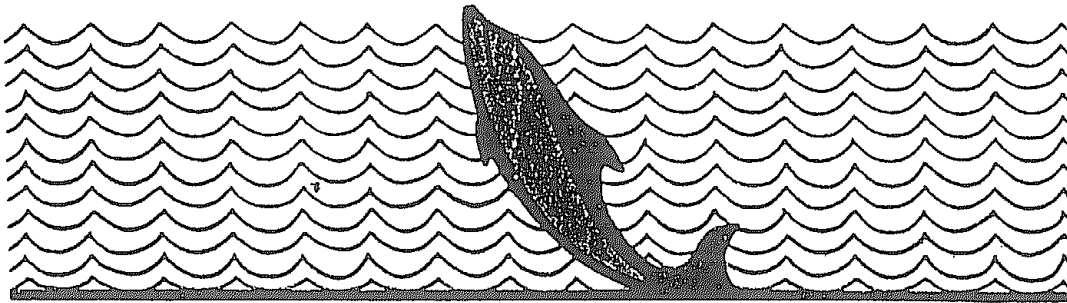


ДИНАМИКА ЖИДКОСТИ СО СВОБОДНЫМИ ГРАНИЦАМИ



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CONTINUOUS DEPENDENCE & STABILIZATION OF SOLUTIONS
OF THE DEGENERATE SYSTEM IN TWO-PHASE FILTRATION

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The filtration of two immiscible fluids in a heterogeneous anisotropic porous medium is described by a system of nonlinear differential equations including an eventual degenerate parabolic equation for the saturation $S(x,t)$ of one of the phases and an elliptic equation for the "reduced" pressure $p(x,t)$:

$$m(x) \frac{\partial s}{\partial t} = \operatorname{div} (K_0(x) \alpha(s) \nabla s + K_1(x,s) \nabla p + \vec{f}_0(x,s)), \quad (1)$$

$$\operatorname{div} (K(x,s) \nabla p + \vec{f}(x,s)) = 0, \quad (2)$$

$$x \in \Omega \in R^n, \quad n = 1, 2, 3, \quad (x,t) \in \Omega_T = \Omega \times (0,T),$$

where Ω is a bounded and multiple-connected domain with the boundary Γ . The derivation of such a system and the meaning of each of the coefficients can be found in [1].

For system (1), (2) we consider the following initial-boundary problem

$$s(x,t) = s^0(x,t), p(x,t) = p^0(x,t), (x,t) \in \Gamma_{1T} \equiv \Gamma_1 \times (0,T), \quad (3)$$

$$(K \nabla p + \vec{f}) \vec{n} = -Q(x,t), (K_0 \operatorname{div} s + K_1 \nabla p + \vec{f}_0) \vec{n} = -Q_1(x,t), \quad (4)$$

$$(x,t) \in \Gamma_{2T} = \Gamma_2 \times (0,T),$$

$$s(x,0) = s^0(x,0), \quad x \in \Omega. \quad (5)$$

Here $\Gamma = \Gamma_1 + \Gamma_2$; \vec{n} is the unit outward normal vector to Γ ; p^0, s^0, Q, Q_1 are the prescribed functions.

The solvability of problem (1)-(5) has been considered by numerous authors: see references in the monograph [1].

The uniqueness of generalized solutions of this problem was proved in [2] and in [3] for $\Gamma = \Gamma_1$.

This paper presents a complete account of the brief article [4].

1. Continuous dependence result.

In [1] the proof is made for existence of generalized solution of the problem (1)-(5) in the following sense:

$s(x,t), p(x,t)$ are the bounded measurable functions, $s(x,t) \in [0,1], p(x,t) \in L_\infty(\Omega_T) \cap L_\infty(0,T; W_2^1(\Omega))$,

$$u(x,t) \equiv \int_0^t \sqrt{a(\xi)} d\xi \in L_2(0,T; W_2^1(\Omega))$$

satisfy (3) in the sense of traces from $W_2^1(\Omega)$;

for all $\varphi(x,t), \psi(x,t), (\varphi, \psi) \in W_2^1(\Omega_T), \varphi(x,t) = \psi(x,t) = 0, (x,t) \in \Gamma_{1T}, \varphi(x,T) = 0, x \in \Omega$ and almost all $t \in (0,T)$ the following integral identities are fulfilled:

$$-(ms, \varphi) \Big|_{t=0}^{t=t} + (ms, \varphi_t)_{\Omega_t} - (K_0 \operatorname{div} s + K_1 \nabla p + \vec{f}_0, \nabla \varphi)_{\Omega_t} - (Q_1, \varphi)_{\Gamma_{2t}} = 0, \quad (6)$$

$$(K \nabla p + \vec{f}, \nabla \psi)_{\Omega} + (Q, \psi)_{\Gamma_2} = 0, \quad (7)$$

where

$$(u, v)_{\Omega} = \int_{\Omega} uv dx, (u, v)_{\Omega_0} = (u(0), v(0))_{\Omega}, \Omega_t = \Omega \times (0, t), \Gamma_{2t} = \Gamma_2 \times (0, t),$$

In the present article we shall assume the extra condition

$$\nabla p \in L_\infty(\Omega_T)$$

which has been obtained in [1] under some additional conditions.

