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A Simple Proof of the Approximate Controllability from the Interior for Nonlinear Evolution Problems

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Abstract—The approximate controllability property for solutions of a large class of nonlinear evolution problems is obtained under some abstract conditions which hold, for instance, when the control is the right hand side of the equation. Our very simple method put in evidence the independence between the solvability of a boundary value problems and the study of the approximate controllability property which takes places in a number of cases. No duality type arguments are used which allows the consideration of very general nonlinear problems.

Keywords—Approximate controllability, Nonlinear evolution problems.

DEFINITIONS AND RESULTS

Let y denote the state of a deterministic dynamical system

$$\frac{\partial y}{\partial t} + \mathcal{A}y = Bu, \qquad y(0) = y_0, \tag{1}$$

where $\mathcal{A}$ is a nonlinear operator (mainly a partial differential operator) and where u denotes the control function. More precisely, let $\mathcal{Y}$, $\mathcal{X}$, $\mathcal{U}$, and $\mathcal{Z}$ be the normed spaces, $\mathcal{Y} \subset \mathcal{X}$, and assume that $y_0 \in \mathcal{Y}$ and

$$y \in C([0,T] : \mathcal{X}), \tag{2}$$

$$\mathcal{N} \equiv \left(\frac{\partial}{\partial t} + \mathcal{A}\right) : D(\mathcal{N}) \to L^1(0,T : \mathcal{Z}), D(\mathcal{N}) \subset C([0,T] : \mathcal{X}), \tag{3}$$

$$B : \mathcal{U} \to L^1(0,T : \mathcal{Z}). \tag{4}$$

DEFINITION. *We say that problem* (1) *has the property of $\mathcal{X}$-approximate controllability with respect to the control space $\mathcal{U}$ if for a given $y_d \in \mathcal{X}$ and for any $\varepsilon > 0$ there exist a couple $(y_\varepsilon, u_\varepsilon)$ with $u_\varepsilon \in \mathcal{U}$ and y_ε solution of* (1) *associated to $u = u_\varepsilon$ such that $\|y_\varepsilon(T) - y_d\|_{\mathcal{X}} \le \varepsilon$.*

The approximate controllability of many different special linear and nonlinear problems has been established in the last years by several authors (see, e.g., [1–6]). This property is usually

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX studied into the scope of the existence and uniqueness theory, and so it is customary to assume structural considerations on the nonlinear operator $\mathcal{A}$ assuring the solvability of problem (1) with an arbitrary fixed $u \in \mathcal{U}$ (i.e., with $(Bu)(t) \in \mathcal{Z}$ for a.e. $t \in (0,T)$). The main goal of this note has a pedagogical character and consists in showing the independence between the existence theory for the boundary value problems and the approximate controllability results. More precisely, we shall show that under additional assumptions, it is possible to give a positive answer to the approximate controllability question even in the case for which the existence (or the uniqueness) of a solution is not assured by the general theory. Indeed, the approximate controllability property leads to the problem of finding a sequence of state-control couples (z_k, u_k) such that $\|z_k(T,\cdot) - y_d\|_{\mathcal{X}} \to 0$ as $k \to \infty$ for a prescribed y_d. Thus, we merely need to justify the existence of a solution z_k corresponding to the control u_k but not for an arbitrary datum $u \in \mathcal{U}$.

The following abstract result shows that if the control operator B has a large range then the approximate controllability property is reduced to a trace type condition.

THEOREM. *Assume the conditions*

$$\text{Image}\, \mathcal{N} \subset B(\mathcal{U}) \tag{5}$$

and

$$\begin{gathered}\text{for any } y_0 \in \mathcal{Y},\ y_d \in \mathcal{X} \text{ and any } \varepsilon > 0, \text{ there exists } z_\varepsilon \in D(\mathcal{N}) \text{ such that} \\ z_\varepsilon(0) = y_0 \quad \text{and} \quad \|z_\varepsilon(T) - y_d\|_{\mathcal{X}} \leqq \varepsilon. \end{gathered} \tag{6}$$

Then problem (1) has the $\mathcal{X}$-approximate controllability property.

PROOF. From (5) and (6), there exists $u_\varepsilon \in \mathcal{U}$ such that $\mathcal{N} z_\varepsilon = B u_\varepsilon$ and so z_ε is a solution of (1) associated to u_ε. ∎

A class of problems for which condition (5) holds includes the case in which (1) is an initial-boundary value problem associated to a parabolic operator in a domain Ω of $\mathbb{R}^n$ and the controls take action on the whole domain Ω. We shall illustrate it by means of a simple example. For the sake of the exposition, we shall assume that Ω is bounded and regular.

