dx\,dt$$
$$+\int_{-r}^{+r}\int_0^{t_1}(C_n-C_{nkr})(u_n-v)\phi\,dx\,dt + \int_{-r}^{+r}\int_0^{t_1}[A_{nkr}\phi_{xx}+\phi_t-C_{nkr}\phi](u_n-v)\,dx\,dt$$
$$+\lambda n^{-q}\int_{-r}^{+r}\int_0^{t_1}\phi(x,t)\,dx\,dt. \tag{3.3}$$

It is well-known (see e.g. [15]) that the uniformly parabolic linear problem

$$\begin{cases}\mathfrak{L}(\phi)\equiv A_{nkr}\phi_{xx}+\phi_t-C_{njkr}\phi=0, & \text{in} \quad P=(-r,r)\times(0,t_1),\\ \phi(\pm r,t)=0, & \text{for} \quad t\in(0,t_1),\\ \phi(x,t_1)=w(x), & \text{for} \quad x\in(-r,r),\end{cases}$$

has a unique linear classical solution $\phi(x,t)$ which satisfies:

(i) $0\leqq\phi(x,t)\leqq\max\{w(x);x\in(-r,r)\}$;

(ii) for any $\gamma>1$, there exist $M_1(\gamma)$, $M_2(\gamma)$ such that

$$|\phi(x,t)|\leqq M_1(1+|x|)^{-\gamma} \quad \text{in } P,$$
$$|\phi_x(\pm r,t)|<M_2r^{-\gamma};$$

(iii) there exists $M_3(n,r)$ such that

$$\int_{-r}^{+r}\int_0^{t_1}(\phi_{xx})^2\,dx\,dt\leqq M_3.$$

Substituting ϕ in [3.3], we deduce that

$$\int_{-r}^{+r}[u_n(x,t_1)-v(x,t_1)]w(x)\,dx\leqq\int_{-r}^{+r}[u_n(x,0)-v(x,0)]\phi(x,0)\,dx$$
$$+2M_2t_1\max_{t\leqq t_1}|u_n^m(\pm r,t)-v_n^m(\pm r,t)|\,r^{-\gamma}$$
$$+\max_P|A_n(x,t)-A_{nkr}(x,t)|\max_P|u_n(x,t)-v(x,t)|\,M_3^{\frac{1}{2}}(2rt_1)^{\frac{1}{2}}$$
$$+\max_P|C_n(x,t)-C_{nkr}(x,t)|\max_P|u_n(x,t)-v(x,t)|\,t_1M_1\int_{-r}^{+r}\frac{dx}{(1+|x|)}\gamma$$
$$+M_1\lambda n^{-q}\int_{-r}^{+r}\int_0^{t_1}\frac{dx}{(1+|x|)}\gamma.$$

Then inequality (3.2) follows by choosing $\gamma=(2m/(m-1))+1$, and by passing to the limit when $k \to +\infty$, $n \to +\infty$ and $r \to +\infty$ (notice that $v \in E_m$, and u_n satisfies (3.1)).

Consider now the case $m = 1$. We can write $I(u - v, \phi, P)$ as

$$\int_{-r}^{+r} [u_n(x, t_1) - v(x, t_1)]\phi(x, t_1)\, dx = \int_{-r}^{+r} [u_n(x, 0) - v(x, 0)]\phi(x, 0)\, dx$$

$$- \int_0^{t_1} (u_n - v)\phi_x \Big]_{-r}^{+r} + \int_{-r}^{+r}\int_0^{t_1} (C_n - C_{nkr})(u_n - v)\phi\, dx\, dt$$

$$+ \int_{-r}^{+r}\int_0^{t_1} [\phi_{xx} - \phi_t - C_{nkr}\phi](u_n - v)\, dx\, dt + \lambda n^{-q}\int_{-r}^{+r}\int_0^{t_1} \phi(x, t)\, dx\, dt. \tag{3.4}$$

Let ϕ be the classical solution of the parabolic linear problem:

$$\begin{cases} \mathfrak{L}(\phi) = \phi_{xx} + \phi_t - C_{nkr}\phi = 0, & \text{in} \quad P = (-r, r) \times (0, t_1), \\ \phi(\pm r, t) = 0, & \text{for} \quad t \in (0, t_1), \\ \phi(x, t_1) = w(x), & \text{for} \quad x \in (-r, r). \end{cases}$$

Since C_{nkr} is uniformly bounded and $w(x)$ is a non-negative bounded function with compact support, we easily deduce the following estimates: there exist $K_1 = K_1(w)$, $\alpha_0 = \alpha_0(t_1) > 0$ such that

$$0 \leqq \phi(x, t) \leqq K_1 e^{-\alpha_0 x^2} \quad \text{in} \quad P = (-r, r) \times (0, t_1)$$

and

$$|\phi_x(\pm r, t)| \leqq K_1 e^{-\alpha_0 r^2} \quad \text{for} \quad t \in [0, t_1].$$

