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Existence for Reaction Diffusion Systems. A Compactness Method Approach

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1. INTRODUCTION

The purpose of this paper is to prove some local and global existence results concerning weak solutions to nonlinear (possibly degenerate and/or possibly multivalued) reaction diffusion systems of the form

$$\begin{cases} u_t - \Delta\varphi(u) \in F(u, v) & \text{in } (0, T) \times \Omega \\ v_t - \Delta\psi(v) \in G(u, v) & \text{in } (0, T) \times \Omega \\ \varphi(u) = \psi(v) = 0 & \text{on } (0, T) \times \partial\Omega \\ u(0, x) = u_0(x),\ v(0, x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\partial\Omega$, φ, ψ: $\mathbb{R} \to \mathbb{R}$ are continuous and nondecreasing with $\varphi(0) = \psi(0) = 0$, F, G: $\mathbb{R}^2 \to 2^{\mathbb{R}}$, and u_0, $v_0 \in L^\infty(\Omega)$.

We note that the formulation above includes some special important cases extensively treated in the literature, such as, for instance the case in which F and/or G are single-valued, φ and/or ψ are linear, and so on. The motivation of considering possibly multivalued reaction terms is

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offered by the study of nonlinear parabolic systems with discontinuous (with respect to the state) right-hand sides.

A special case of (1.1) of great relevance in applications is that in which $\psi \equiv 0$, i.e., the case in which the diffusion process in the second equation is absent. In this situation (1.1) takes the form

$$\begin{cases} u_t - \Delta\varphi(u) \in F(u, v) & \text{in } (0, T) \times \Omega \\ v_t \in G(u, v) & \text{in } (0, T) \times \Omega \\ \varphi(u) = 0, & \text{on } (0, T) \times \partial\Omega \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{1.2}$$

In order to make a distinction between systems of the form (1.1) and those of the form (1.2), in all that follows we shall say that (1.1) is *diffusive* if both φ and ψ are strictly increasing, and *semi-diffusive* if φ is strictly increasing and ψ is only nondecreasing. Obviously, (1.2) represents a special case of semi-diffusive system.

The mathematical modelling of chemical reaction is perhaps one of the best examples in which coupled systems of the form (1.1) or (1.2) occur (see, e.g., Aris [2]). More precisely, the isothermal chemical reaction between two substances of concentrations $u(x, t)$ and $v(x, t)$, respectively, is usually modelled by (1.1) with the special choice of F, G

$$F(u, v) = -f(u)g(v),\ G(u, v) = -\lambda f(u)g(v), \qquad \text{for } u, v \in \mathbb{R},$$

where f and g are real (possible multivalued) functions, $\lambda > 0$, and the functions $\varphi, \psi\colon \mathbb{R} \to \mathbb{R}$ are continuous nondecreasing, with $\varphi(0) = \psi(0) = 0$.

It is clear that in the event that one or both substances diffuses in Ω—and this happens, for instance, whenever the substance in question in a gas or a liquid—then the corresponding diffusion term φ and/or ψ in (1.1) cannot be identically 0. If, instead, one of the two substances does not diffuse in Ω—and this is certainly the case whenever that substance is a solid—then the corresponding diffusion term φ or ψ must be identically 0.

One of the most important instances in applications is that of Freundlich kinetics, or p–q reaction, which corresponds to the special choice of f and g

$$f(u) = u|u|^{p-1},\ g(v) = v|v|^{q-1}, \qquad \text{for } u, v \in \mathbb{R},$$

where p and q are nonnegative real numbers. The limit case $p = 0$ and/or $q = 0$, occurring in zero-order reactions, leads to the particular choice of f and/or g as the Heaviside (maximal monotone) multivalued function

$$H(r) = \begin{cases} 0 & \text{if } r < 0 \\ [0, 1] & \text{if } r = 0 \\ 1 & \text{if } r > 0. \end{cases}$$

Other kinetics, such as, for instance, the so-called Langmuir kinetics, are modelled by means of

$$f(u) = \frac{\alpha u}{1 + \beta u},\ g(v) = \frac{\nu v}{1 + \delta v}, \qquad \text{for } u \geq 0, v \geq 0$$

and $f(u) = g(v) = 0$ for $u < 0$, $v < 0$, and have also been intensively studied (see, e.g., Aris [2]). Here α, β, ν, and δ are positive constants.

We also mention that coupled systems of the form (1.1) arise in chemical engineering as models of single exothermic p-order reactions. In this case the system is diffusive, and u and v represent the concentration and the temperature of the reactant, respectively, while

$$f(u) = -\lambda u|u|^{p-1},\ g(v) = \exp\left(\nu - \frac{v}{|v|}\right), \qquad \text{for } u, v \in \mathbb{R}.$$

See, e.g., Bandle and Stakgold [3] and Díaz and Hernández [12]. The zero-order reaction leads again to a multivalued problem. See, e.g., Gianni and Hulshof [23].

A different type of problem modelled by (1.1) corresponds to the phenomenon of adsorption of a solute of concentration $u(x, t)$ when in solution, and $v(x, t)$ when adsorbed in the porous medium. In this situation the system is semi-diffusive and usually

$$F(u, v) = \lambda[f(u) - v],\ G(u, v) = -\lambda[f(u) - v], \qquad \text{for } u, v \in \mathbb{R},$$

where f is either a Freundlich or a Langmuir kinetic term and $\lambda > 0$. For details on this subject we refer to van Duijn and Knabner [20].

The diffusivity terms φ and ψ in (1.1) are very often linear but different, as for instance in gas–liquid reactions. See Hollis *et al.* [25] and Pao [29]. Nevertheless, in some special cases they depend on the concentration of the reactants and thus they are nonlinear (e.g., $\varphi(u) = u|u|^{m-1}$, $\psi(v) = v|v|^{r-1}$, for u, $v \in \mathbb{R}$, where $m \geq 0$, $r \geq 0$). See Bear [4], Galaktionov *et al.* [22], Kalashnikov [26], and Maddalena [27], [28]. It should be noted that, in gas–liquid reactions, the solid does not diffuse and so $\psi \equiv 0$ (leading to the special case of semi-diffusive system (1.2)). See Díaz and Stakgold [13], DiLiddo and Stakgold [19], and Stakgold [30].

Diffusive systems arise in many other contexts different from those in chemical engineering, such as, for instance, neurophysiology and biology (see Pao [29]). Some models in climatology are also formulated in terms of (1.1), where u and v represent the temperature and the humidity of the surface of the Earth, and the multivalued right-hand sides originate from the modelling of the albedo as a discontinuous function. See Díaz [11] and Hetzer and Schmidt [24].