Introduce the function

$$c(s) \equiv \int a(\xi) d\xi.$$

Let $w^i = (s^i, p^i), i = 1, 2$ denote any generalized solutions of problem (1) - (5) associated to the data $a^i, c^i, m^i, s_0^i, p_0^i, \vec{f}_0^i, \vec{f}_1^i, K^i, K_0^i, K_1^i, Q^i, Q_1^i$. One of the key assumptions of our results is the one concerning the diffusion coefficients and the boundary data on Γ_1 ; it is stated as the following alternative:

$$|c^1(s^2) - c^2(s^2)|^q \leq M \int_0^1 a^1(\tau s^1 + (1-\tau) s^2) d\tau$$

for some $M > 0$ and $q \in (0, 2)$, and

$$c^1(s_0^1) = c^2(s_0^2), (x,t) \in \Gamma_{1T},$$

either there exist $M > 0$ such that

$$a^1(s) \leq M a^2(s), \forall s \in [0,1] \text{ and } s_0^1 = s_0^2, (x,t) \in \Gamma_{1T}. \quad (8')$$

Theorem 1. Assume, that conditions (8) or (8') holds. Assume also, that there exist $M > 0$ such that

$$M^{-1} \leq (m^1, (K_0^1 \vec{\xi}, \vec{\xi}) / |\vec{\xi}|^2) \leq M, (|\vec{f}^1|, |\vec{f}_0^1|) \leq M, \left| \frac{\partial}{\partial s} (K_1^1, K^1, \vec{f}_1^1, \vec{f}_0^1) / \sqrt{a^1} \right| \leq M, 0 \leq a^j \in C[0,1], j = 1, 2, \quad (9)$$

$$K_0^1(x) \in C(\Omega), \operatorname{meas} \Gamma_1 > 0.$$

then for $w^i = (s^i, p^i)$ with $|\nabla p^i| \leq M$ the following estimate holds

$$(s^1 - s^2, c^1(s^1) - c^2(s^2))_{\Omega_T} + \|p^1 - p^2\|_{2, \Omega_T} \leq C_1(M, T) H, \quad (10)$$

where

$$H = \|m^1 - m^2\|_{2, \Omega}^2 + \|s_0^1 - s_0^2\|_{2, \Gamma_{1T} \cup \Omega}^2 + \|p_0^1 - p_0^2\|_{2, \Gamma_{1T}}^2 + \|Q^1 - Q^2\|_{2, \Gamma_{2T}}^2 + \|Q_1^1 - Q_1^2\|_{2, \Gamma_{2T}}^2 + \|\vec{f}^1 - \vec{f}^2\|_{\infty, \nu}^2 + \|\vec{f}_0^1 - \vec{f}_0^2\|_{\infty, \nu}^2 + \|K_0^1 - K_0^2\|_{\infty, \nu}^2 + \|K_1^1 - K_1^2\|_{\infty, \nu}^2 + \|K^1 - K^2\|_{\infty, \nu}^2 + (\|a^1 - a^2\|_{C[0,1]})^{1-q/2}.$$

Here $V = [0, 1] \times \Omega$ and $q=1$ if (8') takes place.

Proof. From the notion of generalized solution we deduce, after integration by parts, that

$$\begin{aligned} & (m^1(s^1 - s^2), \varphi_t)_{\Omega_T} + (c^1(s^1) - c^1(s^2), \operatorname{div}(K_0^1 \nabla \varphi))_{\Omega_T} + \\ & (K_1^1(x, s^2) \nabla p^2 - K_1^1(x, s^1) \nabla p^1 + \vec{f}_0^1(x, s^2) - \vec{f}_0^1(x, s^1), \nabla \varphi)_{\Omega_T} = \\ & - (c^1(s^2) - c^2(s^2), \operatorname{div}(K^1 \nabla \varphi))_{\Omega_T} + (c^1(s^1) - c^2(s^2), K^1 \nabla \varphi \cdot \vec{n})_{\Gamma_{1T}} - \\ & ((m^1 - m^2) s^2, \varphi_t)_{\Omega_T} + ((K_0^1 - K_0^2) \nabla c^2(s^2), \nabla \varphi)_{\Omega_T} + \end{aligned} \quad (11)$$

$$\begin{aligned} & ((K_1^1(x, s^2) - K_1^2(x, s^2)) \nabla p^2, \nabla \varphi)_{\Omega_T} + ((\vec{f}_0^1(x, s^2) - \vec{f}_0^2(x, s^2)), \nabla \varphi)_{\Omega_T} + \\ & ((m^1 - m^2) s_0^1, \varphi)_{\Omega_0} - (m^1(s_0^1 - s_0^2), \varphi)_{\Omega_0} + (Q^1 - Q^2, \varphi)_{\Gamma_{2T}} \equiv E(\varphi), \end{aligned}$$

Analogously,

$$\begin{aligned} & (K^1(x, s^1) \nabla p^1 - K^1(x, s^2) \nabla p^2 + \vec{f}^1(x, s^1) - \vec{f}^1(x, s^2), \nabla \psi)_{\Omega} = \\ & (Q^1 - Q^2, \psi)_{\Gamma_2} - ((K^1(x, s^2) - K^2(x, s^2)) \nabla p^2 + \vec{f}^1(x, s^2) - \vec{f}^2(x, s^2), \nabla \psi)_{\Omega} \equiv \\ & \equiv F(t/\psi). \end{aligned} \quad (12)$$

