ISBN : 0240 - 2955

The Support Shrinking Properties for Solutions of Quasilinear Parabolic Equations with Strong Absorption Terms(*)

STANISLAV ANTONTSEV[1], JESUS ILDEFONSO DÍAZ[2] and SERGUEI I. SHMAREV[3]

RÉSUMÉ. — On étudie par méthode d'énergie certaines propriétés de support des solutions faibles d'équations paraboliques du second ordre avec absorption.

Ces solutions faibles ne sont pas nécessairement de signe constant et des propriétés de non-propagation de perturbations de la donnée initiale, rétrécissement conique de support et formation de *dead-core* sont établies sous hypothèses sur la non linéarité et conditions locales sur les données.

ABSTRACT. — The local energy method is used to study some support shrinking properties of local solutions of second order nonlinear parabolic equations with absorption terms. We deal with local weak solutions, not necessarily having a definite sign, and we establish properties such as the non-propagation of the initial disturbances, support shrinking of cone type, and formation of dead cores (null-level sets with positive measure). The conditions providing these effects are formulated in terms of local assumptions on the data and the character of the nonlinearity terms of the equation under consideration.

(*) Reçu le 26 octobre 94

(1) Lavrentiev Institute of Hydrodynamics, 630090 Novosibirsk (Russia) and Departamento de Matemáticas, Universidad de Oviedo, 33007 Oviedo (Spain)

(2) Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid (Spain)

Partially sponsored by the DGICYT (Spain), projet PB93/0443.

(3) Lavrentiev Institute of Hydrodynamics, 630090 Novosibirsk (Russia) and Departamento de Matemáticas, Universidad de Oviedo, 33007 Oviedo (Spain)

1. Introduction

1.1 Statement of the problem

This paper deals with the propagation and vanishing properties of local weak solutions of nonlinear parabolic equations. Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2, \ldots$, be an open connected domain with smooth boundary $\partial\Omega$, and $T > 0$. We consider the problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left(|u|^{\alpha-1}u\right) = \operatorname{Div}\left(\vec{A}(x,t,u,\nabla u)\right) - B(x,t,u) + f(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in } Q = \Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

assuming that the functions $\vec{A}$ and B are subject to the following structural conditions: there exist constants $\lambda > 0$ and $p > 1$ such that

$$\forall\ (x,t,s,\rho) \in \Omega \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^N,$$
$$M_1|\rho|^p \leq \left(\vec{A}(x,t,s,\rho)\,,\, \rho\right) \leq M_2|\rho|^p \tag{1.2}$$

$$\forall\ (x,t,s) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}, \quad sB(x,t,s) \geq M_3|s|^{\lambda+1}\,. \tag{1.3}$$

In (1.2)-(1.3) M_i, $i = 1, 2, 3$, are positive constants. The term "strong absorption" involved in the title of this article refers to the additional (and crucial) assumption

$$\lambda < \alpha \tag{1.4}$$

The right-hand side $f(x,t)$ of equation (1.1) and the initial data $u_0(x)$ are assumed to satisfy

$$u_0 \in L^{\alpha+1}(\Omega)\,, \quad f \in L^{(1+\lambda)/\lambda}(Q)\,. \tag{1.5}$$

We are interested in the qualitative properties of solutions of problem (1.1), understood in the following sense.

Definition 1.— *A measurable in Q function $u(x,t)$ is said to be a weak solution of problem (1.1) if*

(a) $u \in L^p(0,T;\, W^{1,p}(\Omega)) \cap L^\infty(0,T;\, L^{\alpha+1}(\Omega))$;

(b) $\lim_{t\to 0}\left\|u(x,t) - u_0(x)\right\|_{L^{\alpha+1}(\Omega)} = 0$;

(c) for any test function $\zeta(x,t) \in W^{1,\infty}(0,T;\, W_0^{1,p}(\Omega))$, *vanishing at* $t = T$, *the integral identity holds*

$$\int_Q \left\{|u|^{\alpha-1}u\zeta_t - (\vec{A}, \nabla\zeta) - B\zeta + f\zeta\right\} dx\,dt + \int_\Omega |u_0|^{\alpha-1}u_0\zeta(x,0)\,dx = 0\,. \tag{1.6}$$

Let us note at once that we will never touch any question concerning the solvability of problem (1.1). As we said, the paper deals with qualitative properties of local weak solutions of problem (1.1), regardless the boundary conditions on $\Sigma = (0,T) \times \Omega$ they probably correspond to. Evidently, each of the functions satisfying in a weak sense equation (1.1), some boundary conditions on Σ, and the initial conditions is also a weak solution in the sense of our definition.

So far, the theory of problems of the type (1.1) already accounts for a number of existence results. We refer the reader to papers [1], [9], [12], [17] and their references.

The class of equations of (1.1) includes, in particular, the following equation

$$v_t = \Delta\left(|v|^{m-1}v\right) - M_3|v|^{\gamma-1}v + f(x,t)\,. \tag{1.7}$$

To pass to an equation of the form (1.1) with the parameters $\alpha = 1/m$, $\lambda = \gamma/m$, $p = 2$ amounts to introduce the new unknown $v := |u|^{1/m}\operatorname{sign} u$. Equation (1.7) is usually referred to as the nonlinear heat equation with absorption. If $v(x,t)$ is interpreted as the temperature of some continuum, the first and the second terms of the right-hand side of (1.7) represent, respectively, the diffusion and the volume absorption of heat. The term $f(x,t)$ models an external source or sink of heat. Assumptions (1.4) and $\alpha \leq 1$ are equivalent to:

$$m \geq 1\,, \quad \gamma \in (0,1)\,.$$

The first one of these inequalities means that we study the processes of *linear and/or slow diffusion*, while the second one signifies that we are interested in the case of *strong absorption*. In this choice of the exponents of nonlinearity the disturbances originated by data propagate with *finite speed* (see [16] and references therein). Moreover, it is known [19], [20], [15], that in this range of the parameters the supports of nonnegative weak solutions to equation (1.7) may shrink as t grows. It is known also, [8], [11], that solutions of the Cauchy problem and the Cauchy-Dirichlet problem for equation (1.7) may even vanish on some subset of the problem domain Q despite of the fact that u_0 and the boundary data are stricly positive. These properties were derived by means of comparison of solutions of (1.7) with suitable sub and supersolutions of these problems.

It is to be pointed out here that in our formulation the function $\overrightarrow{A}(x,t,x,\rho)$ is not subject to any monotonicity assumptions neither is s nor in ρ. Next, we are not constrained by any special boundary conditions. Lastly, as follows from Definition 1.1, the solutions of problem (1.1) are not supposed to have a definite sign.

Each of these features complicates and makes it hardly possible to apply to the study of the qualitative properties of solutions of problem (1.1) any of the methods based on comparison of solutions (or sub/super-solutions) through the data.

The purpose of the paper is to generalize the referred results and to describe the dynamics of the supports of solutions of problem (1.1) without having recourse to any comparison method. For this purpose the local energy method is used enabling one to reduce the study of the support behavior to dealing with special nonlinear first order differential inequalities for "the energy" functions, associated with the solution under study.