Other models leading to semi-diffusive systems arise in phase-change with dissipation in thermodynamics (see Blanchard *et al.* [6]) and heat transfer with phase-change in a fissured medium (see DiBenedetto and Showalter [18]).

The existence of weak solutions for diffusive systems has been studied (even in the discontinuous/multivalued case) by means of monotone iterative techniques (see Carl [9] and Pao [29]), or by topological arguments (see Chang [10]). Here we shall follow a different strategy based upon the compactness of the generalized Green operator associated to the nonlinear diffusion equation (see Díaz and Vrabie [16]), which allows us to handle more general situations than those considered until now. This paper expands a former unpublished manuscript by the authors (see Díaz and Vrabie [15]) but already mentioned in several works since 1987: Díaz [11], Díaz and Stakgold [13], [14], and Vrabie [31].

Our results can be easily reformulated to cover the case in which the diffusion operators $-\Delta\varphi$ and $-\Delta\psi$ are replaced either by $\mathcal{L}_1\varphi$, $\mathcal{L}_2\psi$ with $\mathcal{L}_1$, $\mathcal{L}_2$ second order (strongly) linear ellptic operators, or $-\Delta_p\varphi$, $-\Delta_q\psi$, with $-\Delta_p h = \operatorname{div}(|\nabla h|^{p-2}\nabla h)$ and $-\Delta_q$ analogously defined, where $p > 1$ and $q > 1$.

We would like to point out that the uniqueness of weak solutions is not considered here. Some references illustrating the complexity of this problem in the presence of multivalued sources are Díaz [11], Feireisl and Norbury [21], and Gianni and Hulshof [23].

Finally, we note that the case in which both φ and ψ in (1.1) are only nondecreasing will be analyzed in a forthcoming paper (see Díaz and Vrabie [17]), by means of some other much more refined compactness arguments.

2. Statements of the Main Results

We begin by recalling that a *weak solution* of (1.1) is a pair (u, v), satisfying $u, v \in \mathscr{C}([0, T]; L^2(\Omega))$ and $\varphi(u), \psi(v) \in L^1(0, T; L^1(\Omega))$ for which there exists $f, g \in L^1(0, T; L^1(\Omega))$, $f(t, x) \in F(u(t, x), v(t, x))$, $g(t, x) \in G(u(x, t), v(t, x))$ a.e. $(t, x) \in (0, T) \times \Omega$ and such that (u, v) is a solution in the sense of distributions over $(0, T) \times \Omega$ to the system

(2.1) below

$$\begin{cases} u_t - \Delta\psi(u) = f & \text{in } (0, T) \times \Omega \\ v_t - \Delta\psi(v) = g & \text{on } (0, T) \times \Omega \\ \varphi(u) = \psi(v) = 0 & \text{in } (0, T) \times \partial\Omega \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) & \text{in } \Omega. \end{cases} \tag{2.1}$$

By an *upper-semicontinuous* (u.s.c.) mapping from $\mathbb{R}^2$ into $\mathbb{R}$ we mean a (possibly) multivalued mapping F: $\mathbb{R}^2 \to 2^{\mathbb{R}}$ such that for each pair $(u, v) \in \mathbb{R}^2$, $F(u, v)$ is a nonempty compact interval in $\mathbb{R}$, and for each closed subset $\mathscr{C}$ in $\mathbb{R}$ the set

$$F^{-1}(\mathscr{C}) = \{(u, v) \in \mathbb{R}^2; F(u, v) \cap \mathscr{C} \neq \varnothing\}$$

is closed.

A (possibly) multivalued mapping G: $\mathbb{R}^2 \to 2^{\mathbb{R}}$ is called *with separated variables* if there exist a continuous function g: $\mathbb{R} \to \mathbb{R}$ and an m-dissipative mapping H: $\mathbb{R}^2 \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ such that, either g is nonnegative and

$$G(u, v) = g(u) \cdot H(v) \qquad \text{for each } (u, v) \in \mathbb{R}^2,$$

or

$$G(u, v) = g(u) + H(v) \qquad \text{for each } (u, v) \in \mathbb{R}^2.$$

The mapping G: $\mathbb{R}^2 \to \mathbb{R}$ is called *globally Lipschitz with respect to its second variable* if it is continuous and for each bounded subset B in $\mathbb{R}$ there exists $L = L(B) > 0$ such that

$$|G(u, v) - G(u, \bar{v})| \leq L|v - \bar{v}|,$$

for each $u \in B$ and $v, \bar{v} \in \mathbb{R}$.

Let F; G: $\mathbb{R}^2 \to 2^{\mathbb{R}}$. The pair (F, G) is called *positively sublinear* if there exists $a > 0$, $b > 0$, and $m > 0$ such that for each $(u, v) \in \mathbb{R}^2$ with $|u| > m$ or $|v| > m$ for which either there exists $f_0 \in F(u, v)$ or there exists $g_0 \in G(u, v)$ satisfying

$$uf_0 > 0 \qquad \text{or} \qquad vg_0 > 0,$$

we have both

$$|f| \leq a|u| + b|v| + c \qquad \text{and} \qquad |g| \leq a|u| + b|v| + c$$

for each $f \in F(u, v)$ and each $g \in G(u, v)$.

Remark 2.1. The positively sublinear condition has been introduced in an abstract setting in Vrabie [31, Definition 3.2.5, p. 130] in order to bring into a unitary frame both the usual sublinear growth condition and the sign condition widely used in obtaining global existence results.

We may now proceed to the statement of our main results.

THEOREM 2.1. *If* (1.1) *is diffusive and* $F, G: \mathbb{R}^2 \to 2^{\mathbb{R}}$ *are* u.s.c., *then for each pair* $u_0, v_0 \in L^\infty(\Omega)$ *there exists* $T_0 \in (0, T]$ *such that* (1.1) *has at least one weak solution* (u, v) *defined on* $[0, T_0]$ *and satisfying*

$$u, v \in W^{1,2}(0, T_0; H^{-1}(\Omega)) \cap L^\infty(0, T_0, L^\infty(\Omega)) \tag{2.2}$$

$$\varphi(u), \psi(v) \in L^2(0, T_0; H_0^1(\Omega)). \tag{2.3}$$

If, in addition, the pair (F, G) *is positively sublinear, the same conclusion holds true with* $T_0 = T$.

