PERIODIC SOLUTIONS OF A QUASILINEAR PARABOLIC BOUNDARY VALUE PROBLEM ARISING IN UNSATURATED FLOW THROUGH A POROUS MEDIUM.

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Abstract

We prove the existence and uniqueness of periodic solutions of the equation $u_t = \varphi(u)_{xx} + b(u)_x$ with Dirichlet and Neumann boundary periodic data. The equation arises, for instance, in the mathematical modelling of the flow of groundwater in a homogeneous isotropic unsaturated porous medium. Those kinds of boundary data represent different periodic behaviours on the top and bottom of the medium due, for instance, to the seasons in the course of the years. The equation arises in many other contexts. We improve precedent results in the literature limited to the non degenerate case ($\varphi'(0) > 0$) and uniformly bounded convection terms.

Key words: periodic solutions, quasilinear parabolic problems, porous media.

1.INTRODUCTION.

In this paper we study the existence and uniqueness of periodic solutions of the following quasilinear parabolic problem

$$u_t = \varphi(u)_{xx} + b(u)_x \quad \text{in } (-L,0)\times \mathbb{R}, \tag{1}$$

$$\varphi(u(0,t)) = h(t) \qquad \text{for } t \in \mathbb{R}, \tag{2}$$

$$\varphi(u(-L,t))_x + b(u(-L,t)) = g(t) \qquad \text{for } t \in \mathbb{R}, \tag{3}$$

$$u(x,t+\omega) = u(x,t) \text{ and } u \geq 0 \quad \text{in } (-L,0)\times \mathbb{R}. \tag{4}$$

Throughout the remainder of the paper we shall assume that the "coefficients" satisfy the following general hypothesis

(H_φ) $\quad \varphi \in C([0,\infty)) \cap C^{2+\alpha}_{\text{loc}}((0,\infty)), \varphi(0) = 0$ and $\varphi'(s) > 0$ for $s > 0$

(H_b) $\quad b \in C([0,\infty)) \cap C^{2+\alpha}_{\text{loc}}((0,\infty)), b(0) = 0,$

for some $\alpha \in (0,1)$. The boundary data are assumed to be ω-periodic functions

(H_h) $\quad h$ is Lipschitz continuous on $\mathbb{R}$, $h(t) \geq 0$ and $h(t+\omega) = h(t)$.

(H_g) $\quad g$ is Lipschitz continuous on $\mathbb{R}$, $g(t) \leq 0$ and $g(t+\omega) = g(t)$.

Equation (1) is a useful model in many different applications as, for instance, the flow of groundwater in a homogeneous, isotropic, rigid and unsaturated porous medium. If we choose the coordinate x to measure the vertical height from ground level and pointing upwards, the soil is represented by the vertical column $(-L,0)$. If $\theta(x,t)$ denotes the moisture content, defined as the volume of water present per unit volume of soil and $v(x,t)$ is the seepage velocity of the water, the equations of groundwater flow are based on two principles (see [3],[4]): the Darcy's law

$$v = -K(\theta)\phi_x \tag{5}$$

and the continuity equation

$$\theta_t + v_x = 0$$

(the fluid is assumed to be incompressible). In (5), $K(\theta)$ is the hydraulic conductivity of the soil and ϕ is the total potential. When absorption and chemical osmotic and thermal effects are negligible, the total potential may be expressed as $\phi = \psi(\theta) + x$, where $\psi(\theta)$ is the hydrostatic potential due to capillary suction (see Bear [3] p.123). The function $K(\theta)$ is empirically determined and satisfies $K(\theta) > 0$ for $\theta > 0$. The moisture distribution is affected by seasonal fluctuation of precipitations or irrigations of the soil. Combining both equations we obtain

$$\theta_t = (K(\theta)\psi_\theta(\theta)\theta_x + K(\theta))_x = (D(\theta)\theta_x + K(\theta))_x \tag{6}$$

in which $D(\theta) := K(\theta)\psi_\theta(\theta)$ denotes the soil moisture diffusivity. By defining $\varphi(s) := \int_0^s D(r)dr$ and $b(s) := K(s)$, (6) yields (1).

In problem (1)—(4), u denotes the moisture content in the soil, hence we required the condition $u \geq 0$. The boundary condition (2), represents a periodical information on u at the top of the medium, due, for instance, to the rain intensity. A natural boundary condition at the bottom is the one indicating that $x = -L$ is an impervious boundary. According to Bear [3] p.251, this corresponds to the assumption $v(-L,t) = 0$, which leads to condition (3) with $g \equiv 0$. More generally, our treatment will allow prescribed flux boundary conditions with g satisfying (H_g). We also point out that the equation is significant in describing a number of different diffusion-convection processes: movement of a thin viscous film under the influence of gravity, problems in plasma physics, population dynamics, etc.

The existence and uniqueness of a periodic weak solution u with $\varphi(u)_x \in L^2((-L,0)\times \mathbb{R})$ was proved in Kenmochi-Kröner-Kubo [14] under the assumptions $\varphi'(0) > 0$ and b Lipschitz continuous and uniformly bounded on $[0,\infty)$ (in fact they prove that under these assumptions u is also a strong solution *i.e.* $u_t \in L^2((-L,0)\times \mathbb{R})$). Those assumptions correspond to the modelling of a partially saturated porous medium (see, for instance, [1], [10] and [17]). A related paper is DiBenedetto-Friedman [7], in which the existence and uniqueness of periodic solutions was shown for the evolutionary dam problem.

In the present paper we prove the existence of periodic weak solutions to problem (1)—(4) under the general assumptions $(H_\varphi), (H_b), (H_h)$ and (H_g) jointly with some additional conditions (see Theorem 1). Since this set of conditions includes the cases of degenerate equations ($\varphi'(0) = 0$) and singular convection ($b'(0) = \pm\infty$) it is well-known that classical solutions may not exist (see *e.g.* [20] and [8]). Nevertheless we shall prove the continuity of u and that $\varphi(u)_x \in L^\infty((-L,0)\times \mathbb{R})$. Our proof of the existence of such a weak solution uses the Schauder fixed point theorem for the Poincaré map of the associated <u>initial</u> boundary value problem. As a previous step we study this initial boundary problem by showing the existence and uniqueness of solution. To do that we follow the approach initiated by Oleinik, Kalashnikov and Chzhou [20], improved later by different authors (see *e.g.* [12], [8]) and developed in full generality by Gilding [13]. In those works the Cauchy or Cauchy-Dirichlet problems are the object of the theory. Here we extend it to the case of boundary conditions (2) and (3).

One of the main difficulties in proving uniqueness comes from the lack of regularity of the time derivative of periodic weak solutions. Our proof is inspired on some regularizing and duality arguments as introduced in Díaz-Kersner [8] for the correspondent initial boundary value problem. The uniqueness of periodic weak solutions is established under the additional assumptions

$$\left.\begin{array}{l}\varphi'(0)=0 \text{ and there exists a convex function}\\ \mu \in C^0([0,\infty)) \cap C^2((0,\infty)) \text{ such that}\\ \mu(0)=0 \text{ and } 0<\mu'(r)\leq \varphi'(r) \text{ for } r>0,\\ b\circ\varphi^{-1} \text{ is Lipschitz continuous on } [0,\infty).\end{array}\right\} \qquad (7)$$

In contrast with the case of other boundary conditions, the Lipschitz assumption on $b\circ\varphi^{-1}$ is, in some sense, necessary in order to get the uniqueness of a periodic weak solution (see Remark 1). Our uniqueness result is consequence of a more general result which shows the continuous dependence of solutions with respect to the data. As another consequence of this general result we prove a comparison principle for periodic solutions which seems to be new even for φ and b regular.

2. EXISTENCE OF PERIODIC SOLUTIONS FOR AN APPROXIMATE PROBLEM.

For different purposes, it will be useful to reformulate problem (1)—(4) by introducing

$$v(x,t) = \varphi(u(x,t)) \text{ and } c = \varphi^{-1}.$$

Then problem (1)—(4) becomes

$$c(v)_t = v_{xx} + b(c(v))_x, \qquad \text{in } (-L,0)\times \mathbb{R} \tag{8}$$

$$v(0,t) = h(t) \qquad \text{for } t \in \mathbb{R}, \tag{9}$$

$$v_x(-L,t) + b(c(v(-L,t))) = g(t) \qquad \text{for } t \in \mathbb{R}, \tag{10}$$

$$v(x,t+\omega) = v(x,t) \quad \text{and } v \geq 0 \text{ in } (-L,0)\times \mathbb{R}. \tag{11}$$

First of all we give the different notions of periodic solutions we shall use in this section

Definition 1 *A function $v(x,t)$ is said to be a* <u>*periodic generalized solution*</u> *of* (8)—(11) *if $v \in C([-L,0]\times \mathbb{R})$ satisfies* (9) *and* (10) *and*

$$\int_{t_0}^{t_1}\int_{-L}^{0}\{c(v)\zeta_t + v\zeta_{xx} - b(c(v))\zeta_x\}dxdt$$

$$= \int_{-L}^{0}\{c(v(x,t_1))\zeta(x,t_1) - c(v(x,t_0))\zeta(x,t_0)\}dx$$

$$+\int_{t_0}^{t_1} h(t)\zeta_x(0,t)dt + \int_{t_0}^{t_1} g(t)\zeta(-L,t)dt - \int_{t_0}^{t_1} v(-L,t)\zeta_x(-L,t)dt$$

for any ζ such that $\zeta, \zeta_t, \zeta_{xx} \in L^2((-L,0)\times(t_0,t_1)), \zeta(0,t) = 0$ on $[t_0,t_1]$ for any $t_0 < t_1$.