In this paper we propose certain refinements of the energy methods in the literature ([23], [24], [5], [6], [7], [4], [3]). These techniques allow us to obtain certain conclusions about the properties of the supports of local weak solutions to problem (1.1) which rely only on some assumptions about the properties of initial data or even use only the information on the character of the nonlinearity of the equation in (1.1).

The results we obtain below may be illustrated by the following simplified description. Let $v(x,t)$ be a weak solution of the model equation

$$v_t = \Delta_p\big(|v|^{m-1}v\big) - |v|^{\gamma-1}v\,,$$

where $\Delta_p(\,\cdot\,)$ denotes the p-Laplace operator given by

$$\Delta_p v \equiv \mathrm{Div}\big(|\nabla v|^{p-2}\nabla v\big)\,, \quad p>1\,.$$

Then:

(i) if $0<\gamma<1$, $m(p-1)\geq 1$ and $v(x,0)$ is flat enough near the boundary of its support, the so-called "waiting time" of v is complete, *i.e.*

$$\forall\, t\in[\,0\,,T\,]\,,\quad \mathrm{supp}\, v(\,\cdot\,,t)\subseteq \mathrm{supp}\, v(\,\cdot\,,0)$$

(we also may say that there is no dilatation of the initial support);

(ii) if γ, m and p additionally satisfy the relation $m+\gamma\leq p/(p-1)$, we have shrinking of the initial support, *i.e.* the above inclusion is strict:

$$\mathrm{supp}\, v(\,\cdot\,,t)\subset\subset \mathrm{supp}\, v(\,\cdot\,,0)\quad \text{for } t>0 \text{ small enough}\,;$$

(iii) under the assumption of item (ii) on the exponents but without any assumption on the initial datum, a null-set with nonempty interior (or dead core) is formed, *i.e.*

$$\exists\, t^*>0:\quad \forall\, t>t^*\,,\quad \overline{\Omega}\setminus\{\mathrm{supp}\, v(\,\cdot\,,t)\}\neq\emptyset\,.$$

In order to compare these results and the theorems below, recall that $\alpha=1/m$ and $\lambda=\gamma/m$. We also remark that the above results remain true when the diffusion is linear, *i.e.* $p=2$ and $m=1$, but in the presence of the strong absorption term : $\gamma\in(0,1)$.

1.2 Formulation of the results

Let us introduce the following notations: given $T>0$, $t\in[\,0\,,T)$, $x_0\in\Omega$, $\rho\geq 0$, and nonnegative parameters σ and μ,

$$\begin{aligned} P(t,\rho) &\equiv \big\{(x,s)\in Q \mid |x-x_0|<\rho(s)\equiv \rho+\sigma(s-t)^{\mu}\,,\ s\in(t,T)\big\}\\ &\equiv P(t,\rho\,;\sigma,\mu)\,. \end{aligned}$$

It is clear that the choice of the parameters σ, μ, ρ, T determines the shape of the domains $P(t,\rho)$. We distinguish three cases.

(a) $\sigma=0$, $\mu=0$, $\rho>0$; in this case $P(t,\rho)$ is a cylinder $B_\rho(x_0)\times(t,T)$;

(b) $\sigma>0$, $\mu=1$, $\rho>0$; $P(0,\rho)$ renders a truncated cone centered in the point $x_0\in\Omega$ and with base $B_\rho(x_0):=\big\{x\in\Omega \mid |x-x_0|<\rho\big\}$ on the plane $t=0$;

(c) $\sigma>0$, $0<\mu<1$, $\rho=0$; then $P(0,t)$ becomes a paraboloid.

To simplify the notation we will omit the arguments of P wherever possible. Treating separately cases (a), (b), (c) we indicate specially which of the parameters are essential and which are not. These three special cases allow us to obtain qualitative properties of different nature, as is presented in above. The domains of the type $P(t,\rho)$ will play the fundamental role in the definition of the local energy functions

$$E(P) := \int_{P(t,\rho)} \left|\nabla u(x,\tau)\right|^p \,\mathrm{d}x\,\mathrm{d}\tau\,,$$

$$C(P) := \int_{P(t,\rho)} \left|u(x,\tau)\right|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau\,,$$

$$b(P) := \operatorname*{ess\,sup}_{s\in(t,T)} \int_{|x-x_0|<\rho+\sigma(s-t)^\mu} \left|u(x,s)\right|^{\alpha+1} \,\mathrm{d}x\,,$$

associated to any weak solution of problem (1.1). This choice of domains $P(t,\rho)$ is explained by the convenience of having at our disposal the domains of variable form but depending, actually, only on a single variable: ρ in cases (a)-(b) and t in the case (c).

Let us pass to the precise statement of our results. We always assume that conditions (1.2)-(1.4) are fulfilled. The unique global information we need will be formulated in terms of *the global energy function*

$$D(u) := b(T,\Omega) + \int_Q \left(|\nabla u|^p + |u|^{\lambda+1}\right) \mathrm{d}x\,\mathrm{d}t\,,$$

where

$$b(T,\Omega) := \operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \left|u(x,t)\right|^{\alpha+1} \mathrm{d}x\,.$$

Our first result refers to the situation when the support of u (an arbitrary weak solution of (1.1)) does not display the property of dilatation with respect to the initial support $\operatorname{supp} u_0$ and the support of the forcing term $\operatorname{supp} f(\cdot,t)$ (in contrast to the case of the undisturbed equation with $B \equiv 0$). It will be assumed that the data u_0 and f are "flat" enough near the boundary of their supports. For instance, assume that.

$$u_0 \equiv 0 \quad \text{in } B_{\rho_0}(x_0) \text{ for some } x_0 \in \Omega \text{ and } \rho_0 > 0 \tag{1.8}$$

and

$$f \equiv 0 \quad \text{in the cylinder } P = P(0,\rho_0) = P(0,\rho_0\,;\,0,0)\ (= B_{\rho_0}(x_0)\times(0,T)) \tag{1.9}$$

Then *the flatness condition* is stated by the claim of convergence (near $\rho = \rho_0$) of the auxiliary integral

$$\begin{aligned} I := \int_{\rho_0+0} (\rho-\rho_0)^\beta \Big[\|u_0\|^{\alpha+1}_{L^{\alpha+1}(B_\rho(x_0))} + \\ + \|f\|^{(1+\lambda)/\lambda}_{L^{(1+\lambda)/\lambda}(P(0,\rho))} \Big]^{p/(p-1)} \mathrm{d}\rho < \infty \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} \beta &= (1-\delta\widetilde{\theta})(1+\kappa)\,,\\ \delta &= -\left(1+\frac{p-1-\lambda}{p(1+\lambda)}N\right),\\ \widetilde{\theta} &= \frac{pN - r(N-1)}{(N+1)p - Nr}\,, \end{aligned} \tag{1.11}$$

with some

$$\kappa \in \left(0\,,\ \frac{p(1+\alpha)}{(p-1-\lambda)(1-\widetilde{\theta})}\right), \tag{1.12}$$

Note that condition (1.10) implies certain restrictions on the vanishing rates of the functions

$$\|u_0\|_{L^{\alpha+1}(B_\rho(x_0))} \quad \text{and} \quad \|f(\cdot,t)\|_{L^{(\lambda+1)/\lambda}(B_\rho(x_0))} \quad \text{as} \quad \rho\to\rho_0\,.$$

THEOREM 1.— *Assume (1.2), (1.3) and*

$$\lambda < \alpha \le p-1 \tag{1.13}$$

Let u_0 and f satisfy (1.8), (1.9) and (1.10). Then there exists a positive constant M (depending only on the constants in (1.2) and (1.3), ρ_0, and $\operatorname{dist}(x_0,\partial\Omega)$ such that any weak solution of (1.1) with bounded global energy, $D(u)\le M$, possesses the property

$$u(x,t) \equiv 0 \quad \textit{in} \quad B_{\rho_0}(x_0)\times(0,T)\,.$$

Under some additional assumptions on the structural exponents α, λ, p and the function f one may get a stronger result which means that the support of $u(\cdot,t)$ shrinks strictly with respect to the initial support.