THEOREM 2.2. *If* (1.1) *is semi-diffusive,* $F: \mathbb{R}^2 \to 2^{\mathbb{R}}$ *is* u.s.c., *and* $G: \mathbb{R}^2 \to 2^{\mathbb{R}}$ *is either with separated variables or globally Lipschitz with respect to its second variable, then for each* $u_0, v_0 \in L^\infty(\Omega)$ *there exists* $T_0 \in (0, T]$ *such that* (1.1) *has at least one weak solution* (u, v) *defined on* $[0, T_0]$ *and satisfying* (2.2) *and* (2.3). *If, in addition, the pair* (F, G) *is positively sublinear, the same conclusion holds true with* $T_0 = T$.

A very useful instance of Theorem 2.2 is

THEOREM 2.3. *If* (1.2) *is semi-diffusive,* $F: \mathbb{R}^2 \to 2^{\mathbb{R}}$ *is* u.s.c., *and* $G: \mathbb{R}^2 \to 2^{\mathbb{R}}$ *is either with separated variables or globally Lipschitz with respect to its second variable, then for each* $u_0, v_0 \in L^\infty(\Omega)$ *there exists* $T_0 \in (0, T]$ *such that* (1.2) *has at least one weak solution* (u, v) *defined on* $[0, T_0]$ *and satisfying* (2.2) *and* (2.3). *If, in addition, the pair* (F, G) *is positively sublinear, the same conclusion holds true with* $T_0 = T$.

3. PRELIMINARIES

First, let us consider the nonlinear diffusion equation

$$\begin{cases} u_t - \Delta\varphi(u) = f & \text{in } (0, T) \times \Omega \\ \varphi(u) = 0 & \text{on } (0, T) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing, $\varphi(0) = 0$, $f \in L^1(0, T; L^1(\Omega))$, and $u_0 \in L^1(\Omega)$.

By a *weak solution* of (3.1) we mean a function $u \in \mathscr{C}([0, T]; L^1(\Omega))$ such that $\varphi(u) \in L^1(0, T; L^1(\Omega))$ and which satisfies (3.1) in the sense of distributions over $(0, T) \times \Omega$.

It is well-known that, for each $u_0 \in L^1(\Omega)$ and $f \in L^1(0, T; L^1(\Omega))$ the problem (3.1) has a unique weak solution $u = W(u_0, f)$. See Brezis and Crandall [8]. Moreover, if $u_0 \in L^p(\Omega)$ and $f \in L^1(0, T; L^p(\Omega))$ for some $p \in [1, \infty]$, the unique weak solution u of (3.1) satisfies

$$\|u(t)\|_{L^p(\Omega)} \leq \|u_0\|_{L^p(\Omega)} + \int_0^t \|f(s)\|_{L^p(\Omega)}\, ds \tag{3.2}$$

for each $t \in [0, T]$. See Benilan [5]. Furthermore, if u_0 and f are bounded i.e., if $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(0, T; L^\infty(\Omega))$, then

$$u \in W^{1,2}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega)) \tag{3.3}$$

$$\varphi(u) \in L^2(0, T; H_0^1(\Omega)). \tag{3.4}$$

A consequence of the next result, due to Díaz and Vrabie [16], is one of the main ingredients in the proof of our existence theorems.

THEOREM 3.1. *If* $\varphi: \mathbb{R} \to \mathbb{R}$ *is continuous, strictly increasing, and* $\varphi(0) = 0$, *then, for each fixed* $u_0 \in L^1(\Omega)$ *and each weakly relatively compact subset* $\mathcal{K}$ *in* $L^1(0, T; L^1(\Omega))$, *the set of all weak solutions of* (3.1) *when* f *ranges in* $\mathcal{K}$, *i.e.,* $\{W(u_0, f); f \in \mathcal{K}\}$, *is strongly relatively compact in* $\mathscr{C}([0, T]; L^1(\Omega))$.

COROLLARY 3.1. *If* $\varphi: \mathbb{R} \to \mathbb{R}$ *is continuous, strictly increasing, and* $\varphi(0) = 0$, *then, for each fixed* $u_0 \in L^\infty(\Omega)$ *and each bounded set* $\mathcal{K}$ *in* $L^\infty(0, T; L^\infty(\Omega))$, *the mapping* $f \mapsto W(u_0, f)$—*the unique weak solution of* (3.1) *corresponding to* u_0 *and* f—*is sequentially continuous from* $\mathcal{K}$ *endowed with the weak topology of* $L^1(0, T; L^1(\Omega))$ *into* $\mathscr{C}([0, T]; L^1(\Omega))$ *endowed with the strong topology.*

Proof. Let $u_0 \in L^\infty(\Omega)$ be fixed and let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(0, T; L^\infty(\Omega))$ such that $f_n \rightharpoonup f$ in $L^1(0, T; L^1(\Omega))$. Since by Theorem 3.1 the set $\{W(u_0, f); f \in \mathcal{K}\}$, is strongly relatively compact in $\mathscr{C}([0, T]; L^1(\Omega))$, to complete the proof, we have merely to show that the only limit point in $\mathscr{C}([0, T]; L^1(\Omega))$ of $(W(u_0, f_n))_{n\in\mathbb{N}}$ is exactly $W(u_0, f)$. But this is clearly the case because the only limit point of $(W(u_0, f_n))_{n\in\mathbb{N}}$ in the sense of distributions over $(0, T) \times \Omega$ is $W(u_0, f)$. ∎

We recall for our later purposes that an operator $\mathscr{A}: D(\mathscr{A}) \subset L^1(\Omega) \to L^1(\Omega)$ is called *accretive* if for each $u, \tilde{u} \in D(\mathscr{A})$ we have

$$(u - \tilde{u}, \mathscr{A}u - \mathscr{A}\tilde{u})_+ \geq 0,$$

where $(\cdot,\cdot)_+$ stands for the usual semi-inner product in $L^1(\Omega)$, i.e.,

$$(u,v)_+ = \|u\|_{L^1(\Omega)}\left[\int_{(u>0)} v(x)\,dx - \int_{(u<0)} v(x)\,dx + \int_{(u=0)} |v(x)|\,dx\right]$$

for every $u, v \in L^1(\Omega)$. If, in addition, for each $\lambda > 0$, $I + \lambda\mathcal{A}$ is surjective, then $\mathcal{A}$ is called *m-accretive*.

