SIAM J. MATH. ANAL.
Vol. 25, No. 4, pp. 1085–1111, July 1994

003

ON A NONLINEAR PARABOLIC PROBLEM ARISING IN SOME MODELS RELATED TO TURBULENT FLOWS *

JESUS ILDEFONSO DIAZ† AND FRANCOIS DE THELIN‡

Abstract. This paper studies the Cauchy–Dirichlet problem associated with the equation

$$b(u)_t - \text{div } \left(|\nabla u - K(b(u))\,\mathbf{e}|^{p-2} (\nabla u - K(b(u))\,\mathbf{e})\right) + g(x,u) = f(t,x).$$

This problem arises in the study of some turbulent regimes: flows of incompressible turbulent fluids through porous media and gases flowing in pipes of uniform cross sectional areas. The paper focuses on the class of bounded weak solutions, and shows (under suitable assumptions) their stabilization, as $t \to \infty$, to the set of bounded weak solutions of the associated stationary problem. The existence and comparison properties (implying uniqueness) of such solutions are also investigated.

Key words. nonlinear parabolic equations, degenerate parabolic and elliptic equations, stabilization, existence and uniqueness of bounded weak solutions

AMS subject classifications. 35K65, 35K60, 76S05

Introduction. Physical models. The purpose of this paper is the study of the following nonlinear boundary value problem:

$$(0.1)\qquad \begin{cases} b(u)_t - \text{div } \phi(\nabla u - K(b(u))\,\mathbf{e}) + g(x,u) = f(t,x) & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ b(u(0,x)) = b(u_o(x)) & \text{on } \Omega, \end{cases}$$

where Ω is a bounded regular open set of $\mathbb{R}^N$, b is a nondecreasing continuous function, $K(\cdot)$ and $g(x,\cdot)$ are continuous functions satisfying some additional assumptions, and

$$(0.2)\qquad \phi(\zeta) = |\zeta|^{p-2}\,\zeta \quad \text{for some } p>1 \text{ and any } \zeta \in \mathbb{R}^N$$

(in (0.1) $\mathbf{e}$ denotes a given unit vector in $\mathbb{R}^N$).

When b is strictly increasing and $p = 2$ the partial differential equation of (0.1) is of the parabolic type. Nevertheless, it becomes degenerate when $p > 2$ or $b'(0) = +\infty$ (for instance) and singular if $1 < p < 2$ or $b'(0) = 0$ (for example).

Problem (0.1), or some special cases of it, arises in many different physical contexts. Here we shall mention two of them which are related with turbulent flows, thus justifying the title of this article.

Model 1. *Flow through porous media in turbulent regime.* The infiltration of an incompressible fluid in laminar regime through a porous medium (assumed homogeneous for simplicity) is governed by the continuity equation

$$\theta_t + \text{div } \mathbf{v} = 0$$

*Received by the editors August 14, 1991; accepted for publication (in revised form) March 19, 1993.

†Departamento de Matemàtica Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain. This author's research was supported in part by Dirección General de Investigación Cientifica y Technológica project PB90/0620.

‡Laboratoire d'Analyse Numerique, Universitè Paul Sabatier, 31062 Toulouse, France.

and the Darcy law

$$\mathrm{v} = -K(\theta)\,\mathrm{grad}\,\Phi(\theta),$$

where $\theta(x,t)$ is the volumetric moisture content, $K(\theta)$ is the hydraulic conductivity and the total potential Φ is given by $\Phi(\theta) = \psi(\theta) + z$ with $\psi(\theta)$ the hydrostatic potential and z the gravitational potential (obviously we have simplified the exposition by assuming several constants equal to one: see details in Bear [7]). In turbulent regimes (which appear, for instance, in the flow through rock filled dams) the flow rate is different from that which can be predicted by the Darcy law, and so several authors proposed a nonlinear relation between v and $K(\theta)$ grad Φ (nonlinear Darcy's law)

$$|\mathrm{v}|^{q-2}\,\mathrm{v} = -K(\theta)\,\mathrm{grad}\,\Phi(\theta) \quad \text{for some } q > 2 \tag{0.3}$$

(see Ahmed and Sunada [1], Hannoura and Barends [35], and Volker [54]). If e denotes the unit vector in the vertical direction, by introducing

$$\varphi(\theta) = \int_0^\theta K(s)\,\Phi'(s)\,ds, \qquad p = q/(q-1) \tag{0.4}$$

(notice that $1 < p < 2$), we obtain

$$\theta_t - \mathrm{div}\,\left(|\nabla\varphi(\theta) - K(\theta)\,\mathrm{e}|^{p-2}\,(\nabla\varphi(\theta) - K(\theta)\,\mathrm{e})\right) = 0. \tag{0.5}$$

The functions φ and K are given by physical experiments (see the above references). Usually they are nondecreasing functions, being φ strictly increasing for unsaturated media. In the unsaturated case the function $u = \varphi(\theta)$ satisfies the equation of (0.1) with $b = \varphi^{-1}$ and $g = f = 0$. The case of partially saturated media leads to the same equation (for a different unknown u) but with b a strictly increasing function on $(-\infty, u^*)$ and identically constant ($\equiv b(u^*)$) on the set $[u^*,\infty)$ for some $u^* \in \mathbb{R}$ (see Bear [7]). The interest of the presence of the term $g(x,u)$ appears when the action of the roots of plants into soil is taken into account (see Gilding [33] and his references). We mention that if $p = 2$ and $N = 1$ (0.5) is also known as the nonlinear Fokker–Planck equation and has been intensively treated in the mathematical literature (see the works Kalashnikov [39], Diaz and Kersner [25], Gilding [34] and their references). Finally we point out that equation (0.1) also arises if the fluid is assumed to be compressible and (again) turbulent (see Leibenson [44] and Bear [7]).

Model 2. *Turbulent gas flowing in pipelines.* Let ρ, p, v, and T be the density pressure velocity and temperature of a perfect gas flowing in a pipe of uniform cross sectional area. In the practical cases of interest the flow is turbulent, and so ρ, p, v, and T can be assumed to depend on the scalar x (the distance along the pipe) and time t (see, e.g., Shapiro [50]). The conservation of the mass and linear momentum leads to the system

$$\rho_t + (\rho v)_x = 0, \tag{0.6}$$

$$\rho v_t + \rho v v_x + p_x = -\frac{\lambda}{2}\,\rho|v|v, \tag{0.7}$$

to which we add the equation of the conservation of the energy and the constitutive law $p/\rho = T$ (after suitable normalizations). In (0.6) the term $(\lambda/2)\rho|v|v$ models the frictional forces (λ is known as the Darcy-Weissbach coefficient). Using asymptotic methods it was shown in Diaz and Liñan [26] that if the lengh L of the pipeline is considerably greater than the diameter D of the cross section, for large values of time the term $\rho v_t + \rho v v_x$ can be neglected obtaining

$$p_x = -\frac{\lambda}{2}\rho|v|v. \tag{0.8}$$

An easy computation allows to see that $u = |p|p$ satisfies the equation (0.1) with $b(u) = u^{1/2}$ sign $u, K = g = f = 0$ and the exponent p of (0.2) given by $p = 3/2$. The study of incompressible flow leads to a similar equation (0.1) but with a linear ϕ (see Liñan [45]). Finally, we mention that the interest of the presence of the term $g(x,u)$ in this context is motivated by the study of the behavior of solutions near the extinction time (see Diaz and Liñan [26, Thm. 3]).

Problems like (0.1) appear in a variety of different settings (see Bermudez, Durany, and Saguez [10], Diaz and Herrero [24], van Duijn and Hilhorst [28], Martinson and Pavlov [47] and the monographs by Diaz [22], [23]).

This paper deals with the mathematical treatement of problem (0.1) (which sometimes will be referred to as *the model problem*). Motivated by the physical models, we shall resctrict ourselves to the study of *bounded weak solutions*. This class of solutions is introduced in §1, where we also show that under suitable hypothesis those solutions stabilize as $t \to +\infty$ to the set of bounded weak solutions of the associated stationary problem. This is done for a general class of nonlinear equations including the one of (0.1). We extend the result of Langlais and Phillips [43] concerning the special case $p = 2$ and $K \equiv 0$ by passing to the limit by a variant of the already classical Minty argument (see Lions [46]). The rest of the paper is devoted to the study of the model problem (0.1).

The comparison properties (and uniqueness) of bounded weak solutions of (0.1) and its stationary problem is analysed in §2. In the case of problem (0.1) we extend the result of Alt and Lukhaus [3] valid only for $p \geq 2$ by giving a comparison criterion for $1 < p \leq 2$. The results for the stationary problem needs a different type of assumptions depending on whether $g(x,u)$ is a strictly increasing function or not.

The existence of a bounded weak solution of (0.1) is carried out in §3 by coupling regularity and sub-supersolutions arguments. The boundedness result of Boccardo and Giachetti [17] for a general class of stationary problems is shown to be applicable to our case, thus being systematically used in order to formulate our assumptions on the data f and u_0. References to some existence results for similar problems are collected in Remark 6.

We finish the article by coming back to the stabilization question and checking the assumptions of §1 for the concrete case of problem (0.1). In the first part we prove this property by using the comparison principle and the uniqueness of the associated stationary problem. That extends the result of Kröner and Rodrigues [41] concerning the case $p = 2, b$ and K Lipschitz continuous functions (b being also assumed to be bounded). Finally we treat the case of $K(b(s)) = \lambda s$ by purely energy methods generalizing several results in the literature for special cases of b, p, and $K \equiv 0$.

Some notation used through the paper follows: given $p > 1$ we associate to it the exponents $p' = p/(p-1), p^*= Np/(N-p)$ if $p < N$ and p^* arbitrary in $(p, +\infty)$ if $p \geq N$ and finally $p\# = \max(p, 2)$. The symbol $\langle , \rangle$denotes the duality product between the Sobolev space $W_0^{1,p}(\Omega)$ and its dual $(W_0^{1,p}(\Omega))^*= W^{-1,p'}(\Omega)$. Finally, we shall use the common letter C to denote different constants if no other specification is needed in the context.

1. Notion of solution and a stabilization result for a general class of equations. Let Ω be a regular open bounded set of $\mathbb{R}^N$. In this section we consider the following problem:

$$(1.1)\qquad \begin{cases} b(u)_t + Au + g(x,u) = f(t,x) & \text{in } (0,\infty)\times\Omega, \\ u = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ b(u(0,x)) = b(u_0(x)) & \text{in } \Omega, \end{cases}$$

where Au denotes the operator

$$(1.2)\qquad Au = -\text{div}\, \mathbf{A}(x,u,\nabla u)$$

for $\mathbf{A} : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$, a Caratheodory function (i.e., continuous in $(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^N$ and measurable in x) satisfying

$$(1.3)\qquad |\mathbf{A}(x,\eta,\xi)| \le C_0\left(|\eta|^{p*/p'} + |\xi|^{p-1}\right) + k_0(x)$$

for some $C_0 > 0$ and $k_0 \in L^{p'}(\Omega)$ and

$$(1.4)\qquad \left(\mathbf{A}(x,\eta,\xi) - \mathbf{A}\left(x,\eta,\hat{\xi}\right)\right)\cdot\left(\xi - \hat{\xi}\right) > 0,$$

for any $\eta\in\mathbb{R}, \xi,\hat{\xi}\in\mathbb{R}^N, \xi\neq\hat{\xi}$ and almost every $x\in\Omega$. Obviously, the model problem (0.1) corresponds to the special case

$$(1.5)\qquad \mathbf{A}(x,\eta,\xi) = \phi(\xi - K(b(\eta))\,\mathbf{e}), \qquad \phi(\xi) = |\xi|^{p-2}\xi.$$

In that case (1.4) is trivially satisfied and (1.3) holds under an additional condition (see (3.1)).

Here and throughout the rest of the paper we assume the following conditions:

(1.6) $\quad b : \mathbb{R}\to\mathbb{R}$ is a continuous nondecreasing function with $b(0) = 0$,

(1.7)

$$\begin{cases} g : \Omega\times\mathbb{R}\to\mathbb{R} \text{ is a Caratheodory function such that } |g(x,\eta)| \le \beta(|\eta|)(1+d(x)) \\ \text{for some } d\in L^1(\Omega) \text{ and some continuous increasing function } \beta, \end{cases}$$

$$(1.8)\qquad f\in L^1((0,T)\times\Omega) + L^{p'}\left(0,T : W^{-1,p'}(\Omega)\right), \text{ for any } T>0,$$

$$(1.9)\qquad u_0\in L^\infty(\Omega).$$

We shall use the notion of a weak solution introduced in [3]. By a *bounded weak solution* of the problem (1.1) we mean a function $u\in L^p(0,T : W_0^{1,p}(\Omega))\cap L^\infty((0,T)\times\Omega)$, satisfying

(1.10)

$$\begin{cases} b(u)_t \in L^{p'}\left(0,T : W^{-1,p'}(\Omega)\right) \quad \text{and} \quad \int^T \langle b(u)_t, v\rangle + \int^T\int [b(u) - b(u_0)]\, v_t = 0, \\ \text{for any } v\in L^p(0,T : W_0^{1,p}(\Omega))\cap W^{1,1}(0,T : L^1(\Omega)), \quad \text{with} \quad v(T,\cdot) = 0, \end{cases}$$

$$(1.11)\qquad \begin{cases} \int_0^T \langle b(u)_t, v\rangle + \int_0^T \int_\Omega \mathbf{A}(\cdot, u, \nabla u)\cdot\nabla v + \int_0^T\int_\Omega g(\cdot,u)\,v = \int_0^T\int_\Omega fv, \\ \text{for any } v \in L^p(0,T:W_0^{1,p}(\Omega))\cap L^\infty((0,T)\times\Omega), \end{cases}$$

where T is any positive number. In (1.11) we have used the notation

$$(1.12)\qquad \int_0^T\int_\Omega fv = \int_0^T\int_\Omega f_1 v + \int_0^T \langle f_2, v\rangle$$

if $f = f_1 + f_2$ with $f_1 \in L^1((0,T)\times\Omega)$ and $f_2 \in L^{p'}(0,T:W^{-1,p'}(\Omega))$.

