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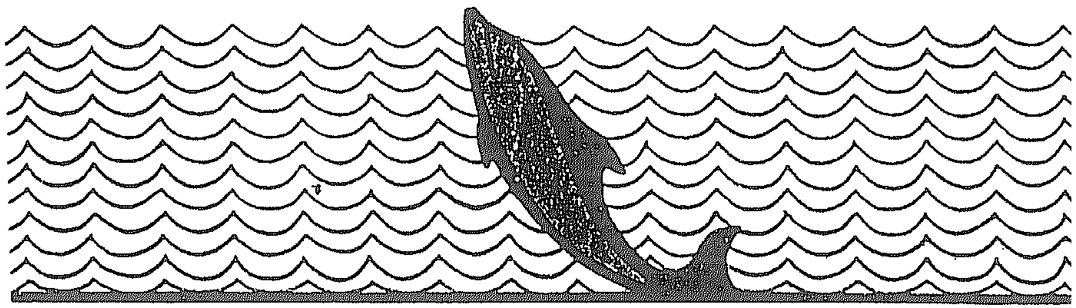

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CONTINUOUS DEPENDENCE & STABILIZATION OF SOLUTIONS OF THE DEGENERATE SYSTEM IN TWO-PHASE FILTRATION

S.N.Antontsev, A.V.Domansky
Lavrentiev Institute of Hydrodynamics
Russian Academy of Sciences
Siberian Division
Novosibirsk 630090, RUSSIA
J.I.Diaz
Universidad Complutense de Madrid
Departamento de Matematica Aplicada 28040-Madrid SPAIN

The filtration of two immiscible fluids in a heterogeneous anisotropic porous medium is described by a system of nonlinear differential equations including an eventual degenerate parabolic equation for the saturation $s(x,t)$ of one of the phases and an elliptic equation for the "reduced" pressure $p(x,t)$:

$$m(x)\frac{\partial s}{\partial t} = div\,(K_0(x)a(s)\nabla s + K_1(x,s)\nabla p + \vec{f}_0(x,s)), \qquad (1)$$

$$div(K(x,s)\nabla p + \vec{f}(x,s)) = 0, \qquad (2)$$

$$x \in \Omega \in R^n, \quad n = 1,2,3, \qquad (x,t) \in \Omega_T = \Omega \times (0,T),$$

where Ω is a bounded and multiple-connected domain with the boundary Γ. The derivation of such a system and the meaning of each of the coefficients can be found in [1].

For system (1), (2) we consider the following initial-boundary problem

$$s(x,t) = s^0(x,t),\ p(x,t) = p^0(x,t),\quad (x,t) \in \Gamma_{1T} \equiv \Gamma_1 \times (0,T), \tag{3}$$

$$(K \nabla p + \vec{f})\,\vec{n} = -\,Q(x,t),\ (K_0 a \nabla s + K_1 \nabla p + \vec{f}_0)\,\vec{n} = -\,Q_1(x,t), \tag{4}$$

$$(x,t) \in \Gamma_{2T} = \Gamma_2 \times (0,T),$$

$$s(x,0) = s^0(x,0), \qquad x \in \Omega. \tag{5}$$

Here $\Gamma = \Gamma_1 + \Gamma_2$; $\vec{n}$ is the unit outward normal vector to Γ; p^0, s^0, Q, Q_1 are the prescribed functions.

The solvability of problem (1)-(5) has been considered by numerous authors: see references in the monograph [1].

The uniqueness of generalized solutions of this problem was proved in [2] and in [3] for $\Gamma = \Gamma_1$.

This paper presents a complete account of the brief article [4].

1. Continuous dependence result.

In [1] the proof is made for existence of generalized solution of the problem (1)-(5) in the following sense:

$s(x,t)$, $p(x,t)$ are the bounded measurable functions, $s(x,t) \in [0,1]$, $p(x,t) \in L_\infty(\Omega_T) \cap L_\infty(0,T;\ W_2^1(\Omega))$,

$$u(x,t) \equiv \int_0^{s(x,t)} \sqrt{a(\xi)}\, d\xi \in L_2(0,T;\ W_2^1(\Omega))$$

satisfy (3) in the sense of traces from $W_2^1(\Omega)$;

for all $\varphi(x,t)$, $\psi(x,t)$, $(\varphi,\ \psi) \in W_2^1(\Omega_T)$, $\varphi(x,t) = \psi(x,t) = 0$, $(x,t) \in \Gamma_{1T}$, $\varphi(x,T) = 0$, $x \in \Omega$ and almost all $t \in (0,T)$ the following integral identities are fulfilled:

$$-(ms,\varphi)\Big|_{t=0}^{t=t} + (ms,\varphi_t)_{\Omega_t} - (K_0 a \nabla s + K_1 \nabla p + \vec{f}_0,\ \nabla\varphi)_{\Omega_t} - (Q_1,\varphi)_{\Gamma_{2t}} = 0, \tag{6}$$

$$(K \nabla p + \vec{f},\ \nabla\psi)_\Omega + (Q,\ \psi)_{\Gamma_2} = 0, \tag{7}$$

where

$(u,v)_\Omega = \int_\Omega uv\,dx$, $(u,v)_{\Omega_0} = (u(0),v(0))_\Omega$, $\Omega_t = \Omega\times(0,t)$, $\Gamma_{2t} = \Gamma_2 \times (0,t)$.