Remark 3.1. The problem (3.1) can be rewritten as an abstract evolution equation in $L^1(\Omega)$ of the form

$$\begin{cases} u'(t) + \mathcal{A}_\varphi u(t) = f(t) & t \in (0,T) \\ u(0) = u_0, \end{cases} \tag{3.5}$$

where $\mathcal{A}_\varphi$: $D(\mathcal{A}_\varphi) \subset L^1(\Omega) \to L^1(\Omega)$ is defined by $\mathcal{A}_\varphi u = -\Delta\varphi(u)$ for each $u \in D(\mathcal{A}_\varphi)$ with $D(\mathcal{A}_\varphi) = \{u \in L^1(\Omega);\ \varphi(u) \in W_0^{1,1}(\Omega),\ \Delta\varphi(u) \in L^1(\Omega)\}$.

It is well-known that $\mathcal{A}_\varphi$ is m-accretive in $L^1(\Omega)$ (see Benilan [5]), and that each weak solution of (3.1) is an integral solution of (3.5) and vice versa (see Brezis and Crandall [8]).

We conclude this section with some results concerning multivalued mappings.

Let $\mathcal{X}$ be a real Banach space and D a Lebesgue measurable subset in $\mathbb{R}^p$, $p \geq 1$.

A mapping E: $D \to 2^{\mathcal{X}}$ is called *measurable* if for each closed subset C in $\mathcal{X}$ the set

$$E^{-1}(C) = \{y \in D;\ E(y) \cap C \neq \emptyset\}$$

is Lebesgue measurable.

By a *selection* of E: $D \to 2^{\mathcal{X}}$ we mean a function f: $D \to \mathcal{X}$ such that $f(y) \in E(y)$, a.e. $y \in D$. In all that follows we denote by

$$\operatorname{Sel} E = \{f, f\colon D \to \mathcal{X}, f \text{ is a measurable selection of } E\}.$$

Of course, this set can be empty, but in certain specific cases is not as is shown by the theorem below.

THEOREM 3.2. *If $\mathcal{X}$ is separable, E: $D \to 2^{\mathcal{X}}$ is measurable, and a.e. for $y \in D$, $E(y)$ is nonempty and close, then the set* Sel E *is nonempty.*

For the proof of this particular form of the Kuratowski–Ryll–Nardzewski' selection theorem see Vrabie [31, Theorem 3.1.1, p. 117].

Next, let $\mathcal{U}$ be a topological space and E: $\mathcal{U} \to 2^{\mathcal{X}}\backslash\{\emptyset\}$. We say that E is upper-semicontinuous or u.s.c. (*weakly upper-semicontinuous* or w.u.s.c.) if for each $u \in \mathcal{U}$, $E(u)$ is nonempty, closed, and convex, and

for each closed (weakly closed) subset C in $\mathcal{X}$, the set

$$E^{-1}(C) = \{u \in \mathcal{U};\ E(u) \cap C \neq \emptyset\}$$

is closed in $\mathcal{U}$.

Remark 3.2. If $\mathcal{X}$, $\mathcal{Y}$ are two real Banach spaces, D is a Lebesgue measurable subset in $\mathbb{R}^p$, $p \geq 1$, g: $D \to \mathcal{Y}$ is strongly measurable, and E: $\mathcal{Y} \to 2^{\mathcal{X}}\backslash\{\emptyset\}$ is u.s.c., then $E \circ g$: $C \to 2^{\mathcal{X}}\backslash\{\emptyset\}$ is measurable.

For the proof of the next important tool in our later analysis see Vrabie [31, Theorem 3.1.2, p. 120].

THEOREM 3.3. *Let D be a nonempty and Lebesgue measurable subset of $\mathbb{R}^p$, $p \geq 1$, $\mathcal{U}$ a topological space, and $\mathcal{X}$ a real Banach space. If E: $\mathcal{U} \to 2^{\mathcal{X}}\backslash\{\emptyset\}$ is* w.u.s.c. *and u_n: $D \to \mathcal{U}$, $f_n \in$ Sel $E(u_n)$ for $n \in \mathbb{N}$ satisfy*

$$f \rightharpoonup f \text{ weakly in } L^1(D;\mathcal{X}) \text{ and } u_n \to u \qquad \text{a.e. in } D$$

then $f \in$ Sel $E(u)$.

Remark 3.3. Since each u.s.c. mapping is w.u.s.c. also, the conclusion of Theorem 3.3 remains unchanged if we assume that E is u.s.c.

We also need the following two results on w.u.s.c. mappings.

THEOREM 3.4. *Let $\mathcal{K}$ be a weakly compact subset in a real Banach space $\mathcal{X}$ and let E: $\mathcal{K} \to 2^{\mathcal{X}}\backslash\{\emptyset\}$ be such that for each $u \in \mathcal{K}$, $E(u)$ is closed and convex and $E(\mathcal{K}) = \bigcup_{u\in\mathcal{K}} E(u)$ is weakly compact in $\mathcal{X}$. Then E is* w.u.s.c. *from $\mathcal{K}$ endowed with the weak topology into $\mathcal{X}$ if and only if its graph is weakly $\times$ weakly sequentially closed.*

Since the proof of Theorem 3.4 follows exactly the same lines as those in the proof of [31, Theorem 3.1.3, p. 121] we do not give details.

THEOREM 3.5. *Let $\mathcal{K}$ be a nonempty and weakly compact subset in a real Banach space $\mathcal{X}$ and let E: $\mathcal{K} \to 2^{\mathcal{X}}\backslash\{\emptyset\}$ be such that for each $u \in \mathcal{K}$, $E(u)$ is closed and convex. If the graph of E is weakly $\times$ weakly sequentially closed, then E has at least one fixed point, i.e., there exists at least one element $u \in \mathcal{K}$ such that $u \in E(u)$.*

Theorem 3.5 is an easy consequence of Theorem 3.4 combined with results of Arino *et al.* [1, Remark 1, p. 274].