The variational formulation of problem (8)—(11) starts by introducing the space

$$V = \{z \in H^1(-L,0) : z(0) = 0\}$$

Definition 2 *A function $v \in h + L^2_{\text{loc}}(\mathbb{R} : V)$ is said to be a* <u>*periodic weak solution*</u> *of* (8)—(11) *if for any compact interval $I = [t_0,t_1]$ in $\mathbb{R}$ we have: $c(v) \in C(I : L^1(-L,0))$ and $c(v)_t \in L^2(I : V')$, $c(v(x,t+\omega)) = c(v(x,t))$ and $c(v(x,t)) \geq 0$ for any $t \in I$ and a.e. $x \in (-L,0)$, v satisfies*

$$\int_I < c(v)_t, z > dt + \int_I\int_{-L}^{0}(v_x + b(c(v)))z_x dxdt = -\int_I g(t)z(-L,t)dt$$

for any $z \in L^2(I : V)$.

It is easy to see that any continuous periodic weak solution is a periodic generalized solution and that any periodic generalized solution v such that $v_x \in L^2_{\text{loc}}(\mathbb{R} : L^2(-L,0))$ is a periodic weak solution (see [8, Theorem 3.2] for a related result). Another class of more regular solutions is the following one:

Definition 3 *A function v is said to be a* <u>*periodic strong solution*</u> *of* (8)—(11) *if v is a periodic weak solution such that $c(v)_t \in L^2((-L,0)\times I)$ for any compact interval $I = [t_0,t_1]$ in $\mathbb{R}$.*

The existence of periodic solutions of problem (8)—(11) will be obtained by using some L^∞-estimates which are related with the existence of nonnegative solutions of the following stationary problem:

$$SP(b\circ c, H, G) \quad \begin{cases} W_{xx} + b(c(W))_x = 0, & \text{in } (-L,0) \\ W(0) = H \\ W_x(-L) + b(c(W(-L))) = G \end{cases}$$

where $G \leq 0 \leq H$ are given numbers.

It is not difficult to see that some assumptions on the function $b(c(.))$ are needed in order to get the existence of a solution of $SP(b\circ c, H, G)$. Take, for example, $c(s) = s, b(s) = \lambda s^\gamma, H > 0$ and $G = 0$. Then, it is easy to prove that if $\gamma > 1$ and $\lambda > 0$ there is a blow-up point in $(-L, 0)$ (*i.e.* there is no global solutions) if $-L < (H^{\gamma-1}\lambda(1-\gamma))^{-1} < 0$.

In [17] the existence of a solution of $SP(b\circ c, H, G)$ is proved under the assumption of boundedness of $b\circ c$. Later (see Proposition 1) we shall generalize the result of [17] by assuming the conditions

$$\overline{H}(b\circ c, H, G) \begin{cases} H = G \text{ or there exists } \overline{\sigma} \in (H, +\infty) \text{ such that} \\ 0 \leq \displaystyle\int_h^\sigma \frac{ds}{(b(c(s)))_+ - G} < +\infty, \quad \forall\sigma \in (H, \overline{\sigma}) \text{ and} \\ \displaystyle\int_h^{\overline{\sigma}} \frac{ds}{(b(c(s)))_+ - G} \geq L \end{cases}$$

and

$$\underline{H}(b\circ c, H, G) \quad \begin{cases} (b(c(s)))_- + G = 0 \ a.e. \text{ or there exist } \epsilon_1, \epsilon_2 \geq 0, \\ H - \epsilon_1 \geq 0 \text{ such that} \\ (b(c(s)))_- + G > 0, \text{ for } a.e.\ s \in (H-\epsilon_1, H] \\ (b(c(s)))_- + G < 0, \text{ for } a.e.\ s \in [H, H+\epsilon_2) \\ 0 \leq \displaystyle\int_s^H \frac{ds}{(b(c(s)))_- - G} \geq L \text{ for } a.e.\ s \in (H-\epsilon_1, H+\epsilon_2) \end{cases}$$

Remark 1 The conditions $\overline{H}(b\circ c, H, G)$ and $\underline{H}(b\circ c, H, G)$ are satisfied if $|b(c(s))| \leq K_b$. In that case, if $G \neq 0$, then

$$0 \leq \int_H^\sigma \frac{ds}{(b(c(s)))_+ - G} \leq \frac{H-\sigma}{G} < +\infty;$$

and

$$\int_H^{\overline{\sigma}} \frac{ds}{(b(c(s)))_+ - G} \geq \int_H^{\overline{\sigma}} \frac{ds}{K_b - G} \geq L$$

if $\overline{\sigma} \geq H + L(K_b - G)$. Whenever $G = 0$ the assumptions are trivially satisfied. To verify $\underline{H}(b\circ c, H, G)$, we consider the following cases:

- If $(b(c(s)))_- + G > 0$, *a.e.* in $(H-\epsilon_1, H]$, then

$$0 \leq \int_s^H \frac{dt}{(b(c(t)))_- + G} \leq \int_{H-\epsilon_1}^H \frac{dt}{(b(c(t)))_- + G} \leq L$$

when we choose ϵ_1 small enough.

- If $(b(c(s)))_- + G < 0$, *a.e.* in $[H, H+\epsilon_2)$, then

$$0 \leq \int_s^H \frac{dt}{(b(c(t)))_- + G} \leq \int_{H+\epsilon_2}^H \frac{dt}{(b(c(t)))_- + G} \leq L$$

if ϵ_2 is small enough. Thus, $\underline{H}(b \circ c, H, G)$ follows. $\square$

Remark 2 We consider again the example $c(s) = s, b(s) = \lambda s^\gamma,\ \gamma > 0,\ \lambda > 0,\ G = 0,\ H > 0$. In this case $\overline{H}(b \circ c, H, 0)$ leads to the condition

$$\int_H^\sigma \frac{ds}{\lambda s^\gamma - G} = \frac{1}{\lambda} \int_H^\sigma \frac{ds}{s^\gamma} < +\infty$$

and thus we can choose

$$L \leq L_0 := \int_H^{\overline{\sigma}} \frac{ds}{\lambda s^\gamma}. \quad \square$$

The main result of this section is the following

Theorem 1 *Assume* $(H_\varphi), (H_b), (H_h)(H_g), \underline{H}(b \circ c, M_h + 1, -M_g)$, *as well as* $\overline{H}(b \circ c, M_h + 1, -M_g)$. *Then there exists a periodic weak solution* v *of problem* (8)—(11). *Moreover* $v \in C([-L, 0] \times \mathbb{R})$ *and* $v_x \in L^\infty((-L, 0) \times \mathbb{R})$.

As mentioned in the Introduction, the proof of Theorem 1 will be obtained by passing to the limit on solutions of some regularized formulations. In fact, given $0 < \epsilon \leq 1$ we shall start by showing the existence of a periodic solution for the approximate problem

$$\begin{cases} c_\epsilon(v_\epsilon)_t = v_{\epsilon xx} + b_\epsilon(c_\epsilon(v_\epsilon))_x & \text{in } (-L, 0) \times \mathbb{R}, \\ v_\epsilon(0, t) = h_\epsilon(t) & \text{for } t \in \mathbb{R}, \\ v_{\epsilon x}(-L, t) + b_\epsilon(c_\epsilon(v_\epsilon(-L, t))) = g_\epsilon(t) & \text{for } t \in \mathbb{R}, \\ v_\epsilon(x, t+\omega) = v_\epsilon(x, t) \quad \text{and } v_\epsilon \geq 0 & \text{in } (-L, 0) \times \mathbb{R}, \end{cases}$$

where the functions c_ϵ and b_ϵ are constructed in such a way that

(H_c^ϵ) $\quad c_\epsilon(s) = c(s)$ if $\dfrac{\epsilon}{2} \leq s$, $\quad \varphi_\epsilon(c_\epsilon)^{-1}$ satisfies (H_φ) and $\varphi_\epsilon \in C^{2+\alpha}_{\rm loc}([0, \infty))$

(H_b^ϵ) $\quad b_\epsilon(s) = b(s)$ if $s \geq c\left(\frac{\epsilon}{2}\right)$, $\quad b_\epsilon$ satisfies (H_b) and $b_\epsilon \in C^{2+\alpha}_{\rm loc}([0, \infty))$.

The regularized boundary data are taken such that $h_\epsilon \downarrow h$ and $g_\epsilon \uparrow g$ uniformly as $\epsilon \downarrow 0$ on any compact interval $[0, T]$ and satisfy the following conditions

$$(H_h^\epsilon) \begin{cases} h^\epsilon \text{ is uniformly Lipschitz continuous on } \mathbb{R} \text{ and } \omega\text{-periodic}, \\ h_\epsilon(t) \geq \epsilon \text{ and } h_\epsilon(t) \leq M_h + \epsilon \text{ and } h_\epsilon(t) \leq h_{\epsilon'}(t) \text{for any } t \in \mathbb{R} \text{ if } \epsilon \leq \epsilon', \\ |h'_\epsilon(t)| \leq K_h \text{ for } a.e.\ t \in \mathbb{R} \text{ and some } K_h \text{ independent on } \epsilon \end{cases}$$

$$(H_g^\epsilon)\quad \begin{cases} g^\epsilon \text{ is Lipschitz continuous } |g_\epsilon'(t)| \leq C \text{ with } C \text{ independent on } \epsilon, \\ g_\epsilon \text{ is } \omega\text{−periodic, } g_\epsilon(t) \leq 0 \text{ and } g_\epsilon(t) \geq -M_g \text{ and} \\ g_\epsilon(t) \geq g_{\epsilon'}(t) \text{for any } t \in \mathbb{R} \text{ if } \epsilon \leq \epsilon' \end{cases}$$

where

$$M_h = \max\{h(t) : t \in [0, \omega]\}, \tag{12}$$

and

$$M_g = \max\{-g(t) : t \in [0, \omega]\}, \tag{13}$$

In order to show that $\{v_\epsilon\}$ is uniformly bounded we start by studying the existence of a nonnegative solution of the problem $SP(b \circ c, H, G)$.