In the present article we shall assume the extra condition

$$\nabla p \in L_\infty(\Omega_T)$$

which has been obtained in [1] under some additional conditions.

Introduce the function

$$c(s) \equiv \int_0^s a(\xi)d\xi.$$

Let $w^i = (s^i,p^i)$, $i = 1,2$ denote any generalized solutions of problem (1) - (5) associated to the data a^i, c^i, m^i, s_0^i, p_0^i, $\vec{f}_0^i$, $\vec{f}^i$, K^i, K_0^i, K_1^i, Q^i, Q_1^i. One of the key assumptions of our results is the one concerning the diffusion coefficients and the boundary data on Γ_1; it is stated as the following alternative:

$$|c^1(s^2) - c^2(s^2)|^q \leqslant M \int_0^1 a^1(\tau s^1 + (1-\tau)\, s^2)\, d\tau$$

for some $M>0$ and $q \in (0,2)$, and (8)

$$c^1(s_0^1) = c^2(s_0^2),\quad (x,t) \in \Gamma_{1T},$$

either there exist $M>0$ such that

$$a^1(s) \leqslant M\, a^2(s),\quad \forall\, s \in [0,1] \quad \text{and} \quad s_0^1 = s_0^2,\ (x,t) \in \Gamma_{1T}. \tag{8'}$$

Theorem 1. Assume, that conditions (8) or (8') holds. Assume also, that there exist $M > 0$ such that

$$M^{-1} \leqslant (m^1,\ (K_0^1\, \vec{\xi},\ \vec{\xi})/|\vec{\xi}|^2\ ) \leqslant M, \qquad (|\vec{f}^1|,\ |\vec{f}_0^1|) \leqslant M,$$

$$\left|\frac{\partial}{\partial s}(\ K_1^1,\ K^1,\ \vec{f}^1,\ \vec{f}_0^1)/\sqrt{a^1}\ \right| \leqslant M,\ 0 \leqslant a^j \in C[0,1],\ j = 1,2, \tag{9}$$

$$K_0^1(x) \in C(\Omega),\ \text{meas}\ \Gamma_1 > 0.$$

then for $\omega^i = (s^i,p^i)$ with $|\nabla p^i| \leqslant M$ the following estimate holds

$$(s^1 - s^2\ ,\ c^1(s^1) - c^2(s^2))_{\Omega_T} + \|p^1 - p^2\|_{2,\Omega_T} \leqslant C_1(M,T)\, H, \tag{10}$$

where

$$H = \|m^1 - m^2\|_{2,\Omega}^2 + \|s_0^1 - s_0^2\|_{2,\Gamma_{1T}\cup\Omega} + \|p_0^1 - p_0^2\|_{2,\Gamma_{1T}} + \|Q^1 - Q^2\|_{2,\Gamma_{2T}}$$
$$+ \|Q_1^1 - Q_1^2\|_{2,\Gamma_{2T}} + \|\vec{f}^1 - \vec{f}^2\|_{\infty,v} + \|\vec{f}_0^1 - \vec{f}_0^2\|_{\infty,v} + \|K_0^1 - K_0^2\|_{\infty,v}$$
$$+ \|K_1^1 - K_1^2\|_{\infty,v} + \|K^1 - K^2\|_{\infty,v} + (\|a^1 - a^2\|_{C[0,1]})^{1-q/2}.$$

Here $V = [0,1]\times\Omega$ and $q=1$ if (8') takes place.

Proof. From the notion of generalized solution we deduce, after integration by parts, that

$$\begin{aligned}
&(m^1(s^1 - s^2),\varphi_t)_{\Omega_T} + (c^1(s^1) - c^1(s^2), div(K_0^1\nabla\varphi))_{\Omega_T} + \\
&(K_1^1(x,s^2)\nabla p^2 - K_1^1(x,s^1)\nabla p^1 + \vec f_0^1(x,s^2) - \vec f_0^1(x,s^1),\nabla\varphi)_{\Omega_T} = \\
&-(c^1(s^2) - c^2(s^2),\ div(K^1\nabla\varphi))_{\Omega_T} + (c^1(s^1) - c^2(s^2),K^1\nabla\varphi\cdot\vec n)_{\Gamma_{1T}} - \\
&((m^1 - m^2)s^2,\varphi_t)_{\Omega_T} + ((K_0^1 - K_0^2)\nabla c^2(s^2),\nabla\varphi)_{\Omega_T} + \\
&((K_1^1(x,s^2) - K_1^2(x,s^2)\nabla p^2,\nabla\varphi)_{\Omega_T} + ((\vec f_0^1\ (x,s^2) - \vec f_0^2(x,s^2),\nabla\varphi)_{\Omega_T} + \\
&((m^1 - m^2)s_0^1,\varphi)_{\Omega_0} - (m^1(s_0^1 - s_0^2),\varphi)_{\Omega_0} + (Q^1 - Q^2,\varphi)_{\Gamma_{2T}} \equiv E(\varphi),
\end{aligned}\qquad(11)$$