4. PROOFS OF THE LOCAL EXISTENCE RESULTS

The idea in the proofs of both Theorems 2.1 and 2.2 is, except for some technical specific considerations (see Remark 4.1), the same. Namely, we will show that a suitably defined multivalued mapping has at least one fixed point whose existence is equivalent with the existence of at least one weak (local) solution of (1.1). We begin with

Proof of the local existence part of Theorem 2.1. Let $u_0, v_0 \in L^\infty(\Omega)$ and choose $m > 0$, such that

$$\|u_0\| + 1 \leq m, \qquad \|v_0\| + 1 \leq m. \tag{4.1}$$

Furthermore, since both F and G are u.s.c., it is always possible to find $M > 0$, $r \geq M$, and $T_0 \in (0, T]$ such that

$$|f| \leq M \qquad \text{and} \qquad |g| \leq M \tag{4.2}$$

for each $f \in F(u, v)$, $g \in G(u, v)$, provided $|u| \leq m$ and $|v| \leq m$, and

$$T_0 r \leq 1. \tag{4.3}$$

Next, let us define the set

$$\mathcal{K} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix}; f, g \in L^1(0, T_0; L^1(\Omega)), \|f\|_{L^\infty(D)} \leq r, \|g\|_{L^\infty(D)} \leq r \right\},$$

where $D = (0, T_0) \times \Omega$. Clearly, $\mathcal{K}$ is nonempty and weakly compact in

$$\begin{matrix} L^1(0, T_0; L^1(\Omega)) \\ \times \\ L^1(0, T_0; L^1(\Omega)). \end{matrix}$$

Let us define the operator

$$P: \mathcal{K} \to \begin{matrix} \mathcal{C}([0, T_0]; L^1(\Omega)) \\ \times \\ \mathcal{C}([0, T_0]; L^1(\Omega)) \end{matrix} \qquad \text{by } P\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix},$$

where $\binom{u}{v}$ is the unique weak solution of the system

$$\begin{cases} u_t - \Delta\varphi(u) = f & \text{in } (0, T) \times \Omega \\ v_t - \Delta\psi(v) = g & \text{in } (0, T) \times \Omega \\ \varphi(u) = \psi(v) = 0 & \text{on } (0, T) \times \partial\Omega \\ u(0, x) = u_0(x), v(0, x) = v_0(x) & \text{in } \Omega. \end{cases} \tag{4.4}$$

From (3.3), (4.3), and (4.1) we easily conclude that

$$\|u(t, \cdot)\|_{L^\infty(\Omega)} \leq m, \|v(t, \cdot)\|_{L^\infty(\Omega)} \leq m, \qquad \text{for } t \in [0, T_0]. \tag{4.5}$$

Finally, let us define the operator

$$\mathcal{S}: \mathcal{K} \to \begin{matrix} \mathcal{C}([0, T_0]; L^1(\Omega)) \\ \times \\ \mathcal{C}([0, T_0]; L^1(\Omega)) \end{matrix} \qquad \text{by } \mathcal{S}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \operatorname{Sel} F(u, v) \\ \operatorname{Sel} G(u, v) \end{pmatrix},$$

where $\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} f \\ g \end{pmatrix}$.

Since F and G are u.s.c., from (4.5), (4.2), Remark 3.2, and Theorem 3.2, we deduce that $\mathcal{S}$ is well-defined on the whole of $\mathcal{K}$, maps $\mathcal{K}$ into $2^{\mathcal{K}} \setminus \{\emptyset\}$, and has closed convex values. Thus, in order to appeal to Theorem 3.5, we have merely to check that the graph of $\mathcal{S}$ is weakly $\times$ weakly sequentially closed. To this end, let us observe first that, by Corollary 3.1 it follows that the operator P is weakly–strongly sequentially continuous from $\mathcal{K}$ into

$$\begin{matrix} \mathcal{C}([0, T_0]; L^1(\Omega)) \\ \times \\ \mathcal{C}([0, T_0]; L^1(\Omega)). \end{matrix}$$

Now Theorem 3.3 comes into play and shows that the graph of $\mathcal{S}$ is weakly $\times$ weakly sequentially closed in

$$\begin{matrix} L^1(0, T_0; L^1(\Omega)) \\ \times \\ L^1(0, T_0; L^1(\Omega)). \end{matrix}$$

Therefore, Theorem 3.5 applies, and consequently $\mathcal{S}$ has at least one fixed point $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{K}$. Since $\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} f \\ g \end{pmatrix}$ is obviously a weak solution of (1.1), and (2.2), (2.3) follow from (3.3), (3.4), this completes the proof of the local existence part of Theorem 2.1. ∎

Before proceeding to the proof of the local existence part of Theorem 2.2 some preliminaries are needed.

First, let us consider the problems

$$\begin{cases} v_t - \Delta\psi(v) \in gH(v) & \text{in } (0, T) \times \Omega \\ \psi(v) = 0 & \text{on } (0, T) \times \partial\Omega \\ v(0, x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{4.6}$$

and

$$\begin{cases} v_t - \Delta\psi(v) \in g + H(v) & \text{in } (0, T) \times \Omega \\ \psi(v) = 0 & \text{on } (0, T) \times \partial\Omega \\ v(0, x) = v_0(x) & \text{in } \Omega. \end{cases} \tag{4.7}$$

We recall that a mapping $H; \mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ is called *m-dissipative* if for each $v, \bar{v} \in \mathbb{R}$ and $w \in H(v)$, $\bar{w} \in H(\bar{v})$, we have

$$(v - \bar{v})(w - \bar{w}) \leq 0,$$

and for $\lambda > 0$, $I - \lambda H$ is surjective. We denote by $\mathcal{J}_\lambda = (I - \lambda H)^{-1}$ and by $H_\lambda = (1/\lambda)(I - \mathcal{J}_{\lambda H})$ the *resolvent* and the *Yosida approximation* of H, respectively. We recall that for each $\lambda > 0$, H_λ is Lipschitzian on $\mathbb{R}$ with Lipschitz constant $1/\lambda$, and also that

$$\lim_{\lambda \downarrow 0} H_\lambda(v) = H^0(v)$$

for each $v \in \mathbb{R}$, where $H^0(v)$ is the element of minimal absolute value in $H(v)$. See Brezis [7, Proposition 2.6, p. 280].

For our later purposes we need the following two lemmas.

LEMMA 4.1. *If H: $\mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ is m-dissipative, then for each nonnegative $g \in L^\infty(0, T; L^\infty(\Omega))$ and each $v_0 \in L^\infty(\Omega)$ the problem* (4.6) *has a unique weak solution $v = W(v_0, g)$ defined on* $[0, T]$. *In addition, for each bounded subset B in $L^\infty(0, T; L^\infty(\Omega)) \times L^\infty(\Omega)$ there exists $C = C(B) > 0$ such that for each $(g, v_0), (\bar{g}, \bar{v}_0) \in B$ with g, $\bar{g}$ nonnegative, the corresponding weak solutions $v = W(v_0, g)$ and $\bar{v} = W(\bar{v}_0, \bar{g})$ satisfy*

$$\|v(t) - \bar{v}(t)\|_{L^1(\Omega)} \leq \|v(s) - \bar{v}(s)\|_{L^1(\Omega)} + C \int_s^t \|g(\tau) - \bar{g}(\tau)\|_{L^1(\Omega)}\, d\tau \tag{4.8}$$

for each $0 \leq s \leq t \leq T$.