The rest of this section is devoted to the study of the stabilization, as $t\to\infty$, of any bounded weak solution u of (1.1). As usual, we define the ω limit set associated to u by

$$\omega(u) = \Big\{u_\infty \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) : \exists t_n \to \infty \text{ such that } u(t_n,.) \to u_\infty \\ \text{in } L^p(\Omega),\ \text{as } n\to\infty\Big\}.$$

In order to state our result we need some additional assumptions on f:

$$(1.13)\qquad \begin{cases} \text{there exists } f_\infty \in L^1(\Omega) + W^{-1,p'}(\Omega) \text{ such that } f(t,\cdot)\to f_\infty \text{ as} \\ t\to\infty \text{ in the sense that } \int_{t+1}^{t-1} \|f(\tau,\cdot) - f_\infty\|_{L^1+W^{-1,p}} \to 0 \quad \text{as } t\to\infty. \end{cases}$$

Finally if $f_\infty \in L^1(\Omega) + W^{-1,p'}(\Omega)$ (i.e., $f_\infty = f_{\infty,1} + f_{\infty,2}, f_{\infty,1}\in L^1(\Omega), f_{\infty,2}\in W^{-1,p'}(\Omega)$) we say that u_∞ is a *bounded weak solution* of the stationary problem

$$(1.14)\qquad \begin{cases} -\operatorname{div} A(x,u_\infty,\nabla u_\infty) + g(x,u_\infty) = f_\infty & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega, \end{cases}$$

if $u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and satisfies

$$(1.15)\qquad \int_\Omega \mathbf{A}(x,u_\infty,\nabla u_\infty)\cdot\nabla v + \int_\Omega g(\cdot,u_\infty)\,v = \int_\Omega f_\infty v$$

for any $v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ (we have used again the same abuse in the notation as in (1.12)).

THEOREM 1. *Let u be a bounded weak solution of* (1.1) *such that*

$$(1.16)\qquad u\in L^\infty(t_0,+\infty;W_0^{1,p}(\Omega)) \quad \text{for some } t_0>0.$$

Then $\omega(u)\neq\phi$. Moreover, if $u_\infty\in\omega(u)$ satisfies

$$(1.17)\ \exists t_n\to+\infty \textit{ such that } u(t_n+s,\cdot)\to u_\infty \textit{ in } L^r(-1,1;L^p(\Omega)) \textit{ for any } r\geq 1,$$

then u_∞ is a bounded weak solution of the stationary problem (1.14).

Proof. Let $t_n\to+\infty$. As $\{u(t_n,\cdot)\}$ is bounded in $W_0^{1,p}(\Omega)$ there is some subsequence (denoted again by t_n) such that $u(t_n,\cdot)$ converges in $L^p(\Omega)$ and so $\omega(u)\neq\phi$. For any $\zeta\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for any $\varphi\in\mathcal{D}(-1,1), \varphi\geq 0$ such that $\int_{-1}^1\varphi(s)\,ds = 1$, we define the function $v(t,x) = \zeta(x)\varphi(t-t_n)$. For $T\geq t_n+1$, we have

$$\int_0^T\int_\Omega [b(u) - b(u_0)]\,v_t = \int_{t_n-1}^{t_n+1}\int_\Omega b(u)\,\zeta\varphi'(t-t_n)$$

and from conditions (1.10) and (1.11) we get

$$\int_{t_n-1}^{t_n+1}\int_\Omega b(u)\,\zeta\varphi'(t-t_n)+\int_{t_n-1}^{t_n+1}\int_\Omega \mathbf{A}(\cdot,u,\nabla u)\cdot\nabla\zeta\varphi(t-t_n)$$
$$=\int_{t_n-1}^{t_n+1}\int_\Omega (f(t,\cdot)-g(\cdot,u))\,\zeta\varphi(t-t_n).$$

Changing variables, namely $s=t-t_n$ and defining $U_n(s,x)=u(t_n+s,x)$ we obtain

$$\begin{aligned}\int_{-1}^{1}\int_\Omega b(U_n)\,\zeta\varphi'(s)+\int_{-1}^{1}\int_\Omega \mathbf{A}(\cdot,U_n,\nabla U_n)\cdot\nabla\zeta\varphi(s)\\=\int_{-1}^{1}\int_\Omega (f(s,\cdot)-g(\cdot,U_n))\,\zeta\varphi(s).\end{aligned}\tag{1.18}$$

Since U_n is bounded in $L^\infty(-1,1;W_0^{1,p}(\Omega))$, by (1.3) and the Sobolev theorem $\mathbf{A}(\cdot,U_n,\nabla U_n)$ is bounded in $L^\infty(-1,1;(L^{p'}(\Omega))^N)$. So there is a subsequence, denoted again by U_n, weakly* convergent to u_∞ in $L^\infty(-1,1;W_0^{1,p}(\Omega))$, weakly convergent to u_∞ in $L^p(-1,1;W_0^{1,p}(\Omega))$ and such that $\mathbf{A}(\cdot,U_n,\nabla U_n)$ converges weakly* to $\mathbf{Y}$ in $L^\infty(-1,1;(L^{p'}(\Omega))^N)$. Moreover, from the assumptions on b and g the sequence $b(U_n)$ converges to $b(u_\infty)$ and $g(\cdot,U_n)$ converges to $g(\cdot,u_\infty)$ in the space $L^r((-1,1)\times\Omega)$ for any $r\in[1,+\infty)$. Moreover, we have

$$\int_\Omega \mathbf{Y}\cdot\nabla\zeta=\int_\Omega (f_\infty-g(\cdot,u_\infty))\,\zeta.$$

Due to the quasilinear character of our operator, the main difficulty is to identify $\mathbf{Y}$ as $\mathbf{A}(\cdot,u_\infty,\nabla u_\infty)$. We shall show that by means of the following inequality which is a variant of the well-known Minty argument (see [46])

$$\int_\Omega [\mathbf{Y}-\mathbf{A}(\cdot,u_\infty,\nabla\chi)]\cdot\nabla(u_\infty-\chi)\geq 0\quad\text{for any }\chi\in W_0^{1,p}(\Omega).\tag{1.19}$$

If (1.19) is verified taking $\chi=u_\infty+\lambda\xi$, with $\lambda>0$ and arbitrary $\xi\in W_0^{1,p}(\Omega)$ and letting $\lambda\to 0$ we obtain

$$\int_\Omega [\mathbf{Y}-\mathbf{A}(\cdot,u_\infty,\nabla u_\infty)]\cdot\nabla\xi\geq 0.$$

Hence $\mathcal{A}u_\infty=-\operatorname{div}\mathbf{Y}$ and the conclusion of the theorem holds. The proof of (1.19) follows from the next two lemmas. □

LEMMA 1.

$$\lim_{n\to+\infty}\int_{-1}^{1}\int_\Omega \mathbf{A}(\cdot,U_n,\nabla U_n)\cdot\nabla(U_n-u_\infty)\,\varphi(s)=0.$$

Proof of Lemma 1. Let $v(t,x)=u(t,x)\varphi(t-t_n)$. By (1.10) and (1.11) we get

$$\int_{t_n-1}^{t_n+1}\langle b(u)_t,u\rangle\,\varphi(t-t_n)+\int_{t_n-1}^{t_n+1}\int_\Omega \mathbf{A}(\cdot,u,\nabla u)\cdot\nabla u\varphi(t-t_n)$$
$$=\int_{t_n-1}^{t_n+1}\int_\Omega [f(t,\cdot)-g(\cdot,u)]\,u\varphi(t-t_n).$$

Following Alt and Luckhaus [3], we define the real function

$$B(u) = \int_0^u [b(u) - b(s)]\,ds \quad \forall u \in \mathbb{R},$$

and the time variable function

$$z_u(t) = \int_\Omega B(u(t,\cdot)).$$

As $B[u(\cdot,\cdot)]$ is bounded, then $z_u \in L^1(0,T)$ and Lemma 2 of Bamberger [5] gives

$$\begin{aligned}\int_{t_n-1}^{t_n+1}\int_\Omega \langle b(u)_t, u\rangle \varphi(t-t_n) &= -\int_{t_n-1}^{t_n+1}\int_\Omega z_u(t)\varphi'(t-t_n)\\ &= -\int_{-1}^1 \varphi'(s)\int_\Omega\int_0^{U_n(s,x)} \{b[U_n(s,x)] - b(\sigma)\}.\end{aligned}$$

By the dominated convergence theorem, the last term converges to

$$-\int_{-1}^1 \varphi'(s)\int_\Omega\int_0^{u_\infty(x)} \{b[u_\infty(x)] - b(\sigma)\}$$

which is identically equal to zero. Since U_n (and, therefore, u_∞) is uniformly bounded and since $f_{\infty,1} \in L^1(\Omega)$ from the Egorov theorem, we deduce that

$$\lim_{n\to\infty}\int_{-1}^1\int_\Omega f_{\infty,1}(U_n - u_\infty)\varphi(s) = 0.$$

Then, by the previous results on the convergence of U_n we have

$$\lim_{n\to+\infty}\int_{-1}^1\int_\Omega [f(t_n+s,\cdot) - g(\cdot,U_n)]\,U_n\varphi(s) = \int_\Omega [f_\infty - g(\cdot,u_\infty)]\,u_\infty.$$

Then we get

$$\lim_{n\to+\infty}\int_{-1}^1\int_\Omega \mathbf{A}(\cdot,U_n,\nabla U_n)\cdot\nabla U_n\varphi(s) = \int_\Omega [f_\infty - g(\cdot,u_\infty)]\,u_\infty = \int_\Omega \mathbf{Y}\cdot\nabla u_\infty. \qquad \square$$

LEMMA 2.

$$\mathbf{Y} = \mathbf{A}(\cdot,u_\infty,\nabla u_\infty).$$

Proof of Lemma 2. For any $\chi \in W_0^{1,p}(\Omega)$ we have

$$\int_{-1}^1\int_\Omega [\mathbf{A}(\cdot,U_n,\nabla U_n) - \mathbf{A}(\cdot,u_\infty,\nabla\chi)]\cdot\nabla(u_\infty - \chi)\varphi(s) = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \int_{-1}^1\int_\Omega \mathbf{A}(\cdot,U_n,\nabla U_n)\cdot\nabla(u_\infty - U_n)\varphi(s) \to 0 \quad \text{by Lemma 1},$$
$$I_2 = \int_{-1}^1\int_\Omega [\mathbf{A}(\cdot,U_n,\nabla U_n) - \mathbf{A}(\cdot,U_n,\nabla\chi)]\cdot\nabla(U_n - \chi)\varphi(s) \ge 0 \quad \text{by (1.4)},$$

and

$$I_3 = \int_{-1}^{1} \int_{\Omega} \mathbb{A}\left(\cdot, U_n, \nabla \chi\right) \cdot \nabla \left(u_n - u_\infty\right) \varphi\left(s\right).$$

By Lemma 2.1 of [46] $\mathbb{A}(\cdot, U_n, \nabla\chi)$ converges strongly to $\mathbb{A}(\cdot, u_\infty, \nabla\chi)$ in $(L^{p'}(\Omega))^n$, and by (1.3) there exists $C > 0$, independent of n such that

$$\underset{s\in[-1,1]}{\text{ess}} \underset{\Omega}{\sup} \int_{\Omega} \left|\mathbb{A}\left(\cdot, U_n, \nabla \chi\right) \cdot \nabla \left(U_n - u_\infty\right)\right| \leq C < +\infty.$$

Whence, by the dominated convergence theorem, $\lim I_3 = 0$. By the same reason $\lim I_4 = 0$, where

$$I_4 = \int_{-1}^{1} \int_{\Omega} \left[\mathbb{A}\left(\cdot, U_n, \nabla \chi\right) - \mathbb{A}\left(\cdot, u_\infty, \nabla \chi\right)\right] \cdot \nabla \left(u_\infty - \chi\right) \varphi\left(s\right).$$

That proves the inequality (1.19) and the conclusion of the theorem follows. □

We point out that the proof of Theorem 1 also gives the information that $u(t_n, \cdot) \to u_\infty(x)$ weakly in $W_0^{1,p}(\Omega)$. The next result shows that this convergence can be improved under the following additional coercivity assumption:

(1.20)

$$\begin{cases} C\left|\xi - \hat{\xi}\right|^p \\ \leq \left\{\left[\mathbb{A}\left(x, \eta, \xi\right) - \mathbb{A}\left(x, \eta, \hat{\xi}\right)\right] \cdot \left[\xi - \hat{\xi}\right]\right\}^{\alpha/2} \left\{k_1(x) + k_2\left(|\eta|^{p*} + |\xi|^p + \left|\hat{\xi}\right|^p\right)\right\}^{1-(\alpha/2)} \\ \text{for any } \eta \in \mathbb{R}, \xi, \hat{\xi} \in \mathbb{R}^N, \text{ a.e. } x \in \Omega \text{ and for some } C_1 > 0, k_1 \in L^1(\Omega), \\ k_2 > 0 \text{ and some } \alpha \in (1, 2]. \end{cases}$$

THEOREM 2. *Assume the same conditions as in Theorem* 1 *and also* (1.20). *Then, for any* $u_\infty \in \omega(u)$, *there exists a sequence* $\{\tilde{t}_n\}, \tilde{t}_n \to +\infty$ *as* $n \to +\infty$, *such that* $u(\tilde{t}_n, \cdot) \to u_\infty$ *strongly in* $W_0^{1,p}(\Omega)$.