THEOREM 2.— *Assume (1.2)-(1.3), (1.13) and let*

$$1+\lambda \le \alpha \frac{p}{p-1}. \tag{1.14}$$

Let u_0 satisfy (1.8). Assume

$$f \equiv 0 \quad \text{in the truncated cone } P \equiv P(0,\rho_0\,;\sigma,1) \text{ for some } \sigma > 0 \tag{1.15}$$

and let (1.10) be true. Then there exist positive constants M and t^* such that each weak solution of problem (1.1) with global energy satisfying the inequality $D(u) \le M$, possesses the property

$$u(x,t) \equiv 0 \quad \text{in} \quad P(0,\rho_0\,;\sigma,1) \cap \{t \le t^*\}.$$

Remark 1.— It is curious to observe that the assertion of Theorem 2 has a local character in the sense that different parts of the boundary of supp u_0 may originate pieces of the boundary of the null-set of $u(x,t)$, which display different shrinking properties. Having a possibility to control the rate of vanishing of u_0 and $f(x,t)$, on may design solutions of problem (1.1) which have prescribed shapes of supports. For the model equation (1.7) this phenomenon is already known as "the heat crystal" [22, Ch. 3, Sec. 3].

The last of our main results refers to the case when the initial datum need not vanish, that is, the parameter ρ_0 in the conditions of Theorems 1 and 2 is assumed to be zero. Assuming $f \equiv 0$ we show how the strong absorption term causes the formation of the null-set of the solution.

THEOREM 3.— *Assume (1.2)-(1.3), (1.13)-(1.14). Let $f \equiv 0$. Then there exist positive constants M, t^*, and $\mu \in (0,1)$ such that any weak solution of problem (1.1) satisfying the inequality $D(u) \le M$ possesses the property*

$$u(x,t) \equiv 0 \quad \text{in} \quad P(t^*,0\,;1,\mu).$$

To get the above results one has to work with weak solutions of suitably bounded global energy. In section 4 we demonstrate how the value of the global energy may be estimated through the data of the problem under consideration: i.e. problem (1.1) supplemented with some boundary conditions. Also, we send the reader to section 4 (final remarks) for certain commentaries on previous results in the literature.

2. Differential inequalities

2.1 Formula of integration by parts

Given $x_0 \in \Omega$, $t \in [0\,,T]$, $\sigma \ge 0$ and $\mu \in (0,1)$, we define the following cutting function on the set $P(t,\rho)$

$$\zeta(x,\theta) := \psi_\varepsilon\left(|x-x_0|\,,\theta\right)\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m\left(u(x,s)\right)\mathrm{d}s\,, \quad h>0\,,$$

where

$$T_m\left(u(x,t)\right) := \operatorname{sign} u(x,s)\min\left\{m\,,\ \left|u(x,s)\right|\right\}, \quad m \in \mathbb{N}\,,$$

and

$$\zeta_k(\theta) := \begin{cases} 1 & \text{if } \theta \in \left[t\,,T-\dfrac{1}{k}\right] \\ k(T-\theta) & \text{for } \theta \in \left[T-\dfrac{1}{k}\,,T\right] \\ 0 & \text{otherwise, } k \in \mathbb{N}\,, \end{cases}$$

$$\psi_\varepsilon\left(|x-x_0|\,,\theta\right) := $$
$$:= \begin{cases} 1 & \text{if } \operatorname{dist}\left((x,\theta)\,,\partial_\ell P(t,\rho)\right) > \varepsilon \\ \dfrac{1}{\varepsilon}\operatorname{dist}\left((x,\theta)\,,\partial_\ell P(t,\rho)\right) & \text{if } \operatorname{dist}\left((x,\theta)\,,\partial_\ell P(t,\rho)\right) < \varepsilon \\ 0 & \text{if } (x,\theta) \in Q \setminus P(t,\rho),\ \varepsilon > 0. \end{cases}$$

Here and in the what follows $\partial_\ell P$ denotes the lateral boundary of P, *i.e.*

$$\partial_\ell P := \left\{(x,s)\ \middle|\ |x-x_0| = \rho + \sigma(s-t)^\mu\,,\ s \in (t,T)\right\}.$$

By construction we have $\operatorname{supp}\zeta(x,\theta) \equiv P(t,\rho)$. It is known, [13], that for every natural m, k and positive real numbers h, ε

$$\zeta\,,\ \frac{\partial\zeta}{\partial t} \in L^\infty\left((0,T)\times\Omega\right), \quad \frac{\partial\zeta}{\partial x_i} \in L^p\left((0,T)\times\Omega\right).$$

These properties allow one to substitute $\zeta(x,\theta)$ into the integral identity (1.6) as a test function. Passing to the limits in the appearing equality and taking into account conditions (1.2)-(1.3), one has:

$$\begin{aligned}
i_1+i_2+i_3 &:= \frac{\alpha}{\alpha+1}\int_{P\cap\{t=T\}} |u|^{\alpha+1}\,\mathrm{d}x + \int_P (\overrightarrow{A},\nabla u)\,\mathrm{d}x\,\mathrm{d}\theta + \\
&\quad + \int_P uB\,\mathrm{d}x\,\mathrm{d}\theta \\
&= \int_{\partial_\ell P} (\overrightarrow{n}_x,\overrightarrow{A})u\,\mathrm{d}\Gamma\,\mathrm{d}\theta - \frac{\alpha}{\alpha+1}\int_{\partial_\ell P} n_\tau |u|^{\alpha+1}\,\mathrm{d}\Gamma\,\mathrm{d}\theta + \\
&\quad + \frac{\alpha}{\alpha+1}\int_{P\cap\{t=0\}} |u_0|^{\alpha+1}\,\mathrm{d}x + \int_P uf\,\mathrm{d}x\,\mathrm{d}\theta \\
&:= j_1+j_2+j_3+j_4\,.
\end{aligned} \tag{2.1}$$

Here $\mathrm{d}\Gamma$ is the differential form on the hypersurface $\partial_\ell P\cap\{t=\mathrm{const}\}$, $\overrightarrow{n}_x$ and n_τ are the components of the unit normal vector to $\partial_\ell P$, $|\overrightarrow{n}_x|^2 + |n_\tau|^2 = 1$.