LEMMA 4.2. *If H: $\mathbb{R} \to 2^{\mathbb{R}} \backslash \{\varnothing\}$ is m-dissipative, then for each $g \in L^\infty(0, T; L^\infty(\Omega))$ and each $v_0 \in L^\infty(\Omega)$ the problem* (4.7) *has a unique weak solution $v = W(v_0, g)$ defined on* $[0, T]$. *In addition, for each $g, \bar{g} \in L^\infty(0, T; L^\infty(\Omega))$ and each $v_0, \bar{v}_0 \in L^\infty(\Omega)$ the corresponding weak solutions $v = W(v_0, g)$ and $\bar{v} = W(\bar{v}_0, \bar{g})$ satisfy*

$$\|v(t) - \bar{v}(t)\|_{L^1(\Omega)} \leq \|v(s) - \bar{v}(s)\|_{L^1(\Omega)} + \int_s^t \|g(\tau) - \bar{g}(\tau)\|_{L^1(\Omega)}\, d\tau \tag{4.9}$$

for each $0 \leq s \leq t \leq T$.

Since the proofs of both lemmas are quite similar, we confine ourselves only with the proof of the first one.

Proof of Lemma 4.1. Let $g \in L^\infty(0, T; L^\infty(\Omega))$, $g \geq 0$ and $v_0 \in L^\infty(\Omega)$. For each $\lambda > 0$ let us consider the approximation equation

$$\begin{cases} v_t^\lambda - \Delta\psi(v^\lambda) \in gH_\lambda(v^\lambda) & \text{in } (0, T) \times \Omega \\ \psi(v^\lambda) = 0 & \text{on } (0, T) \times \partial\Omega \\ v^\lambda(0, x) = v_0^\lambda(x) & \text{in } \Omega, \end{cases} \tag{4.10}$$

where H_λ is the Yosida approximation of H. Since H_λ is globally Lipschitz, we conclude by standard arguments that (4.10) has a unique weak solution v^λ defined in $[0, T]$.

Taking an arbitrary $p \in [1, \infty)$, multiplying both sides in (4.10) by $\|v^\lambda\|_{L^p(\Omega)}^{2-p} |v^\lambda(t, x)|^{p-2} v(t, x)$, integrating over Ω, taking into account the positivity of g and the dissipativity of H_λ, and integrating over $[0, t]$ we get

$$\|v^\lambda(t)\|_{L^p(\Omega)}^2 \leq \|v_0\|_{L^p(\Omega)}^2 + 2 \int_0^t \|g(s) H_\lambda(0)\|_{L^p(\Omega)} \|v^\lambda(s)\|_{L^p(\Omega)}\, ds$$

for $p \in [1, \infty)$ and $t \in (0, T)$. According to Brezis [7, Lemma A5, p. 157], it then follows that

$$\|v^\lambda(t)\|_{L^p(\Omega)} \leq \|v_0\|_{L^p(\Omega)} + |H_\lambda(0)| \int_0^t \|g(s)\|_{L^p(\Omega)}\, ds$$

for $p \in [1, \infty)$ and $t \in [0, T]$. Since $g \in L^\infty(0, T; L^\infty(\Omega))$, $v_0 \in L^\infty(\Omega)$, and $|H_\lambda(v)| \leq |H^0(v)|$, the last inequality shows that there exists $m_1 > 0$ such that

$$\|v^\lambda(t)\|_{L^\infty(\Omega)} \leq m_1 \qquad \text{for each } \lambda > 0 \text{ and } t \in [0, T]. \tag{4.11}$$

From (4.11), inasmuch as $|H_\lambda(v(t, x))| \leq \|H^0(v(t, x))|$, a.e. for $(t, x) \in (0, T) \times \Omega$, and H maps bounded sets in $\mathbb{R}$ into bounded subsets in $\mathbb{R}$ (see Brezis [7, Proposition 2.9, p. 32]), it follows that there exists $m_2 > 0$ such that

$$\|H_\lambda(v^\lambda(t, \cdot))\|_{L^\infty(\Omega)} \leq m_2 \qquad \text{for each } \lambda > 0 \text{ and } t \in [0, T]. \tag{4.12}$$

Recalling that the operator $\mathscr{A}_\psi$ is m-accretive in $L^1(\Omega)$ (see Remark 3.1) by standard computations we deduce

$$\begin{aligned}
&\|v^\lambda(t) - v^\mu(t)\|^2_{L^1(\Omega)} \\
&\quad \le 2\int_0^t (v^\lambda(s) - v^\mu(s), g(s)H_\lambda(v^\lambda(s)) - g(s)H_\mu(v^\mu(s)))_+ \, ds \\
&\quad \le 2\int_0^t (\mathscr{J}_\lambda v^\lambda(s) - \mathscr{J}_\mu v^\mu(s), g(s)H_\lambda(v^\lambda(s)) - g(s)H_\mu(s)))_+ \, ds \\
&\qquad + 2\int_0^t (\lambda H_\lambda(v^\lambda(s)) - \mu H_\mu(v^\mu(s)), g(s)H_\lambda(v^\lambda(s)) - g(s)H_\mu(v^\mu(s)))_+ \, ds
\end{aligned}$$

for each $\lambda > 0$, $\mu > 0$, and $t \in [0, T]$. This inequality, in view of (4.12), yields

$$\|v^\lambda(t) - v^\mu(t)\|^2_{L^1(\Omega)} \le M(\lambda + \mu)$$

for each $\lambda > 0$, $\mu > 0$, and $t \in [0, T]$, where $M > 0$ does not depend on λ, μ, and t. Clearly, this shows that there exists

$$\lim_{\lambda \downarrow 0} v^\lambda = v \qquad \text{strongly in } \mathscr{C}([0, T]; L^1(\Omega)).$$

Moreover, from (4.11), we conclude that

$$\|v(t)\|_{L^\infty(\Omega)} \le m_1 \qquad \text{for } t \in [0, T]. \tag{4.13}$$

In view of (4.12), at least on a subsequence, $(H_\lambda(v_\lambda))_{\lambda > 0}$ converges weakly in $L^2(0, T; L^2(\Omega))$ to some element in Sel $H(v)$. See Brezis [7, Proposition 2.6, p. 28]. Passing to the limit in (4.10) for $\lambda \to 0$, we deduce that v is a weak solution of (4.6). Since the uniqueness of v and (4.8) follow in a usual way thanks to dissipativity of H, the proof of Lemma 4.1 is complete. ∎