Substituting ϕ in (3.4), we obtain that

$$\int_{-r}^{+r} [u_n(x, t_1) - v(x, t_1)]w(x)\, dx \leqq \int_{-r}^{+r} [u_n(x, 0) - v(x, 0)]\phi(x, 0)\, dx$$

$$+ 2t_1 \max_{t \leqq t_1} |u_n(\pm r, t) - v(\pm r, t)|\, K_1 e^{-\alpha_0 r^2}$$

$$+ \max_P |C_n(x, t) - C_{nkr}(x, t)| \max_P |u_n(x, t) - v(x, t)|\, t_1 \int_{-r}^{+r} K_1 e^{-\alpha_0 x^2}\, dx$$

$$+ \lambda n^{-q} \int_{-r}^{+r}\int_0^{t_1} K_1 e^{-\alpha_0 x^2}\, dx.$$

Finally, the inequality (3.2) follows when $k \to +\infty$, $n \to +\infty$ and $r \to +\infty$. □

Remark 3.6. Notice that the uniqueness of solutions in the class of functions E_m is now a trivial consequence of the above comparison principle.

Remark 3.7. The choice of the class of functions E_m in order to obtain global existence and uniqueness of the solutions of (1.4) is optimal in the sense that if u_0 satisfies (1.7) in the limit case (i.e. if $u_0(x) = (K + |x|)^{2/(m-1)}$ for some $K > 0$, when $m > 1$ or if $u_0(x) = Ke^{\alpha|x|^2}$ for some K and $\alpha > 0$ when $m = 1$), we can easily get subsolutions of the Cauchy problem (1.4) with initial data $u_0(x)$ such that they blow up in a finite time.

3.3. Regularity

The regularity in the x-variable of solutions of equation (1.1) is a direct consequence of the following result:

PROPOSITION 3.8 [**28**]. *Let $u(x, t)$ be a generalised solution of* (1.1) *in E_m. Let $\tau > 0$, $n \in \mathbb{N}$, $Q_n = (-n, n) \times (0, T)$ and $Q_n^\tau = (-n, n) \times (\tau, T)$.*

(a) *If $m + q \geqq 2$, then $(u^{m-1})_x$ is bounded in Q_n^τ. Moreover, if $\sup\{|(u_0^{m-1})_x|: x \in (-n, n)\} < \infty$, then $(u^{m-1})_x$ is bounded in Q_n.*

(b) *If $m + q < 2$, then $(u^{(m-q)/2})_x$ is bounded in Q_n^τ. Moreover, if $\sup\{|(u_0^{(m-1)/2})_x|: x \in (-n, n)\} < \infty$ then $(u^{(m-1)/2})_x$ is bounded in Q_n.*

It is not difficult to show that this result implies the continuity of the solution of (1.1) in $\mathbb{R} \times (0, \infty)$ (see [**28**]). As a final step, we shall prove the continuity of u near $t = 0$.

PROPOSITION 3.9. *Let $u(x, t)$ be a generalised solution of* (1.4) *in E_m, where the initial datum u_0 is assumed to be a non-negative continuous function. Then $u(x_0, t)$ converges to $u_0(x_0)$ when $t \to 0^+$ for any $x_0 \in \mathbb{R}$.*

Proof. Let $x_0 \in \mathbb{R}$, and $Q_r = (x_0 - r, x_0 + r) \times (0, T)$. Let $M > 0$ such that $0 \leqq u(x_0 \pm r, t) \leqq M$ for all $t \in [0, T]$. Two different cases arise.

(i) Assume $u_0(x_0) = 0$. Let $\bar{v}(x, t)$ be the solution of the porous medium equation $\mathfrak{L}(\bar{v}) = \bar{v}_t - \bar{v}^m_{xx} = 0$ in Q_n, such that $\bar{v}(x_0 \pm r, t) = M$ and $\bar{v}(x, 0) = u_0(x)$. Then it is well known from the theory for the porous medium equation that $\bar{v}(x, t)$ is a non-negative function in Q_r, continuous in $t = 0$, and that:

$$\begin{aligned} &\mathfrak{L}(u) \leqq \mathfrak{L}(\bar{v}) \quad \text{in } Q_r;\\ &u(x_0 \pm r, t) \leqq \bar{v}(x_0 \pm r, t), \quad \text{for} \quad t \in [0, T];\\ &u(x, 0) = \bar{v}(x, 0), \quad \text{for} \quad x \in [x_0 - r, x_0 + r]. \end{aligned}$$

Therefore by the comparison principle we have that

$$0 \leqq u(x, t) \leqq \bar{v}(x, t) \quad \text{in } Q_r.$$

Then the continuity in $t = 0$ follows from the continuity of $\bar{v}(x, t)$ in $t = 0$.