Proposition 1 *Assume* $(H_\varphi), (H_b), \underline{H}(b \circ c, H, G)$ *and* $\overline{H}(b \circ c, H, G)$. *Then, there exists a unique solution* $W \in C^1([-L, 0])$ *of* $SP(b \circ c, H, G)$. *Moreover* $W(x) \geq 0$ *in* $[-L, 0]$.

Proof. Integrating on $(-L, x)$, we see that any solution W of $SP(b \circ c, H, G)$ must satisfies the Cauchy Problem

$$(CP) \quad \begin{cases} W' + b(c(W)) = G \text{ in } (-L, 0) \\ W(0) = H \end{cases}$$

The existence and uniqueness of a C^1-solution W of (CP) is well known, since $H \geq 0$ and $b \circ c$ is locally Lipschitz function near $s = H$ (see (H_φ) and (H_b)). We shall show that this local solution is globally defined on $[-L, 0]$ and it is nonnegative. From the obvious inequality

$$-(b(c(s)))_- \leq b(c(s)) \leq (b(c(s)))_+$$

and standard arguments, we know that if $\underline{W}, \overline{W}$ are C^1-functions such that

$$(\overline{CP}) \quad \begin{cases} \overline{W}' + b(c(\overline{W}))_+ = G \text{ in } (-L, 0) \\ \overline{W}(0) = H \end{cases}$$

and

$$(\underline{CP}) \quad \begin{cases} \underline{W}' - b(c(\underline{W}))_- = G \text{ in } (-L, 0) \\ \underline{W}(0) = H \end{cases}$$

then, at least, for $x \in (-\epsilon, 0]$, for some $\epsilon > 0$, we have

$$\underline{W}(x) \leq W(x) \leq \overline{W}(x).$$

Our conclusion, will be obtained by constructing $\underline{W}$ and $\overline{W}$ on the whole interval $[-L, 0]$ and with $\underline{W} \geq 0$. Then, by usual ODE arguments, it follows that W is defined on $[-L, 0]$, the above inequality holds for any $x \in [-L, 0]$ and W is a solution of $SP(b \circ c, H, G)$.

To construct $\overline{W}$ solution of $(\overline{CP})$, we define the function

$$\psi(r) := \int_H^r \frac{ds}{(b(c(s)))_+ - G}.$$

It is clear that $\psi(H) = 0$ and by the assumption $\overline{H}(b \circ c, H, G)$, we deduce that $\psi(r) \geq 0$ and $\psi'(r) > 0$ for any $r \in (H, \overline{\sigma})$ and that $\psi(\overline{\sigma}^-) \geq L$. It is easy to check that the function $\overline{W}(x) := \psi^{-1}(-x)$ is defined at least on $[-L, 0]$, satisfies $(\overline{CP})$ and $H \leq \overline{W}(x) < \overline{\sigma}$ for any $x \in [-L, 0]$.

The construction of $\underline{W}$ is similar. If we assume $(b(c(s)))_- \not\equiv -G$ on $(H - \epsilon_1, H + \epsilon_2)$, we define the function

$$\Phi(r) := \int_r^H \frac{ds}{(b(c(s)))_- + G}$$

Obviously $\Phi(H) = 0$. By the assumption $\underline{H}(b \circ c, H, G)$, we deduce that $0 \leq \Phi(r) \leq L$ for any $r \in (H - \epsilon_1, H + \epsilon_2)$ and that $\Phi((H + \epsilon_2) \leq L$. Besides the condition $\underline{H}(b \circ c, H, G)$ implies that $\Phi'(r) \neq 0$ in $(H - \epsilon_1, H + \epsilon_2)$ and so we can define the inverse function Φ^{-1}. Finally, we introduce $\underline{W}(x) := \Phi^{-1}(-x)$ and again it is easy to check that $\underline{W}$ is defined on $[-L, 0]$. $\underline{W}$ satisfies $(\underline{CP})$ and $0 \leq \underline{W}(x)$ for any $x \in [-L, 0]$. To conclude, if $(b(c(s)))_- \equiv G$, then $\underline{W}(x) \equiv H$ satisfies all the required conditions.

In order to show that $\{v_\epsilon\}$ is uniformly bounded we introduce now the auxiliary functions $W_\epsilon(x)$, solutions of the approximate problem

$$SP(b_\epsilon \circ c_\epsilon, M_h + \epsilon, -M_g) \begin{cases} W_{\epsilon xx} + b_\epsilon(c_\epsilon(W_\epsilon))_x = 0, \text{ in } (-L, 0) \\ W_\epsilon(0) = M_h + \epsilon \\ W_{\epsilon x}(-L) + b_\epsilon(c_\epsilon(W_\epsilon))(-L) = -M_g \end{cases}$$

Since we assume $\overline{H}(b \circ c, M_h + 1, -M_g)$ then $\overline{H}(b_\epsilon \circ c_\epsilon, M_h + \epsilon, -M_g)$ holds for any ϵ small. The same is true when we assume $\underline{H}(b \circ c, M_h + 1, -M_g)$. Then by Proposition 1 there exists a unique of $W_\epsilon \in H^1(-L, 0)$ (and so $W_\epsilon \in L^\infty(-L, 0)$) satisfying $SP(b_\epsilon \circ c_\epsilon, M_h + \epsilon, -M_g)$. In addition, since $b_\epsilon(c_\epsilon(s))$ is a locally Lipschitz continuous function the comparison principle holds (Lemma 3.6 of [14]) and so

$$0 \leq \epsilon \leq W_\epsilon(x) \leq W_{\epsilon'}(x) \leq M \text{ for any } x \in [-L, 0] \tag{14}$$

assumed $\epsilon \leq \epsilon'$ and

$$M := \|W_1\|_{L^\infty(-L,0)} \tag{15}$$

(here W_1 denotes W_ϵ with $\epsilon = 1$). Notice that $M < +\infty$ by Proposition 1. □

Concerning the approximate problem we have

Theorem 2 *Assume* $(H_c^\epsilon), (H_b^\epsilon), (H_h^\epsilon), (H_g^\epsilon), \underline{H}(b \circ c, M_h + 1, -M_g)$ *as well as* $\overline{H}(b \circ c, M_h + 1, -M_g)$. *Then there exists a unique periodic strong solution* v_ϵ *of problem* (8)—(11). *Moreover* $v_\epsilon \in C([-L, 0] \times \mathbb{R})$, $(v_\epsilon)_x \in L^\infty((-L, 0) \times \mathbb{R})$,

$$|(v_\epsilon)_x| \leq K \text{ on } (-L, 0) \times \mathbb{R} \tag{16}$$

for some $K > 0$ *independent of* ϵ *and*

$$\epsilon \leq v_\epsilon(x, t) \leq v_{\epsilon'}(x, t) \leq W_{\epsilon'}(x) \leq M \text{ for any } (x, t) \in [-L, 0] \times \mathbb{R} \tag{17}$$

and for any $\epsilon \leq \epsilon'$. *Finally, given* $\delta \in (0, L/2)$ *there exists* $C = C(\delta) > 0$ *such that*

$$\left.\begin{array}{r} c([v_\epsilon(x_1,t_1) - C|x_1 - x_2| - C|t_1 - t_2|^{\frac{1}{2}}]_+) \\ \leq c(\min\{M, v_\epsilon(x_2,t_2) + C|x_1 - x_2| + C|t_1 - t_2|^{1/2}]) + C|t_1 - t_2|^{\frac{1}{2}} \end{array}\right\} \quad (18)$$

for any $(x_i, t_i) \in [-L + \delta, -\delta] \times \mathbb{R},\ i = 1, 2.$

With this information on v_ϵ we can return to the proof of Theorem 1.