We may now proceed to the proof of the local existence part of Theorem 2.2 in the case in which G is with separated variables. We begin with the following

Remark 4.1. It is clear that under the hypotheses of Theorem 2.2 we may construct the set $\mathscr{K}$ and the operator S: $\mathscr{K} \to 2^{\mathscr{K}} \setminus \{\varnothing\}$ as in the proof of Theorem 2.1. Nevertheless, in this case, the graph of S is *no longer* weakly × weakly sequentially closed in $\mathscr{K} \times \mathscr{K}$ because Corollary 3.1 *does not* apply to the second equation in (4.4). We recall that in the case of Theorem 2.2 ψ is only nondecreasing and not strictly increasing as required by the use of Corollary 3.1. This explains why, in the proof of

Theorem 2.2, we have been forced to adopt a different strategy, even though it is also based on the fixed point device offered by Theorem 3.5.

Proof of the local existence part of Theorem 2.2. Here we confine ourselves only to the case in which G is of the form $G(u, v) = g(u) \cdot H(v)$. The case in which $G(u, v) = g(u) + H(v)$ is quite similar, with the sole exception that instead of Lemma 4.1 we have to use Lemma 4.2, and therefore we do not give details.

Thus, let $u_0, v_0 \in L^\infty(\Omega)$ and choose $m > 0$ satisfying

$$\|u_0\|_{L^\infty(\Omega)} + 1 \le m, \qquad \|v_0\|_{L^\infty(\Omega)} + 1 \le m. \tag{4.14}$$

Since F is u.s.c., g is continuous, and H everywhere defined and m-dissipative on $\mathbb{R}$, it is always possible to choose $M > 0$, $r \ge M$, and $T_0 \in (0, T]$ such that

$$|f| \le M \qquad \text{for each } f \in F(u, v) \text{ provided } |u| \le m, |v| \le M \tag{4.15}$$

$$|g| \le M \qquad \text{for } u \in \mathbb{R}, |u| \le m \tag{4.16}$$

$$T_0 r \le 1 \text{ and } T_0 M|h| \le 1 \qquad \text{for each } h \in H(0). \tag{4.17}$$

Next, let us define the subset $\mathscr{K}$ in $L^1(0, T_0; L^1(\Omega))$ by

$$\mathscr{K} = \{f \in L^1(0, T_0; L^1(\Omega)); \|f\|_{L^\infty(D)} \le r\},$$

where $D = (0, T_0) \times \Omega$. Clearly, $\mathscr{K}$ is nonempty and weakly compact in $L^1(0, T_0; L^1(\Omega))$. We define the operator P: $\mathscr{K} \to \mathscr{C}([0, T_0]; L^1(\Omega))$ by

$$Pf = u,$$

for $f \in \mathscr{K}$, where u is the unique weak solution of the problem

$$\begin{cases} u_t - \Delta\psi(u) = f & \text{in } (0, T_0) \times \Omega \\ \psi(u) = 0 & \text{on } (0, T_0) \times \partial\Omega \\ u(0, x) = u_0(x) & \text{in } \Omega. \end{cases}$$

By (3.2), (4.17), and (4.14) we easily deduce

$$\|u(t)\|_{L^\infty(\Omega)} \le m \qquad \text{for } t \in [0, T_0]. \tag{4.18}$$

Thus P maps $\mathscr{K}$ into $L^\infty(0, T_0; L^\infty(\Omega))$. Furthermore, let us define the operator Q: $\mathscr{K} \to L^\infty(0, T_0; L^\infty(\Omega))$ by

$$Qf = u$$

for $f \in \mathcal{H}$, where v is the unique weak solution of the problem

$$\begin{cases} u_t - \Delta\psi(u) \in g(Pf)H(v) & \text{in } (0, T_0) \times \Omega \\ \psi(v) = 0 & \text{on } (0, T_0) \times \partial\Omega \\ v(0, x) = v_0(x) & \text{in } \Omega. \end{cases}$$

By virtue of Lemma 4.1 (see also (4.13)) the operator Q is well-defined on the whole $\mathcal{H}$. On the other hand, from Corollary 3.1 combined with (4.8) in Lemma 4.1, we conclude that Q is weakly–strongly sequentially continuous from $\mathcal{H}$ into $\mathcal{C}([0, T_0]; L^1(\Omega))$. In addition, from (4.18), (4.16), (4.14), and (4.17), we get

$$\|Q(f)(t)\|_{L^\infty(\Omega)} \leq m \qquad \text{for } t \in [0, T_0]. \tag{4.19}$$

Finally, let us define the operator $S: \mathcal{H} \to 2^{L^1(0,T_0;L^1(\Omega))}$ by

$$S(f) = \operatorname{Sel} F(Pf, Qf)$$

for each $f \in \mathcal{H}$. Since F is u.s.c., from Theorem 3.2 combined with (4.18), (4.19), we deduce that S is well-defined and has nonempty and weakly compact values. Thus, in order to appeal to Theorem 3.5, we have to show that S maps $\mathcal{H}$ into $2^{\mathcal{H}}$ and its graph is weakly $\times$ weakly sequentially closed. We start by proving that S maps $\mathcal{H}$ into $2^{\mathcal{H}}$. To this end, let us observe that, in view of (4.18), (4.19), and (4.15) we have

$$|g(t, x)| \leq M \leq r \qquad \text{for each } g \in S(f) \text{ and a.e. } (t, x) \in (0, T_0) \times \Omega.$$

Hence $S: \mathcal{H} \to 2^{\mathcal{H}}$. To prove that the graph of S is weakly $\times$ weakly sequentially closed take $((f_n, g_n))_{n \in \mathbb{N}}$ a sequence in the graph of S such that

$$f_n \rightharpoonup f \text{ and } g_n \rightharpoonup g \qquad \text{weakly in } L^1(0, T_0; L^1((\Omega)).$$

Since P and Q are weakly–strongly sequentially continuous, we may assume with no loss of generality—taking a subsequence if necessary—that

$$Pf_n \to Pf \qquad \text{and} \qquad Qg_n \to Qg \qquad \text{a.e. in } (0, T_0) \times \Omega.$$