(ii) Assume $u_0(x_0) > 0$. Then by the continuity of $u_0(x)$ there exist $r, \varepsilon > 0$ such that $u(x, t) > \varepsilon$ for any $(x, t) \in (x_0 - r, x_0 + r) \times [0, T]$ (notice that $u(x, t)$ is the limit of a decreasing sequence of classical solutions $u_n(x, t)$ defined in Lemma 3.2). Let $\bar{v}(x, t)$ be as in (i) and $\underline{v}(x, t)$ be the classical solution of the equation: $\mathfrak{L}(\underline{v}) = \underline{v}_t - \underline{v}^m_{xx} = -bM^q$ in Q_r, with $\underline{v}(x_0 \pm r, t) = \varepsilon$, and $\underline{v}(x, 0) = u_0(x)$. Then $\underline{v}(x, t)$ is continuous in $t = 0$ and we have

$$\begin{aligned} &\mathfrak{L}(\underline{v}) \leqq \mathfrak{L}(u) \leqq \mathfrak{L}(\bar{v}) \quad \text{in } Q_r;\\ &\underline{v}(x_0 \pm r, t) \leqq u(x_0 \pm r, t) \leqq \bar{v}(x_0 \pm r, t) \quad \text{for} \quad t \in [0, T];\\ &\underline{v}(x, 0) = u(x, 0) = \bar{v}(x, 0) \quad \text{for} \quad x \in [x_0 - r, x_0 + r]. \end{aligned}$$

Then by the comparison principle we have that $\underline{v}(x, t) \leqq u(x, t) \leqq \bar{v}(x, t)$ in Q_r and thus the continuity in $t=0$ follows from the same property for $\underline{v}(x,t)$ and $\bar{v}(x,t)$ in $t=0$. □

Remark 3.10. The study of solutions of equation (1.1) with unbounded initial data started with the pioneering work of Kalashnikov [**24**] where the exponent q is assumed to be $q>1$. More recent results (showing the nonuniqueness of solutions when $1<q<m$) can be found in [**26**]. We also remark that if $q\in(0,1)$ the growth condition (1.7) coincides with that of solutions of the unperturbed equation: i.e. the porous media equation if $m>1$ and the linear heat equation if $m=1$. As it was shown in [**7**] (for the porous media equation) it seems possible to generalise assumption (1.7) by expressing it in terms of a weighted average.

4. Proof of Theorem 1.1

Let u be any continuous solution of the equation (1.1). Assume that there exist $x_0\in(-L,0)$ and $C\in(0,C_0)$ such that

$$u(x,0)\leqq C(-x)_+^{2/(m-q)} \quad \text{if} \quad x\in[x_0,0]. \tag{4.1}$$

Let C_1 such that $C<C_1<C_0$, and let $\bar{v}(x,t)$ be the solution of the Cauchy problem (1.4) with initial datum

$$\bar{v}(x,0)=C_1(-x)_+^{2/(m-q)} \quad \text{for} \quad x\in(-\infty,+\infty).$$

From the continuity of u and $\bar{v}$ on the set $(-L,L)\times[0,T]$ and (4.1), we deduce the existence of a time $t_0\in(0,T]$ such that

$$u(x_0,t)\leqq\bar{v}(x_0,t) \quad \text{for any } t\in[0,t_0].$$

Then we are allowed to apply the comparison principle for (bounded) solutions of the equation (1.1) on the set $Q=(x_0,+\infty)\times(0,t_0)$ and thus we deduce that:

$$u(x,t)\leqq\bar{v}(x,t) \quad \text{for any} \quad (x,t)\in\bar{Q}. \tag{4.2}$$

Conclusion (i) of Theorem 1.1 is now a direct consequence of Theorem 2.2 and inequality (4.2). The assertion (ii) follows a similar argument. □

We shall end this article by giving two direct applications of Theorem 1.1.

COROLLARY 4.1. *Let $u_0\in C^{0,\beta}(-\infty,\infty)\cap L^\infty(-\infty,\infty)$ with $\beta\geqq 1/m$ and such that $u_0\geqq 0$ on $(-\infty,\infty)$. Assume that $u_0(x)\leqq C(-x)_+^{2/(m-q)}$ for any $x\in[x_0,+\infty)$, for some $x_0<0$ and $C<C_0$. Let u be the solution of the Cauchy problem* (1.4) *with initial datum u_0, and let $\zeta(t)$ be defined by*

$$\zeta(t)=\sup\{x\in(-\infty,\infty): u(x,t)>0\}.$$

Then there exists a $t_0>0$ and $a>0$ such that

$$\zeta(t)\leqq -at^{(m-q)/2(1-q)} \quad \textit{for any} \quad t\in[0,t_0].$$

If on the contrary, $u_0(x)$ satisfies

$$u_0(x)\geqq C(-x)_+^{2/(m-q)} \quad \textit{for any } x\in[x_0,+\infty), \textit{ for some } x_0<0 \textit{ and } C>C_0,$$

then

$$\zeta(t)\geqq at^{(m-q)/2(1-q)} \quad \textit{for any} \quad t\in[0,t_0]$$

for some $t_0>0$ and $a>0$.