Proof. Taking $\chi = u_\infty$ in the proof of Lemma 2 we have

$$\lim_{n\to+\infty} \int_{-1}^{1} \int_{\Omega} \mathbb{A}\left(\cdot, U_n, \nabla u_\infty\right) \cdot \nabla \left(U_n - u_\infty\right) \varphi\left(s\right) = 0.$$

So, by Lemma 1, we obtain $\lim_{n\to+\infty} I_n = 0$, where

$$I_n = \int_{-1}^{1} \int_{\Omega} \left[\mathbb{A}\left(\cdot, U_n, \nabla U_n\right) - \mathbb{A}\left(\cdot, u_\infty, \nabla u_\infty\right)\right] \cdot \nabla \left(U_n - u_\infty\right) \varphi\left(s\right).$$

Moreover, by (1.20) and Hölder's inequality

$$C \int_{-1}^{1} \int_{\Omega} \left|\nabla U_n - \nabla u_\infty\right|^p \varphi\left(s\right)$$
$$\leq \left\{I_n\right\}^{\frac{\alpha}{2}} \left\{\int_{-1}^{1} \int_{\Omega} \left[k_1 + k_2\left(\left|U_n\right|^{p*} + \left|\nabla U_n\right|^p + \left|\nabla u_\infty\right|^p\right)\right] \varphi\left(s\right)\right\}^{1-\frac{\alpha}{2}}.$$

So, as U_n is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$, for any $\varphi \in \mathcal{D}(-1,1)$ $\varphi \geq 0$ such that $\int_{-1}^{1} \varphi(s) = 1$, we have

$$\lim_{n\to+\infty} \int_{-1}^{1} \int_{\Omega} \left|\nabla u\left(t_n+s,\cdot\right) - \nabla u_\infty\right|^p \varphi\left(s\right) = 0.$$

But that is impossible if for some $\varepsilon \geq 0$,

$$\int_{\Omega} \left|\nabla u\left(t_n+s,\cdot\right) - \nabla u_\infty\right|^p \geq \varepsilon$$

for almost every $s \in (-1,1)$. Then there is a sequence $\{s_n\}, s_n \in [-1,1]$, such that

$$\lim_{n\to+\infty} \int_{\Omega} \left|\nabla u\left(t_n+s_n,\cdot\right) - \nabla u_\infty\right|^p = 0. \qquad \square$$

Remark 1. The results of this section generalize previous results in the literature: the case of $\mathcal{A} = -\Delta$ (the Laplacian operator) was treated by Langlais and Phillips [43] who showed convergence in $L^2(\Omega)$. Convergence in $L^s(\Omega)$ for some $s \geq 1$ was given in the papers Berryman and Holland [12] and Diaz and Liñan [26] for the special case of the model equation with $b(s) = s^{1/m}, K \equiv 0, (p-1)m \leq 1, p = 2$, and $p \neq 2$, respectively. The convergence in $L^1(\Omega)$ was shown in Chipot and Rodrigues [20] for $b(s) = s$ and $\mathcal{A}$ satisfying a coercivity condition stronger than (1.20). Concerning strong convergence, our result improves the one by Kröner and Rodrigues [41] (for the model equation, $p = 2$ and b Lipschitz and bounded) and the results of El Hachimi and de Thelin [29], [30] (for the model equation with $b(u) = u$ and $K \equiv 0$).

Remark 2. If $\omega(u)$ consists of a discrete number of points it is easy to see that in Theorems 1 and 2 the convergence holds for *any* subsequence t_n, i.e., when $t \to +\infty$. A more difficult task is to prove such conclusions when there is a continuum of equilibrium solutions. Some results in this direction are due to Matano [48], Alikakos and Bates [2] and Diaz and Veron [27].

The assumptions (1.16) and (1.17) hold under additional conditions on the formulation of the problem. Concerning the condition (1.17), we shall verify it (in §3) by using suitable comparison arguments. Energy type arguments also lead to this condition once we have suitable additional information on the solution. This is contained in the following result.

PROPOSITION 1. *Let $u \in L^\infty((0,\infty) \times \Omega)$. Assume that there is a continuous strictly increasing function k from $\mathbb{R}$ to $\mathbb{R}$ with $k(0) = 0$ such that $k(u) \in L^1_{\text{loc}}(0,\infty : L^q(\Omega))$ for some $q \geq 1$ and*

$$\lim_{t\to+\infty} \int_{t-1}^{t+1} \int_{\Omega} |k(u)_t|^q = 0. \tag{1.21}$$

Then, if there exists a sequence $t_n \to +\infty$ satisfying

$$\lim_{n\to+\infty} u\left(t_n,\cdot\right) = u_\infty \quad \text{in } L^p\left(\Omega\right), \tag{1.22}$$

condition (1.17) *holds.*

Proof. Let $u_\infty = \lim_{n\to+\infty} u(t_n,\cdot)$ in $L^p(\Omega)$. Then there exists a subsequence (denoted again by t_n) such that $u(t_n,\cdot)$ converges almost everywhere to u_∞. For

$s \in]-1,1[$ and $x \in \Omega$, we define $U_n(s,x) = u(t_n+s,x)$. As u is bounded, by the dominated convergence theorem, $k[u(t_n,\cdot)]$ converges to $k(u_\infty)$ in $L^r(\Omega)$ for any $r \in [1,\infty)$. Moreover,

$$\begin{aligned} |k[u(t_n+s,\cdot)] - k[u(t_n,\cdot)]| &= \left| \int_{t_n}^{t_n+s} k(u)_t(\tau,\cdot)\, d\tau \right| \\ &\leq \left\{ \int_{t_n-1}^{t_n+1} |k(u)_t(\tau,\cdot)|^q \, d\tau \right\}^{\frac{1}{q}} 2^{\frac{1}{q'}}. \end{aligned}$$

Thus using (1.21)

$$\|k(U_n) - k[u(t_n,\cdot)]\|_{L^q(\Omega)} \to 0 \quad \text{as } t_n \to \infty,$$

so, $k(U_n)$ converges to $k(u_\infty)$ in $L^q(\Omega)$. Moreover $k(U_n)$, and U_n converge in almost everywhere point to $k(u_\infty)$ and u_∞, respectively. Finally, as all these sequences are bounded the convergence holds in the space $L^r((-1,1)\times\Omega)$, for any $r \geq 1$. □

2. Comparison and continuous dependence results for the model equation. In this section, we give several results on the comparison (and then uniqueness) and continuous dependence of solutions of the model problem (0.1), i.e., (1.1) with A given by (1.5). We make the following additional assumptions:

$$\text{(2.1)} \qquad \begin{cases} |K[b(\eta)] - K[b(\hat{\eta})]| \leq C|\eta - \hat{\eta}|^\gamma \\ \text{for any } \eta, \hat{\eta} \in \mathbb{R} \text{ with } \gamma \geq \frac{1}{p} \quad \text{if } 1 < p < 2, \gamma \geq \frac{1}{p'}, \quad \text{if } p > 2, \end{cases}$$

(2.2)

$$g(\cdot,\eta) - g(\cdot,\hat{\eta}) \geq -C^*(b(\eta) - b(\hat{\eta})) \quad \text{for some } C^* \geq 0 \text{ and any } \eta, \hat{\eta} \in \mathbb{R}, \eta > \hat{\eta}.$$

Our operator is coercive in the sense that it satisfies the relation (1.20). This is a direct consequence of the well-known inequality

$$\text{(2.3)} \qquad C\left|\theta - \hat{\theta}\right|^p \leq \left\{ \left[|\theta|^{p-2}\theta - \left|\hat{\theta}\right|^{p-2}\hat{\theta} \right] \cdot \left[\theta - \hat{\theta}\right] \right\}^{\frac{\alpha}{2}} \left\{ |\theta|^p + \left|\hat{\theta}\right|^p \right\}^{1-\frac{\alpha}{2}},$$

which holds for any $\theta, \hat{\theta} \in \mathbb{R}^N$ and $p > 1$ with $\alpha = p$ if $1 < p \leq 2$ and $\alpha = 2$ if $p \geq 2$ (see Simon [51]). Inequality (2.3) generalizes the following one (sometimes referred as Tartar's inequality)

$$\text{(2.4)} \qquad C\left|\theta - \hat{\theta}\right|^p \leq \left[|\theta|^{p-2}\theta - \left|\hat{\theta}\right|^{p-2}\hat{\theta} \right] \cdot \left[\theta - \hat{\theta}\right]$$

which only holds for $p \geq 2$. Moreover taking $\theta = \phi(\zeta), \hat{\theta} = \phi(\hat{\zeta})$ and changing p by p' we obtain the following inequality for any ζ and $\hat{\zeta}$ in $\mathbb{R}^N$

$$\text{(2.5)} \qquad \begin{cases} \left|\phi(\zeta) - \phi\left(\hat{\zeta}\right)\right|^{p'} \leq C\left\{ \left(\zeta - \hat{\zeta}\right)\left[\phi(\zeta) - \phi\left(\hat{\zeta}\right)\right] \right\}^{\frac{\beta}{2}} \left\{ |\zeta|^p + \left|\hat{\zeta}\right|^p \right\}^{1-\frac{\beta}{2}}, \\ \text{where } \beta = 2 \text{ if } 1 < p \leq 2 \text{ and } \beta = p' \text{ if } p \geq 2 \text{ and } \phi(\xi) = |\xi|^{p-2}\xi. \end{cases}$$

THEOREM 3. *Assume* (2.1) *and* (2.2). *Let* (f, u_0) *and* $(\hat{f}, \hat{u}_0)$ *be a pair of data satisfying* (1.8) *and* (1.9). *Let* u *and* $\hat{u}$ *be bounded weak solutions of problem* (0.1) *corresponding to* $(f, u_0), (\hat{f}, \hat{u}_0)$, *respectively, and such that*

$$b(u)_t, b(\hat{u})_t \in L^1((0,T) \times \Omega). \tag{2.6}$$

Then

(i) *if the data are ordered* [*i.e.*, $f \leq \hat{f}, u_0 \leq \hat{u}_0$] *we have* $u \leq \hat{u}$ *in* $(0,T) \times \Omega$,

(ii) *if* $f = f_1 + f_2, \hat{f} = \hat{f}_1 + \hat{f}_2$ *with* $f_1, \hat{f}_1 \in L^1((0,T)\times\Omega)$ *and* $f_2 = \hat{f}_2 \in L^{p'}(0,T : W^{-1,p'}(\Omega))$, *we have*

$$\|b(u(t,\cdot)) - b(\hat{u}(t,\cdot))\|_{L^1(\Omega)} \leq e^{C^*t} \left(\|b(u_0) - b(\hat{u}_0)\|_{L^1(\Omega)} + \int_0^t e^{-C^*t} \left\| f_1(s,\cdot) - \hat{f}_1(s,\cdot) \right\|_{L^1(\Omega)} \right).$$

Remark 3. Conclusion (i) of Theorem 3 for $p \geq 2$ is a direct consequence of Theorem 2.2 of [3] (see also Artola [4] and Chipot and Rodrigues [20] for $b(u) = u$, and $p = 2$ and $p \geq 2$, respectively). Indeed, from (2.5) we deduce that if $p \geq 2$, then

$$\left|\phi(\zeta) - \phi\left(\hat{\zeta}\right)\right| \leq C \left|\zeta - \hat{\zeta}\right| \left(|\zeta|^p + \left|\hat{\zeta}\right|^p\right)^{2-p/p}.$$

Applying this inequality to $\zeta = \xi - K(b(\eta)), \hat{\zeta} = \xi - K(b(\hat{\eta}))$ and using the assumption (2.1) we arrive to the hypothesis of the mentioned result (remark that $2 - p' \in (0,1)$). The case $1 < p < 2$ needs a new treatment because the assumptions of the mentioned papers are not satisfied.