2.2 The energy differential inequalities. Domains of type (c)

Here we derive some differential inequalities for the energy function $E+C$ which later on will be utilized for the proofs of Theorem 1.3. We begin with the most complicated case (c) where the domain P is a paraboloid determined by the parameters $\mu\in(0,1)$, $\sigma>0$ and t:

$$P=P(t)=\left\{(x,\tau)\;\middle|\; |x-x_0|\le\rho(t)\equiv\sigma(\tau-t)^\mu,\ \tau\in(t,T)\right\},\ t\in(0,T).$$

We assume that $f\equiv 0$ and that P does not touch the initial plane $\{t=0\}$. These assumptions simplify the basic energy equality (2.1)

$$i_1+i_2+i_3=j_1+j_2\,.$$

Let us estimate the first term j_1. It is easy to see that

$$\overrightarrow{n}\equiv(\overrightarrow{n}_x,n_\tau)=\frac{1}{\left(\sigma^2\mu^2+(\theta-t)^{2(1-\mu)}\right)^{1/2}}\left((\theta-t)^{1-\mu}\overrightarrow{e}_x-\mu\sigma\overrightarrow{e}_\tau\right),$$

where $\overrightarrow{e}_\tau$ and $\overrightarrow{e}_x$ are unit vectors orthogonal to the hyperplane $t=0$ and the axis t respectively.

Let (ρ,ω), $\rho>0$, $\omega\in\partial B_1$, be the polar coordinate system in $\mathbb{R}^N$. Given an arbitrary function $F(x,t)$, we use the notation $x=(\rho,\omega)$ and $F(x,t)=\Phi(\rho,\overrightarrow{\omega},t)$. There holds the equality

$$\begin{aligned}
I(t) &:= \int_P F(x,\theta)\,\mathrm{d}x\,\mathrm{d}\theta \\
&\equiv \int_t^T \mathrm{d}\theta \int_0^{\rho(\theta,t)} \rho^{N-1}\,\mathrm{d}\rho \int_{\partial B_1} \Phi(\rho,\overrightarrow{\omega},\theta)|J|\,\mathrm{d}\omega\,,
\end{aligned}$$

where J is the Jacobi matrix and, due to the definition of P, $\rho(\theta,t)=\sigma(\theta-t)^\mu$. It is easy to check that:

$$\begin{aligned}
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= -\int_0^{\rho(\theta,t)} \rho^{N-1}\,\mathrm{d}\rho\int_{\partial B_1}\Phi(\rho,\overrightarrow{\omega},\theta)|J|\,\mathrm{d}\omega\Big|_{\theta=t} + \\
&\quad + \int_t^T \rho_t(\theta,t)\rho^{N-1}(\theta,t)\,\mathrm{d}\theta\int_{\partial B_1}\Phi(\rho,\overrightarrow{\omega},t)|J|\,\mathrm{d}\omega \\
&= \int_{\partial_\ell P}\rho_t F(x,\theta)\,\mathrm{d}\Gamma\,\mathrm{d}\theta\,.
\end{aligned} \tag{2.2}$$

Treating the energy function E as a function of t, with the use of (1.2), (2.2), and the Hölder inequality, we have now:

$$\begin{aligned}
&\left|\int_{\partial_\ell P}(\overrightarrow{n}_x,\overrightarrow{A})u\,\mathrm{d}\Gamma\,\mathrm{d}\theta\right| \le \\
&\le M_2\int_{\partial_\ell P}|\overrightarrow{n}_x|\,|\nabla u|^{p-1}|u|\,\mathrm{d}\Gamma\,\mathrm{d}\theta \\
&\le M_2\left(\int_{\partial_\ell P}|\rho_t|\,|\nabla u|^p\,\mathrm{d}\Gamma\,\mathrm{d}\theta\right)^{\frac{p-1}{p}}\left(\int_{\partial_\ell P}\frac{|\overrightarrow{n}_x|^p}{|\rho_t|^{p-1}}|u|^p\,\mathrm{d}\Gamma\,\mathrm{d}\theta\right)^{\frac1p} \\
&= M_2\left(-\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\frac{p-1}{p}}\left(\int_t^T\frac{|\overrightarrow{n}_x|^p}{|\rho_t|^{p-1}}\left(\int_{\partial B_{\rho(\theta,t)}}|u|^p\,\mathrm{d}\Gamma\right)\mathrm{d}\theta\right)^{\frac1p}.
\end{aligned} \tag{2.3}$$

To estimate the right-hand side of (2.3) we use the following interpolation inequality: given $v\in W^{1,p}(B_\rho)$ and $\lambda\le p-1$,

$$\|v\|_{p,S_\rho}\le L_0\left(\|\nabla v\|_{p,B_\rho}+\rho^\delta\|v\|_{\lambda+1,B_\rho}\right)^{\widetilde{\theta}}\cdot\left(\|v\|_{\tau,B_\rho}\right)^{1-\widetilde{\theta}} \tag{2.4}$$

with a universal constant $L_0 > 0$ independent of $v(x)$ and the exponents

$$r \in [1+\lambda, 1+\alpha], \quad \widetilde{\theta} = \frac{pN - r(N-1)}{(N+1)p - Nr}, \quad \delta = -\left(1 + \frac{p-1-\lambda}{p(1+\lambda)} N\right)$$

(see, e.g., Díaz-Veron [13]). Let us introduce the notation

$$E_*(t,\rho) := \int_{B_\rho} |\nabla u|^p \, dx, \quad C_*(t,\rho) := \int_{B_\rho} |u|^{\lambda+1} \, dx,$$

so that

$$E = \int_t^T E_*(\theta, \rho(\theta,t)) \, d\theta, \quad C = \int_t^T C_*(\theta, \rho(\theta,t)) \, d\theta,$$

and make use of the Hölder inequality

$$\left(\int_{B_\rho} |u|^r \, dx\right)^{\frac{1}{r}} \le \left(\int_{B_\rho} |u|^{1+\lambda} \, dx\right)^{\frac{1+\lambda}{qr}} \left(\int_{B_\rho} |u|^{\alpha+1} \, dx\right)^{\frac{(1+\alpha)(q-1)}{qr}},$$