As in [2] we integrate (12) on $(0, t)$ and sum up it with (11); then, after substitution $t \rightarrow T-t$, we obtain the following identities for the functions $s = s^1 - s^2$, $p = p^1 - p^2$:

$$\begin{aligned} & (m^1 s, L_1(\varphi, \psi))_{\Omega_T} - (p, L_2(\varphi, \psi))_{\Omega_T} = \\ & = -\varepsilon (m^1 s, \operatorname{div}(K_0^1 \nabla \varphi))_{\Omega_T} + E(\varphi) - \int_0^T F(t/\psi) dt, \end{aligned} \quad (13)$$

where for $\varepsilon > 0$ the operators L_1 and L_2 are defined by

$$L_1(\varphi, \psi) \equiv -\frac{\partial \varphi}{\partial t} + A(x, t) \operatorname{div}(K_0^1 \nabla \varphi) + A_1(x, t) \nabla \varphi + C_1(x, t) \nabla \psi,$$

$$L_2(\varphi, \psi) \equiv \operatorname{div}(E(x, t) \nabla \psi) - D(x, t) \nabla \varphi,$$

and the coefficients and the boundary conditions are given by

$$A = (c^1(s^1) - c^1(s^2))/m^1(s^1 - s^2) + \varepsilon \equiv A_0 + \varepsilon, \quad \varepsilon > 0,$$

$$-A_1 = \nabla p^1 ((K_1^1(x, s^1) - K_1^1(x, s^2) + \vec{f}_0^1(x, s^1) - \vec{f}_0^1(x, s^2))/m^1(s^1 - s^2),$$

$$-C_1 = \nabla p^1 (K^1(x, s^1) - K^1(x, s^2) + \vec{f}^1(x, s^1) - \vec{f}^1(x, s^2))/m^1(s^1 - s^2),$$

$$E = K^1(x, s^2), \quad D = K_1^1(x, s^2).$$

$$\varphi \Big|_{\Gamma_{1T}} = \psi \Big|_{\Gamma_{1T}} = K_0^1 \nabla \varphi \cdot \vec{n} \Big|_{\Gamma_{2T}} = K_0^1 \nabla \psi \cdot \vec{n} \Big|_{\Gamma_{2T}} = \varphi \Big|_{t=0} = \psi \Big|_{t=0} = 0. \quad (14)$$

Now let us assume that

$$L_1(\varphi, \psi) = h(x, t), \quad L_2(\varphi, \psi) = g(x, t) \quad (15)$$

and consider initial boundary-value problem (14)-(15). It is possible to show that this problem has a unique solution when $\varepsilon > 0$ and conditions of Theorem 1 takes place. As in [2] it may be shown that if there exist $M_0 > 0$ such that

$$(|g|, (|h| + |A_1| + |C_1|)/\sqrt{A_0}) \leq M_0,$$

then for its solution the following estimate holds

$$\|\nabla \varphi\|_{2, \infty, \Omega}^2 + \|\nabla \psi\|_{2, \infty, \Omega}^2 + \|\varphi_t\|_{2, \Omega}^2 + \varepsilon \|\operatorname{div}(K_0^1 \nabla \varphi)\|_{2, \Omega}^2 \leq C_2(M, M_0, T). \quad (16)$$

$$\text{with } h = (1 - \int_{\Gamma_{1T}} (s_0^1 - s_0^2) d\Gamma) (c^1(s^2) - c^1(s^2)), \quad g = -(p - \int_{\Gamma_{1T}} (p_0^1 - p_0^2) d\Gamma)$$

the equality (13) takes the form

$$(m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T} + \|p\|_{2, \Omega_T}^2 = E(\varphi) + \int_0^T F(t/\psi) dt + (p, 1)_{\Omega_T} +$$

$$\int_{\Gamma_{1T}} (p_0^1 - p_0^2) d\Gamma + (m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T} \int_{\Gamma_{1T}} (s_0^1 - s_0^2) d\Gamma -$$

$$(m^1 s, \varepsilon \operatorname{div}(K_0^1 \nabla \varphi))_{\Omega_T} \equiv E(\varphi) + \int_0^T F(t/\psi) dt + J_1 + J_2 + J_3.$$