Remark 4.2. The existence and uniqueness of a continuous generalised solution of the Cauchy problem (1.4) was obtained in [**25**] and [**28**] (see also [**29**] and [**21**]). The study of the initial behaviour of the interfaces was started in [**27**], where Kersner also gave an explicit self-similar solution for the case $m+q=2$. He obtained sufficient conditions for the existence of heating and cooling fronts which are special cases of the assumptions of Corollary 4.1, but no growth estimates on $\zeta(t)$ are given in that work. Our estimates also improve the ones given in [**29**]. The study of the initial growth of the interfaces via asymptotic methods started with the work of Rosenau and Kamin [**32**] and was continued by Grundy and Peletier [**18, 19**] for $(m=1)$ and Grundy [**17**]. Finally, we mention the work of G. Díaz [**11**], where the study of the initial growth of the interface was carried out for the N-dimensional semilinear equation by using stochastic methods.

Remark 4.3. The comparison of the initial datum $u_0(x)$ and the stationary solution $u_\infty(x)$ is useful for many other purposes (see e.g. [**23**] where the authors study the finite extinction time phenomena for solutions of the semilinear heat equation with initial data in the class E_1).

Concerning the case of bounded domains we have the following corollary:

COROLLARY 4.4. *Let* $u_0 \in C^{0,\beta}([-L, L])$ *and* $h_+, h_- \in C^{0,\beta}([0, T])$ *with* $\beta \geqq 1/m$ *and such that* $u_0(-L)=h_-(0)$, $u_0(L)=h_+(0)$, $u_0 \geqq 0$ *and* $h_+, h_- \geqq 0$. *Let* $u \in C([-L, L]\times[0, T])$ *be the (unique) generalised solution of the problem*:

$$\begin{cases} u_t=(u^m)_{xx}-\lambda u^q, & \text{in} \quad Q=(-L, L)\times(0, T), \\ u(-L, t)=h_-(t), \quad u(L, t)=h_+(t), & \text{for} \quad t\in(0, T), \\ u(x, 0)=u_0(x), & \text{for} \quad x\in(-L, L). \end{cases} \tag{4.3}$$

Assume that

$$u_0(x)\leqq C(-x)_+^{2/(m-q)} \quad \text{for any} \quad x\in(-x_0, x_0)$$

for some $x_0\in(0, L)$, *and* $C\in(0, C_0)$. *Then, if* $\zeta(t)$ *denotes the interface*

$$\zeta(t)=\sup\{x\in(-L, L): u(x, t)>0\},$$

there exists $t_0\in(0, T]$ *and* $a>0$ *such that*

$$\zeta(t)\leqq -at^{(m-q)/2(1-q)} \quad \text{for any} \quad t\in[0, t_0].$$

If on the contrary $u_0(x)$ *satisfies*

$$u_0(x)\geqq C(-x)_+^{2/(m-q)} \quad \text{for any} \quad x\in(-x_0, x_0),$$

for some $x_0\in(0, L)$ *and* $C>C_0$, *then*

$$\zeta(t)\geqq at^{(m-q)/2(1-q)} \quad \text{for any} \quad t\in[0, t_0],$$

for some $t_0\in(0, T]$ *and* $a>0$.

Remark 4.5. The existence and uniqueness of a generalised solution of the Dirichlet problem (4.3) has been obtained by different authors (see e.g. [**30, 9**] and their references). It seems that the study of the formation of the free boundary by means of local methods and its application to the Dirichlet problem (4.3) was started in [**13**] (see also [**14, 6, 16**]). Our results also apply to formulations with an explicit (moving) free boundary (see [**10, 20**] and their references).

Remark 4.6. The initial growth of the interface $\zeta(t)$ is very close to the behaviour of the interface for the elliptic problem

$$-(u^m)_{xx} + \lambda u^q - \mu u = f(x), \quad x \in (-L, L).$$

The nondiffusion of the support property (see [**12**]) leads to results on the existence of a waiting time, and the dilation of the support property ([**2, 1**]) leads to the existence of heating fronts as in part (ii) of Theorem 1.1. The connection between the parabolic and elliptic problems is established by means of implicit time discretisation schemes.

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