where

$$q = \frac{\alpha - \lambda}{\alpha - r + 1}.$$

Then, by virtue of (2.4),

$$\begin{aligned}
\int_{\partial B_\rho} |u|^p \, d\Gamma &\le \\
&\le L_0 \left(\int_{B_\rho} |\nabla u|^p \, dx + \rho^{\delta p} \left(\int_{B_\rho} |u|^{\lambda+1} \, dx\right)^{\frac{p}{\lambda+1}}\right)^{\widetilde{\theta}} \left(\int_{B_\rho} |u|^r \, dx\right)^{\frac{p(1-\widetilde{\theta})}{r}} \\
&\le L_0 \rho^{\delta \widetilde{\theta} p} \left(\int_{B_\rho} |\nabla u|^p \, dx + \int_{B_\rho} |u|^{\lambda+1} \, dx\right)^{\widetilde{\theta}} \times \\
&\quad \times \left(\int_{B_\rho} |u|^{\lambda+1} \, dx\right)^{\frac{p(1-\widetilde{\theta})}{qr}} \left(\int_{B_\rho} |u|^{\alpha+1} \, dx\right)^{\frac{p(q-1)(1-\widetilde{\theta})}{qr}} \\
&\le K \rho^{\delta \widetilde{\theta} p} (E_* + C_*)^{\widetilde{\theta}} C_*^{(1-\widetilde{\theta})p/qr} b^{(q-1)(1-\widetilde{\theta})p/qr} \\
&\le K \rho^{\delta \widetilde{\theta} p} (E_* + C_*)^{\widetilde{\theta} + (1-\widetilde{\theta})p/qr} b^{(q-1)(1-\widetilde{\theta})p/qr},
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
K &= L_0 \max\left(1, \left(\operatorname*{ess\,sup}_{(t,T)} \int_{B_{\rho(\theta)}} |u|^{\lambda+1} \, dx\right)^{\frac{p}{\lambda+1} - 1}\right)^{\frac{\widetilde{\theta}}{p}} \\
&\le L_0 \max\left(1, (\text{meas}\, B_{\rho(T)})^{\frac{\alpha-\lambda}{\alpha+1}\left(\frac{p}{\lambda+1}-1\right)} (b(T))^{\frac{\lambda+1}{\alpha+1}\left(\frac{p}{\lambda+1}-1\right)}\right)^{\frac{\widetilde{\theta}}{p}}.
\end{aligned}$$

Returning to (2.3) and applying once again the Hölder inequality, we have from (2.5) that if

$$\frac{\widetilde{\theta}}{p} + \frac{1-\widetilde{\theta}}{qr} < 1,$$

then

$$\begin{aligned}
|j_1| &\le L \left(-\frac{dE}{dt}\right)^{\frac{p-1}{p}} \times \\
&\quad \times \left(\int_t^T \frac{|\overrightarrow{n}_x|^p}{|\rho_t|^{p-1}} K \rho^{\delta \widetilde{\theta} p} (E_* + C_*)^{\widetilde{\theta} + (1-\widetilde{\theta})p/qr} b^{(q-1)(1-\widetilde{\theta})p/qr} \, d\tau\right)^{\frac{1}{p}} \\
&\le L \left(-\frac{dE}{dt}\right)^{\frac{p-1}{p}} b^{(q-1)(1-\widetilde{\theta})p/qr} \times \\
&\quad \times \left(\int_t^T (E_* + C_*) \, d\tau\right)^{\frac{\widetilde{\theta}}{p} + \frac{1-\widetilde{\theta}}{qr}} \left(\int_t^T \left(\frac{|\overrightarrow{n}_x|^p}{|\rho_t|^{p-1}} \rho^{\delta \widetilde{\theta} p}(\tau)\right)^{\mu} d\tau\right)^{\frac{1}{\mu p}} \\
&\le M \Lambda(t) \left(-\frac{d(E+C)}{dt}\right)^{\frac{p-1}{p}} b^{(q-1)(1-\widetilde{\theta})/qr} (E + C)^{\frac{\widetilde{\theta}}{p} + \frac{1-\widetilde{\theta}}{qr}}
\end{aligned} \tag{2.6}$$

for a suitable positive constant L and the exponent

$$\mu = \frac{1}{(1-\widetilde{\theta})(1-(p/qr))}.$$

To obtain (2.6) we have assumed

$$\frac{\widetilde{\theta}}{p} + \frac{1-\widetilde{\theta}}{qr} < 1, \tag{2.7}$$

$$\frac{p}{\lambda+1} > 1 \tag{2.8}$$

$$\Lambda(t) := \left(\int_t^T \left(\frac{1}{|\rho_t|^{p-1}}\, \rho^{\delta\widetilde{\theta}p}(\tau)\right)^{\mu} \mathrm{d}\tau\right)^{\frac{1}{\mu p}} < \infty\,. \tag{2.9}$$

Inequality (2.7) is safely fulfilled if $1 < qr$, which is always true. We have:

$$(\alpha+1-r) < r(\alpha-\lambda) \Leftrightarrow r(\alpha+1-\lambda) > \alpha+1 \Leftrightarrow (r-1)(\alpha+1) > r\lambda\,.$$

The last inequality holds just due to the choice of r. Inequality (2.8) follows from (1.4). To satisfy (2.9) one has to take μ small enough, since the condition of convergence of the integral $\Lambda(t)$ is:

$$(1-\mu)(2p-1) + \mu\delta\widetilde{\theta}p > -(1-\widetilde{\theta})\left(1 - \frac{p(\alpha-r+1)}{(\alpha-\lambda)r}\right).$$

So, we have obtained an estimate of the following type:

$$|j_1| \leq L_1 \Lambda(t) D(u)^{(q-1)(1-\widetilde{\theta})/qr} (E+C)^{1-\gamma}\left(-\frac{\mathrm{d}(E+C)}{\mathrm{d}t}\right)^{\frac{p-1}{p}} \tag{2.10}$$

where L_1 is a universal positive constant, $D(u)$ is the total energy of the solution under investigation and

$$\gamma := 1 - \frac{\widetilde{\theta}}{p} - \frac{1-\widetilde{\theta}}{qr} \in (0,1)\,.$$

Let us estimate j_2. For this purpose we use the interpolation inequality

$$\|v\|_{\alpha+1,\partial B_\rho} \leq L_0\left(\|\nabla v\|_{p,B_\rho} + \rho^{\delta}\|v\|_{\lambda+1,B_\rho}\right)^{s}\cdot\|v\|^{1-s}_{r,B_\rho} \tag{2.11}$$

with a universal positive constant $L_0 > 0$, the exponent

$$s = \frac{(\alpha+1)N - r(N-1)}{(N+r)p - Nr}\,\frac{p}{\alpha+1}\,,$$

and δ from (2.4), which holds for each $v \in W^{1,p}(B_\rho)$. Similarly to the previous estimate, using (2.11) we have:

$$\int_{\partial B_\rho} |u|^{\alpha+1}\,\mathrm{d}x \leq$$

$$\leq L\left(\int_{B_\rho}|\nabla u|^p\,\mathrm{d}x + \int_{B_\rho}|u|^{\lambda+1}\,\mathrm{d}x\right)^{\frac{s(\alpha+1)}{p}} \times$$

$$\times\left[\left(\int_{B_\rho}|u|^{\lambda+1}\,\mathrm{d}x\right)^{\frac{1}{qr}}\left(\int_{B_\rho}|u|^{\alpha+1}\,\mathrm{d}x\right)^{\frac{q-1}{qr}}\right]^{(1-s)(\alpha+1)} K^{s(\alpha+1)/\widetilde{\theta}p}.$$