Taking into account the assumptions of Theorem 1 and (6), the last three terms in the right-hand part of (17) may be estimated as follows

$$|J_1| \leq \frac{\delta}{2} \|p\|_{2, \Omega_T}^2 + \frac{1}{2\delta} \|p_0^1 - p_0^2\|_{2, \Gamma_{1T}}^2, \quad \delta > 0$$

$$|J_2| \leq \frac{\delta}{2} (m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T} + C_3(\delta, M) \|s_0^1 - s_0^2\|_{2, \Gamma_{1T}}^2$$

$$|J_3| \leq \sqrt{\varepsilon} C_4(M_0, T).$$

Assume that condition (8) is valid. Then the second term in the right-hand part of (11) is equal to zero and on the basis of the method suggested in [5] the first term is estimated as follows

$$\begin{aligned} & |(c^1(s^2) - c^2(s^2)), \operatorname{div}(K_0^1 \nabla \varphi)|_{\Omega_T} \leq \\ & \leq \left[\frac{|c^1(s^2) - c^2(s^2)|}{\sqrt{A}} |c^1(s^2) - c^2(s^2)|^{q/2} \cdot |\sqrt{A} \operatorname{div}(K_0^1 \nabla \varphi)| \right]_{\Omega_T} \leq \\ & \leq (\|a^1 - a^2\|_{C(0,1)})^{1-q/2} C_2^{1/2} M^{1/2} (\operatorname{meas} \Omega_T)^{1/2}. \end{aligned}$$

For the case of (8') the sum of the first two terms in $E(\varphi)$ may be represented as

$$(\nabla(c^1(s^2) - c^2(s^2)), K_0^1 \nabla \varphi)_{\Omega_T} = J.$$

From the facts that

$$u^t = \int_0^t \sqrt{a^t(\xi)} d\xi \in L_2(0, T; W_2^1(\Omega))$$

and

$$|\nabla c^1(s^2) - \nabla c^2(s^2)| = \frac{\sqrt{|a^1(s^2) - a^2(s^2)|}}{\sqrt{a^2(s^2)}} |\nabla u^2| \sqrt{|a^1(s^2) - a^2(s^2)|}$$

we obtain

$$|J| \leq C_5^*(M) (\|a^1 - a^2\|_{C(0,1)})^{1/2} \|\nabla u^2\|_{2, \Omega_T} \|\nabla \varphi\|_{2, \Omega_T}.$$

The other terms in $E(\varphi)$ and $F(t/\varphi)$ are estimated using (16). The obtained estimates make it possible to go over to the limit when $\varepsilon \rightarrow 0$ in (19). Theorem 1 is proved.

Let us study now stability of $S(x, t)$ in $\|\cdot\|_{\Omega}$ with respect only to the variation of the initial data (5).

Let two solutions ω^i , $i=1,2$ of the problem (1)-(5) differs only by the initial data (5) and S, p are defined analogously theorem 1.

Theorem 2. Let ω^i , $i=1,2$, be generalized solutions of the problems (1)-(5), differing only by the initial data, $m=1$, $K_0^t = (\delta_{ij})$ and the conditions of theorem 1 hold. Then there hold estimate

$$\|s^1(x, t) - s^2(x, t)\|_{H^1(\Omega)} \leq C \|s^1(x, 0) - s^2(x, 0)\|_{H^1(\Omega)} \quad \forall t \leq T.$$

Proof. Analogously theorem 1 we deduce equality

$$(s, \varphi)_{\Omega_t} \Big|_{t=0}^{t=T} + (s, L_1(\varphi, \psi))_{\Omega_t} - (p, L_2(\varphi, \psi))_{\Omega_t} = -\varepsilon (s, \operatorname{div}(K_0^1 \nabla \varphi))_{\Omega_t}, \quad (16)$$

where L_1, L_2 are defined as in theorem 1.

Consider initial-boundary problem (14)-(15) with

$$h = 0, \quad g = 0, \quad \varphi(x, 0) = \varphi_0(x).$$

Similarly [2] may be shown that for this conditions the next estimates takes place (after returning to the starting time $t \rightarrow T-t$)

$$\|\nabla \varphi(x, t)\|_{2, \Omega}^2 \leq C \|\nabla \varphi(x, T)\|_{2, \Omega}^2, \quad \forall t \leq T.$$

Letting $\varepsilon \rightarrow 0$ in (16) we get the equality

$$(s(x, 0), \varphi(x, 0))_{\Omega} = (s(x, T), \varphi(x, T))_{\Omega}.$$

Let Δ denotes the Laplacian and $g_k(x)$ $k=1,2,\dots$ be the solutions of the problems

$$\begin{aligned} (\Delta + \lambda_k) g_k &= 0, \quad x \in \Omega; & (g_k, g_j) &= \delta_{kj}, \\ g_k|_{\Gamma_1} &= 0, & \nabla g_k \cdot \vec{n}|_{\Gamma_2} &= 0. \end{aligned} \quad (17)$$

Then any function $f \in L_2(\Omega)$ may be represented in the form

$$f(x) = \sum_{k=1}^{\infty} f_k g_k(x), \quad f_k = (f, g_k)_{\Omega}.$$

Let Δ^{-1} denotes operator inverse to Δ under the boundary conditions (17). Then there valid the formulas

$$-(s, \Delta^{-1} s)_{\Omega} = \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k} = \|s\|_{H^{-1}(\Omega)}^2 = \|\nabla(\Delta^{-1} s)\|_{2, \Omega}^2.$$