Analogously,

$$\begin{aligned}
&(K^1(x,s^1)\nabla p^1 - K^1(x,s^2)\nabla p^2 + \vec f^1(x,s^1) - \vec f^1(x,s^2),\nabla\psi)_\Omega = \\
&(Q^1-Q^2,\psi)_{\Gamma_2} - ((K^1(x,s^2)-K^2(x,s^2))\nabla p^2+\vec f^1(x,s^2)-\vec f^2(x,s^2),\nabla\psi)_\Omega \equiv \\
&\equiv F(t/\psi).
\end{aligned}\qquad(12)$$

As in [2] we integrate (12) on (0,t) and sum up it with (11); then, after substitution t → T-t, we obtain the following identities for the functions $s = s^1 - s^2$, $p = p^1 - p^2$:

$$(m^1 s, L_1(\varphi,\psi))_{\Omega_T} - (p, L_2(\varphi,\psi))_{\Omega_T} = $$
$$= -\varepsilon(m^1 s; div(K_0^1\nabla\varphi))_{\Omega_T} + E(\varphi) - \int_0^T F(\tau/\psi)d\tau, \qquad (13)$$

where for $\varepsilon>0$ the operators L_1 and L_2 are defined by

$$L_1(\varphi,\psi) \equiv -\frac{\partial\varphi}{\partial t} + A(x,t)div(K_0^1\nabla\varphi) + A_1(x,t)\nabla\varphi + C_1(x,t)\nabla\psi,$$
$$L_2(\varphi,\psi) \equiv div(E(x,t)\nabla\psi - D(x,t)\nabla\varphi),$$

and the coefficients and the boundary conditions are given by

$$A = (c^1(s^1) - c^1(s^2))/m^1(s^1 - s^2) + \varepsilon \equiv A_0 + \varepsilon,\ \varepsilon > 0,$$
$$-A_1 = \nabla p^1((K_1^1(x,s^1) - K_1^1(x,s^2) + \vec f_0^1(x,s^1) - \vec f_0^1(x,s^2))/m^1(s^1 - s^2),$$
$$-C_1 = \nabla p^1(K^1(x,s^1) - K^1(x,s^2) + \vec f^1(x,s^1) - \vec f^1(x,s^2))/m^1(s^1 - s^2),$$

$$E = K^1(x,s^2),\ D = K_1^1(x,s^2).$$
$$\varphi\Big|_{\Gamma_{1T}} = \psi\Big|_{\Gamma_{1T}} = K_0^1\nabla\varphi\cdot\vec n\Big|_{\Gamma_{2T}} = K_0^1\nabla\psi\cdot\vec n\Big|_{\Gamma_{2T}} = \varphi\Big|_{t=0} = 0. \qquad (14)$$

Now let us assume that

$$L_1(\varphi,\psi) = h(x,t), \qquad L_2(\varphi,\psi) = g(x,t) \qquad (15)$$

and consider initial boundary-value problem (14)-(15). It is possible to show that this problem has a unique solution when $\varepsilon > 0$ and conditions of Theorem 1 takes place. As in [2] it may be shown that if there exist $M_0 > 0$ such that

$$(|g|,(|h| + |A_1| + |C_1|)/\sqrt{A_0}) \leqslant M_0,$$

then for its solution the following estimate holds

$$\|\nabla\varphi\|^2_{2,\infty,\Omega} + \|\nabla\psi\|^2_{2,\infty,\Omega} + \|\varphi_t\|^2_{2,\Omega} + \varepsilon\|div(K_0^1(\nabla\varphi)\|^2_{2,\Omega} \leqslant C_2(M,M_0,T). \qquad (16)$$

With $h = (1-\int_{\Gamma_{1T}}(s_0^1-s_0^2)d\Gamma)(c^1(s^2)-c^1(s^2))$, $g = -(p-\int_{\Gamma_{1T}}(p_0^1-p_0^2)d\Gamma)$

the equality (13) takes the form

$$\begin{aligned}
&(m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T} + \|p\|^2_{2,\Omega_T} = E(\varphi) + \int_0^T F(t/\Psi)\,dt + (p,1)_{\Omega_T}\times \\
&\int_{\Gamma_{1T}}(p_0^1 - p_0^2)d\Gamma + (m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T}\int_{\Gamma_{1T}}(s_0^1 - s_0^2)d\Gamma - \\
&(m^1 s,\varepsilon\ div(K_0^1\nabla\varphi))_{\Omega_T} \equiv E(\varphi) + \int_0^T F(t/\psi)\,dt + J_1 + J_2 + J_3.
\end{aligned}$$

Taking into account the assumptions of Theorem 1 and (6), the last three terms in the right-hand part of (17) may be estimated as follows

$$|J_1| \leqslant \frac{\delta}{2}\|p\|^2_{2,\Omega_T} + \frac{1}{2\delta}\|p_0^1 - p_0^2\|^2_{2,\Gamma_{1T}}, \qquad \delta > 0$$
$$|J_2| \leqslant \frac{\delta}{2}(m^1 s, c^1(s^1) - c^1(s^2))_{\Omega_T} + C_3(\delta,M)\|s_0^1 - s_0^2\|^2_{2,\Gamma_{1T}}$$
$$|J_3| \leqslant \sqrt{\varepsilon}\,C_4(M_0,T).$$