Here K is defined as before. If

$$\frac{s(\alpha+1)}{p} + \frac{(1-s)(\alpha+1)}{qr} < 1\,,$$

$$\frac{s(\alpha+1)}{p} + \frac{(1-s)(\alpha+1)}{qr} + \frac{(q-1)(1-s)(\alpha+1)}{qr} \geq 1\,,$$

and since always $|n_\tau| \leq 1$, this inequality implies

$$|j_2| = \left|\int_t^T |n_\tau|\,\mathrm{d}\tau \int_{\partial B_{\rho(\tau)}} |u|^{\alpha+1}\,\mathrm{d}\Gamma\right|$$

$$\leq L\big(b(T)\big)^{\frac{(q-1)(1-s)(\alpha+1)}{qr}} \times$$

$$\times\left(\int_t^T K^{s(\alpha+1)/\widetilde{\theta}p}(E_*+C_*)^{\frac{s(\alpha+1)}{p}+\frac{(1-s)(\alpha+1)}{qr}}|n_\tau|\,\mathrm{d}\tau\right)$$

$$\leq L\big(E+C+b(T,\Omega)\big)\big(b(T,\Omega)\big)^{\kappa}\int_t^T\left(K^{s(\alpha+1)/\widetilde{\theta}p}\right)^{\varepsilon}\mathrm{d}\tau$$

with the exponents

$$\kappa := 1 - \frac{s(\alpha+1)}{p} + \frac{(1-s)(\alpha+1)}{qr} + \frac{(q-1)(1-s)(\alpha+1)}{qr}\,,$$

$$\varepsilon = \frac{1}{1 - \frac{s(\alpha+1)}{p} - \frac{(1-s)(\alpha+1)}{qr}}\,.$$

To perform this estimate we have assumed that

$$\frac{s(\alpha+1)}{p}+\frac{(1-s)(\alpha+1)}{qr}<1$$

and
$$\frac{s(\alpha+1)}{p}+\frac{(1-s)(\alpha+1)}{qr}+\frac{(q-1)(1-s)(\alpha+1)}{qr}\geq 1\,.$$

The first one of these two inequalities is a simplified version of (2.7). As for the second one, a direct computation shows that it is equivalent to the inequality

$$r\leq \alpha\frac{p}{p-1}\,.$$

Recalling the choice of r, we have to claim:

$$1+\lambda\leq \alpha\frac{p}{p-1}\,,$$

which is the hypothesis (1.14).

We now turn to estimating the left-hand side of (2.1). By (1.2)-(1.3) we have at once that

$$i_2+i_3\geq M_4\left(E+C+\frac{\alpha}{\alpha+1}\int_{B_{\rho(T)}}|u|^{\alpha+1}\,\mathrm{d}x\right)\,,$$
$$M_4=\min\left\{M_1\,;\,M_3\,;\,\frac{\alpha}{\alpha+1}\right\}.$$

Since the right-hand side of (2.1) is an increasing function of T, we may always replace i_1 by $(\alpha/(\alpha+1))b(T)$ in the left-hand side of (2.1). Now, assuming that $T-t$ and $D(u)$ are so small that

$$L\big(b(T,\Omega)\big)^{\kappa}\int_t^T\big(K^{s(\alpha+1)/\widetilde{\theta}p}\big)^{\epsilon}\,\mathrm{d}\tau<\frac{M_4}{2}\,,$$

we arrive at the inequality

$$E+C\leq E+C+b(T,\Omega)$$
$$\leq L_1\Lambda(t)D(u)^{(q-1)(1-\widetilde{\theta})/qr}(E+C)^{1-\gamma}\left(-\frac{\mathrm{d}(E+C)}{\mathrm{d}t}\right)^{\frac{p-1}{p}}\,,$$

whence we get the desired differential inequality for the energy function $Y(t):=E+C$

$$Y^{\gamma p/(p-1)}(t)\leq c(t)\big(-Y(t)\big)'\,, \tag{2.12}$$

where

$$c(t)=\big(L_1(M_*)^{(q-1)(1-\widetilde{\theta})/qr}\Lambda(t)\big)^{\frac{p}{p-1}}\,,\quad L_1=\mathrm{const}>0\,,$$

for $M_*:=D(u)$. Note that $c(t)\to 0$ as $t\to T$. Moreover, the exponent $\gamma p/(p-1)$ always belongs to the interval $(0,1)$. Indeed, the inequality $\gamma p/(p-1)<1$ is equivalent to $qr<p$ which, in its turn, is equivalent to our basic assumption $p>\lambda+1$.

2.3 The energy differential inequalities. Domains of types (a), (b)

In these cases the differential inequality for the energy function $E+C$ is derived in same way that in the case (c) but with certain simplifications due to the choice of the domain P.

Let us begin with the case (b). Let

$$P=\big\{(x,t)\;\big|\;|x-x_0|<\rho+\sigma\theta\,,\;\sigma\in(0,T)\big\}\,,\quad \rho\geq\rho_0>0\,.$$

The unit outer normal to $\partial_\ell P$ has the form

$$\overrightarrow{n}=\frac{1}{\sqrt{1+\sigma^2}}(1,-\sigma)$$

and if we treat now the energy function $Y:=E+C$ as a function of ρ, we have:

$$\begin{aligned}\frac{\mathrm{d}Y(\rho)}{\mathrm{d}\rho}&=\\ &=\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\int_0^T\mathrm{d}\theta\int_0^{\rho+\sigma\theta}\tau^{N-1}\,\mathrm{d}\tau\int_{\partial B_1}|J|\,\big(|\nabla u|^p+|u|^{\lambda+1}\big)\Big|_{x=(\tau,\omega)}\,\mathrm{d}\omega\right\}\\ &=\int_0^T\mathrm{d}\theta\int_{\partial B_1}\left\{(\rho+\sigma\theta)^{N-1}|J|\big(|\nabla u|^p+|u|^{\lambda+1}\big)\Big|_{x=(\rho+\sigma\theta,\omega)}\right\}\mathrm{d}\omega\\ &=\int_{\partial_\ell P}\big(|\nabla u|^p+|u|^{\lambda+1}\big)\,\mathrm{d}\Gamma\,\mathrm{d}\theta\,.\end{aligned} \tag{2.13}$$

Following the above scheme for estimating the term j_1 in (2.1) and applying (2.13), we arrive at the following inequality

$$|j_1| \le \frac{K}{\sqrt{1+\sigma^2}}\left(\frac{\mathrm{d}E}{\mathrm{d}\rho}\right)^{\frac{p-1}{p}} \rho^{\delta\widetilde{\theta}}\left(b(T)\right)^{\frac{(q-1)(1-\widetilde{\theta})}{qr}}\left(\int_0^T (E_*+C_*)^{\widetilde{\theta}+\frac{p(1-\widetilde{\theta})}{qr}}\right)^{\frac{1}{p}}.$$

Let r be such that

$$\widetilde{\theta}+\frac{(1-\theta)p}{qr}=1.$$

Such a choice is always possible, since

$$\widetilde{\theta}+\frac{(1-\theta)p}{qr}=1 \quad \Leftrightarrow \quad r=\frac{p(1+\alpha)}{p+\alpha-\lambda},$$

and the last equality is compatible with the conditions $p>1+\lambda$, $\alpha>\lambda$, and the starting choice of r: $r\in[1+\lambda\,,\,1+\alpha]$. The estimate for j_1 the takes the form

$$|j_1| \le \frac{K\rho^{\delta\widetilde{\theta}}}{\sqrt{1+\sigma^2}}\left(\frac{\mathrm{d}E}{\mathrm{d}\rho}\right)^{\frac{p-1}{p}}\left(b(T)\right)^{\frac{(q-1)(1-\widetilde{\theta})}{qr-\varepsilon}}(E+C)^{\varepsilon+\frac{1}{p}}.$$

with an arbitrary $\varepsilon\in\left(0\,,\,(q-1)(1-\widetilde{\theta})/qr\right)$.