Indeed,

$$\begin{aligned} -(s, \Delta^{-1} s)_{\Omega} &= - \left(\sum_{k=1}^{\infty} s_k g_k, \sum_{l=1}^{\infty} s_l \Delta^{-1} g_l \right)_{\Omega} = \left(\sum_{k=1}^{\infty} s_k g_k, \sum_{l=1}^{\infty} \frac{1}{\lambda_l} s_l g_l \right)_{\Omega} = \\ &= \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k}, \end{aligned}$$

$$\|s\|_{H^{-1}(\Omega)}^2 = \sup_{\|\nabla h\|_{2,\Omega} \leq 1} |(s,h)|_{\Omega}^2 = \sup_{\|\nabla h\|_{2,\Omega} \leq 1} \left| \sum_{k=1}^{\infty} \frac{s_k}{\sqrt{\lambda_k}} \sqrt{\lambda_k} h_k \right|^2 \leq \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k} \sum_{k=1}^{\infty} \lambda_k h_k^2 \leq \sum_{k=1}^{\infty} \frac{s_k^2}{\lambda_k},$$

$$\|\nabla h\|_{2,\Omega}^2 = (\nabla h, \nabla h)_{\Omega} = - (h, \Delta h)_{\Omega} = \sum_{k=1}^{\infty} \lambda_k h_k^2 \leq 1,$$

$$\|\nabla(\Delta^{-1}s)\|_{2,\Omega}^2 = (\nabla \Delta^{-1}s, \nabla \Delta^{-1}s)_{\Omega} = - (\Delta \Delta^{-1}s, \Delta^{-1}s)_{\Omega} = \|s\|_{H^{-1}(\Omega)}^2.$$

Let us set now

$$\varphi(x,T) = \Delta^{-1}s(x,T).$$

Then, taking into account the last formulas we obtain the following chain of inequalities

$$(s(x,T), \varphi(x,T))_{\Omega} = (s(x,T), \Delta^{-1}s(x,T))_{\Omega} = \|s(x,T)\|_{H^{-1}(\Omega)}^2 = |(s(x,0), \varphi(x,0))_{\Omega}| \leq C \|s(x,0)\|_{H^{-1}(\Omega)} \|\nabla \varphi(x,T)\|_{2,\Omega} = C \|s(x,0)\|_{H^{-1}(\Omega)} \|s(x,T)\|_{H^{-1}(\Omega)}.$$

Theorem 2 is proved.

2. Stabilization of generalized solutions.

Consider stationary solution of problem (1)-(5). This solution $(s_0(x), p_0(x))$ for all $(\varphi, \psi) \in W_2^1(\Omega)$, $\varphi = 0$, $\psi = 0$, $x \in \Gamma_1$ satisfy the integral identities

$$(K_0(x) \nabla(s_0) + K_1(x, s_0) \nabla p_0 + \vec{f}_0(x, s_0), \nabla \varphi)_{\Omega} - (R_1, \varphi)_{\Gamma_2} = 0, \quad (18)$$

$$(K(x, s_0) \nabla p_0 + f(x, s_0), \nabla \psi)_{\Omega} + (R, \psi)_{\Gamma_2} = 0, \quad (19)$$

and boundary conditions

$$s_0(x) = 1, \quad p_0(x) = p^0(x), \quad x \in \Gamma_1$$

in the sense of traces from $W_2^1(\Omega)$. The functions R , R_1 are analogies of the functions Q , Q_1 in (4).

If $R = R_1$, there exists the solution in the form

$$s_0(x) \equiv 1, \quad x \in \Omega, \quad (20)$$

and function p_0 is a generalized solution of the boundary-value

problem

$$\begin{aligned} \operatorname{div}(K(x,1) \nabla p_0 + \vec{f}(x,1)) &= 0, \\ (K(x,1) \nabla p_0 + \vec{f}(x,1)) \cdot \vec{n} &= -R, \quad x \in \Gamma_2, \\ p_0 &= p^0(x), \quad x \in \Gamma_1, \end{aligned} \quad (21)$$

since the coefficients equations of system (1)-(2) have the properties [1, p.212]

$$K_1(x,1) = K(x,1), \quad \vec{f}(x,1) = \vec{f}_0(x,1). \quad (22)$$

Note 1. The solution of (20), (21) corresponds in terms of physics to the case of a complete displacement of oil initially occupying a part of the volume Ω , by water.

Here it is proved that the generalized solution (s,p) of degenerate problem (1)-(5) is converged to the solution $(1, p_0)$ from (20)-(21) when t infinitely increasing.

Lemma 1. Let the non-negative functions $A_1(t) \in L_1(0, \infty)$ and $y(t)$ be connected by the differential inequality

$$\frac{dy}{dt} + g(y) \leq A_1(t), \quad y(0) = y_0 > 0,$$

where the function $g(z)$ is such that

$$g(z) \geq 0, \quad z \geq 0, \quad \frac{dH}{dz} \equiv \frac{d}{dz} \left(\frac{g(z)}{z} \right) > 0, \quad \int_z^{z_0} \frac{d\xi}{g(\xi)} \equiv \Phi(z) \rightarrow \infty, \quad z \rightarrow 0, \quad z_0 \leq y_0.$$

Then

$$y \leq \Phi^{-1}(t) \left[1 + \int_0^t \frac{A_1(\tau)}{\Phi^{-1}(\tau)} d\tau \right] \equiv E(t) \rightarrow 0, \quad t \rightarrow \infty.$$

where Φ^{-1} is the symbol of the inverse function to Φ .

The proof of Lemma 1 may be found in [6].