Assume that condition (8) is valid. Then the second term in the right-hand part of (11) is equal to zero and on the basis of the method suggested in [5] the first term is estimated as follows

$$|(c^1(s^2) - c^2(s^2), div(K_0^1\nabla\varphi)_{\Omega_T}| \leqslant$$

$$\leqslant \Big(\frac{|c^1(s^2)-c^2(s^2)|}{\sqrt{A}}^{q/2} |c^1(s^2)-c^2(s^2)|^{1-q/2} \cdot |\sqrt{A}\, div(K_0^1\nabla\varphi)|\Big)_{\Omega_T} \leqslant$$

$$\leqslant (\|a^1 - a^2\|_{C[0,1]})^{1-q/2}\, C_2^{1/2}\, M^{1/2} (meas\,\Omega_T)^{1/2}.$$

For the case of (8') the sum of the first two terms in $E(\varphi)$ may be represented as

$$(\nabla(c^1(s^2) - c^2(s^2)), K_0^1\nabla\varphi)_{\Omega_T} = J.$$

From the facts that

$$u^i = \int_0^{s^i} \sqrt{a^i(\xi)}\, d\xi \in L_2(0,T;W_2^1(\Omega))$$

and

$$|\nabla c^1(s^2)-\nabla c^2(s^2)| = \frac{\sqrt{|a^1(s^2)-a^2(s^2)|}}{\sqrt{a^2(s^2)}}\, |\nabla u^2|\, \sqrt{|a^1(s^2)-a^2(s^2)|}$$

we obtain

$$|J| \leqslant \tilde{C}_5(M)\, (\|a^1 - a^2\|_{C[0,1]})^{1/2}\, \|\nabla u^2\|_{2,\Omega_T} \|\nabla\varphi\|_{2,\Omega_T}.$$

The other terms in $E(\varphi)$ and $F(t/\psi)$ are estimated using (16). The obtained estimates make it possible to go over to the limit when $\varepsilon\to 0$ in (19). Theorem 1 is proved.

Let us study now stability of $s(x,t)$ in $\|\cdot\|_{\Omega}$ with respect only to the variation of the initial data (5).

Let two solutions ω^i, $i=1,2$ of the problem (1)-(5) differs only by the initial data (5) and s,p are defined analogously theorem 1.

Theorem 2. Let ω^i, $i=1,2$, be generalized solutions of the problems (1)-(5), differing only by the initial data, $m = 1$, $K_0^i = (\delta_{ij})$ and the conditions of theorem 1 hold. Then there hold estimate

$$\|s^1(x,t) - s^2(x,t)\|_{H^{-1}(\Omega)} \leqslant C\, \|s^1(x,0) - s^2(x,0)\|_{H^{-1}(\Omega)} \qquad \forall\, t \leqslant T.$$

Proof. Analogously theorem 1 we deduce equality

$$(s,\varphi)_{\Omega_t}\Big|_{t=0}^{t=t} + (s, L_1(\varphi,\psi))_{\Omega_t} - (p, L_2(\varphi,\psi))_{\Omega_t} = -\varepsilon(s, div(K_0^1\nabla\varphi))_{\Omega_t}, \quad (16)$$

where L_1, L_2 are defined as in theorem 1.

Consider initial-boundary problem (14)-(15) with

$$h = 0, \quad g = 0, \quad \varphi(x,0) = \varphi_0(x).$$

Similarly [2] may be shown that for this conditions the next estimates takes place (after returning to the starting time $t\to T-t$)

$$\|\nabla\varphi(x,t)\|^2_{2,\Omega} \leqslant C\, \|\nabla\varphi(x,T)\|^2_{2,\Omega}, \qquad \forall\, t \leqslant T.$$

Letting $\varepsilon\to 0$ in (16) we get the equality

$$(s(x,0),\varphi(x,0))_{\Omega} = (s(x,T),\varphi(x,T))_{\Omega}.$$

Let Δ denotes the Laplacian and $g_k(x)$ $k=1,2,\dots$ be the solutions of the problems

$$(\Delta + \lambda_k)\, g_k = 0, \quad x \in \Omega\,; \qquad (g_k, g_j) = \delta_{kj},$$
$$g_k\Big|_{\Gamma_1} = 0, \qquad \nabla g_k \cdot \vec{n}\Big|_{\Gamma_2} = 0. \tag{17}$$

Then any function $f \in L_2(\Omega)$ may be represented in the form

$$f(x) = \sum_{k=1}^{\infty} f_k\, g_k(x), \qquad f_k = (f, g_k)_{\Omega}.$$

Let Δ^{-1} denotes operator inverse to Δ under the boundary conditions (17). Then there valid the formulas

$$-(s,\Delta^{-1}s)_{\Omega} = \sum_{k=1}^{\infty}\frac{s_k^2}{\lambda_k} = \|s\|^2_{H^{-1}(\Omega)} = \|\nabla(\Delta^{-1}s)\|^2_{2,\Omega}.$$