Proof of Theorem 1. Given h and g satisfying (H_h) and (H_g), by using convolution with mollifier functions it is possible to show (see *e.g.* the proof of Theorem 1 of [13]) the existence of a family of functions $\{h_\epsilon\}$ and $\{g_\epsilon\}$ which satisfy (H_h^ϵ) and (H_g^ϵ), (in particular

$$h_\epsilon(t) \leq h_{\epsilon'}(t) \text{ and } g_\epsilon(t) \geq g_{\epsilon'}(t) \text{ for } t \in [0, \omega] \tag{19}$$

if $\epsilon \leq \epsilon' \leq 1$) and such that $h_\epsilon \downarrow h$ and $g_\epsilon \uparrow g$ uniformly on $[0, \omega]$. Let v_ϵ be the correspondent periodic strong solution of the relative approximate problem. From (17) we deduce that for any $(x, t) \in [-L, 0] \times \mathbb{R}$ we can define the function

$$v(x,t) = \lim_{\epsilon \downarrow 0} v_\epsilon(x,t) \tag{20}$$

and that

$$0 \leq v(x,t) \leq W_1(x) \leq M$$

Analogously, from (16) we conclude that

$$|v_x| \leq K \text{ in the sense of } \mathcal{D}'((-L,0) \times \mathbb{R}) \tag{21}$$

and so $(v_x) \in L^\infty((-L, 0) \times \mathbb{R})$. From (20) we can apply the Lebesgue Theorem and deduce that

$$\{v_\epsilon\} \to v \quad \text{ in } L^2((-L,0) \times \mathbb{R})$$

$$\{(v_\epsilon)_x\} \rightharpoonup (v)_x \quad \text{ in } L^2_{\text{loc}}(\mathbb{R} : L^2(-L,0))$$

$$\{c(v_\epsilon)\} \to c(v) \text{ and } \{b(c(v_\epsilon))\} \to b(c(v)) \quad \text{ in } L^2_{\text{loc}}(\mathbb{R} : L^2(-L,0))$$

and

$$\{c(v_\epsilon)_t\} \rightharpoonup c(v)_t \quad \text{ in } L^2_{\text{loc}}(\mathbb{R} : V').$$

Then we can pass to the limit in the integral identity of the condition of periodic weak solution of v_ϵ and thus v is a periodic weak solution of the approximate problem. It remains to prove that v is continuous in $\mathbb{R} \times [-L, 0]$. Letting $\epsilon \downarrow 0$ in (18) we conclude that the same inequality also holds with v_ϵ replaced by v, for all $(x_1, t_1), (x_2, t_2) \in (-L, 0) \times \mathbb{R}$. Taking $|t_2 - t_1|$ and $|x_1 - x_2|$ small enough we obtain the continuity of v in $(-L, 0) \times \mathbb{R}$. Finally, the continuity of v at the points $(-L, t)$ and $(0, t), t \in \mathbb{R}$, is implied by the continuity of $h(t), g(t)$, and the fact that for any fixed t, the function $v(\cdot, t)$ is a Lipschitz continuous function on $[-L, 0]$, as we derive from (21). □

Now we pass to consider the proof of Theorem 2. It will follow from the application of the Schauder fixed point theorem on the space $\mathcal{X} = C([-L,0])$ for the Poincaré map

$$F(w_{0,\epsilon}(\cdot)) := w_\epsilon(\cdot,\omega)$$

where w_ϵ denotes the weak solution of the following <u>*initial*</u> boundary value problem: Let $T > \omega$ arbitrary and denote $Q_T = (-L,0) \times (0,T)$; find $w_\epsilon \geq 0$ solution of

$$(IBVP_\epsilon) \qquad \begin{cases} c_\epsilon(w_\epsilon)_t = (w_\epsilon)_{xx} + b_\epsilon(c_\epsilon(w_\epsilon))_x & \text{in } Q_T \\ w_\epsilon(0,t) = h_\epsilon(t) & \text{for } t \in (0,T) \\ (w_\epsilon)_x(-L,t) + b_\epsilon(c_\epsilon(w_\epsilon(-L.t))) = g_\epsilon(t) & \text{for } t \in (0,T) \\ w_\epsilon(x,0) = w_{0,\epsilon}(x) & \text{for } x \in (-L,0). \end{cases}$$

The definition of weak solution of $(IBVP_\epsilon)$ is an obvious extension of the similar definition for the periodic problem. In order to apply the Schauder fixed point we need to find a (nonempty) closed convex $K_\epsilon \subset \mathcal{X}$ such that:

1. $F(K_\epsilon) \subset K_\epsilon$,
2. $F(K_\epsilon)$ is relatively compact in $\mathcal{X}$,
3. $F|_{K_\epsilon}$ is continuous.

We define

$$K_\epsilon = \{w \in C([-L,0]) : w(0) = h_\epsilon(0), \epsilon \leq w(x) \leq W_\epsilon(x) \text{ for any } x \in [-L,0]\}$$

It is clear that K_ϵ is a closed convex set of $\mathcal{X}$. Moreover it is also clear that K_ϵ is not empty because $\epsilon \leq h_\epsilon(0) \leq W_\epsilon(0)$ and then the function $w(x) := \min\{h_\epsilon(0), W_\epsilon(x)\}$ belongs to K_ϵ.

The following result shows the invariance of the set K_ϵ by the application F.

Lemma 1 *Assume the same conditions than in Theorem 2. Let $w_{0,\epsilon} \in K_\epsilon$ arbitrary. Then there exists a unique weak solution w_ϵ of $(IBVP_\epsilon)$. Moreover $w_\epsilon(t) \in K_\epsilon$ for any $t \in [0,T]$.*

Proof. Let $w_{0,\epsilon}^k \in K_\epsilon \cap H^1(-L,0)$ such that $w_{0,\epsilon}^k \to w_{0,\epsilon}$ in $L^2(\Omega)$ (such a sequence can be constructed by standard regularizing arguments). Then we are in a position to apply Theorems 1.7, 2.2 and 2.3 of Alt-Luckhaus [2] and so, for any fixed $k \in \mathbb{N}$, there exists a unique function $w_\epsilon^k \in L^2(0,T : H^1(-L,0))$ with $c(w_\epsilon^k) \in C([0,T] : L^2(-L,0))$ and $c(w_\epsilon^k)_t \in L^2(Q_T)$ satisfying the equation in $\mathcal{D}'(Q_T)$ and the boundary conditions, and such that $w_\epsilon^k(x,0) = w_{0,\epsilon}^k(x)$ on $(-L,0)$. From the above regularity we also deduce that $w_\epsilon^k(\cdot,t) \in C([-L,0])$ for any $t \in [0,T]$. On the other hand, if we call $w_1(x,t) = \epsilon$ and $w_2(x,t) = W_\epsilon(x)$ we have that w_i satisfy the equation in $\mathcal{D}'(Q_T)$ and

$$w_1 \leq w_\epsilon^k \leq w_2 \text{ in } \{0\} \times (0,T)$$

$$-(w_1)_x - b(c(w_1)) \leq -(w_\epsilon^k)_x - b(c(w_\epsilon^k)) \leq -(w_2)_x - b(c(w_2)) \text{ in } (-L,0)\times\{0\}$$
$$w_1 \leq w_{0,\epsilon}^k \leq w_2 \text{ in } (-L,0)\times\{0\}.$$

Then, as $c(w_\epsilon^k)_t \in L^2(Q_T)$ we can apply the comparison theorem (Theorem 2.2) of [2] and we conclude that

$$\epsilon \leq w_\epsilon^k(t,x) \leq W_\epsilon(x) \text{ in } Q_T. \tag{22}$$

Moreover, we have the estimate

$$|w_\epsilon^k|_{L^2(0,T:H^1(-L,0))} + \text{ess.sup}_{[0,t]}|t^{1/2}w_\epsilon^k(t)|_{H^1(-L,0)} \leq M$$

where M is a positive constant depending on $|w_{0,\epsilon}^k|_{L^2(-L,0)}$, $|g_\epsilon|_{W^{1,2}(0,T)}$ and $|h_\epsilon|_{W^{1,2}(0,T)}$ (see (3.2) of [14]). Now it is easy to see that $w_\epsilon^k \rightharpoonup w_\epsilon$ in $L^2(0,T: H^1(-L,0))$, and that w_ϵ is a weak solution of $(IBVP_\epsilon)$ satisfying $w_\epsilon(t,\cdot)\in K_\epsilon$ for any $t\in[0,T]$. Finally, as $b_\epsilon(c_\epsilon(s))$ is a Lipschitz continuous function, we can apply Theorem 2.4 of [2] which shows the uniqueness of a weak solution of $(IBVP_\epsilon)$. □

The continuity of $F\,|_{K_\epsilon}$ is shown in the next result.

Lemma 2 *Assume the same conditions than in Theorem 2. Let $w_{0,\epsilon}, w_{0,\epsilon}^n \in K_\epsilon$ such that $w_{0,\epsilon}^n \to w_{0,\epsilon}$ uniformly in $[-L,0]$ as $n\to\infty$. Then, if w_ϵ and w_ϵ^n are the weak solutions of $(IBVP_\epsilon)$ of initial data $w_{0,\epsilon}$ and $w_{0,\epsilon}^n$ respectively we have that $w_\epsilon^n(x,t)\to w_\epsilon(x,t)$, uniformly in $[-L,0]$, as $n\to\infty$, for any $t\in[0,T]$.*

Proof. We start by showing that

$$\int_{-L}^0 |c(w_\epsilon^n(x,t)) - c(w_\epsilon(x,t))|dx \leq \int_{-L}^0 |w_{0,\epsilon}^n(x) - w_{0,\epsilon}(x)|dx. \tag{23}$$

Indeed, as w_ϵ^n and w_ϵ are obtained as limits of strong solutions, and as this inequality is stable by convergence in L^1, we can assume, without lost of generality, that $c(w_\epsilon^n)_t$, and $c(w_\epsilon)_t \in L^1(Q_T)$. Multiplying by $\text{sign}(w_\epsilon^n(x,t)-w_\epsilon(x,t))$, integrating by parts and using that

$$\int_0^T\int_{-L}^0 -(w_\epsilon^n - w_\epsilon)_{xx}\text{sign}(w_\epsilon^n - w_\epsilon)dxdt \geq 0$$

we obtain (23) (for a justification of the above argument see Bénilan [5] or Díaz-de Thelin [9]). Then $c(w_\epsilon^n(\cdot,t))$ converges to $c(w_\epsilon(\cdot,t))$ strongly in $L^1(-L,0)$ as $n\to\infty$ and $w_\epsilon^n(x,t)$ converges to $w_\epsilon(x,t)$ for *a.e.* $x\in(-L,0)$. Since $w_\epsilon^n(\cdot,t)\leq M$ (M given by (10)), we conclude, by the Lebesgue Theorem, that $w_\epsilon^n(\cdot,t)\to w_\epsilon(\cdot,t)$ in $L^p(-L,0)$ for any $1\leq p\leq\infty$. Finally, as $w_\epsilon^n(\cdot,t), w_\epsilon(\cdot,t)\in C([-L,0])$ (since $(w_\epsilon^n)_x, (w_\epsilon)_x \in L^2$) the uniform convergence holds. □

By the Ascoli-Arzelá theorem, $F(K_\epsilon)$ is relatively compact in $C([-L,0])$ if and only if the set $F(K_\epsilon)$ is *equicontinuous* on $[-L,0]$. This will be proved by means of a uniform estimate on the derivative $(w_\epsilon)_x$.