Consider the problem

$$\frac{d\varphi}{dt} + A(x,t) \operatorname{div}(K_0 \nabla \varphi) + \vec{B}(x,t) \nabla \varphi = -\lambda A \varphi, \quad (23)$$

$$\varphi = 0, \quad (x,t) \in \Gamma_{1T}, \quad K_0 \nabla \varphi \cdot \vec{n} = 0, \quad (x,t) \in \Gamma_{2T}, \quad \varphi(x,T) = \varphi_0(x) \geq 0,$$

$$\varphi_0 \in C^{2+\alpha}(\Omega), \quad A, \vec{B} \in C^{\alpha, \alpha/2}(\Omega_T), \quad \alpha \in (0,1), \quad A = A_0(x,t) + \varepsilon \geq \varepsilon > 0.$$

The function φ_0 satisfies the conditions of zero and first order

compatibility.

Let the function $\omega(x)$ be the solution of problem

$$\begin{aligned} \operatorname{div}(K \nabla \omega) &= \alpha_1, \quad x \in \Omega, \quad \alpha_1 > 0, \\ \omega &= 0, \quad x \in \Gamma_1, \quad K_0 \nabla \omega \cdot \vec{n} = -\alpha_1, \quad x \in \Gamma_2. \end{aligned} \quad (24)$$

By the maximum principle $\omega \leq 0$, $x \in \Omega$ [7].

Lemma 2. Assume that $|B|/A \leq M_1$, where the constant M_1 is independent of t, x, ε , $\operatorname{meas} \Gamma_1 > 0$. Then there exist a number $\lambda_0 > 0$, such that for all $\lambda \leq \lambda_0$ any solution of problem (23) satisfies the maximum principle in the form

$$0 \leq \varphi(x, t) \leq \|\varphi_0\|_{C(\Omega)} + 1.$$

If the function $\varphi_0(x)$ satisfies the inequality

$$\varphi_0 \leq \omega (1 - \exp(\alpha_1 \min_{\Omega} \omega)) / \min_{\Omega} \omega, \quad (25)$$

where ω is taken from (24), then for $\lambda \leq \lambda_0$ next estimate is held

$$-\alpha_1 \exp(\alpha_1 \omega) K_0 \nabla \omega \cdot \vec{n} \leq K_0 \nabla \varphi \cdot \vec{n} \leq 0, \quad (26)$$

and φ_0 can be chosen such that $\varphi_0^{-1} \in L_q(\Omega)$, $q \in (0, 1)$.

In addition, for all $\eta \in L_2(\Omega_T)$ and $T < \infty$ holds

$$\begin{aligned} (K_0 \nabla \varphi, \nabla \varphi)_{\Omega} &\leq M_0(T) < \infty, \\ \varepsilon_{\eta} (|\eta|, |\operatorname{div}(K_0 \nabla \varphi)|)_{\Omega_T} &\leq \sqrt{\varepsilon} M(T) \|\eta\|_{2, \Omega_T}. \end{aligned} \quad (27)$$

Proof. Substituting $t \rightarrow T-t$ from (23) results in the problem

$$L\varphi \equiv \varphi_t - A \operatorname{div}(K_0 \nabla \varphi) - B \nabla \varphi - \lambda A \varphi = 0,$$

$$\varphi = 0, \quad (x, t) \in \Gamma_{1T}, \quad K_0 \nabla \varphi \cdot \vec{n} = 0, \quad (x, t) \in \Gamma_{2T}, \quad \varphi(x, 0) = \varphi_0(x) \geq 0.$$

Its unique solvability in a class of functions $C^{2+\alpha, 1+\alpha/2}(\Omega_T)$ under assumption from (23) was proved in [8], [9].

It will be shown that $\varphi \geq 0$. Determine the function $u_1(x, t)$ by the equality $\varphi = \exp(\delta t + \beta) u_1$, where β is the solution of the problem

$$\operatorname{div}(K_0 \nabla \beta) = \gamma_1, \quad K_0 \nabla \beta \cdot \vec{n} = \gamma, \quad x \in \Gamma, \quad \gamma_1 \operatorname{meas} \Omega = \gamma \operatorname{meas} \Gamma$$

with constants $\gamma_1, \gamma, \delta > 0$. According to the maximum principle for solutions of parabolic equations [8] may be shown that $u_1 \geq 0$

in Ω for sufficiently large values of δ and, therefore, $\varphi \geq 0$.

Introduce the function

$$\varphi_* = \exp(\alpha_1 \omega) - \exp(\alpha_1 R),$$

where ω satisfies (24) and the constant $R > 0$. It is obvious that

$$L\chi + \delta \chi = \exp(-\delta t) L\varphi_*, \quad \chi = \exp(-\delta t)(\varphi + \varphi_*).$$

Let us prove that for sufficiently large values of δ, α_1, R the function $\chi \leq 0$, $(x, t) \in \Omega_T$. Calculate $L\varphi_*$ and estimate the above expression

$$\begin{aligned} \exp(-\alpha_1 \omega) L\varphi_* &\leq -A (\alpha_1^2 + \alpha_1 \frac{B}{A} \nabla \omega + \alpha_1^2 K_0 \nabla \omega \cdot \nabla \omega - \lambda \exp(\alpha_1 (R - \omega))) \leq \\ &\leq -A (\alpha_1^2 - \frac{1}{2} M M_1^2 - 1) < 0, \quad \alpha_1^2 > 1 + \frac{1}{2} M M_1^2. \end{aligned}$$