The estimate for j_2 is the same that of the case (c). The only difference is that now we need not claim that T is small. The value of the coefficient in the estimate for j_2 is controlled now by the choice of σ, since $n_\tau=-\sigma/\sqrt{1+\sigma^2}$. Due to (1.8) we have $j_3=0$. At last, we estimate j_4 with the help of the Young inequality

$$|j_4|\le \tau C+L(\tau)\int_P |f|^{(\lambda+1)/\lambda}\,\mathrm{d}x\,\mathrm{d}\theta.$$

Gathering these estimates, we arrive to the inequality

$$Y(\rho)\le c(\rho)y^{\varepsilon+1/p}(\rho)\left(Y'(\rho)\right)^{(p-1)/p}+F(\rho),\quad \rho>\rho_0$$

with the coefficient

$$c(\rho)=\rho^{\delta\widetilde{\theta}}K\left(D(u)\right)^{(q-1)(1-\widetilde{\theta})/qr-\varepsilon}$$

and the right-hand side term

$$F(\rho)=\frac{\alpha}{\alpha+1}\int_{R_\rho(x_0)}|u_0|^{\alpha+1}\,\mathrm{d}x+L(\tau)\int_P|f|^{(\lambda+1)/\lambda}\,\mathrm{d}x\,\mathrm{d}\theta.$$

It is easy to see now that the function

$$Z:=Y^{p(1-\varepsilon)/(p-1)}(\rho)$$

satisfies the inequality

$$Z^{\gamma}(\rho)\le\frac{p-1}{p(1+\varepsilon)}c^{p/(p-1)}Z'(\rho)+F^{p/(p-1)}(\rho),\quad \rho>\rho_0\,,\ \gamma=\frac{1}{1+\varepsilon}. \tag{2.14}$$

In the case (a), the desired inequality (2.14) for the energy function $Z(\rho):=(E+C)^{p(1+\varepsilon)/(p-1)}$ defined on the cylinders

$$P=\left\{(x,t)\mid |x-x_0|<\rho\,,\ t>0\right\}$$

is a by-product of the previous consideration, since the term j_2 of the right-hand side of (2.1) vanishes.

3. Analysis of the Differential Inequalities

3.1 The main lemma

The proofs of Theorems 1-3 are framed by the following general assertion.

LEMMA 1.— *Let a function $U(\rho)$ be defined for $\rho\in(\rho_0,R)$, $\rho_0\ge 0$ and possesses the properties:*

$$0\le U(\rho)\le M=\text{const},\quad U'(\rho)\ge 0$$

and

$$AU^s(\rho)\le G\rho^{-\delta}U'(\rho)+\varphi(\rho)\quad \text{as}\quad \rho\in(\rho_0,R) \tag{3.1}$$

where $R<\infty$, $s\in(0,1)$, A, G, δ are finite positive constants, and $\varphi(\rho)$ is a given function. If the integral

$$i(\rho):=\int_{\rho_0}^{\rho}\sigma^\delta(\sigma-\rho_0)^{-(1+\delta)/(1-s)}\varphi(\sigma)\,\mathrm{d}\sigma$$

converges and the equation

$$(\rho-\rho_0)^{(1+\delta)/(1-s)}\left\{\left(\frac{A(1-s)}{G(1+\delta)}\right)^{\frac{1}{1-s}}-\frac{1}{G}\,i(\rho)\right\}=M \tag{3.2}$$

has a root $\rho_*\in(\rho_0,R)$, then $U(\rho_0)=0$.

Proof.— Let us consider the function

$$z(\rho) = \left(\frac{A(1-s)}{G(1+\delta)}\right)^{\frac{1}{1-s}} (\rho-\rho_0)^{(1+\delta)/(1-s)},$$

satisfying the conditions

$$Az^s = G\rho^{-\delta} z' \quad \text{as} \quad \rho \in (\rho_0, R),\ z(\rho_0) = 0. \tag{3.3}$$

Introduce the function

$$\Phi(\rho) := \exp\left(-\frac{sA}{G}\int_{\rho_0}^{\rho} \sigma^\delta \, d\sigma \int_0^1 (\theta U + (1-\theta z))^{s-1} \, d\theta\right)$$

and observe that always

$$U^s - z^s \equiv \int_0^1 (\theta U + (1-\theta z))^{s-1} \, d\theta (U-z).$$

Subtracting now termwise equality (3.3) from inequality (3.1) and multiplying the result by the function $\rho^{-\delta} G^{-1}\Phi(\rho)$, we get:

$$\frac{d}{d\rho}\{(U-z)\Phi\} \geq -\frac{\rho^\delta}{B}\Phi\varphi. \tag{3.4}$$

Integrate inequality (3.4) over the interval (ρ_0, ρ):

$$U(\rho) \geq z(\rho) + \frac{1}{\Phi(\rho)} U(\rho_0) - \frac{1}{G\Phi(\rho)}\int_{\rho_0}^{\rho} \sigma^\delta \Phi(\sigma)\varphi(\sigma) \, d\sigma. \tag{3.5}$$

Let us relax (3.5), having rewritten it in the form

$$M \geq U(\rho_0) - \frac{1}{G}\int_{\rho_0}^{\rho} \sigma^\delta \varphi(\sigma) \times$$
$$\times \exp\left(\frac{sA}{G}\int_{\sigma}^{\rho} \tau^\delta \, d\tau \int_0^1 (\theta U(\tau) + (1-\theta) z(\tau))^{s-1} \, d\theta\right) d\sigma$$

and then make use of the following relations:

$$\exp\left(\frac{sA}{G}\int_{\sigma}^{\rho} \tau^\delta \, d\tau \int_0^1 (\theta U(\tau) + (1-\theta) z(\tau))^{s-1} \, d\theta\right) \leq$$
$$\leq \exp\left(\frac{sA}{G}\int_1^0 (1-\theta)^{s-1} \, d\theta \int_{\sigma}^{\rho} \tau^\delta z^{s-1}(\tau) \, d\tau\right)$$
$$= \exp\left(\int_{\sigma}^{\rho} \frac{d(z(\tau))}{z(\tau)}\right) = \exp\left(\ln\left(\frac{z(\rho)}{z(\sigma)}\right)\right) = \frac{z(\rho)}{z(\sigma)}.$$

In the result we have

$$\begin{aligned} 0 \leq U(\rho_0) &\leq M - z(\rho)\left\{1 - \frac{1}{G}\int_{\rho_0}^{\rho} \sigma^\delta \frac{\varphi(\sigma)}{z(\sigma)} \, d\sigma\right\} \\ &\equiv M - (\rho-\rho_0)^{(1+\delta)/(1-s)} \left\{\left(\frac{A(1-s)}{G(1+\delta)}\right)^{\frac{1}{1-s}} - \frac{i(\rho)}{G}\right\} \\ &:= F(\rho). \end{aligned} \tag{3.6}$$

Assuming existence of some $\rho_* \in (\rho_0, R)$ such that $F(\rho_*) = 0$, we get $U(\rho_0) = 0$.