EXAMPLE. For a given continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $p \geq 2$, we consider the nonlinear parabolic control problem

$$\begin{aligned} \frac{\partial y}{\partial t} - \Delta_p y + f(y) &= u, && \text{in } (0,T) \times \Omega, \\ y &= 0, && \text{on } (0,T) \times (\partial\Omega), \\ y(0,x) &= y_0(x), && \text{on } \Omega, \end{aligned} \tag{7}$$

where $\Delta_p y = \operatorname{div}(|\nabla y|^{p-2} \nabla y)$ (notice that the elliptic operator Δ_p is nonlinear except for the case $p = 2$ which corresponds to the usual Laplacian operator). In order to formulate (7) as an abstract Cauchy problem, we take, for instance, $\mathcal{Y} = L^\infty(\Omega)$, $\mathcal{X} = L^2(\Omega)$, $\mathcal{Z} = W^{-1,p'}(\Omega)$, $\mathcal{U} = L^1(0,T : W^{-1,p'}(\Omega))$, where $p' = p/(p-1)$, and $Bu = u$.

COROLLARY. *Problem (7) has the L^2-approximate controllability property.*

PROOF. Defining, for instance $D(\mathcal{N}) = \{y \in L^{p-1}(0,T : W_0^{1,p}(\Omega)) \cap L^\infty((0,T) \times \Omega) : \frac{\partial y}{\partial t} \in L^1(0,T : W^{-1,p'}(\Omega))\}$ it is clear that conditions (3) and (5) hold. In order to verify the "trace type" condition (6), we can proceed, for instance, as follows: by well-known results (see e.g., [1,3,4]) there exist two regular functions v_ε and z_ε (with $v_\varepsilon \in L^\infty((0,T) \times \Omega)$) satisfying

$$\begin{aligned} \frac{\partial z_\varepsilon}{\partial t} - \Delta z_\varepsilon &= v_\varepsilon, && \text{in } (0,T) \times \Omega, \\ z_\varepsilon &= 0, && \text{on } (0,T) \times \partial\Omega, \\ z_\varepsilon(0,x) &= y_0(x), && \text{on } \Omega, \end{aligned}$$

and

$$\|z_\varepsilon(T,\cdot) - y_d(\cdot) - y_d(\cdot)\|_{L^2(\Omega)} \leq \varepsilon.$$

Standard regularity results imply that $z_\varepsilon \in D(\mathcal{N})$ and the proof is complete. ∎

REMARK 1. We notice that the general theory for the existence of global solutions of (7) requires either some sign condition (e.g., $f(s)s \geq 0$, $\forall s \in \mathbb{R}$) or some growth condition ($|f(s)| \leq C|s|^{p-1} + b$, $\forall s \in \mathbb{R}$). None of them are needed here. ∎

REMARK 2. If Assumption (6) is satisfied for $\varepsilon = 0$ (which is the case if y_d is smooth enough) then problem (1) has the exact controllability property. In particular, if $y_d, y_0 \in C_0^1(\bar{\Omega})$, i.e., they satisfy zero Dirichlet condition on $\partial\Omega$, we easily construct $z \in C^1([0,T] \times \bar{\Omega}) \cap D(N)$ such that $z(T,x) = y_d(x)$, $z(0,x) = y_0(x)$ and $z|_{(0,T)\times\partial\Omega} = 0$. Then z satisfies (6), with $\varepsilon = 0$, in the case of problem (7). ∎

REMARK 3. Assumption (5) fails if the controls take action exclusively on a strict subdomain ω of Ω. This is the case when $Bu = \mathcal{X}_\omega u$ ($\mathcal{X}_\omega$ represents the characteristic function). The answer to the approximate controllability question is already negative for the special case of $p = 2$ and $f(s) = |s|^{m-1}s$ with $m > 1$ (see [3,4]). A different approach is also needed if the control v is subject to some sign conditions (see [7]). ∎

REMARK 4. The abstract theorem can be applied to many different classes of nonlinear evolution equations with controls on the whole domain. This list includes the Navier-Stokes equation, parabolic or hyperbolic equations which do not have divergence form, systems of equations, higher order equations and so on. Different types of boundary conditions are also possible. ∎

REMARK 5. The independence between the general existence theory and the study of the controllability question was already pointed out by Imanuvilov [8] where a problem closed to the exact controllability of the hyperbolic problem

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} - y^3 = 0, \qquad \text{in } (0,T) \times (0,l),$$

$$y(t,0) = v_0(t), \qquad y(t,L) = v_1(t), \qquad t \in (0,T)$$

$$y(0,x) = y_0(x), \qquad y_t(0,x) = y_1(x), \qquad x \in (0,L),$$

was studied. Some analogous situations take place in the theory of optimal control of singular distributed systems [2,9]. ∎

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