Proof. (i) We only consider the case $1 < p < 2$. For small $\delta > 0$, we introduce the test function

$$v = \psi_\delta (u - \hat{u}), \quad \text{where } \psi_\delta(\eta) = \min\left(1, \max\left(0, \frac{\eta}{\delta}\right)\right) \quad \text{for } \eta \in \mathbb{R}.$$

We get

$$\begin{aligned} &\int_0^t \int_\Omega [b(u)_t - b(\hat{u})_t] \psi_\delta(u - \hat{u}) + I_1(\delta) + I_2(\delta) + \int_0^t \int_\Omega [g(\cdot,u) - g(\cdot,\hat{u})] \psi_\delta(u - \hat{u}) \\ &\quad = \int_0^t \int_\Omega \left(f - \hat{f}\right) \psi_\delta(u - \hat{u}), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} I_1(\delta) &= \frac{1}{\delta} \int_0^t \int_{A_\delta} \{\phi[\nabla u - K(b(u))\mathrm{e}] - \phi[\nabla \hat{u} - K(b(\hat{u}))\mathrm{e}]\} \\ &\quad \cdot [\nabla u - K(b(u))\mathrm{e} - \nabla\hat{u} + K(b(\hat{u}))\mathrm{e}] \\ I_2(\delta) &= \frac{1}{\delta} \int_0^t \int_{A_\delta} \{\phi[\nabla u - K(b(u))\mathrm{e}] - \phi[\nabla\hat{u} - K(b(\hat{u}))\mathrm{e}]\} \cdot \mathrm{e}[K(b(u)) - K(b(\hat{u}))], \end{aligned}$$

with $A_\delta = \{(t,x) : 0 < u(t,x) - \hat{u}(t,x) < \delta\}$. By Young's inequality, we have that for any $\varepsilon > 0$

$$\begin{aligned} I_2(\delta) \le & \frac{\varepsilon}{\delta p'} \int_0^t \int_{A_\delta} \left|\phi\left[\nabla u - K\left(b\left(u\right)\right)\mathrm{e}\right] - \phi\left[\nabla \hat{u} - K\left(b\left(\hat{u}\right)\right)\mathrm{e}\right]\right|^{p'} \\ & + \frac{C}{\delta \varepsilon p} \int_0^t \int_{A_\delta} \left|K\left[b(u)\right] - K\left[b\left(\hat{u}\right)\right]\right|^p . \end{aligned}$$

From (2.5) and (2.1) we obtain

$$I_2(\delta) \le \frac{\varepsilon}{p'} I_1(\delta) + \frac{C}{\varepsilon p} \delta^{p\gamma - 1}.$$

Hence, if $p\gamma > 1$, we have

$$\lim_{\delta \to 0} \left[I_1(\delta) + I_2(\delta)\right] \ge 0. \tag{2.8}$$

In the case when $p\gamma = 1$, we obtain the same result because we integrate on a set whose measure goes to 0. From (2.1), (2.7), and (2.8), letting $\delta \to 0$, we have

$$\begin{aligned} \int_\Omega \max\left\{b(u(t)) - b(\hat{u}(t)), 0\right\} &= \int_0^t \int_{\{u > \hat{u}\}} \left[b\left(u\right) - b\left(\hat{u}\right)\right]_t \\ &\le C^* \int_0^t \int_\Omega \max\left\{b(u) - b(\hat{u}), 0\right\}. \end{aligned} \tag{2.9}$$

From Gronwall's lemma we deduce that $b(u) \le b(\hat{u})$. Using again (2.9) we also obtain that $b(u) = b(\hat{u})$ in the set A_δ. So $I_2(\delta) = 0$ and (2.7) gives $I_1(\delta) \le 0$. From (2.3) we obtain

$$\int_0^t \int_\Omega \frac{\left|\nabla \psi_\delta\left(u - \hat{u}\right)\right|^2}{\left|\nabla u - K\left(b\left(u\right)\right)\mathrm{e}\right|^{2-p} - \left|\nabla \hat{u} - K\left(b\left(\hat{u}\right)\right)\mathrm{e}\right|^{2-p}} \le 0.$$

Hence max $(0, \min\,(u - \hat{u}, \delta))$ is constant and this implies $u \le \hat{u}$ since it is true on $(0,T) \times \partial\Omega$.

(ii) Suppose first that $C^* = 0$. Using (2.9) and that $f_2 = \hat{f}_2$ we have

$$\begin{aligned} \int_\Omega \left[b\left[u\left(t,\cdot\right)\right] - b\left[\hat{u}\left(t,\cdot\right)\right]\right]_+ \le & \int_\Omega \left[b\left(u_0\right) - b\left(\hat{u}_0\right)\right]_+ \\ & + \int_0^t \int_\Omega \left|f_1\left(s,\cdot\right) - \hat{f}_1\left(s,\cdot\right)\right| \operatorname{sign}_+\left(u - \hat{u}\right) ds. \end{aligned}$$

Adding this inequality with the similar estimate obtained for $[b(u) - b(\hat{u})]_-$ the result follows. Suppose now that $C^* > 0$. Multiplying the equations by e^{-C^*t} (in the sense that we multiply the previous test functions by e^{-C^*t}) the integrand in $I_j(\delta)$ is also multiplied by e^{-C^*t} and we can apply (2.8). Hence, we have

$$\begin{aligned} & \int_0^t \int_\Omega e^{-C^*t} \left[b(u)_t - b\left(\hat{u}_t\right)\right] \operatorname{sign}_+\left(u - \hat{u}\right) ds \\ & \quad \le \int_0^t \int_\Omega C^* e^{-C^*t} \left[b\left[u\left(s,\cdot\right)\right] - b\left[\hat{u}\left(s,\cdot\right)\right]\right]_+ ds \\ & \quad\quad + \int_0^t \int_\Omega e^{-C^*t} \left(f_1 - \hat{f}_1\right) \operatorname{sign}_+\left(u - \hat{u}\right) ds. \end{aligned}$$

Define $v = e^{-C^* t} b(u)$ and $\hat{v} = e^{-C^* t} b(\hat{u})$. As $\text{sign}_+ (v - \hat{v}) = \text{sign}_+ (u - \hat{u})$ and $v_t = -C^* v + e^{-C^* t} b(u)_t$ we obtain

$$\int_0^t \int_\Omega [v_t - \hat{v}_t] \,\text{sign}_+ (v - \hat{v})\, ds \leq \int_0^t e^{-C^* t} \left[\int_\Omega \left(f_1 - \hat{f}_1 \right) \text{sign}_+ (v - \hat{v}) \right] ds.$$

As before we obtain in conclusion that

$$\begin{aligned} \|v(t,\cdot) - \hat{v}(t,\cdot)\|_{L^1(\Omega)} &\leq \|v_0 - \hat{v}_0\|_{L^1(\Omega)} \\ &\quad + \int_0^t e^{-C^* s} \left\| f_1(s,\cdot) - \hat{f}_1(s,\cdot) \right\|_{L^1(\Omega)} ds. \end{aligned}$$

Coming back to $b(u)$ and $b(\hat{u})$, we get the final estimate. □

We shall end this section by studying the comparison of solutions of the stationary problem

$$-\text{div}\, \phi(\nabla u_\infty - K(b(u_\infty))\,\mathbf{e}) + g(x, u_\infty) = f_\infty \quad \text{in } \Omega, \tag{2.10}$$

$$u_\infty = 0 \quad \text{on } \partial\Omega. \tag{2.11}$$

As a consequence, we shall prove the uniqueness of solutions of (2.10) and (2.11): a result which will be useful for the stabilization of bounded weak solutions of the model problem.

PROPOSITION 2. *Assume* (1.6) *and* (2.1) *and suppose that one of the following assumptions holds:*

(2.12) *$g(\cdot, \eta)$ is a strictly increasing function on η,*

(2.13) *$g(\cdot, \eta) = \hat{g}(\cdot, b(\eta))$ with $\hat{g}(\cdot, s)$ a strictly increasing function on s,*

(2.14) *$g(\cdot, \eta)$ is a nondecreasing function on η and we have one of the additional conditions:*

(a) *$p = 2$ and $N \geq 2$ or K is also a monotone function,*

(b) *$K(b(\eta)) = \lambda \eta$ for some $\lambda \in \mathbb{R}$,*

(c) *$1 < p \leq 2$ and there exists a constant*

$$\begin{aligned} &|\phi(\xi - K(b(\eta))\,\mathbf{e}) - \phi(\xi - K(b(\hat{\eta}))\,\mathbf{e})| \\ &\quad \leq |\eta - \hat{\eta}| \left(C + |\xi|^{p-1} + |\eta|^{p-1} + |\hat{\eta}|^{p-1} \right) \end{aligned}$$

for any $\zeta \in \mathbb{R}^N$ and $\eta, \hat{\eta} \in \mathbb{R}$.

Let $f_\infty, \hat{f}_\infty \in L^1(\Omega) + W^{-1,p'}(\Omega)$ such that $f_\infty \leq \hat{f}_\infty$ on Ω. Then for any $u_\infty, \hat{u}_\infty$ bounded weak solutions of the associated problems (2.10) *and* (2.11) *we have $u_\infty \leq \hat{u}_\infty$ on Ω. Moreover, in any case, if $f_\infty - \hat{f}_\infty \in L^1(\Omega)$ then*

$$\|g(\cdot, u_\infty) - g(\cdot, \hat{u}_\infty)\|_{L^1(\Omega)} \leq \left\| f_\infty - \hat{f}_\infty \right\|_{L^1(\Omega)}. \tag{2.15}$$

Proof. For the sake of the notation we drop the subindex ∞ in the data and solutions. Arguing as in the proof of Theorem 3 we have

$$I_1(\delta) + I_2(\delta) + \int_\Omega [g(\cdot, u) - g(\cdot, \hat{u})]\psi_\delta(u - \hat{u}) = \int_\Omega (f - \hat{f})\psi_\delta(u - \hat{u}) \leq 0,$$

where

$$I_1(\delta) = \frac{1}{\delta}\int_{A_\delta} \{\phi[\nabla u - K(b(u))\mathrm{e}] - \phi[\nabla \hat{u} - K(b(\hat{u}))\mathrm{e}] \cdot [\nabla u - K(b(u))\mathrm{e} - \nabla \hat{u} + K(b(\hat{u}))\mathrm{e}]\}$$

$$I_2(\delta) = \frac{1}{\delta}\int_{A_\delta} \{\phi[\nabla u - K(b(u))\mathrm{e}] - \phi[\nabla \hat{u} - K(b(\hat{u}))\mathrm{e}] \cdot \mathrm{e}\,[K(b(u)) - K(b(\hat{u}))]\},$$

and $A_\delta = \{0 < u - \hat{u} < \delta\}$.

(i) Assume first $1 < p \leq 2$. From (2.1) we have (as in the proof of (2.8)) that

$$\lim_{\delta \to 0}[I_1(\delta) + I_2(\delta)] \geq 0 \tag{2.16}$$

and so

$$\int_\Omega (g(\cdot, u) - g(\cdot, \hat{u}))\,\mathrm{sign}_+(u - \hat{u}) \leq \int_\Omega \left(f - \hat{f}\right)\mathrm{sign}_+(u - \hat{u}), \tag{2.17}$$

where $\mathrm{sign}_+(r) = 0$ if $r \leq 0$ and $\mathrm{sign}_+(r) = 1$ if $r > 0$. If (2.12) is satisfied we have that $\mathrm{sign}_+(u - \hat{u}) = \mathrm{sign}_+(g(\cdot, u) - g(\cdot, \hat{u}))$ and the conclusion is clear. Now suppose that (2.13) is verified. From (2.17) and the fact that $\mathrm{sign}_+(b(u) - b(\hat{u})) \leq \mathrm{sign}_+(u - \hat{u})$ we have

$$\int_\Omega (\hat{g}(\cdot, b(u)) - \hat{g}(\cdot, b(\hat{u})))\,\mathrm{sign}_+(b(u) - b(\hat{u})) \leq 0.$$

As $\hat{g}$ is strictly increasing we conclude that $b(u) \leq b(\hat{u})$. In particular $b(u) = b(\hat{u})$ on the set A_δ. Then $I_1(\delta) \leq 0$ and similarly to the evolution case the inequality $I_1(\delta) \leq 0$ and (2.3) imply that $u \leq \hat{u}$.

(ii) Assume now that $p > 2$. The proof of (2.17) is the following: using (2.3) we have

$$\begin{aligned}&\frac{C}{\delta}\int_{A_\delta} |\nabla(u - \hat{u})|^p + \int_\Omega (g(\cdot, u) - g(\cdot, \hat{u}))\,\psi_\delta(u - \hat{u}) \\ &\leq \frac{1}{\delta}\int_{A_\delta} \{\phi(\nabla \hat{u} - K(b(\hat{u})\mathrm{e})) - \phi(\nabla \hat{u} - K(b(u)\mathrm{e}))\} \cdot \{\nabla(u - \hat{u})\} = I_3(\delta).\end{aligned} \tag{2.18}$$

Using the Young inequality and the inequality of the Remark 3, we have

$$\begin{aligned}|I_3(\delta)| &\leq \frac{\varepsilon}{\delta}\int_{A_\delta} |\nabla(u - \hat{u})|^p + \frac{C(\varepsilon)}{\delta}\int_{A_\delta} |K(b(\hat{u})) - K(b(u))|^{p'} \\ &\quad \times \left(1 + \|K(b(u))\|^p_{L^\infty(\Omega)} + \|K(b(\hat{u}))\|^p_{L^\infty(\Omega)} + |\nabla \hat{u}|^p\right) \\ &\leq \frac{\varepsilon}{\delta}\int_{A_\delta} |\nabla(u - \hat{u})|^p + \frac{C(\varepsilon)}{\delta}\,\delta^{p'\gamma}\int_{A_\delta} H\end{aligned} \tag{2.19}$$

for some $H \in L^1(\Omega)$. As $p'\gamma - 1 \geq 0$, letting $\delta \to 0$ we obtain (2.17).