Lemma 3 *Assume* $(H_c^\epsilon), (H_b^\epsilon), (H_h^\epsilon)$ *and* (H_g^ϵ). *Let* $w_{0,\epsilon} \in K_\epsilon$ *arbitrary and let* w_ϵ *be the weak solution of* $(IBVP_\epsilon)$. *Then* $(w_\epsilon)_x \in L^\infty(Q_T)$. *More explicitly, given any* $\tau > 0$ *there exists a constant* $C > 0$ *which depends only on* M_h, M_g, K_h, L *and* τ *such that*

$$|(w_\epsilon)_x(x,t)| \leq C \max\{t^{-1/2}, \tau^{-1/2}\} + M^* \quad \text{in } Q_T \tag{24}$$

where

$$M^* = \max\{b(c(s)) : \ s \in [0, M]\} \text{ with } M \text{ given by } (15). \tag{25}$$

Proof. We shall start by proving (24) for a sequence of approximate classical solutions. In a final step we shall obtain (24) by passing to the limit. By using regularizing arguments (convolution with mollifier functions, etc; see [13], Theorem 1) it is possible to approximate h_ϵ, g_ϵ and $w_{0,\epsilon}$ satisfying the following properties:

$$w_{0,\epsilon}^k \in C^{2+\alpha_k}([-L,0]), \text{ for some } \alpha_k \in (0,1],$$
$$\epsilon \leq w_{0,\epsilon}^k(x) \leq W_\epsilon(x) \text{ for any } x \in [-L,0]$$
$$h_\epsilon^k \in C^{1+\alpha_k}([0,T]), \epsilon \leq h_\epsilon^k(t) \leq M_h + \epsilon \text{ for any } t \in [0,T],$$
$$|(h_\epsilon^k)'(t)| \leq K_h \text{ for any } t \in [0,T] \text{ and any } k \in \mathbb{N} \ (K_h \text{ given in } (H_h^\epsilon)),$$
$$w_{0,\epsilon}^k(0) = h_\epsilon^k(0), (w_{0,\epsilon}^k)''(0) + b(c(w_{0,\epsilon}^k))'(0) = c(h_\epsilon^k(0))$$
$$g_\epsilon^k \in C^{1+\alpha_k}([0,T]), -M_g \leq g_\epsilon^k(t) \leq 0 \text{ for any } t \in [0,T]$$
$$(w_{0,\epsilon}^k)'(-L) + b(c(w_{0,\epsilon}^k))(-L) = g_\epsilon^k(0).$$

Moreover the sequences $\{w_{0,\epsilon}^k\}, \{h_\epsilon^k\}$ and $\{g_\epsilon^k\}$ are monotone in k and

$$\left.\begin{array}{l} \{w_{0,\epsilon}^k\} \downarrow w_{0,\epsilon}, \ \{h_\epsilon^k\} \downarrow h_\epsilon \text{ and } \{g_\epsilon^k\} \uparrow g \ \text{ uniformly on } [-L,0] \\ \text{and } [0,T] \text{ respectively, as } k \to \infty \ . \end{array}\right\} \tag{26}$$

Under the above conditions, there exists a unique classical solution $w_\epsilon^k \in C^{2+\alpha_k}(\overline{Q}_T)$ of the problem

$$(IBVP_\epsilon^k) \qquad \left\{ \begin{array}{ll} c_\epsilon(w_\epsilon^k)_t = (w_\epsilon^k)_{xx} + b_\epsilon(c_\epsilon(w_\epsilon^k)) & \text{in } Q_T \\ w_\epsilon^k(0,t) = h_\epsilon^k(t) & \text{for } t \in (0,T) \\ (w_\epsilon^k)_x(-L,t) + b_\epsilon(c_\epsilon(w_\epsilon^k(-L.t)))_x = g_\epsilon^k(t) & \text{for } t \in (0,T) \\ w_\epsilon^k(x,0) = w_{0,\epsilon}^k(x) & \text{for } x \in (-L,0). \end{array} \right.$$

Indeed, this conclusion is shown in [18] (Theorem 7.4) under a set of assumptions that are fulfilled in our case except the one concerning the growth of the nonlinear function $b_\epsilon(c_\epsilon(\cdot))$ in the boundary condition on $x = -L$ (see assumption (7.34) of [18]). Nevertheless, such a condition is only used in order to assure the boundness of any solution. In our case, any classical solution is also a weak solution and so, due to the special coupling between the equation (8) and this boundary condition we do not need any additional assumption on $b_\epsilon(c_\epsilon(\cdot))$. From Lemma 1 we conclude that

$$\epsilon \leq w_\epsilon^k(x,t) \leq W_\epsilon(x) \quad \text{for any } (x,t) \in Q_T$$

and thus the existence and uniqueness of a classical solution follows from the arguments of [18]. Moreover, from the regularity assumed on b_ϵ and φ_ϵ we can apply Theorem 10 of [11] to conclude that $(w_\epsilon^k)_x \in C^{2,1}(Q_T)$.

The estimate (24) for the classical solution w_ϵ^k is obtained by the combination of the Bernstein technique and a barrier argument near $x = 0$. That was carried out by Gilding [13] (Lemma 2.b and Lemma 3) who proved the following:

Lemma 4 [13]. *Assume $(H_c^\epsilon), (H_b^\epsilon), (H_h^\epsilon)(H_g^\epsilon)$. Let w_ϵ^k be any classical solution of the equation* (8) *such that $(w_\epsilon^k)_x \in C^{2,1}(Q_T)$. Assume also that*

$$L > \rho \quad \text{for some } \rho > 0. \tag{27}$$

Let $\tau > 0$ arbitrary. Then there exists $C_1 > 0$, which depends only on K_h, $\|w_\epsilon^k\|_\infty$ and τ, such that

$$|(w_\epsilon^k)_x(0,t) + b_\epsilon(c_\epsilon(w_\epsilon^k))(0,t)| \leq C_1 \max\{t^{-1/2}, \tau^{-1/2}\} \text{ for all } t \in (0,T].$$

Moreover if we assume

$$|(w_\epsilon^k)_x(-L,t) + b_\epsilon(c_\epsilon(w_\epsilon^k))(-L,t)| \leq K_g \tag{28}$$

for any $t \in (0,T]$ and for some $K_g > 0$ then there exists $C_2 > 0$, which depends only on $C_1, K_g, \|w_\epsilon^k\|_\infty$ and τ, such that

$$|(w_\epsilon^k)_x(x,t) + b_\epsilon(c_\epsilon(w_\epsilon^k))(x,t)| \leq C_2 \max\{t^{-1/2}, \tau^{-1/2}\} \text{ for all } (x,t) \in Q_T. \tag{29}$$

Proof of Lemma 3. (Continuation). First of all we shall prove that assumption (27) is not important in our case. Indeed, if $L \leq \rho$ we introduce

$$y = 2\rho\frac{x}{L} \quad \text{and } w_\epsilon^k(y,t) = w_\epsilon^k(x,t).$$

Then $y \in [-2\rho, 0]$ when $x \in [-L, 0]$ and $\tilde{w}_\epsilon^k$ is a classical solution of the equation

$$c_\epsilon(\tilde{w}_\epsilon^k)_t = \frac{2\rho}{L}\left(\frac{2\rho}{L}(\tilde{w}_\epsilon^k)_{yy} + b_\epsilon(c_\epsilon(\tilde{w}_\epsilon^k))_y\right) \quad \text{in } (-2\rho, 0) \times (0,T)$$

leading to a similar conclusion (in this case C_1 and C_2 depend also on $2\rho/L$ but this is not relevant for our arguments). On the other hand, assumption (28) is equivalent to

$$|\ g_\epsilon^k(t)\ | \leq K_g$$

which trivially holds by taking $K_g = M_g + M^*$, with M^* given by (25). Thus, by Lemma 4, estimate (29) holds. Using that $w_\epsilon^k \geq \epsilon$ and the definition of b_ϵ, c_ϵ and M^* we conclude that

$$|(w_\epsilon^k)_x(x,t)| \leq C \max\{t^{-1/2}, \tau^{-1/2}\} + M^* \quad \text{for all } (x,t) \in Q_T \tag{30}$$

where C depends only on M_h, M_g, K_h, L and $\tau > 0$. Finally, from the monotonicity with respect to k of $h^k_\epsilon, g^k_\epsilon$ and $w^k_{0,\epsilon}$, and the comparison principle (see Theorem 2.2 of [2]) we deduce that

$$w^{k+1}_\epsilon(x,t) \leq w^k_\epsilon(x,t) \quad \text{for all } (x,t) \in Q_T.$$

Hence, we can define

$$\mathbf{w}_\epsilon(x,t) = \lim_{k\uparrow\infty} w^k_\epsilon(x,t) \text{ for all } (x,t) \in Q_T$$

and in fact, $w^k_\epsilon \to \mathbf{w}_\epsilon$ in $L^P(Q_T)$ for any $1 \leq p \leq +\infty$. Now, it is a standard matter to see that $\mathbf{w}_\epsilon$ is a solution of $(IBVP_\epsilon)$. Then, by the uniqueness of weak solutions (since $b_\epsilon(c_\epsilon(\cdot))$ is Lipschitz continuous) we deduce that $\mathbf{w}_\epsilon = w_\epsilon$. Finally, as the estimate (30) is stable by weak convergence in $L^2(Q_T)$ we obtain (24). □

The estimate (18) of Theorem 2 will be a consequence of a similar inequality for solutions w_ϵ of the (IBVP$_\epsilon$).