Indeed,

$$-(s,\Delta^{-1}s)_{\Omega} = -\Big(\sum_{k=1}^{\infty} s_k g_k, \sum_{l=1}^{\infty} s_l \Delta^{-1} g_l\Big)_{\Omega} = \Big(\sum_{k=1}^{\infty} s_k g_k, \sum_{l=1}^{\infty}\frac{1}{\lambda_l} s_l g_l\Big)_{\Omega} =$$
$$= \sum_{k=1}^{\infty}\frac{s_k^2}{\lambda_k},$$

$$\|s\|^2_{H^{-1}(\Omega)} = \sup_{\|\nabla h\|_{2,\Omega}\le 1} |(s,h)_\Omega|^2 = \sup_{\|\nabla h\|_{2,\Omega}\le 1} \Big|\sum_{k=1}^{\infty} \frac{s_k}{\sqrt{\lambda_k}}\sqrt{\lambda_k}\, h_k\Big|^2 \le \sum_{k=1}^{\infty}\frac{s_k^2}{\lambda_k}\sum_{k=1}^{\infty}\lambda_k h_k^2 \le \sum_{k=1}^{\infty}\frac{s_k^2}{\lambda_k},$$

$$\|\nabla h\|^2_{2,\Omega} = (\nabla h,\nabla h)_\Omega = -(h,\Delta h)_\Omega = \sum_{k=1}^{\infty}\lambda_k h_k^2 \le 1,$$

$$\|\nabla(\Delta^{-1} s)\|^2_{2,\Omega} = (\nabla\Delta^{-1}s,\nabla\Delta^{-1}s)_\Omega = -(\Delta\Delta^{-1}s,\Delta^{-1}s)_\Omega = \|s\|^2_{H^{-1}(\Omega)}.$$

Let us set now

$$\varphi(x,T) = \Delta^{-1}s(x,T).$$

Then, taking into account the last formulas we obtain the following chain of inequalities

$$(s(x,T),\varphi(x,T))_\Omega = (s(x,T),\Delta^{-1}s(x,T))_\Omega = \|s(x,T)\|^2_{H^{-1}(\Omega)} = |(s(x,0),\varphi(x,0)_\Omega|$$

$$\le C\|s(x,0)\|_{H^{-1}(\Omega)}\|\nabla\varphi(x,T)\|_{2,\Omega} = C\,\|s(x,0)\|_{H^{-1}(\Omega)}\|s(x,T)\|_{H^{-1}(\Omega)}.$$

Theorem 2 is proved.

2. Stabilization of generalized solutions.

Consider stationary solution of problem (1)-(5). This solution $(s_0(x),\ p_0(x))$ for all $(\varphi,\psi)\in W_2^1(\Omega)$, $\varphi = 0$, $\psi = 0$, $x\in\Gamma_1$ satisfy the integral identities

$$(K_0(x)a(s_0)\nabla s_0 + K_1(x,s_0)\nabla p_0 + \vec f_0(x,s_0),\nabla\varphi)_\Omega - (R_1,\varphi)_{\Gamma_2} = 0, \qquad (18)$$

$$(K(x,s_0)\nabla p_0 + \vec f(x,s_0),\nabla\psi)_\Omega + (R,\psi)_{\Gamma_2} = 0, \qquad (19)$$

and boundary conditions

$$s_0(x) = 1, \qquad p_0(x) = p^0(x), \qquad x\in\Gamma_1$$

in the sense of traces from $W_2^1(\Omega)$. The functions R, R_1 are analogies of the functions Q, Q_1 in (4).

If $R = R_1$, there exists the solution in the form

$$s_0(x)\equiv 1, \qquad x\in\Omega, \qquad (20)$$

and function p_0 is a generalized solution of the boundary-value

problem

$$div(K(x,1)\nabla p_0 + \vec f(x,1)) = 0,$$

$$(K(x,1)\nabla p_0 + \vec f(x,1))\cdot\vec n = -R, \qquad x\in\Gamma_2, \qquad (21)$$

$$p_0 = p^0(x), \qquad x\in\Gamma_1,$$

since the coefficients equations of system (1)-(2) have the properties [1, p.212]

$$K_1(x,1) = K(x,1), \qquad \vec f(x,1) = \vec f_0(x,1). \qquad (22)$$

Note 1. The solution of (20),(21) corresponds in terms of physics to the case of a complete displacement of oil initially occupying a part of the volume Ω, by water.

Here it is proved that the generalized solution (s,p) of degenerate problem (1)-(5) is converged to the solution $(1,p_0)$ from (20)-(21) when t infinitely increasing.

Lemma 1. Let the non-negative functions $A_1(t)\in L_1(0,\infty)$ and $y(t)$ be connected by the differential inequality

$$\frac{dy}{dt} + g(y) \le A_1(t), \qquad y(0) = y_0 > 0,$$

where the function $g(z)$ is such that

$$g(z)\ge 0,\ z\ge 0,\ \frac{dH}{dz} \equiv \frac{d}{dz}\Big(\frac{g(z)}{z}\Big) > 0,\ \int_z^{z_0}\frac{d\xi}{g(\xi)} \equiv \Phi(z)\to\infty,\ z\to 0,\ z_0\le y_0.$$

Then

$$y \le \Phi^{-1}(t)\Big(1 + \int_0^t \frac{A_1(\tau)}{\Phi^{-1}(\tau)}d\tau\Big) \equiv E(t)\to 0,\ t\to\infty.$$

where Φ^{-1} is the symbol of the inverse function to Φ.