Then, according to Theorem 3.3, we have $g \in S(f)$, and consequently the graph of S is weakly $\times$ weakly sequentially closed. Finally, Theorem 3.5 shows that S has at least one fixed point $f \in \mathcal{H}$. Obviously, $(u, v) = (Pf, Qf)$ is a weak solution of (1.1) defined in $[0, T_0]$. Since (2.2) and (2.3) follow from (3.3) and (3.4), the proof of the local existence part of Theorem 2.2 is complete. ∎

5. Proof of the Global Existence Results

First, let us observe that under the hypotheses of either Theorem 2.1 or Theorem 2.2, each local weak solution of (1.1) can be continued up to a noncontinuable one (u, v) defined either on $[0, T]$ or on $[0, T_m)$ for some $T_m \leq T$. To complete the proof it suffices to show that the latter situation cannot occur. To this end, let us assume by contradiction that (u, v) is defined on $[0, T_m)$, where $T_m \leq T$. Taking an arbitrary $p \in [1, \infty)$, multiplying both sides in the first equation of (2.1) by $\|u(t)\|_{L^p(\Omega)}^{2-p} u(t, x)$, and integrating over Ω and over $[0, t] \subset [0, T_m)$, we get

$$\|u(t)\|_{L^p(\Omega)}^2 \leq \|u_0\|_{L^p(\Omega)}^2 + 2 \int_0^t \int_\Omega f(s, x) \|u(s)\|_{L^p(\Omega)}^{2-p} |u(s, x)|^{p-2} u(s, x)\, dx\, ds$$

for each $p \in [1, \infty)$ and $t \in [0, T_m)$. Since the pair (F, G) is positively sublinear and $u_0 \in L^\infty(\Omega)$, from the inequality above we deduce that there exists $k > 0$ which does not depend on $p \in [1, \infty)$ and $t \in [0, T_m)$, such that

$$\begin{aligned} \|u(t)\|_{L^p(\Omega)}^2 &\leq k^2 \\ &+ 2 \int \int_D [a|u(s, x)| + b|v(s\, x)| + c] \|u(s)\|_{L^p(\Omega)}^{2-p} |u(s, x)|^{p-2} u(s, x)\, dx\, ds, \end{aligned}$$

where D is a subset of all $(s, x) \in [0, T_m) \times \Omega$ such that either $|u(s, x)| > m$ or $|v(s, x)| > m$ and either $u(s, x) f_0(s, x) > 0$ or $v(s, x) g_0(s, x) > 0$ for some $f_0(s, x) \in F(u(s, x), v(s, x))$, or some $g_0(s, x) \in G(u(s, x), v(s, x))$.

Since on the complement of D in $[0, T_m) \times \Omega$ we have either $|u(s, x)| \leq m$ and $|v(s, x)| \leq m$, or $u(s, x) f(s, x) \leq 0$ and $v(s, x) f(s, x) \leq 0$ for each $f(s, x) \in F(u(s, x), v(s, x))$ and $g(s, x) \in G(u(s, x), v(s, x))$, the last inequality implies

$$\|u(t)\|_{L^p(\Omega)}^2 \leq k_1^2 + 2 \int_0^t [a\|u(s)\|_{L^p(\Omega)} + b\|v(s)\|_{L^p(\Omega)} + c] \|u(s)\|_{L^p(\Omega)}\, ds$$

for each $p \in [1, \infty)$ and $t \in [0, T_m)$, where $k_1 > 0$ does not depend on $p \in [1, \infty)$ and $t \in [0, T_m)$. According to Brezis [7, Lemma A6, p. 157], we conclude that there exists $M > 0$ which does not depend on $p \in [1, \infty)$ and $t \in [0, T_m)$ such that

$$\|u(t)\|_{L^p(\Omega)} \leq M + \int_0^t [a\|u(s)\|_{L^p(\Omega)} + b\|v(s)\|_{L^p(\Omega)}]\, ds$$

for each $p \in [1, \infty)$ and $t \in [0, T_m)$.

Similarly, we get

$$\|v(t)\|_{L^p(\Omega)} \leq \tilde{M} + \int_0^t [a\|u(s)\|_{L^p(\Omega)} + b\|v(s)\|_{L^p(\Omega)}]\, ds$$

for each $p \in [1, \infty)$ and $t \in [0, T_m)$, where $\tilde{M} > 0$ does not depend on $p \in [1, \infty)$ and $t \in [0, T_m)$. Adding these inequalities, and denoting by $\alpha = M + \tilde{M}$ and $\beta = 2\max(a, b)$, we get

$$\|u(t)\|_{L^p(\Omega)} + \|v(t)\|_{L^p(\Omega)} \leq \alpha + \beta \int_0^t [\|u(s)\|_{L^p(\Omega)} + \|v(s)\|_{L^p(\Omega)}]\, ds$$

for each $p \in [1, \infty)$ and $t \in [0, T_m)$, where $\alpha > 0$ and $\beta > 0$ do not depend on $p \in [1, \infty)$ and $t \in [0, T_m)$. From Gronwall's inequality it follows that there exists $\ell > 0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq \ell \qquad \text{and} \qquad \|v(t)\|_{L^\infty(\Omega)} \leq \ell \qquad \text{for } t \in [0, T_m).$$

Since F and G map bounded subsets in $\mathbb{R}^2$ into bounded subsets in $\mathbb{R}$, there exists $L > 0$ such that

$$|f(s, x)| \leq L \qquad \text{and} \qquad |g(s, x)| \leq L$$

for each $(s, x) \in (0, T_m) \times \Omega$, each $f(s, x) \in F(u(s, x), v(s, x))$, and each $g(s, x) \in G(u(s, x), v(s, x))$.

Coming back to (2.1), we easily conclude that for each $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, T_m)$ such that for each $t, \bar{t} \in (T_m - \delta(\varepsilon), T_m)$ we have

$$\|u(t) - \bar{u}(t)\|_{L^1(\Omega)} < \varepsilon \qquad \text{and} \qquad \|v(t) - \bar{v}(t)\|_{L^1(\Omega)} < \varepsilon.$$

Hence there exist both

$$\lim_{t \to T_m} u(t) = u^* \qquad \text{and} \qquad \lim_{t \to T_m} v(t) = v^*$$

and moreover $u^*, v^* \in L^\infty(\Omega)$. In view of the local existence part of either Theorem 2.1 or Theorem 2.2, (u, v) can be continued to the right of T_m if $T_m < T$, or at least to T_m if $T_m = T$, and consequently (u, v) is not noncontinuable. This contradiction shows that the initial supposition is false, and therefore (u, v) is defined on $[0, T]$, as claimed. ∎

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