Lemma 5 . *Assume $(H^\epsilon_c), (H^\epsilon_b), (H^\epsilon_h)$ and (H^ϵ_g). Let $w_{0,\epsilon} \in K_\epsilon$ arbitrary and let w_ϵ be the weak solution of $(IBPV_\epsilon)$. Then given $\delta \in (0, L/2), \tau > 0$ and $\lambda \in (0,T)$ there exists a constant C_3 which depends on $\delta, \tau, \lambda, C_2$ (given in (29)), M and M^* such that*

$$\left.\begin{array}{r} c([w_\epsilon(x_1,t_1) - C_3|x_1 - x_2| - C_3|t_1 - t_2|^{1/2}]_+) \\ \leq c(\min[M, w_\epsilon(x_2,t_2) + C_3|x_1 - x_2| + C_3|t_1 - t_2|^{1/2}]) + C_3|t_1 - t_2|^{1/2} \end{array}\right\} \quad (31)$$

for all $(x_1,t_1), (x_2,t_2) \in [-L+\delta, -\delta] \times (\lambda, T]$.

Proof. Let w^k_ϵ be the classical solution of $(IBVP^k_\epsilon)$ given in the proof of Lemma 3. Given $\delta \in (0, L/2)$ define

$$\mathcal{R}^\lambda_\delta = (-L+\delta, -\delta) \times (\lambda, T].$$

By (29) we have

$$|(w^k_\epsilon)_x(x,t) + b(c(w^k_\epsilon))(x,t)| \leq C_1(\lambda,\tau) \quad \text{for all } (x,t) \in (-L,0) \times (\lambda, T),$$

with

$$C_1(\lambda,\tau) = C_2 \max\{\lambda^{-1/2}, \tau^{-1/2}\}$$

and where we have used that $b_\epsilon(c_\epsilon(w^k_\epsilon)) = b(c(w^k_\epsilon))$ since $w^k_\epsilon \geq \epsilon$. Applying Lemma 4 of [13] (see also [18]) we conclude that

$$c([w^k_\epsilon(x_1,t_1) - C_3|x_1 - x_2| - C_3|t_1 - t_2|^{1/2}]_+)$$

$$\leq c(\min[M, w^k_\epsilon(x_2,t_2) + C_3|x_1 - x_2| + C_3|t_1 - t_2|^{1/2}]) + C_3|t_1 - t_2|^{1/2}$$

for all $(x_1,t_1), (x_2,t_2) \in \overline{\mathcal{R}}^\lambda_\delta$, where

$$C_3 = \max\{C_1(\lambda,\tau) + M^*, \frac{M}{\delta}\}$$

Taking the limit as $k \uparrow \infty$, inequality (31) holds. □

We now have all the ingredients necessary to prove Theorem 2.

Proof of Theorem 2. We define the closed convex set K_ϵ and the Poincaré map as before. By Lemmas 1 and 2 we have that $F(K_\epsilon) \subset K_\epsilon$ and that $F\mid_{K_\epsilon}$ is continuous. Lemma 3 shows that $(w_\epsilon(\cdot, \omega))$ is Lipschitz continuous of coefficient independent of the initial datum $w_{0,\epsilon} \in K_\epsilon$. Thus $F(K_\epsilon)$ is relatively compact in $C([-L, 0])$ and we can apply the Schauder fixed point theorem. To prove the estimate $|(v_\epsilon)_x| \leq C$, with C independent on ϵ, it is enough to take the limit in (24) and to use the ω-periodicity of the function $v(t - t_0, x)$ for any $t_0 \neq 0$. On the other hand, applying Lemma 5 with $\lambda = \omega/2$ and $T = 2\omega$ to the solution $v_\epsilon = w_\epsilon$ of $(IBVP_\epsilon)$ whose initial datum is the fixed point of F on K_ϵ we obtain (18) for any $(x_i, t_i) \in [-L + \delta, -\delta] \times (\frac{\omega}{2}, 2\omega)$. By the periodicity of v_ϵ this inequality holds for any $(x_i, t_i) \in [-L + \delta, -\delta] \times \mathbb{R}$. Finally, if $\epsilon \leq \epsilon'$ the comparison $v_\epsilon(x,t) \leq v_{\epsilon'}(x,t)$ for any $(x,t) \in [-L, 0] \times \mathbb{R}$ is a consequence of the properties assumed in (19) and the comparison principle given in Corollary 3 (the proof of this result for periodic weak solutions of (8)—(12) is obviously independent of the proof of Theorem 1). □

Remark 3 Estimate (16) and thus the x-modulus of continuity of v can be improved under some additional information on φ and b. Moreover, it is possible to show that w_ϵ is in fact a strong solution of (IBVP_ϵ). See Bénilan-Díaz [6]. □

Remark 4 The existence result given in Theorem 1 may be easily generalized to equations with an absorption term

$$u_t = \varphi(u)_{xx} + b(u)_x + f(u). \quad \square$$

Remark 5 It seems interesting to point out that the boundary condition (3) may be the origin of important difficulties when proving the boundness of v. In our case this property was obtained from the special coupling between (3) and the differential equation. Nevertheless it is well known that the solutions of

$$\begin{cases} u_t = \varphi(u)_{xx}, & \text{in } (-L, 0) \times (0, \infty) \\ \varphi(u)_x + b(u) = g(t), & \text{on } (\{-L\} \cup \{0\}) \times (0, \infty) \\ u(x, 0) = u_0(x), & \text{on } (-L, 0) \end{cases}$$

blows-up after a finite time if, for instance $b(u) = u^q$ with $q > 1$ (see *e.g.* Levine-Payne [19]). □

3. A CONTINUOUS DEPENDENCE INEQUALITY: UNIQUENESS AND COMPARISON OF PERIODIC WEAK SOLUTIONS.

In this section we shall use the construction of the periodic weak solution of (1)—(4) made in Section 1 in order to show that, at least under the additional assumption (7), this solution is the unique periodic weak solution.

Remark 6 The Lipschitz condition assumed on $b \circ c$ in (7) is, in some sense, a necessary condition in order to get the uniqueness of the periodic weak solution. Indeed, if we take $g \equiv 0$ and $h(t) \equiv H$ a nonnegative constant, any stationary function $V(x)$ satisfying

$$(ODE) \qquad \begin{cases} V_x + b(c(V)) = 0, & -L < x < 0 \\ V(0) = H \end{cases}$$

is a weak periodic solution of problem (8)—(11). Then, if for instance we assume $H = 0$ and $b(c(s)) = \lambda s^\gamma$ with $\lambda > 0$ and $\gamma \in (0,1)$ the uniqueness result fails ($V \equiv 0$ and $V(x) = (1-\gamma)\lambda(-x)^{1/(1-\gamma)}$ are two different solutions). It would be interesting to know if the uniqueness of periodic weak solutions of (1)—(4) can be obtained merely under an Osgood's type condition on $b \circ c$ (notice that this implies the uniqueness of solutions of the ODE). □

To get our uniqueness result we start by proving some inequalities concerning v_ϵ(the periodic weak solution of (8)—(11)) and v^* any periodic weak solution of (8)—(11) of boundary data $h^*(t), g^*(t)$. We define

$$u_\epsilon = c(v_\epsilon) \quad \text{ and } u^* = c(v^*).$$

Throughout the remainder of the paper we shall assume $(H_\varphi), (H_b), (H_h)$ and (H_g) when dealing with problem (8)—(11) and $(H_c^\epsilon), (H_b^\epsilon), (H_h^\epsilon)$ and (H_g^ϵ) when dealing with problem (8)—(11).