(iii) Assume now (2.14), i.e., $g(\cdot, \eta)$ is merely a nondecreasing function on η. Again we only have to prove that $u \leq \hat{u}$ because the inequality (2.15) is then a consequence of (2.17). For the special case $p = 2$ the uniqueness of the solution of (2.10), (2.11) was obtained in Carrillo and Chipot [18] under the assumption (2.1) (notice then that $\gamma \geq 1/2$) and $N \geq 2$ (see Theorem 6 of [18], [32], and [19]) or $N = 1$ and $K(b(\eta))$ a merely *continuous* monotone function (see Theorem 3, (ii) of [18]). Some easy modifications of the proofs lead to the comparison $u \leq \hat{u}$. In case (b), without loss of generality we can assume that $\mathbf{e} = \mathbf{e}_1$ where $\mathbf{e}_1$ is the first term of a orthonormal base of $\mathbb{R}^N$. Then we have

$$\phi(\nabla u - \lambda u \mathbf{e}_1) = e^{\lambda(p-1)x_1}\phi(\nabla(ue^{-\lambda x_1})).$$

Using $e^{-\lambda x_1}(u-\hat{u})_+$ as test function we obtain

$$\int_{\{u>\hat{u}\}} [\phi(\nabla(ue^{-\lambda x_1})) - \phi(\nabla(\hat{u}e^{-\lambda x_1}))] \cdot [\nabla(ue^{-\lambda x_1}) - \nabla(\hat{u}e^{-\lambda x_1})]\, e^{\lambda(p-1)x_1} \leq 0.$$

Applying (2.3) we conclude that $u \leq \hat{u}$. Assume finally the conditions of case (c). Let $\psi_\delta(u-\hat{u})$ the same test function of the proof of Theorem 3. Then, if we denote by C^* the constant in (2.3) we have

$$\begin{aligned}
&\frac{C^*}{\delta}\int_{A_\delta} \frac{|\nabla(u-\hat{u})|^2}{|\nabla u|^{2-p} + |\nabla \hat{u}|^{2-p}} \\
&\leq \int_\Omega [\phi(\nabla u - K(b(u))\mathbf{e}) - \phi(\nabla\hat{u} - K(b(u))\mathbf{e})] \cdot [\nabla\psi_\delta(u-\hat{u})] \\
&\leq \frac{1}{\delta}\int_{A_\delta} [\phi(\nabla\hat{u} - K(b(\hat{u}))\mathbf{e}) - \phi(\nabla\hat{u} - K(b(u))\mathbf{e})] \cdot \nabla(u-\hat{u}) \\
&\leq \frac{1}{\delta}\int_{A_\delta} |u-\hat{u}|\left(C + |\nabla u|^{p-1} + |u|^{p-1} + |\hat{u}|^{p-1}\right)|\nabla(u-\hat{u})|.
\end{aligned}$$

As in Boccardo, Gallouët, and Murat [16] we notice that for any $\tau > 0$

$$\begin{aligned}
|u-\hat{u}|\,|\nabla u|^{p-1}\,|\nabla(u-\hat{u})| &\leq \frac{C^*}{\tau}\left(|\nabla u|^{p-2} + |\nabla\hat{u}|^{p-2}\right)|\nabla(u-\hat{u})|^2 \\
&\quad + C(\tau)\,|u-\hat{u}|^2\,|\nabla u|^p \\
C\,|u-\hat{u}|\,|\nabla(u-\hat{u})| &\leq \frac{C^*}{\tau}\left(|\nabla u|^{p-2} + |\nabla\hat{u}|^{p-2}\right)|\nabla(u-\hat{u})|^2 \\
&\quad + C(\tau)\,|u-\hat{u}|^2\,|\nabla u|^{2-p} \\
|u-\hat{u}|\left(|u|^{p-1} + |\hat{u}|^{p-1}\right)|\nabla(u-\hat{u})| &\leq \frac{C^*}{\tau}\left(|\nabla u|^{p-2} + |\nabla\hat{u}|^{p-2}\right)|\nabla(u-\hat{u})|^2 \\
&\quad + C(\tau)\,|u-\hat{u}|^2\,|\nabla u|^{2-p}.
\end{aligned}$$

In consequence, taking τ large enough, we have that

$$\int_{A_\delta} \frac{|\nabla(u-\hat{u})|^2}{|\nabla u|^{2-p} + |\nabla\hat{u}|^{2-p}} \leq C\int_{A_\delta} |u-\hat{u}|^2 H_1 \leq C\delta^2 \int_{A_\delta} H_1$$

with

$$H_1 = |\nabla u|^p + |\nabla u|^{2-p} + |\nabla\hat{u}|^{2-p}.$$

Introducing the function

$$H_2 = |\nabla u|^{2-p} + |\nabla\hat{u}|^{2-p}$$

is not difficult to show (see [16]) that the condition $1 < p \leq 2$ implies that $H_1, H_2 \in L^1(\Omega)$. Then, by Cauchy–Schwarz we get

$$\delta \int_\Omega |\nabla \psi_\delta (u - \hat{u})| = \int_{A_\delta} |\nabla (u - \hat{u})| \leq \left(C\delta^2 \int_{A_\delta} H_1 \right)^{1/2} \left(\int_{A_\delta} H_2 \right)^{1/2}.$$

Finally, using the Poincarè inequality, for any fixed $\mu, \mu > \delta$, we obtain that

$$\begin{aligned} \operatorname{meas} \{u - \hat{u} \geq \mu\} \leq \int_\Omega |\psi_\delta (u - \hat{u})| &\leq C \int_\Omega |\nabla \psi_\delta (u - \hat{u})| \\ &\leq C \left(\int_{A_\delta} H_1 \right)^{1/2} \left(\int_{A_\delta} H_2 \right)^{1/2}. \end{aligned}$$

Letting $\delta \to 0$ we get the conclusion since $\int_{A_\delta} H_i \to 0$ as $\delta \to 0$, for $i = 1, 2$. □

Remark 4. When $N = 1$ the assumption (2.1) in Theorems 3 and 4 can be generalized to the mere assumption that $K(b(\eta))$ be a continuous function of η. This is an easy adaptation of a result due to Benilan [9] (see also Wolanski [55]). We mention the papers Kalashnikov [39], Ishii [36], and Yin Jingxue [57], [58] where the authors prove the uniqueness of the solution of different special cases of the model equation on $\Omega = \mathbb{R}^N$ and without the regularity condition (2.6). The comparison properties of some special bounded solutions without condition (2.6) will be given in the next section.

We also point out that when $g(\cdot, \eta) = \lambda\eta$ with $\lambda > 0$, the estimate (2.15) proves that the abstract operator associated to $\mathcal{A}$ is an accretive operator in $L^1(\Omega)$ (see Benilan [8], [9] and Crandall [21] for the theory and applications of this class of operators). The proof of the case (c) of Proposition 2 is inspired in Boccardo, Gallouët, and Murat [16] (his result cannot be directly applied because their assumption (1) is not satisfied in our case). Finally we remark that a revision of the proof of part (c) shows that the conclusion still holds if the assumed inequality is verified merely for any $\xi \in R_\infty \equiv \{\zeta = \nabla u_\infty(x)$, for some $x \in \Omega\}$ and any $\eta, \hat{\eta} \in [-M, M]$ with $M = \max\{\|u_\infty\|, \|\hat{u}_\infty\|_\infty\}$. In particular it holds if we assume $u_\infty \in W^{1,\infty}(\Omega)$ [or $\hat{u}_\infty \in W^{1,\infty}(\Omega)]$, $K(b(\cdot))$ locally Lipschitz continuous and $\xi - K(b(\eta))\mathbf{e} \neq \mathbf{0}$ for any $\xi \in R_\infty$ and $\eta \in [-M, M]$. We point out that in the case of Model 1 of the Introduction the function K is usually taken as a regular function such that $K(s) > c > 0$ for any $s \in \mathbb{R}$ and some $c > 0$ (see Bear [7], p. 492).

3. Existence of bounded weak solutions for the model problem. In order to obtain the existence of bounded weak solutions of the model problem (0.1) we shall need to assume some additional conditions on K, g, and f besides the already explicit ones in §2. So, the structure assumption (1.3) will require to assume that

$$\text{(3.1)} \qquad \begin{cases} K \text{ is a continuous real function such that} \\ |K(b(\eta))| \leq C\left(1 + |\eta|^\lambda\right) \text{ for all } \eta \in \mathbb{R} \text{ and some } \lambda \in [0, p^*/p]. \end{cases}$$

Moreover the boundedness condition of the solution under investigation will be obtained by assuming that $g(\cdot, u)$ satisfies (1.7), (2.2), and also

$$\text{(3.2)} \qquad g(x, \eta)\,\eta \geq 0 \quad \text{for any } \eta \in \mathbb{R} \quad \text{and a.e. } x \in \Omega.$$

Concerning the right-hand side term, we have

(3.3)

$$\begin{cases} f \in L^1((0,T)\times\Omega) + L^p\left(0,T : W^{-1,p'}(\Omega)\right) \text{ and } |f(t,x)| \le \bar{f}(x) := \operatorname{div} \mathbf{c}(x), \\ \text{for some } \mathbf{c} \in (L^q(\Omega))^N \text{ with } q > N/(p^{\#}-1) \\ \text{if } p^{\#} \le N, q = \max\left(p', (p^{\#})'\right) \text{ if } p^{\#} > N, \\ \text{where } p^{\#} = \max(p,2). \end{cases}$$

We shall obtain the existence of a bounded weak solution of (0.1) by using comparison arguments and suitable super- and subsolutions of the associated stationary problem. The concrete statement requires previously the next result.

LEMMA 3. *Let $\overline{f}$ be given by* (3.3). *Assume that g satisfies* (1.7) *and* (3.2). *Then there exists $\underline{u}, \overline{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $\underline{u} \le 0 \le \overline{u}$ satisfying*

$$-\operatorname{div} \phi\left(\nabla \underline{u} - K(b(\underline{u}))\,\mathbf{e}\right) + g(x,\underline{u}) = -\bar{f}(x) \quad \text{in } \Omega, \tag{3.4}$$

and

$$-\operatorname{div} \phi\left(\nabla \bar{u} - K(b(\bar{u}))\,\mathbf{e}\right) + g(x,\bar{u}) = \bar{f}(x) \quad \text{in } \Omega, \tag{3.5}$$

where again $\phi(\xi) = |\xi|^{p-2}\xi$ and $p > 1$.

We postpone the proof to later and state our existence result on bounded weak solutions of (0.1).

THEOREM 4. *Assume that the hypothesis* (1.6), (1.7), (2.1), (2.2), and (3.1)–(3.3) *are satisfied. Suppose $u_0 \in L^\infty(\Omega)$ be such that*

$$\underline{u}(x) \le u_0(x) \le \bar{u}(x) \quad \text{a.e. } x \in \Omega. \tag{3.6}$$

Then there exists a bounded weak solution u of problem (0.1).

Proof of Lemma 3. Define $\mathbf{a} : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$\mathbf{a}(\eta,\xi) = \phi\left(\xi - K(b(\eta))\,\mathbf{e}\right) + \phi\left(K(b(\eta))\,\mathbf{e}\right).$$

From (3.1) we deduce that

$$|\mathbf{a}(\eta,\xi)| \le C\left(|\eta|^{p^*/p} + |\xi|^{p-1}\right)$$

and from (2.3)

$$C|\xi|^p \le \left(\mathbf{a}(\eta,\xi)\cdot\xi\right)^{\alpha/2}\left(|\eta|^{p^*} + |\xi|^p\right)^{1-\alpha/2}$$

with $\alpha = 2$ if $p \ge 2$ and $\alpha = p$ if $1 < p \le 2$. Equation (3.5) can be equivalently written as

$$-\operatorname{div} \mathbf{a}(\bar{u}, \nabla\bar{u}) + g(x,\bar{u}) = \operatorname{div}\left(\mathbf{c}(x) + \mathbf{h}(u)\right),$$

with

$$\mathbf{h}(u) = \phi\left(K(b(\bar{u}))\,\mathbf{e}\right).$$

Assumption (3.1) implies that

$$|\mathbf{h}(\eta)| \le C\left(1 + |\eta|^{\mu}\right) \quad \text{with } \mu \in [0, p^*/p']$$

and so the existence of $\overline{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying (3.5) is assured (if $p \geq 2$) by Theorem 2 of Boccardo and Giachetti [17]. The case $1 < p < 2$ can be obtained by obvious modifications of the mentioned work. This function $\overline{u}$ also satisfies that $\overline{u} \geq 0$ on Ω as we deduce by standard methods once we are assuming (3.2) and div $\mathbf{c}(x) \geq 0$. The existence of $\underline{u}$ is proved analogously. □

Proof of Theorem 4. Let $M > 0, M \geq \max\left(\|\overline{u}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)}\right), \varepsilon > 0$ and $k, j, m, n \in \mathbb{N}^*$. We consider the regularized equation:

(3.7)

$$b_m(u)_t - \varepsilon\Delta u - \text{div}\, \phi\left[\nabla u - K_j\left(b_m(u)\right)\mathbf{e}\right] + g_n(x,u) = f_{k,1}(t,x)\,\theta(u) + f_{k,2}(t,x),$$

where

$$\begin{cases} b_m(\eta) = \frac{1}{m}\eta + \bar{b}_m(\eta) \text{ with } \bar{b}_m \text{ the Yosida approximation of } b \text{ (it is well known} \\ \text{that } \bar{b}_m \text{ is a Lipschitz nondecreasing function such that } |\bar{b}_m| \leq |b| \text{ and} \\ \bar{b}_m \to b; \text{ see, e.g., Benilan [8], [9]);} \end{cases}$$

$$\begin{cases} K_j \in C^\infty(\mathbb{R}) \text{ satisfies } \|K_j\|_{L^\infty} \leq \hat{K}, \text{ where} \\ \hat{K} = \sup\limits_{s\in[-2M,2M]} |K[b_1(s)]| \text{ and } K_j \to K\mathbf{1}_{[b_1(-2M),b_1(2M)]} \text{ as } j \to +\infty; \end{cases}$$

$$\begin{cases} g_n \in C^\infty(\Omega \times \mathbb{R}) \text{ satisfies (uniformly on } l) \text{ (1.7), (2.2), and} \\ g_n(x,\eta) \to g(x,\eta) \text{ in } L^1(\Omega) \text{ for any fixed } \eta \text{ and in } \mathbb{R} \text{ for a.e. } x \in \Omega \text{ as } n \to \infty; \end{cases}$$

$$\begin{cases} f_k \in C^\infty\left((0,T) \times \bar{\Omega}\right) \text{ satisfies (uniformly on } n) \text{ the inequality (3.3),} \\ f_k = f_{k,1} + f_{k,2} \text{ and } f_{k,1} \to f_1 \text{ in } L^1\left((0,T)\times\Omega\right), \\ f_{k,2} \to f_2 \text{ in } L^{p'}\left((0,T) : W^{-1,p'}(\Omega)\right) \quad \text{as } k \to \infty; \end{cases}$$

$$\begin{cases} \theta \text{ is a truncation function satisfying } \theta \in C^\infty(\mathbb{R}), \quad 0 \leq \theta \leq 1, \\ \theta(\eta) = 1 \text{ for } |\eta| \leq M \text{ and } \theta(\eta) = 0 \quad \text{for } |\eta| \geq 2M. \end{cases}$$

We also consider the regularized stationary equation

$$-\varepsilon\Delta u - \text{div}\, \phi\left[\nabla u - K_j\left[b_m(u)\right]\mathbf{e}\right] + g_n(x,u) = \bar{f}(x) \text{ in } \Omega. \tag{3.8}$$

Applying Lemma 3 we obtain a function $\overline{u}_{\varepsilon,j,m,n} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ satisfying (3.8). Analogously we get the existence of the associated function $u_{\varepsilon,j,m,n}$. Finally we regularize the initial condition by considering $u_{0,q} \in C_0^\infty(\Omega)$ such that $\underline{u}_{\varepsilon,j,m,n} \leq u_{0,q} \leq \overline{u}_{\varepsilon,j,m,n}$ and $u_{0,q} \to u_0$ in $L^\infty(\Omega)$ as $q \to \infty$. Equation (3.7) is uniformly parabolic and so by well-known results (see, e.g., Ladyzenskaya, Solonnikov, and Uraltceva [42, Chapt. V]) there exists a unique classical solution $U = u_{\varepsilon,m,j,n,k}$ of (3.7) satisfying

$$\begin{cases} U = 0 & \text{on } (0,T) \times \partial\Omega, \\ b_m(U(0,x)) = b_m\left(u_{0,q}(x)\right) & \text{in } \Omega. \end{cases} \tag{3.9}$$

In order to study the convergence of $u_{\varepsilon,m,j,k,n}$ we need the following result.

LEMMA 4. *The solution U of* (3.7) *and* (3.9) *is bounded in $L^p(0,T : W_0^{1,p}(\Omega))$ and this bound does not depend on ε, j, k, m, n.*

Proof of Lemma 4. We use again the notation $U = u$. Multiplying (3.7) by u, we have

$$\begin{aligned}
&\int_0^T \int_\Omega b_m(u)_t u + \varepsilon \int_0^T \int_\Omega |\nabla u|^2 + \int_0^T \int_\Omega |\nabla u - K_j [b_m(u] \mathrm{e}|^p \\
&= -\int_0^T \int_\Omega \phi [\nabla u - K_j [b_m(u)] \mathrm{e}] \cdot K_j [b_m(u)] \mathrm{e} - \int_0^T \int_\Omega g_n(x,u)u + \int_0^T \int_\Omega f_k(t,x)u \\
&\le \frac{1}{p} \int_0^T \int_\Omega |\nabla u - K_j [b_m(u)] \mathrm{e}|^p + \frac{1}{p} \int_0^T \int_\Omega |K_j (b_m(u)) \mathrm{e}|^p \\
&\quad + \int_0^T \int_\Omega f_{k,1}(t,x)\theta(u)u + \int_0^T \langle f_{k,2}(t,x), u \rangle .
\end{aligned}$$

But

$$\int_0^T \int_\Omega f_{k,1}\theta(u)u \le M \|f_{k,1}\|_{L^1((0,T)\times\Omega)}$$

and using Young's inequality there exists a constant $C > 0$ such that

$$\begin{aligned}
\int_0^T \langle f_{k,2}, u \rangle &\le \|f_{k,2}\|_{L^{p'}(0,T:W^{-1,p'}(\Omega))} \left[\int_0^T \int_\Omega |\nabla u|^p \right]^{1/p} \\
&\le C \|f_{k,2}\|^{p'}_{L^{p'}(0,T:W^{-1,p'}(\Omega))} \\
&\quad + \frac{1}{2} \left(\int_0^T \int_\Omega |\nabla u - K_j [b_m(u)] \mathrm{e}|^p + \int_0^T \int_\Omega |K_j [b_m(u)] \mathrm{e}|^p \right) .
\end{aligned}$$

So, we obtain

$$\int_\Omega B_m [u(T)] + \int_0^T \int_\Omega |\nabla u - K_j [b_m(u)] \mathrm{e}|^p \le M^*(T),$$

for some $M^*(T) > 0$. Hence the result. □

End of the proof of Theorem 4. As $b_m(U)_t \in L^1((0,T)\times\Omega)$ we can apply Theorem 3 and conclude the inequality $\underline{u}_{\varepsilon,j,m,n}(x) \le U(t,x) \le \overline{u}_{\varepsilon,j,m,n}(x)$ for $(t,x) \in (0,T)\times\Omega$. Moreover, a careful revision of the proof of Theorem 2 of [17] allows to check that $\|\overline{u}_{\varepsilon,j,m,n}\|_{L^\infty(\Omega)}$ is bounded by a constant independent of ε, j, m, and n. Using this fact, Lemma 4, and proceeding as in Theorem 1, we can pass to the limit as $\varepsilon \to 0$ and $m, j, k, n \to +\infty$, obtaining that $\overline{u}_{\varepsilon,j,m,n} \to \overline{u}, \underline{u}_{\varepsilon,j,m,n} \to \underline{u}$ at least weakly* in $L^\infty(\Omega)$, weakly in $W_0^{1,p}(\Omega)$, and also that $U \to u$ weakly in $L^p(0,T : W_0^{1,p}(\Omega))$ with u as a bounded weak solution of (0.1), satisfying

$$\underline{u}(x) \le u(t,x) \le \bar{u}(x) \quad \text{for } (t,x) \in (0,T) \times \Omega. \tag{3.10}$$

□

Remark 5. When b is assumed to be *strictly increasing* the existence of a bounded weak solution of (0.1) can be obtained for *any* $u_0 \in L^\infty(\Omega)$ (i.e., not necessarily satisfying (3.6)) if we suppose $f \in L^1(0,T : L^\infty(\Omega))$. Indeed, in that case we can repeat the proof of Theorem 3 but replacing $\overline{u}(x)$ by the supersolution

$$\bar{u}(t,x) = b^{-1} \left(\|u_0\|_{L^\infty(\Omega)} + \int_0^t \|f(s,\cdot)\|_{L^\infty(\Omega)} \, ds \right) .$$

The process followed in the proof of Theorem 4 is useful to obtain general comparison results. Indeed, Theorem 3 assumes the regularity condition (2.6) which is very hard to check in some cases. We shall show in §4 that this condition is verified by any bounded weak solution of (0.1) if we additionally suppose that b is a locally Lipschitz function. Nevertheless we have the following result.

COROLLARY 1. *Assume the same hypotheses on b, K, and g given in Theorem 4. Let $(f,u_0),(\hat{f},\hat{u}_0)$ be a couple of data satisfying* (3.3), (3.6), *and the analogous versions for $\hat{f}$ and $\hat{u}_0$. Assume also that $f \leq \hat{f}$ and $u_0 \leq \hat{u}_0$. Then there exist u and $\hat{u}$ weak solutions of the associated problems* (0.1) *such that $u \leq \hat{u}$ in $(0,T) \times \Omega$.*

Proof. Let $U = u_{\varepsilon,m,j,n,k}$ and $\hat{U} = \hat{u}_{\varepsilon,m,j,n,k}$ be the classical solutions obtained by the process described in the proof of Theorem 4 associated to the regularization of the data $f_k, \hat{f}_k, u_{0,q}$ and $\hat{u}_{0,q}$. Without loss of generality we can assume that $f_k \leq \hat{f}_k$ and $u_{0,q} \leq \hat{u}_{0,q}$. Then, as $b_m(U)_t, b_m(\hat{U})_t \in L^1((0,T) \times \Omega)$ we can apply Theorem 3 and obtain $U \leq \hat{U}$. Finally, the conclusion follows by passing to the limit as $\varepsilon \to 0$ and $m,j,n,k \to +\infty$. □

Remark 6. As far as we know the existence of solutions for the model problem has not be treated in the literature. Nevertheless there are many papers which obtain the existence of solutions for some similar problems. We mention explicitly the important work by Alt and Lukhaus [3] and their generalization made in Kaçur [37], [38]. Another point of view is presented in Blanchard and Francfort [13], [14]. Other related works are due to Bermudez, Durany, and Saguez [10], Bernis [11], Esteban and Vazquez [31], Simondon [52], Tsutsumi [53] and Xu [56] (see also the references in the mentioned papers). The comparison between solutions which are limits of sequences of more regular solutions is already an old argument (see Benilan [8], Bamberger [5], [6], and Blanchard and Francfort [14]).

4. Stabilization results for the model problem. Theorem 1 reduces the stabilization of bounded weak solutions to the study of conditions (1.16) and (1.17). We shall start this section by showing that the comparison principle (Corollary 1) and the uniqueness of solutions of the stationary problem (1.14) allows reduction of the stabilization property to the study of condition (1.16) for solutions that are monotone in time.

PROPOSITION 3. *Assume the hypotheses* (1.6) *on b,* (2.1) *and* (3.1) *on K, and* (1.7) *and* (2.2) *on g. Assume also that the stationary problem* (2.10) *and* (2.11) *has a unique bounded weak solution. Let f and f_∞ satisfy* (1.8), (1.13), *and* (3.3), *and assume that there exists $f_+(t,x), f_-(t,x)$ satisfying* (1.8) *with f_+ (respectively, f_-) monotone nonincreasing in t (respectively, nondecreasing) and such that*

$$-\bar{f}(x) \leq f_-(t,x) \leq f(t,x) \leq f_+(t,x) \leq \bar{f}(x) \quad \text{in } (0,\infty) \times \Omega, \tag{4.1}$$

($\bar{f}$ given in (3.3)*) and also satisfying*

$$\lim_{t\to\infty} f_+(t,\cdot) = \lim_{t\to\infty} f_-(t,\cdot) = f_\infty(\cdot) \quad \text{in } L^1(\Omega) + W^{-1,p'}(\Omega). \tag{4.2}$$

Let u, u_+ and u_- be the bounded weak solutions of (0.1) *associated to the data (f,u_0), $(f_+,\overline{u})$, and $(f_-,\overline{u})$, respectively, assured by Corollary* 1 *(with $\overline{u}, \underline{u}$ given in Lemma* 3*). Then if u_+, u_- satisfies* (1.16) *for any $u_\infty \in \omega(u)$ we deduce that u_∞ is a bounded solution of the stationary problem and in fact*

$$u(t,\cdot) \to u_\infty \quad \text{in } L^r(\Omega), \text{ as } t \to \infty, \text{ for any } r \in [1,\infty).$$

Proof. We shall follows closely an argument already used in Kröner and Rodrigues [41] (Theorem 6). First of all we point out that the assumptions made on g imply the condition (3.2) and then by Corollary 1 and the proof of Theorem 1 we deduce the existence of the mentioned bounded weak solutions u, u_+, and u_-. Moreover, we have

$$\underline{u}(x) \leq u_-(t,x) \leq u(t,x) \leq u_+(t,x) \leq \bar{u}(x) \qquad (t,x) \in (0,\infty) \times \Omega. \tag{4.3}$$

By comparison on the related regularized solutions and using (3.4) and (3.5) it is easy to see that $u_+(t,\cdot)$ (respectively, $u_-(t,\cdot)$) is monotone nonincreasing in t (respectively, nondecreasing). Then there exists $u_{\infty,+}(x), u_{\infty,-}(x)$ such that

$$u_+(t,\cdot) \to u_{\infty,+}, u_-(t,\cdot) \to u_{\infty,-} \text{ in } L^r(\Omega)\,, \text{ for any } r \in [1,\infty) \text{ as } t \to \infty. \tag{4.4}$$

From (4.3) we deduce that

$$u_{\infty,-}(x) \leq u_\infty(x) \leq u_{\infty,+}(x) \quad \text{in } \Omega. \tag{4.5}$$

Now as u_+, u_- satisfies (1.16) and (1.17) (due to (4.4)) then Theorem 1 shows that $u_{\infty,+}$ and $u_{\infty,-}$ are bounded weak solutions of the same associated stationary problem (i.e., (1.14) with **A** given by (1.5); recall (4.2)). Finally, from the uniqueness of the bounded weak solution of the stationary problem and (4.5) we deduce that $u_{\infty,-} = u_\infty = u_{\infty,+}$ and so we have the conclusion. □

The important assumption (1.16) will be obtained in the two following results.