The proof of Lemma 1 may be found in [6].

Consider the problem

$$\frac{d\varphi}{dt} + A(x,t)\,div(K_0\nabla\varphi) + \vec B(x,t)\,\nabla\varphi = -\lambda A\varphi, \qquad (23)$$

$$\varphi = 0,\ (x,t)\in\Gamma_{1T},\ K_0\nabla\varphi\cdot\vec n = 0,\ (x,t)\in\Gamma_{2T},\ \varphi(x,T) = \varphi_0(x)\ge 0,$$

$$\varphi_0\in C^{2+\alpha}(\Omega),\ A,\vec B\in C^{\alpha,\alpha/2}(\Omega_T),\ \alpha\in(0,1),\ A = A_0(x,t) + \varepsilon \ge \varepsilon > 0.$$

The function φ_0 satisfies the conditions of zero and first order compatibility.

Let the function $\omega(x)$ be the solution of problem

$$div(K\ \nabla\omega) = \alpha_1, \quad x \in \Omega,\ \alpha > 0, \qquad (24)$$
$$\omega = 0, \quad x \in \Gamma_1, \quad K_0\nabla\omega\cdot\vec{n} = -\alpha_1, \quad x \in \Gamma_2.$$

By the maximum principle $\omega \leqslant 0$, $x \in \Omega$ [7].

Lemma 2. Assume that $|\vec{B}|/A \leqslant M_1$, where the constant M_1 is independent of t, x, ε, meas $\Gamma_1 > 0$. Then there exist a number $\lambda_0 > 0$, such that for all $\lambda \leqslant \lambda_0$ any solution of problem (23) satisfies the maximum principle in the form

$$0 \leqslant \varphi(x,t) \leqslant \|\varphi_0\|_{C(\Omega)} + 1.$$

If the function $\varphi_0(x)$ satisfies the inequality

$$\varphi_0 \leqslant \omega\ (1-exp(\alpha_1 \min_{\Omega} \omega))/\min_{\Omega} \omega, \qquad (25)$$

where ω is taken from (24), then for $\lambda \leqslant \lambda_0$ next estimate is held

$$-\ \alpha_1 exp(\alpha_1\omega)\ K_0\nabla\omega\cdot\vec{n} \leqslant K_0\nabla\varphi\cdot\vec{n} \leqslant 0, \qquad (26)$$

and φ_0 can be chosen such that $\varphi_0^{-1} \in L_q(\Omega)$, $q \in (0,1)$.

In addition, for all $\eta \in L_2(\Omega_T)$ and $T < \infty$ holds

$$(K_0\nabla\varphi, \nabla\varphi)_\Omega \leqslant M_0(T) < \infty, \qquad (27)$$
$$\varepsilon_{\varphi}\,(|\eta|, |div(K_0\nabla\varphi)|)_{\Omega_T} \leqslant \sqrt{\varepsilon}\ M(T)\ \|\eta\|_{2,\Omega_T}.$$

Proof. Substituting $t \to T-t$ from (23) results in the problem

$$L\varphi \equiv \varphi_t - A\ div(K_0\nabla\varphi) - \vec{B}\ \nabla\varphi - \lambda\ A\ \varphi = 0,$$
$$\varphi = 0,\ (x,t) \in \Gamma_{1T},\ K_0\nabla\varphi\cdot\vec{n} = 0,\ (x,t) \in \Gamma_{2T},\ \varphi(x,0) = \varphi_0(x) \geqslant 0.$$

Its unique solvability in a class of functions $C^{2+\alpha,1+\alpha/2}(\Omega_T)$ under assumption from (23) was proved in [8],[9].

It will be shown that $\varphi \geqslant 0$. Determine the function $u_1(x,t)$ by the equality $\varphi = exp(\delta t + \beta)\ u_1$, where β is the solution of the problem

$$div(K_0\nabla\beta) = \gamma_1, \quad K_0\nabla\beta\cdot\vec{n} = \gamma, \quad x \in \Gamma, \quad \gamma_1 meas\ \Omega = \gamma\ meas\ \Gamma$$

with constants γ_1, γ, $\delta > 0$. According to the maximum principle for solutions of parabolic equations [8] may be shown that $u_1 \geqslant 0$

in Ω for sufficiently large values of δ and, therefore, $\varphi \geqslant 0$.