In this estimate the Cauchy inequality was used and it was assumed that

$$\lambda \leq \lambda_0 \equiv \exp(-\alpha_1 h), \quad h \geq R - \min_{\Omega} \omega.$$

Thus, if $\delta > \lambda_0 \|A\|_{C(\Omega_T)}$, there is no the positive maximum of

χ inside Ω_T . For $t = 0$ and $R = \alpha_1^{-1} \ln(\|\varphi_0\|_{C(\Omega)} + 1)$ the function $\chi < 0$. Since $\omega = 0$ on Γ_1 , then $\chi < 0$ on Γ_{1T} . As $K_0 \nabla \chi \cdot \vec{n} < 0$ on Γ_{2T} , on this part of the boundary the function χ has no maximum. The first estimate of lemma has been proved.

The right-hand side inequality in (26) follows from the fact that $\varphi \geq 0$ and $\varphi = 0$ on Γ_{1T} .

Let us prove that under condition (25) the maximum of the function $\varphi + \varphi_*$ is attained on Γ_{1T} . Actually, since

$$L(\varphi + \varphi_*) = L\varphi < 0, \quad K_0 \nabla(\varphi + \varphi_*) \cdot \vec{n} < 0, \quad (x, t) \in \Gamma_{2T}$$

the function $\varphi + \varphi_*$ has the negative maximum at $t = 0$ or on Γ_{2T} . From (25) it follows that $\varphi_0 \leq 1 - \exp(\alpha_1 \omega)$, hence $(\varphi + \varphi_*)|_{t=0} \leq -\|\varphi_0\|_{C(\Omega)} = (\varphi + \varphi_*)|_{\Gamma_{1T}}$.

Thus, the function $\varphi + \varphi_*$ has the maximum value on Γ_{1T} . Since this value is independent of $(x, t) \in \Gamma_{1T}$, we have $K_0 \nabla(\varphi + \varphi_*) \cdot \vec{n} \geq 0$ for $(x, t) \in \Gamma_{1T}$. This proves the estimate (26).

The estimates (27) may be received analogously [2].

Equation (24) multiplied by $(-\omega)^{1-q}$ and integrated over Ω

$$(1-q)(K_0 \nabla \omega \cdot \nabla \omega, (-\omega)^{1-q})_{\Omega} = \alpha_1 (1, (-\omega)^{1-q})_{\Omega} \cup \Gamma_2.$$

It follows from (24) that $\omega < 0$ inside Ω . Further, the boundary condition on Γ_2 and the Zaremba-Giraud principle shows that $|\nabla \omega| \neq 0$ on Γ . Hence the summation of the function $(-\omega)^{-q}$, $q \in (0,1)$ follows from the latter equality.

Thus, the function $\varphi_0(x)$ can be chosen such that it satisfies (25) and $\varphi_0^{-1} \in L_q(\Omega)$.

Let us define

$$W \equiv \int_s^1 \alpha(\xi) d\xi.$$

Let

$$W(s^0(x,t)) \in L_1(\Gamma_1 \times (0,\infty)), Q_1 - R_1, Q - R \in L_1(\Gamma_1 \times (0,\infty)), R = R_1,$$

$$\lim_{t \rightarrow \infty} \|Q - R\|_{1, \Gamma_2} = 0, \left| \left(\frac{\partial f_0}{\partial s}, \frac{\partial f}{\partial s}, \frac{\partial K_2}{\partial s} \right) \right| / \alpha \leq N_0, |p|, |\nabla p| \leq N_1, \quad (28)$$

$$s(x,t) \in C^{\alpha, \alpha/2}(\Omega_T), K_0(x) \in C^{\alpha}(\Omega), f, f_0 \in C^{\alpha}(\Omega) \times C[0,1], \alpha \in C[0,1],$$

and there exist such functions f, g that

$$W(s) \geq f(1-s), \frac{d^2 f(y)}{dy^2} \geq 0, x f(y/x) \geq g(y), x \in [0, N_2], \quad (29)$$

where $N_i, i=0,1,2$ are the constant and the function $g(y)$ satisfies conditions of lemma 1.

Theorem 3. Let conditions (2), (28), (29) be fulfilled. Then for any generalized solutions of problems (1)-(5) and (20), (21) the following equality holds

$$\lim_{t \rightarrow \infty} (\|1-s\|_{1, \Omega} + \|p-p_0\|_{2, \Omega} + \|\nabla(p-p_0)\|_{2, \Omega}) = 0.$$

Proof. Differentiation of identity (6) in t and subsequent subtraction of (18) give the following equality for almost all $t \in (0, T)$

$$\frac{d}{dt} (m\varphi, 1-s)_{\Omega} - (m\varphi_t, 1-s)_{\Omega} + (K_0 \nabla W(s), \nabla \varphi)_{\Omega} - (Q_1 - R_1, \varphi)_{\Omega} + \quad (30)$$

$$(\vec{f}_0(x,1) - \vec{f}_0(x,s), \nabla \varphi)_{\Omega} + ((K_1(x,1) - K_1(x,s)) \nabla p + K_1(x,1) \nabla(p-p_0), \nabla \varphi)_{\Omega} = 0.$$

Here we assume that $S_0 = 1$.