The main result of this section is the following

Theorem 3 *Assume* (7) *and that* $h_\epsilon \downarrow h^*$ *(and so* $h^* = h$*) uniformly on* $[0,T]$ *as* $\epsilon \downarrow 0$*. Then, for every* $f \in C^\infty(\overline{Q}_T); Q_T = (-L,0) \times (0,T)$*, and every* $\theta \in C_0^\infty([-L,0])$ *with* $0 \leq \theta(x) \leq 1$ *there exists a constant* $C > 0$ *depending on* $\|f\|_{\infty,\overline{Q}_T}$ *such that*

$$\begin{aligned} &\int_{-L}^0 (u_\epsilon(x,T) - u^*(x,T))\theta(x)dx - \int_{-L}^0 \int_0^T (u_\epsilon(x,t) - u^*(x,t))f(x,t)dtdx \\ &\leq C\left(\int_0^T |g_\epsilon(t) - g^*(t)|dt + \int_{-L}^0 |u_\epsilon(x,0) - u^*(x,0)|dx\right) + O(\epsilon). \end{aligned} \tag{32}$$

In addition, if $f \leq 0$ *on* Q_T*, then*

$$\begin{aligned} &\int_{-L}^0 (u_\epsilon(x,T) - u^*(x,T))\theta(x)dx - \int_{-L}^0 \int_0^T (u_\epsilon(x,t) - u^*(x,t))f(x,t)dtdx \\ &\leq C\left(\int_0^T [g_\epsilon(t) - g^*(t)]_+ dt + \int_{-L}^0 [u_\epsilon(x,0) - u^*(x,0)]_+ dx\right) + O(\epsilon). \end{aligned} \tag{33}$$

Finally, if $f \equiv 0$*,* C *it may be chosen as* $C \equiv 1$*.*

Before giving the proof of this theorem, we shall derive several consequences from inequalities (32) and (33).

Corollary 1 *Assume* (7). *Let u^* be any periodic weak solution of* (1)—(4) *of boundary data $h^*(t) = h(t)$ and $g^*(t)$ satisfying (H_h) and (H_g). Let $u = c(v)$ with v the periodic weak solution of* (8)—(11) *obtained in Theorem 1. Then, for every $T > 0$ we have*

$$\begin{aligned} &\int_{-L}^{0}[u(x,T) - u^*(x,T)]_+dx \\ \le &\int_0^T [g^*(t) - g(t)]_+dt + \int_{-L}^{0}[u(x,0) - u^*(x,0)]_+dx \end{aligned} \tag{34}$$

and

$$\begin{aligned} &\int_{-L}^{0}\int_0^T |u(x,t) - u^*(x,t)|^2 dtdx \\ \le\, &C\left(\int_0^T |g(t) - g^*(t)|dt + \int_{-L}^{0}|u(x,0) - u^*(x,0)|dx\right) \end{aligned} \tag{35}$$

for some positive constant C which does not depend on T.

Corollary 2 *Assume* (7). *Then problem* (1)—(4) *has a unique periodic weak solution.*

Proof of Corollary 1. To prove (34) it suffices to choose $f(x,t) \equiv 0$ and $\theta = \theta_k \in C_0^\infty([-L,0])$ with $\theta_k(x) \to \mathrm{sign}_+(u_\epsilon(x,T) - u(x,T))$ in $L^1(-L,0)$ as $k \to \infty$. Thus, (33) gives

$$\begin{aligned} &\int_{-L}^{0}[u_\epsilon(x,T) - u^*(x,T)]_+dx \\ \le &\int_0^T [g_\epsilon(t) - g^*(t)]_+dt + \int_{-L}^{0}[u_\epsilon(x,0) - u^*(x,0)]_+dx + O(\epsilon). \end{aligned}$$

Letting $\epsilon \to 0^+$, we obtain the conclusion. The proof of (35) is obtained in a similar way but now with $\theta \equiv 0$ and $f = f_k \in C^\infty(\overline{Q}_T), f_k \to -(u_\epsilon - u^*)$ in $L^2(Q_T)$ as $k \to \infty$. As u^* is ω-periodic, we have that $\|f_k\|_{\infty,\overline{Q}_T} < \|u_\epsilon - u^*\|_{\infty,Q_\omega} + O(\frac{1}{k})$ and so C may be chosen independently of T. $\square$

Proof of Corollary 2. Let u^* be a periodic weak solution of (1)—(4). Since (35) holds for any $T > 0$, we choose $T = n\omega,\ n \in \mathbb{N}$. Due to of the periodicity of u and u^*, one has

$$\begin{aligned} n\int_{-L}^{0}\int_0^\omega |u(x,t) - u^*(x,t)|^2 dtdx &= \int_{-L}^{0}\int_0^{n\omega} |u(x,t) - u^*(x,t)|^2 dtdx \\ \le C\int_{-L}^{0}&|u(x,0) - u^*(x,0)|dx \le 2CLN, \quad \forall n \in \mathbb{N} \end{aligned} \tag{36}$$

where $N > \max\{\|u\|_{\infty,Q_T}, \|u^*\|_{\infty,Q_T}\}$. Hence, $u = u^*$. $\square$

Proof of Theorem 3. Let $v_\epsilon(x,t)$ and $v^*(x,t)$ be as above, then

$$\begin{aligned} 0 = &\int_0^T v_\epsilon(0,t)\zeta_x(0,t)dt - \int_0^T v_\epsilon(-L,t)\zeta_x(-L,t)dt \\ -\int_{-L}^{0}\int_0^T v_\epsilon\zeta_{xx}dtdx &+ \int_{-L}^{0}\int_0^T b(c(v_\epsilon))\zeta_x dtdx - \int_{-L}^{0}\int_0^T c(v_\epsilon)\zeta_t dtdx \\ &+ \int_{-L}^{0} c(v_\epsilon)\zeta|_0^T dx + \int_0^T g_\epsilon(t)\zeta(-L,t)dt \end{aligned} \tag{37}$$

and, analogously

$$
\begin{aligned}
0 = \int_0^T v^*(0,t)\zeta_x(0,t)dt - \int_0^T v^*(-L,t)\zeta_x(-L,t)dt \\
- \int_{-L}^0 \int_0^T v^*\zeta_{xx}dtdx + \int_{-L}^0 \int_0^T b(c(v^*))\zeta_x dtdx - - \int_{-L}^0 \int_0^T c(v^*)\zeta_t dtdx \\
+ \int_{-L}^0 c(v^*)\zeta dx|_0^T + \int_0^T g^*(t)\zeta(-L,t)dt
\end{aligned} \tag{38}
$$

for any $\zeta \in L^2(Q_T)$ such that $\zeta_t, \zeta_{xx} \in L^2(Q_T)$ and $\zeta(0,t) = 0$ on $[0,T]$. Subtracting (37) from (38) one has

$$
\begin{aligned}
\int_{-L}^0 \int_0^T -(u_\epsilon - u^*)[\zeta_t + e_\epsilon\zeta_{xx} - B_\epsilon\zeta_x]dtdx = \int_0^T (h^*(t) - h_\epsilon(t))\zeta_x(0,t)dt \\
+ \int_0^T (\varphi(u_\epsilon(-L,t)) - \varphi(u^*(-L,t)))\zeta_x(-L,t)dt - \int_{-L}^0 (u_\epsilon - u^*)\zeta dx|_0^T \\
+ \int_0^T (g_\epsilon(t) - g^*(t))\zeta(-L,t)dt
\end{aligned} \tag{39}
$$

where

$$
e_\epsilon = e_\epsilon(x,t) := \frac{\varphi(u_\epsilon) - \varphi(u^*)}{u_\epsilon - u^*} = \int_0^1 \varphi'(\lambda u_\epsilon(x,t) + (1-\lambda)u^*(x,t))d\lambda,
$$

$$
B_\epsilon = B_\epsilon(x,t) := \frac{b(u_\epsilon) - b(u^*)}{u_\epsilon - u^*} = \int_0^1 b'(\lambda u_\epsilon(x,t) + (1-\lambda)u^*(x,t))d\lambda.
$$

As u_ϵ and u^* are bounded, by (17) we have

$$
0 < c(\epsilon) \leq \max\{u_\epsilon(x,t), u^*(x,t)\} \leq \overline{M}, \quad \forall (x,t) \in \overline{Q}_T.
$$

where

$$
\overline{M} = \max\{M, \|u\|_\infty\} \quad (M \text{ given by (15)}). \tag{40}
$$

From [13, Lemma 5], if follows that there exist some positive constants $\underline{\alpha}$, $\overline{\alpha}$, $\overline{\beta}$ which depend only on ϵ and M such that

$$
0 < \underline{\alpha}(\epsilon) \leq e_\epsilon(x,t) \leq \overline{\alpha}(\overline{M}), \forall (x,t) \in \overline{Q}_T \tag{41}
$$

$$
|B_\epsilon(x,t)| \leq \overline{\beta}(\overline{M}), \quad \forall (x,t) \in \overline{Q}_T. \tag{42}
$$

Now, consider the backward linear parabolic problem with smooth coefficients

$$
(\zeta_{\epsilon m})_t + e_{\epsilon m}(\zeta_{\epsilon m})_{xx} - B_{\epsilon m}(\zeta_{\epsilon m})_x = f, \quad \text{in } Q_T, m \in \mathbb{N} \tag{43}
$$

$$
\zeta_{\epsilon m}(x,T) = \theta(x) \geq 0, \quad \text{for } x \in (-L,0) \tag{44}
$$

$$
\zeta_{\epsilon m}(0,t) = 0, \quad \text{for } t \in (0,T) \tag{45}
$$

$$
(\zeta_{\epsilon m})_x(-L,t) = 0, \quad \text{for } t \in (0,T), \tag{46}
$$

with $f, e_{\epsilon m}, B_{\epsilon m} \in C^\infty(\overline{Q}_T), e_{\epsilon m} \to e_\epsilon, B_{\epsilon m} \to B_\epsilon$ uniformly in $\overline{Q}_T$ when $m \to \infty$, and $\theta \in C_0^\infty([-L,0]), 0 \leq \theta(x) \leq 1$. From (41) and (42), one has

$$
0 < \underline{\alpha}(\epsilon) \leq e_{\epsilon m}(x,t) \leq \overline{\alpha}(\overline{M}), \; \forall (x,t) \in \overline{Q}_T \tag{47}
$$

$$
||B_{\epsilon m}(x,t)| \leq \overline{\beta}(\overline{M}), \; \forall (x,t) \in \overline{Q}_T. \tag{48}
$$