THEOREM 5. *Assume*

$$1 < p \leq 2, \tag{4.6}$$

g satisfies (1.7), (2.2), (3.2), *and f verifies* (3.3) *and*

(4.7)

$$\begin{cases} f \in L^\infty\left(0,\infty : L^1(\Omega) + W^{-1,p'}(\Omega)\right) \cap W^{1,1}_{\text{loc}}\left(0,\infty : L^1(\Omega) + W^{-1,p'}(\Omega)\right) \text{ and} \\ \displaystyle\int_t^{t+1}\int_\Omega \left\|\frac{\partial f}{\partial t}\right\|_{L^1(\Omega)+W^{-1,p'}(\Omega)} \leq C, \text{ for any } t > 0 \text{ and some } C \text{ independent on } t. \end{cases}$$

Let u_0 satisfy (3.6) *and also*

$$u_0 \in W_0^{1,p}(\Omega)\,. \tag{4.8}$$

Finally, assume one of the following set of hypothesis:

(A) $\begin{cases} (4.9) & b \text{ is a nondecreasing locally Lipschitz function,} \\ (4.10) & K \text{ is a locally Lipschitz function satisfying (3.1),} \end{cases}$

or

(B) $\begin{cases} (4.11) & b^{-1} \text{ is a nondecreasing locally Lipschitz function,} \\ (4.12) & K(b(\cdot)) \text{ is a locally Lipschitz function satisfying (3.1) holds.} \end{cases}$

Then if u is the bounded weak solution given in Theorem 4 *we have $u \in L^\infty(0,\infty : W_0^{1,p}(\Omega))$. Moreover, for any $t > 0$ and some $C > 0$ independent of t we have*

$$\int_t^{t+1}\int_\Omega |b(u)_t| \leq C \tag{4.13}$$

when (A) *is satisfied, and*

$$\int_t^{t+1}\int_\Omega |u_t| \le C \tag{4.14}$$

if (B) *holds.*

Proof. Assume that case (A) holds. Multiplying (0.1) by u, we have for any τ, σ satisfying $\tau > \sigma \ge \tau - 1 > 0$

$$\int_\sigma^\tau \int_\Omega |\nabla u - K[b(u)]\mathrm{e}|^p \le C. \tag{4.15}$$

This is obtained by an easy adaptation of the proof of Lemma 4. Let us define

$$E(t) = \int_\Omega |\nabla u(t) - K[b(u(t))]\mathrm{e}|^p .$$

Assume that u_t is regular enough (otherwise we first work with the approximate solution $u_{\epsilon,m,j,k}$ and then pass to the limit). Taking $v = u_t$ in (1.11) we get

(4.16)

$$\int_\sigma^\tau \int_\Omega b(u)_t u_t + E(\tau) - E(\sigma) + J + \int_\Omega G(\cdot, u(\tau)) - \int_\Omega G(\cdot, u(\sigma)) = \int_\sigma^\tau \int_\Omega f u_t,$$

where $G(x,\cdot)$ is the primitive of $g(x,\cdot)$ and

$$J = \int_\sigma^\tau \int_\Omega \phi\left[\nabla u - K\left[b\left(u\right)\right]\mathrm{e}\right] \frac{dK}{ds}[b(u)]b(u)_t.$$

Then, using (4.10) and (4.6) and Young's inequality

$$\begin{aligned} J &\le \varepsilon \int_\sigma^\tau \int_\Omega |b(u)_t|^2 + C\left(\varepsilon\right) \int_\sigma^\tau \int_\Omega |\nabla u - K[b(u)]\mathrm{e}|^{2(p-1)} \\ &\le \varepsilon \int_\sigma^\tau \int_\Omega |b(u)_t|^2 + C\left(\varepsilon\right)\left[\left(\tau - \sigma\right)|\Omega|\right]^{1/q'} \left\{ \int_\sigma^\tau \int_\Omega |\nabla u - K(b(u))\mathrm{e}|^p \right\}^{1/q} \end{aligned}$$

with $q = p/(2(p-1))$. From (4.15) we obtain

$$J \le \varepsilon \int_\sigma^\tau \int_\Omega |b(u)_t|^2 + C'\left(\varepsilon\right).$$

Moreover, by (4.7) and (3.10) if $f = f_1 + f_2$ with $f_1 \in L^\infty(0,\infty : L^1(\Omega))$ and $f_2 \in L^\infty(0,\infty : W^{-1,p'}(\Omega))$ we have

$$\begin{aligned} \int_\sigma^\tau \int_\Omega f_1 u_t &= \int_\Omega f_1(\tau,\cdot)u(\tau) - \int_\Omega f_1(\sigma,\cdot)u(\sigma) - \int_\sigma^\tau \int_\Omega \frac{\partial f_1}{\partial t} \\ &\le C_1 + \left(\left(\tau - \sigma\right)|\Omega|\right)^{1/2} M \left(\int_\sigma^\tau \int_\Omega \left| \frac{\partial f_1}{\partial t} \right|^2 \right)^{1/2} \le C_2. \end{aligned}$$

The term $\int_\sigma^\tau \langle f_2, u_t\rangle$ is treated in an analogous way to the proof of Lemma 4. From (4.9) we have

$$\frac{1}{L}\int_\sigma^\tau\int_\Omega |b(u)_t|^2 \le \int_\sigma^\tau\int_\Omega b(u)_t u_t$$

for some $L = L(M) > 0$. Then, we obtain (for ϵ small enough)

$$\frac{1}{2L}\int_\sigma^\tau\int_\Omega |b(u)_t|^2 + E(\tau) - E(\sigma) \le C. \tag{4.17}$$

In case (B) we use (4.11) and conclude

$$\frac{1}{L}\int_\sigma^\tau\int_\Omega (u_t)^2 \le \int_\sigma^\tau\int_\Omega b(u)_t u_t.$$

Moreover, by (4.10)

$$\begin{aligned}\int_\sigma^\tau\int_\Omega &|\nabla u - K[b(u)]\mathrm{e}|^{p-1}\left|\frac{d}{ds}(K\circ b)(u)\right||u_t|\\ &\le \frac{1}{4L}\int_\sigma^\tau\int_\Omega (u_t)^2 + C\int_\sigma^\tau\int_\Omega |\nabla u - K(b(u))\mathrm{e}|^{2(p-1)}\end{aligned}$$

and as in the case (A) we arrive to

$$\frac{1}{2L}\int_\sigma^\tau\int_\Omega (u_t)^2 + E(\tau) - E(\sigma) \le C. \tag{4.18}$$

The result follows from the following well-known result. □

LEMMA 5 (Nakao [49]). *Let $\varphi(t) \ge 0$ be a locally bounded function satisfying*

$$\varphi(t+1) \le C\,[\varphi(t) - \varphi(t+1)] + \rho(t) \quad \textit{for } t > 0,$$

where C is a positive constant and $\rho > 0$ for large t. Then as $t \to +\infty$ one has:

$$\varphi(t) = 0(1)\ [\textit{respectively},\ o(1)]\ \textit{if}\ \rho(t) = 0(1)\ [\textit{respectively}, o(1)]. \quad □$$

When $p > 2$ we shall prove that condition (1.16) holds at least for the super- and subsolutions u_+ and u_-.

THEOREM 6. *Assume*

$$p \ge 2 \tag{4.19}$$

and suppose the same hypothesis than in Proposition 3 but with $g(\cdot, u)$ merely a nondecreasing function satisfying (1.7). Then $u_+, u_- \in L^\infty(0,\infty : W_0^{1,p}(\Omega))$.

Proof. From the proof of Proposition 3 we know that $u_+(t,\cdot)$ is monotone and nonincreasing in t and then $b(u_+(t,\cdot))$ satisfies this same property. Taking as test function $v = \overline{u} - u_+(t,\cdot)$ in the conditions (1.11) for u_+ and (1.15) for $\overline{u}$ we obtain

$$\begin{aligned}I_1(t) &= \int_\Omega [\phi(\nabla\bar{u} - K(b(\bar{u}))\mathrm{e}) - \phi(\nabla u_+(t,\cdot) - K(b(u_+(t,\cdot)))\mathrm{e})]\cdot(\nabla\bar{u} - \nabla u_+(t,\cdot))\\ &\le \int_\Omega (\bar{f} - f_+(t,\cdot))(\bar{u} - u_+(t,\cdot)),\end{aligned}$$

where we have used that $v(t,\cdot) \geq 0$ for almost every $t > 0$ (see (4.3)). As in the proof of Theorem 3 we write

$$I_1(t) = I_2(t) + I_3(t)$$

with

$$\begin{aligned} I_2(t) = \int_\Omega &\{\phi\left[\nabla\bar{u} - K\left(b\left(\bar{u}\right)\right)\mathrm{e}\right] - \phi\left[\nabla u_+(t,\cdot) - K\left(b\left(u_+\left(t,\cdot\right)\right)\right)\mathrm{e}\right]\} \\ &\cdot \{\nabla\bar{u} - K\left(b\left(\bar{u}\right)\right)\mathrm{e} - \nabla u_+(t,\cdot) + K\left(b\left(u_+(t,\cdot)\right)\right)\mathrm{e}\} \\ I_3(t) = \int_\Omega &\{\phi\left[\nabla\bar{u} - K\left(b\left(\bar{u}\right)\right)\mathrm{e}\right] - \phi\left[\nabla u_+(t,\cdot) - K\left(b\left(u_+\left(t,\cdot\right)\right)\right)\mathrm{e}\right]\} \\ &\cdot \mathrm{e}\left[K\left(b\left(\bar{u}\right)\right) - K\left(b\left(u_+(t,\cdot)\right)\right)\right]. \end{aligned}$$

By Young's inequality we have that for any $\varepsilon > 0$

$$\begin{aligned} I_3(t) \leq &\frac{\varepsilon}{p'}\int \left|\phi\left[\nabla\bar{u} - K\left(b\left(\bar{u}\right)\right)\mathrm{e}\right] - \phi\left[\nabla u_+(t,\cdot) - K\left(b\left(u_+(t,\cdot)\right)\right)\mathrm{e}\right]\right|^{p'} \\ &+ \frac{C}{\varepsilon p}\int_\Omega \left|K\left(b\left(\bar{u}\right)\right) - K\left(b\left(u_+(t,\cdot)\right)\right)\right|^p. \end{aligned}$$

Using (2.5) and (4.3) we have that

$$I_3(t) \leq \frac{\varepsilon}{p'} I_2(t) + C_2$$

for some $C_2 > 0$ independent of t. From (2.3), (4.2), and (4.3) we get

$$\int_\Omega \left|\nabla\bar{u} - \nabla u_+(t,\cdot) - \left(K\left(b\left(\bar{u}\right)\right) - K\left(b\left(u_+(t,\cdot)\right)\right)\mathrm{e}\right)\right|^p \leq C_3$$

for some $C_3 > 0$ independent of t. Finally using again that $u_+ \in L^\infty((0,\infty)\times\Omega)$ and that $\overline{u} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ the conclusion follows for u_+. The proof for u_- is analogous. □

COROLLARY 2. *Assume the conditions of Proposition* 3 *and also* (4.7), (4.8) *and* [(4.9), (4.10)] *or* [(4.11), (4.12)] *if* $1 < p \leq 2$. *Then if* u *is the bounded weak solution of* (0.1) *associated to the data* (f, u_0) *assured by Corollary* 1, *for any* $u_\infty \in \omega(u)$ *we have that* u_∞ *is a bounded weak solution of the stationary problem and in fact* $u(t,\cdot) \to u_\infty$ *in* $L^r(\Omega)$, *as* $t \to \infty$, *for any* $r \in [1,\infty)$.

Our last result proves the condition (1.21) for the special case of

$$K\left(b\left(s\right)\right) = \lambda s \quad \text{for some } \lambda \in \mathbb{R} \text{ and any } s \geq 0. \tag{4.20}$$

We point out that similar properties to (1.21) have been proved in the literature when the elliptic operator $\mathcal{A}$ is assumed to be the gradient (or subdifferential) of some potential functional (see, e.g., Langlais and Phillips [43], Tsutsumi [53] and El Hachimi and de Thelin [29], [30]) but we also remark that when $K \not\equiv 0$ the associated elliptic operator $\mathcal{A}$ does not satisfy this structure condition.