Introduce the function

$$\varphi_* = exp(\alpha_1\omega) - exp(\alpha_1 R),$$

where ω satisfies (24) and the constant $R > 0$. It is obvious that

$$L\chi + \delta\ \chi = exp(-\delta t)\ L\varphi_*, \quad \chi = exp(-\delta t)(\varphi + \varphi_*).$$

Let us prove that for sufficiently large values of δ, α_1, R the function $\chi \leqslant 0$, $(x,t) \in \Omega_T$. Calculate $L\varphi_*$ and estimate the above expression

$$exp(-\alpha_1\omega)L\varphi_* \leqslant -A\ (\alpha_1^2 + \alpha_1\frac{\vec{B}}{A}\nabla\omega + \alpha_1^2 K_0\nabla\omega\cdot\nabla\omega - \lambda\ exp(\alpha_1(R-\omega))) \leqslant$$
$$\leqslant -A\ (\alpha_1^2 - \frac{1}{2}MM_1^2 - 1) < 0, \quad \alpha_1^2 > 1 + \frac{1}{2}MM_1^2.$$

In this estimate the Cauchy inequality was used and it was assumed that

$$\lambda \leqslant \lambda_0 \equiv exp(-\alpha_1 h), \quad h \geqslant R - \min_{\Omega}\omega.$$

Thus, if $\delta > \lambda_0\|A\|_{C(\Omega_T)}$, there is no the positive maximum of χ inside Ω_T. For $t = 0$ and $R = \alpha_1^{-1}ln(\|\varphi_0\|_{C(\Omega)} + 1)$ the function $\chi < 0$. Since $\omega = 0$ on Γ_1, then $\chi < 0$ on Γ_{1T}. As $K_0\nabla\chi\cdot\vec{n} < 0$ on Γ_{2T}, on this part of the boundary the function χ has no maximum. The first estimate of lemma has been proved.

The right-hand side inequality in (26) follows from the fact that $\varphi \geqslant 0$ and $\varphi = 0$ on Γ_{1T}.

Let us prove that under condition (25) the maximum of the function $\varphi + \varphi_*$ is attained on Γ_{1T}. Actually, since

$$L(\varphi + \varphi_*) = L\varphi < 0, \quad K_0\nabla(\varphi + \varphi_*)\cdot\vec{n} < 0, \quad (x,t) \in \Gamma_{2T}$$

the function $\varphi + \varphi_*$ has the negative maximum at $t = 0$ or on Γ_{2T}. From (25) it follows that $\varphi_0 \leqslant 1 - exp(\alpha_1\omega)$, hence $(\varphi + \varphi_*)|_{t=0} \leqslant -\|\varphi_0\|_{C(\Omega)} = (\varphi + \varphi_*)|_{\Gamma_{1T}}$.

Thus, the function $\varphi + \varphi_*$ has the maximum value on Γ_{1T}. Since this value is independent of $(x,t) \in \Gamma_{1T}$, we have $K_0\nabla(\varphi + \varphi_*)\cdot\vec{n} \geqslant 0$ for $(x,t) \in \Gamma_{1T}$. This proves the estimate (26).

The estimates (27) may be received analogously [2].

Equation (24) multiplied by $(-\omega)^{1-q}$ and integrated over Ω

$$(1 - q)\ (K_0 \nabla\omega\cdot\nabla\omega, (-\omega)^{1-q})_\Omega = \alpha_1 (1, (-\omega)^{1-q})_{\Omega\ \cup\ \Gamma_2}.$$

It follows from (24) that $\omega < 0$ inside Ω. Further, the boundary condition on Γ_2 and the Zaremba-Giraud principle shows that $|\nabla\omega| \neq 0$ on Γ. Hence the summation of the function $(-\omega)^{-q}$, $q \in (0,1)$ follows from the latter equality.

Thus, the function $\varphi_0(x)$ can be chosen such that it satisfies (25) and $\varphi_0^{-1} \in L_q(\Omega)$.

Let us define

$$W \equiv \int_s^1 a(\xi)d\xi.$$

Let

$$W(s^0(x,t)) \in L_1(\Gamma_1\times(0,\infty)),\ Q_1 - R_1,\ Q - R \in L_1(\Gamma_1\times(0,\infty)),\ R = R_1,$$

$$\lim_{t\to\infty} \|Q - R\|_{1,\Gamma_2} = 0,\ |(\frac{\partial \vec f_0}{\partial s}, \frac{\partial \vec f}{\partial s}, \frac{\partial K_2}{\partial s})|/a \leqslant N_0,\ |p|, |\nabla p| \leqslant N_1, \tag{28}$$

$$s(x,t)\in C^{\alpha,\alpha/2}(\Omega_T),\ K_0(x)\in C^{1+\alpha}(\Omega),\ \vec f, \vec f_0 \in C^{\alpha}(\Omega)\times C^{1+\alpha}[0,1],\ a\in C^{\alpha}[0,1],$$

and there exist such functions f, g that

$$W(s) \geqslant f(1-s),\ \frac{d^2 f(y)}{dy^2} \geqslant 0,\ x\, f(y/x) \geqslant g(y),\ x \in [0, N_2], \tag{29}$$

where N_i, $i=0,1,2$ are the constant and the function $g(y)$ satisfies conditions of lemma 1.

Theorem 3. Let conditions (2),(28),(29) be fulfilled. Then for any generalized solutions of problems (1)-(5) and (20),(21) the following equality holds

$$\lim_{t\to\infty} (\|1-s\|_{1,\Omega} + \|p-p_0\|_{2,\Omega} + \|\nabla(p-p_0)\|_{2,\Omega}) = 0.$$

Proof. Differentiation of identity (6) in t and subsequent subtraction of (18) give the following equality for almost all $t \in (0,T)$

$$\frac{d}{dt}(m\varphi, 1-s)_\Omega - (m\varphi_t, 1-s)_\Omega + (K_0\nabla W(s), \nabla\varphi)_\Omega - (Q_1 - R_1, \varphi)_\Omega + (\vec f_0(x,1) - \vec f_0(x,s), \nabla\varphi)_\Omega + ((K_1(x,1) - K_1(x,s))\nabla p + K_1(x,1)\nabla(p - p_0), \nabla\varphi)_\Omega = 0. \tag{30}$$

Here we assume that $s_0 = 1$.