The existence, uniqueness and regularity of $\zeta_{\epsilon m}(x,t)$ follows from [18]. Moreover, $\zeta_{\epsilon m}(x,t)$ has the following properties:

Lemma 6 *Assume (7). Let $\zeta(x,t) := \zeta_{\epsilon m}(x,t)$ be the solution to (43)—(46) then*

$$\max_{\overline{Q}_T} |\zeta(x,t)| \leq C \tag{49}$$

If $f(x,t) \leq 0$, then

$$\zeta(x,t) \geq 0 \quad in\ \overline{Q}_T \tag{50}$$

$$\epsilon\zeta_x(0,t) = 0(\epsilon), \quad in\ [0,T] \tag{51}$$

$$\int_{-L}^{0}\int_0^T |\zeta_x(x,t)|^2 dtdx \leq K \quad and \quad \int_{-L}^{0}\int_0^T |\zeta_{xx}(x,t)|^2 dtdx \leq K \tag{52}$$

where $C := C(\|f\|_{\infty,\overline{Q}_T})$ and $K := K(\epsilon, \overline{\beta}, \|f\|_{2,Q_T})$.

Proof. Inequalities (49) and (50) are a straightforward consequence of the maximum principle with $C(\|f\|_{\infty,\overline{Q}_T}) \geq 1$. The proof of property (51) is an easy adaptation (use that $\|f\|_{L^\infty(Q_T)} \leq C$ with C independent on ϵ) of Lemma 4.2 of Díaz-Kersner [8] where (51) is obtained for the case $f \equiv 0$ and b satisfying a set of assumptions that hold trivially under condition (7). We point out the local character of this result.

To justify (52), we multiply (43) by ζ_{xx} and integrate by parts over $[-L,0] \times [s,T]$, $\forall s \in [0,T)$. This yields,

$$\begin{aligned} &2\int_s^T\int_{-L}^0 e_{\epsilon m}|\zeta_{xx}|^2 dxdt + \int_{-L}^0 |\zeta_x(x,s)|^2 dx \\ &= \omega^* + 2\int_s^T\int_{-L}^0 B_{\epsilon m}\zeta_x\zeta_{xx} dxdt + 2\int_s^T\int_{-L}^0 f\zeta_{xx} dxdt, \end{aligned} \tag{53}$$

where $\omega^* := \displaystyle\int_{-L}^0 |\theta'(x)|^2 dx$. Therefore,

$$\begin{aligned} &2\underline{\alpha}(\epsilon)\int_s^T\int_{-L}^0 |\zeta_{xx}|^2 dxdt + \int_{-L}^0 |\zeta_x(x,s)|^2 dx \\ &= \omega^* + 2\overline{\beta}(M)\int_s^T\int_{-L}^0 |\zeta_x\zeta_{xx}| dxdt + +2\int_s^T\int_{-L}^0 |f\zeta_{xx}| dxdt. \end{aligned} \tag{54}$$

By Young's inequality,

$$2|\zeta_x\zeta_{xx}| \leq \frac{\underline{\alpha}(\epsilon)}{\overline{\beta}(\overline{M})}|\zeta_{xx}|^2 + \frac{\overline{\beta}(\overline{M})}{\underline{\alpha}(\epsilon)}|\zeta_x|^2 \tag{55}$$

and

$$2|f\zeta_x| \leq \frac{\underline{\alpha}(\epsilon)}{2}|\zeta_{xx}|^2 + \frac{2}{\underline{\alpha}(\epsilon)}|f|. \tag{56}$$

Substituting (55) and (56) into (54), we obtain

$$\begin{aligned} &\frac{\underline{\alpha}(\epsilon)}{2}\int_s^T\int_{-L}^0 |\zeta_{xx}|^2 dxdt + \int_{-L}^0 |\zeta_x(x,s)|^2 dx \\ &\leq \omega^* + \frac{\overline{\beta}(\overline{M})}{\underline{\alpha}(\epsilon)}\int_s^T\int_{-L}^0 |\zeta_x|^2 dxdt + \frac{2}{\underline{\alpha}(\epsilon)}\|f\|^2_{2,Q_T} \end{aligned} \tag{57}$$

Applying Gronwall's inequality and ignoring the first term on the left-hand side, we have

$$\int_{-L}^{0} |\zeta_x(x,s)|^2 dx \le (\omega^* + \frac{2}{\underline{\alpha}(\epsilon)}\|f\|^2_{2,Q_T}) \exp\left(\frac{(\overline{\beta}(\overline{M}))^2}{\underline{\alpha}(\epsilon)}(T-s)\right), \ \forall s \in [0,T). \tag{58}$$

Integrating (58) with respect to s from 0 to T, one has the first assertion in (52). The other estimate, is obtained substituting (58) in (57). □

Proof of Theorem 3 (continuation). By substituting the solution of (43)—(46) in (39), we obtain

$$\begin{aligned}
&\int_{-L}^{0}(u_\epsilon(x,T)-u^*(x,T))\theta(x)dx - \int_{-L}^{0}\int_0^T (u_\epsilon(x,t)-u^*(x,t))f(x,t)dtdx \\
&\quad = \int_0^T (h^*(t)-h_\epsilon(t))\zeta_x(0,t)dt + \int_{-L}^{0}(u_\epsilon(x,0)-u^*(x,0))\zeta(x,0)dx \\
&\qquad + \int_0^T (g^*(t)-g_\epsilon(t))\zeta(-L,t)dt \\
&\qquad + \int_{-L}^{0}\int_0^T (u_\epsilon(x,t)-u^*(x,t))(e_\epsilon - e_{\epsilon m})\zeta_{xx}(x,t)dtdx \\
&\qquad - \int_{-L}^{0}\int_0^T (u_\epsilon(x,t)-u^*(x,t))(B_\epsilon - B_{\epsilon m})\zeta_x(x,t)dtdx.
\end{aligned} \tag{59}$$

Using Lemma 3 we conclude

$$\begin{aligned}
&\int_{-L}^{0}(u_\epsilon(x,T)-u^*(x,T))\theta(x)dx - \int_{-L}^{0}\int_0^T (u_\epsilon(x,t)-u^*(x,t))f(x,t)dtdx \\
&\quad \le \int_0^T |h^*(t)-h_\epsilon(t)|\cdot|\zeta_x(0,t)|dt \\
&\quad +C_1\left(\int_0^T\int_0^T |g^*(t)-g_\epsilon(t)|dt + \int_{-L}^{0}|u_\epsilon(x,0)-u^*(x,0)|dx\right) \\
&\quad + \max_{\overline{Q}_t}|u_\epsilon(x,t)-u^*(x,t)|\left\{\max_{\overline{Q}_t}|e_\epsilon - e_{\epsilon m}|| + \max_{\overline{Q}_t}|B_\epsilon - B_{\epsilon m}||\right\}(LTK)^{1/2}
\end{aligned} \tag{60}$$

Letting $m \to \infty$ in (60), we obtain (33). Moreover, inequality (32) follows easily from (59) because of the nonnegativity of $\zeta(x,t)$. □

Remark 7 It would be interesting to extend the results of this section to a more general class of functions φ (see the results of [13] for a related Cauchy-Dirichlet problem). We also point out that Theorem 3 and Corollary 1 remain valid when u^* is a weak solution of the associated initial boundary value problem on Q_T (in this case C may depend on T) □ .

We conclude by proving the following comparison result for periodic weak solutions of (1)—(4).

Corollary 3 *Assume* (7). *Let* u *and* u^* *be weak periodic solutions of* (1)—(4) *with data* $h(t), g(t), h^*(t), g^*(t)$ *respectively. Then, if* $g^*(t) \le g(t)$ *and* $h^*(t) \ge h(t), \forall t \in [0,\omega]$, *we have* $u^*(x,t) \ge u(x,t)$ *in* $(-L,0)\times \mathbb{R}$.

Proof. It is not difficult to show that (33) holds also assumed $h(t) \leq h^*(t), \forall t \in [0, \omega]$. Then taking $\theta(x) \equiv 0$ and $f_k \in L^\infty(Q_T)$ such that $f_k \to f$ in $L^\infty(Q_T)$ as $k \to \infty$, with $f(x,t)$=-sign$_+(u(x,t) - u^*(x,t))$ one has

$$\int_0^T \int_{-L}^0 (u(x,t) - u^*(x,t))_+ dxdt \leq C \int_{-L}^0 (u(x,0) - u^*(x,0))_+ dx. \qquad (61)$$

If we choose $T = n\omega,\ n \in \mathbb{N}$, (61) gives

$$n \int_{-L}^0 \int_0^T (u(x,t) - u^*(x,t))_+ dt \leq 2C\overline{M}L,$$

for any $n \in \mathbb{N}$, where $\overline{M} \geq \max(\|u\|_{\infty,Q_T}, \|u^*\|_{\infty,Q_T})$. Thus, $u^*(x,t) \geq u(x,t)$ in Q_ω. □

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