THEOREM 7. *Assume that* g *satisfies* (1.7), (2.2), (3.2), *and* f *verifies* (3.3) *and* (4.7). *Let* u_0 *satisfying* (3.6) *and* (4.8). *Assume* (4.20) *and that* b *satisfies* (4.9) *or* (4.11). *Then if* u *is the bounded weak solution given in Corollary* 1, u *satisfies that* $u \in L^\infty(0,\infty : W_0^{1,p}(\Omega))$. *Moreover we have that*

$$b(u)_t \in L^2\left((0,\infty)\times\Omega\right) \quad \textit{if } b \textit{ satisfies } (4.9) \tag{4.21}$$

and

$$u_t \in L^2((0,\infty)\times\Omega) \quad \textit{if (4.11) holds.} \tag{4.22}$$

Proof. Assume that b satisfies (4.9). Without loss of generality we can assume $\mathbf{e} = \mathbf{e_1}$ (the first term of the orthonormal base of $\mathbb{R}^N$); otherwise, it is enough to make a change of base on $\mathbb{R}^N$. Multiplying by $e^{-\lambda x_1} u_t$ we have (assuming u_t is regular) that for any $T > 0$ we have

$$\int_0^T \int_\Omega b(u)_t u_t e^{-\lambda x_1} + \frac{1}{p}\int_0^T \int_\Omega e^{\lambda(p-1)x_1} \frac{\partial}{\partial t}\left|\nabla e^{-\lambda x_1} u\right|^p$$

$$+ \int_0^T \int_\Omega e^{-\lambda x_1} \frac{\partial}{\partial t} G(\cdot, u) = \int_0^T \int_\Omega e^{-\lambda x_1} f u_t.$$

The same kind of arguments of the proof of Theorem 5 leads to the conclusion

$$\int_0^T \int_\Omega |b(u)_t|^2 + E(T) - E(0) \le C$$

with C independent of T and then (4.21) follows. The proof of (4.22) is similar. □

COROLLARY 3. *Assume the hypothesis of Theorem 7 and also b strictly increasing if (4.9) holds. Then if u is the bounded weak solution of (0.1) assured by Corollary 1 we have that $\omega(u) \neq \phi$ and any $u_\infty \in \omega(u)$ is a bounded weak solution of the stationary problem. Moreover, there exists $\tilde{t}_n \to +\infty$ such that $u(\tilde{t}_n, \cdot) \to u_\infty$ strongly in $W_0^{1,p}(\Omega)$.*

Proof. Taking $k(s) = b(s)$ if (4.9) holds and $k(s) = s$ if b satisfies (4.11), from Proposition 1 and (2.3) we have that Theorem 2 can be applied, leading to the conclusion. □

Acknowledgments. The authors thank the referees for several observations on a preliminary version of the paper. They also thank L. Boccardo for pointing out reference [17] to the first author.

REFERENCES

[1] N. AHMED AND D.K. SUNADA, *Nonlinear flow in porous media*, J. Hydraulics Div. Proc. Amer. Soc. Civil Engrg., 95 (1969), pp. 1847–1857.

[2] N.D. ALIKAKOS AND P.W. BATES, *Stabilization of solutions for a class of degenerate equations in divergence form in one space dimension*, J. Differential Equations, 73 (1988), pp. 363–393.

[3] H.W. ALT AND S. LUCKHAUS, *Quasilinear Elliptic Parabolic Differential Equations*, Math. Z, 183 (1983), pp. 311–341.

[4] M. ARTOLA, *Sur une classe de problémes paraboliques quasilinaires*, Boll. Un. Mat. Ital., 5-B (1986), pp. 51–70.

[5] A. BAMBERGER, *Etude d'une èquation doublement non linèaire*, J. Funct. Anal., 24 (1977), pp. 148–155.

[6] ———, *Etude d'une èquation doublement non linèaire*, Rapport du Centre de Mathematiques Appliquées, Ecole Polytechnique, 1977. (Extended version of [5].)

[7] J. BEAR, *Dynamics of Fluids in Porous Media*. Elsevier, New York, 1972.

[8] PH. BENILAN, *Equations d'èvolution dans un espace de Banach quelconque et applications*, thesis, Univ. d'Orsay, Orsay, France, 1972.

[9] ———, *Evolution equations and accretive operators*, Lecture Notes, Univ. of Kentucky, Lexington, KY, 1981.

[10] A. BERMUDEZ, J. DURANY, AND C. SAGUEZ, *An existence theorem for an implicit nonlinear evolution equation,* Collect. Math., 35 (1984), pp. 19–34.

[11] F. BERNIS, *Existence results for double nonlinear higher order parabolic equations on unbounded domains,* Math. Ann., 279 (1988), pp. 373–394.

[12] J.G. BERRYMAN AND C.J. HOLLAND, *Stability of the separable solution for fast diffusion,* Arch. Rational Mech. Anal., 74 (1980), pp. 379–388.

[13] D. BLANCHARD AND G. FRANCFORT, *Study of a double nonlinear heat equation with no growth assumptions on the parabolic term,* SIAM J. Math. Anal., 19 (1988), pp. 1032–1056.

[14] ———, *A few results on degenerate parabolic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 18 (1991), pp. 213–279.

[15] L. BOCCARDO, J.I. DIAZ, D. GIACHETTI, AND F. MURAT, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms,* J. Differential Equations, to appear.

[16] L. BOCCARDO, TH. GALLOUËT, AND F. MURAT, *Unicité de la solution de certaines équations elliptiques nonlinèaires,* C.R. Acad. Sci. Paris, 315 (1992), pp. 1159–1164.

[17] L. BOCCARDO AND D. GIACHETTI, *Existence results via regularity for some nonlinear elliptic problems,* Comm. Partial Differential Equations, 14 (1989), pp. 663–680.

[18] J. CARRILLO AND M. CHIPOT, *On some nonlinear elliptic equations involving derivatives of the nonlinearity,* Proc. Roy. Soc. Edinburgh Ser. A, 100 (1985), pp. 281–294.

[19] M. CHIPOT AND G. MICHAILLE, *Uniqueness results and monotonicity properties for strongly nonlinear elliptic variational inequalities,* Ann. Scuola Norm. Sup. Pisa, Cl. Sci., (1989), pp. 137–166.

[20] M. CHIPOT AND J.F. RODRIGUES, *Comparison and stability of solutions to a class of quasilinear parabolic problems,* Proc. Royal Soc. of Edinburgh Ser. A, 110 (1988), pp. 275–285.

[21] M.G. CRANDALL, *Nonlinear semigroups and evolution governed by accretive operators,* in Nonlinear Functional Analysis and Its Applications, F.E. Browder, ed., Proc. of Symposia in Pure Math., Vol. 45 (1986), pp. 305–338.

[22] J.I. DIAZ, *Nonlinear pde's and free boundaries, Vol.* 1, *Elliptic Equations,* Research Notes in Math. 106, Pitman, London, 1985.

[23] ———, *Nonlinear pde's and free boundaries, Vol.* 2, *Parabolic and Hyperbolic Equations,* in preparation.

[24] J.I. DIAZ AND M.A. HERRERO, *Estimates on the support of the solution of some nonlinear elliptic and parabolic problems,* Proc. Royal Soc. Edinburgh Ser., 89 (1981), pp. 249–258.

[25] J.I. DIAZ AND R. KERSNER, *On a nonlinear degenerate parabolic equation in infiltration or evaporation,* J. Differential Equations, 69 (1987), pp. 368–403.

[26] J.I. DIAZ AND A. LIÑÀN, *Tiempo de descarga en oleoductos o gaseoductos largos: Modelizaciòn y estudio de una ecuación parabólica doblemente no lineal,* in Actas de la Reunión Matemática en Honor a A. Dou, J.I. Diaz and J.M. Vegas, eds., Univ. Complutense, Madrid (1989), pp. 95–120.

[27] J.I. DIAZ AND L. VERON, in preparation.

[28] C. J. VAN DUIJN AND D. HILHORST, *On a doubly nonlinear equation in hydrology,* Nonlinear Anal. T.M.A.A., 11 (1987), pp. 305–333.

[29] A. EL HACHIMI AND F. DE THELIN, *Supersolutions and stabilization of the solutions of the equation* $\partial u/\partial t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u)$, Nonlinear Anal. TMA, 12 (1988), pp. 1385–1398.

[30] ———, *Supersolutions and stabilization of the solutions of the equation* $\partial u/\partial u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(x,u)$, Part. II, Publ. Mat., 35 (1981), pp. 347-362.

[31] J.R. ESTEBAN AND J.L. VAZQUEZ, *Homogeneous diffusion in* $\mathbb{R}$ *with power-like nonlinear diffusivity,* Arch. Rational Mech. Anal., 103 (1988), pp. 39–80.

[32] G. GAGNEUX AND F. GUERFI, *Approximations de la fonction de Heaviside et résultats d' unicité pour une classe de problèmes quasi-linéaires elliptiques-paraboliques,* Rev. Mat. Univ. Complut. Madrid, 3 (1990), pp. 59–87.

[33] B.H. GILDING, *The soil-moisture zone in a physically-based hydrologic model,* Advances in Water Resources, 6 (1983), pp. 36–43.

[34] ———, *Improved theory for a nonlinear degenerate parabolic equation,* Ann. Scuola Norm. Sup. Pisa, Cl. Sci. 14 (1989), pp. 165–224.

[35] A.A. HANNOURA AND F.B.J. BARENDS, *Non Darcy flow: a state of the art,* in Flow and Transport in Porous Media, A. Verruijt and F.B.J. Barends, eds., (1982), pp. 37–51.

[36] H. ISHII, *Asymptotic stability and blowing up of solutions of some nonlinear equations,* J. Differential Equations, 26 (1977), pp. 291–319.

[37] J. Kačur, *On a solution of degenerate elliptic-parabolic systems in Orlicz-Sobolev spaces* I., Math. Z., 203 (1990), pp. 153–171.

[38] ———, *On a solution of degenerate elliptic-parabolic systems in Orlicz-Sobolev spaces* II. Math. Z., 203 (1990), pp. 569–579.

[39] A.S. Kalashnikov, *Some problems of the qualitative theory of nonlinear degenerate second-order parabolic equations,* Russian Math. Surveys, 42 (1987), pp. 169–222.

[40] S. Kichenassamy and J. Smoller, *On the existence of radial solutions of quasilinear elliptic equations,* Nonlinearity, 3 (1990), pp. 677–694.

[41] D. Kröner and J.F. Rodrigues, *Global behaviour for bounded solutions of a porous media equation of elliptic parabolic type,* J. Math. Pures Appl., 64 (1985), pp. 105–120.

[42] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Uraltceva, *Linear and Quasi-Linear Equations of Parabolic Type,* Trans. Amer. Math. Soc., Providence, RI, 1968.

[43] M. Langlais and D. Phillips, *Stabilization of solutions of nonlinear and degenerate evolution equations,* Nonlinear Anal. TMA, 9 (1985), pp. 321–333.

[44] L.S. Leibenson, *General problem of the movement of a compressible fluid in a porous medium,* Izv. Akad. Navk. SSSR, Geography and Geophysics, 9 (1945), pp. 7–10. (In Russian.)

[45] A. Liñàn, *Line packing and surge attenuation in long pipelines,* unpublished work.

[46] J.L. Lions, *Quelques mèthodes de rèsolution de problèmes aux limites non linèaires,* Dunod, Paris, 1969.

[47] L.K. Martinson and K.B. Pavlov, *Unsteady shear flows of a conducting fluid with a rheological power law,* Magnit. Gidrodinamika, 2 (1971), pp. 30–58. (In Russian.)

[48] H. Matano, *Existence of nontrivial unstable sets for equilibriums of strongly order-preserving systems,* J. Fac. Sci. Univ. Tokyo, Sec. 1A, 30 (1984), pp. 645–673.

[49] M. Nakao, *A difference inequality and its application to nonlinear evolution equations,* J. Math. Soc. Japan, 30 (1978), pp. 747–762.

[50] A.S. Shapiro, *Compressible Fluid Flow, Vol.* II, Ronald Press, New York, 1954.

[51] J. Simon, *Régularité de la solution d'un problème aux limites non linéaire,* Ann. Fac. Sci. Toulouse Math. (5), 3 (1981), pp. 247–274.

[52] F. Simondon, *Etude de l'èquation* $\partial_t b(u) - \operatorname{div} a(b(u), \nabla u) = 0$, Publ. Mat., Univ. Besançon, France, 1982.

[53] M. Tsutsumi, *On solutions of some doubly nonlinear degenerate parabolic equations with absorption,* J. Math. Anal. Appl., 60 (1987), pp. 543–549.

[54] R.E. Volker, *Nonlinear flow in porous media by finite elements,* J. Hydraulics Div. Proc. Amer. Soc. Civil. Eng., 95 (1969), pp. 2093–2114.

[55] N.I. Wolanski, *Flow through a porous column,* J. Math. Anal. Appl., 109 (1985), pp. 140–159.

[56] X. Xu, *Existence and Convergence Theorems for Doubly Nonlinear Partial Differential Equations of Elliptic-Parabolic Type,* J. Math. Anal. Appl., 150 (1990), pp. 205–223.

[57] J. Yin, *On a class of quasilinear parabolic equations of second order with double-degeneracy,* J. Partial Differential Equations, 3 (1990), pp. 49–64.

[58] ———, *Solutions with compact support for nonlinear diffusion equations,* Nonlinear Anal. TMA, 19 (1992), pp. 309–321.