Analogously

$$((K(x,1) - K(x,s))\nabla p + K(x,1)\nabla(p_0 - p) + (\vec f(x,1) - \vec f(x,s), \nabla\psi)_\Omega + (R-Q, \psi)_{\Gamma_2} = 0. \tag{31}$$

Let $\varphi = \psi$ and $K_0\nabla\varphi\cdot\vec n = 0$ on Γ_{2T}. Then subtraction of (31) from (30) with the allowance for (22) and integration by parts in the third term of (30) give the equality

$$\frac{d}{dt}(m\varphi, 1-s)_\Omega = (m(1-s), \varphi_t + A\ div(K_0\nabla\varphi) + \vec B\ \nabla\varphi)_\Omega + (Q_1 - R_1, \varphi)_{\Gamma_2} + (R-Q, \varphi)_{\Gamma_2} + (K_0\nabla\varphi\cdot\vec n, W(s))_{\Gamma_1} - \varepsilon\ (m(1-s), div(K_0\nabla\varphi))_\Omega,$$

$$A = W(s)/m(1-s) + \varepsilon,\ \varepsilon > 0,\qquad K_2(x,s) \equiv K(x,s) - K_1(x,s),$$

$$\vec B = ((K_2(x,1) - K_2(x,s))\nabla p + \vec f(x,1) - \vec f(x,s) + \vec f_0(x,s) - \vec f_0(x,1))/m(1-s).$$

The function $\vec B$ is approximated by $\vec B_\nu$ at substituting ∇p by its smooth averaging ∇p_ν. The solution of problem (23) with the coefficients $A, \vec B_\nu$ is substituted into the latter equality for φ. Then, using the results of lemma 2, we estimate the integrals in the right-hand side of the obtained equality. As a result, we derive the inequality

$$\frac{dy}{dt} + \lambda M^{-1}(m\varphi, W(s))_\Omega \leqslant A_1(t), \tag{32}$$

where

$$A_1 = N_3(\|Q_1 - R_1\|_{1,\Gamma_2} + \|Q-R\|_{1,\Gamma_2} + \|W(s^0)\|_{1,\Gamma_1}) + M\|\vec B - \vec B_\nu\|_{2,\Omega}\|\nabla\varphi\|_{2,\Omega} + \sqrt{\varepsilon}\ MM_0(T),\quad y \equiv (m\varphi, 1-s)_\Omega,\ \lambda \leqslant \lambda_0,\ N_3 = const.$$

From (29),(32) and the Jensen inequality for the function $f(y)$ it follows that

$$\frac{dy}{dt} + \lambda M^{-1} g(y) \leqslant A_1(t),$$

$$0 < y_0 \leqslant z_0 = M \text{ meas } \Omega \ (\|\varphi_0\|_{C(\Omega)} + 1).$$

Taking into account estimates of lemma 2 in coefficient A_1 the limiting transition in parameters ν, ε are made. After it according to lemma 1 we get

$$y(T) = (m\varphi_0, 1-s(x,T))_\Omega \leqslant E(T).$$

From the Holder inequality and the latter estimate at $\varphi_0^{-1} \in L_q(\Omega)$ it follows that

$$\|1-s(x,T)\|_{1,\Omega} \leqslant \left[M \|\varphi_0^{-1}\|_{q,\Omega} E(T)\right]^{\frac{q}{q+1}} \to 0, \quad T \to \infty.$$

Assume that $\psi = p_0 - p$ in (31). Using the Cauchy inequality and thefact that $|p|, |\nabla p|$ are bounded uniformly, we have

$$\|\nabla(p_0 - p\|^2_{2,\Omega} \leqslant N_4(\|1-s\|_{1,\Omega} + \|Q-R\|_{1,\Gamma_2}).$$

If $\text{meas } \Gamma_1 > 0$, then $\|p-p_0\|_{2,\Omega} \leqslant N_5 \|\nabla(p-p_0)\|_{2,\Omega}$. Here N_4, N_5 are constant independent on t. Theorem has been proved.

Remark 1. Conditions of Lemma 1 and (29) are fulfilled, for example, by such functions $g(y)$, $f(y)$ that

$$g, f \sim cy^{r_0}, \ r_0 \geqslant 2, \ r_1 \int_0^y \exp\left(-\frac{r_2}{\xi}\right) d\xi, \ r_3 \int_0^y \left(\ln\left(\frac{r_4}{\xi}\right)\right)^{-1} d\xi, \ y \to 0,$$

where r_i, $i = 0,1,2,3,4$, are constant

A behaviour of the coefficient

$$a(s) \sim (1-s)^{r_0 - 1}, \ \exp\left(-\frac{1}{1-s}\right), \ |\ln(1-s)|^{-1}, \ s \to 1$$

corresponds to the mentioned functions.

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