Analogously

$$((K(x,1) - K(x,s)) \nabla p + K(x,1) \nabla(p_0 - p) + (\vec{f}(x,1) - \vec{f}(x,s), \nabla \psi)_{\Omega} + (R - Q, \psi)_{\Gamma_2} = 0. \quad (31)$$

Let $\varphi = \psi$ and $K_0 \nabla \varphi \cdot \vec{n} = 0$ on Γ_{2T} . Then subtraction of (31) from (30) with the allowance for (22) and integration by parts in the third term of (30) give the equality

$$\frac{d}{dt} (m\varphi, 1-s)_{\Omega} = (m(1-s), \varphi_t) + A \operatorname{div}(K_0 \nabla \varphi) + \vec{B} \nabla \varphi)_{\Omega} + (Q_1 - R_1, \varphi)_{\Gamma_2} +$$

$$(R - Q, \varphi)_{\Gamma_2} + (K_0 \nabla \varphi \cdot \vec{n}, W(s))_{\Gamma_1} - \varepsilon (m(1-s), \operatorname{div}(K_0 \nabla \varphi))_{\Omega},$$

$$A = W(s)/m(1-s) + \varepsilon, \varepsilon > 0, K_2(x,s) = K(x,s) - K_1(x,s),$$

$$\vec{B} = ((K_2(x,1) - K_2(x,s)) \nabla p + \vec{f}(x,1) - \vec{f}(x,s) + \vec{f}_0(x,s) - \vec{f}_0(x,1))/m(1-s).$$

The function \vec{B} is approximated by \vec{B}_y at substituting ∇p by its smooth averaging ∇p_y . The solution of problem (23) with the coefficients A, \vec{B}_y is substituted into the latter equality for φ . Then, using the results of lemma 2, we estimate the integrals in the right-hand side of the obtained equality. As a result, we derive the inequality

$$\frac{dy}{dt} + \lambda M^{-1} (m\varphi, W(s))_{\Omega} \leq A_1(t), \quad (32)$$

where

$$A_1 = N_3 (\|Q_1 - R_1\|_{1, \Gamma_2} + \|Q - R\|_{1, \Gamma_2} + \|W(s^0)\|_{1, \Gamma_1}) + M \|\vec{B} - \vec{B}_y\|_{2, \Omega} \|\nabla \varphi\|_{2, \Omega} + \sqrt{\varepsilon} M M_0(T), y \equiv (m\varphi, 1-s)_{\Omega}, \lambda \leq \lambda_0, N_3 = \text{const}.$$

From (29), (32) and the Jensen inequality for the function $f(y)$ it follows that

$$\frac{dy}{dt} + \lambda M^{-1} g(y) \leq A_1(t),$$

$$0 < y_0 \leq z_0 = M \operatorname{meas} \Omega (\|\varphi_0\|_{C(\Omega)} + 1).$$

Taking into account estimates of lemma 2 in coefficient A_1 the limiting transition in parameters ν, ε are made. After it according to lemma 1 we get

$$y(T) = (\pi\varphi_0, 1-s(x, T))_{\Omega} \leq E(T).$$

From the Holder inequality and the latter estimate at $\varphi_0^{-1} \in L_q(\Omega)$ it follows that

$$\|1-s(x, T)\|_{1, \Omega} \leq \left[M \|\varphi_0^{-1}\|_{q, \Omega} E(T) \right]^{\frac{q}{q+1}} \rightarrow 0, \quad T \rightarrow \infty.$$

Assume that $\Phi = p_0 - p$ in (31). Using the Cauchy inequality and the fact that $|p|, |\nabla p|$ are bounded uniformly, we have

$$\|\nabla(p_0 - p)\|_{2, \Omega}^2 \leq N_4 (\|1-s\|_{1, \Omega} + \|Q-R\|_{1, \Gamma_2}).$$

If $\operatorname{meas} \Gamma_1 > 0$, then $\|p - p_0\|_{2, \Omega} \leq N_5 \| \nabla(p - p_0) \|_{2, \Omega}$. Here N_4, N_5 are constant independent on t . Theorem has been proved.

Remark 1. Conditions of Lemma 1 and (29) are fulfilled, for example, by such functions $g(y), f(y)$ that

$$g, f \sim cy^{r_0}, \quad r_0 \geq 2, \quad r_1 \int_0^y \exp\left(-\frac{r_2}{\xi}\right) d\xi, \quad r_3 \int_0^y \left(\ln\left(\frac{r_4}{\xi}\right)\right)^{-1} d\xi, \quad y > 0,$$

where $r_i, i = 0, 1, 2, 3, 4$, are constant

A behaviour of the coefficient

$$a(s) \sim (1-s)^{r_0-1}, \quad \exp\left(-\frac{1}{1-s}\right), \quad |\ln(1-s)|^{-1}, \quad s > 1$$

corresponds to the mentioned functions.

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