3.2 Proofs of Theorems 1-3

We begin with the proof of Theorem 1. One has just to verify that the conditions of Lemma 1 are fulfilled. Assume $u_0(x) \equiv 0$ in a ball $B_{\rho_0}(x_0)$ and $f \equiv 0$ in the cylinder $P(0,\rho)$, having this ball as the down-base. Let $R > 0$ be such that $P(0,R) \subset Q$, and the integral I defined in the conditions of Theorem 1 is convergent. Assuming the restrictions on the structural constants listed in the conditions of Theorem 1, we derive for the corresponding energy function inequality (2.14). By Lemma 1, we see that it is sufficient to point out a threshold value of the total energy $M_*^{p(1+\epsilon)/(p-1)}$ such that equation (3.2) would have a solution ρ_*. Recall that in the case of inequality (2.14) the coefficient G of inequality (3.1) depends only on structural constants and the energy $M_*^{p(1+\epsilon)/(p-1)}$, but does not depend on t. So for the function $F(\rho)$ defined in (3.6) satisfies

$$F(\rho) \to -\infty \quad \text{as} \quad M \to 0,$$

for each $\rho \in (\rho_0, R)$ fixed. Further, G is a linear function of the argument $M_*^{p(1+\epsilon)/(p-1)}$ so that $G \to \infty$ as $M \to \infty$, $G \to 0$ as $M \to 0$. Then from (3.6), having just compared the orders of M of positive and negative terms of $F(\rho)$, that

$$F(\rho) > 0 \quad \text{for large } M.$$

This means that $F(R)$, being viewed as a function of M, is always nonnegative for small M, which proves the theorem. □

The proof of Theorem 2 literally repeats the arguments just presented. The only difference is that now one has to add condition (1.14), needed for the derivation of (2.14).

For the proof of Theorem 3 we assume that the value of T is taken so as to satisfy $P \subset Q$. Remind that the coefficient $c(t)$ in inequality (2.12) may be estimated from above by $\ell := c(0)$. Introduce the function

$$z(T-t) := Y(t)\,.$$

Since it satisfies the inequality

$$z^{\gamma p/(p-1)}(t) \le \ell z'(t) \quad \text{as } t \in (0,T)\,, \quad z(0)=0\,,\ z(t) \in \big[\,0\,,\ D(u)\big]\,, \quad (3.7)$$

there remains to apply Lemma 1 with $i(\rho) \equiv 0$ to complete the proof of Theorem 3. □

4. Final remarks

1) The conclusion made in Theorem 3 via Lemma 1 is too implicit. In order to make evident the dependence between the parameters t^*, T and M, let us integrate inequality (3.7) over the interval $t \in (t_1, t_2) \subset (0,T)$. Then

$$z^{1-\kappa}(t_1) \le z^{1-\kappa}(t_2) - \frac{1-\kappa}{\ell}\,(t_2 - t_1)\,,$$

where

$$\kappa = \frac{p\nu}{p-1}\,,$$
$$\ell = \big(L_1 \Lambda(0) M_*^{(q-1)(1-\theta)/qr}\big)^{\frac{p}{p-1}} \equiv \ell_0 M_*^{\kappa_0}\,,$$
$$\kappa_0 = \frac{p}{p-1}\,\frac{(q-1)(1-\theta)}{qr}\,.$$

It follows then

$$z^{1-\kappa}(t_1) \le \frac{1-\kappa}{\ell_0 M_*^{\kappa_0}} \left(\frac{\ell_0}{1-\kappa}\, M_*^{1+\kappa+\kappa_0} - t_2 + t_1 \right)$$

and, hence, $z(t) = 0$ if

$$t \ge t^* \equiv \left(T - \frac{\ell_0}{1-\kappa}\, M_*^{1+\kappa_0-\kappa} \right), \quad 0 < t^* < T\,. \qquad (4.1)$$

Therefore, for each $0 < T < \infty$ and M_* such that

$$M_* < \left(\frac{1-\kappa}{\ell_0}\, T \right)^{\frac{1}{1+\kappa_0-\kappa}}$$

we get that $u(x,t) \equiv 0$ in $P(t^*,0)$ if $D\big(u(x,t)\big) \le M_*$ and t^* is given by (4.1). On the other hand, it is clear that if u is weak solution of equation (1.1) with the property:

$$\sup_{0<t<\infty} D\big(u(\,\cdot\,,t)\big) \le M = \text{const} < \infty$$

there always exists $t^* < \infty$ such that $u(x,t) \equiv 0$ in $P(t^*,0)$.

2) The estimates for the total energy $D(u)$ may be obtained by adding some additional conditions for defining u, say, the boundary conditions $\partial\Omega \times (0,T)$. As an example, one can consider, for instance, the boundary-value problem (P) which consists in finding a function u satisfying the following conditions

$$\frac{\partial\big(|u|^{\alpha-1}u\big)}{\partial t} - \operatorname{Div} \overrightarrow{A}(x,t,u,\nabla u) + B(x,t,u) = f(x,t) \quad \text{in } Q\,,$$
$$u(x,0) = u_0(x) \quad \text{on } \Omega\,,$$
$$u(x,t) = g(x,t) \quad \text{on } \Gamma_D \times (0,T)\,,$$
$$\overrightarrow{A}(x,t,u,\nabla u)\cdot \overrightarrow{n} = 0 \quad \text{on } \Gamma_N \times (0,T)\,,$$

where

$$\Gamma_G \cup \Gamma_N = \partial\Omega\,, \quad f \in L^{(1+\lambda)/\lambda}(Q)\,,$$
$$u_0 \in L^{\alpha+1}(\Omega)\,, \quad g \in L^p\big(0,T;\, W^{1,p}(\Omega)\big) \cap W^{1,\infty}(Q) \quad \text{(for simplicity)}.$$

To establish the existence of a solution for this problem one needs additional structural assumptions on A and B. For instance, let us assume that A and B are monotone

$$\big(A(x,t,s,\rho_1) - A(x,t,s,\rho_2)\big)\cdot(\rho_1 - \rho_2) \ge 0\,,$$
$$\big(B(x,t,s_1) - B(x,t,s_2)\big)(s_1 - s_2) \ge 0$$

for each s, s_1, $s_2 \in \mathbb{R}$, ρ_1, $\rho_2 \in \mathbb{R}^N$ and $\overrightarrow{A}$ and B satisfy certain growth conditions. Under these assumptions the existence of a weak solution to problem (P) was proved in [17].

3) With the use of Theorem 2 and the estimate of the total energy in the case when $\Omega \equiv \mathbb{R}^N$ one may obtain the so-called property of "instantaneous

shrinking of support" which was found by the first time in the paper by Evans-Knerr [14] for a very particular case of equation (1.1) (see also [18] for certain generalization of this result).

4) As follows from of Theorem 3, solutions of problem (P) possess the property of "formation of a dead core", already known due to [8] and [11] for a special class of problems like (P). It was assumed that $\Gamma_N = 0$, $g > 0$ and $u_0 > 0$, proving that the vanishing region (the dead core where $u = 0$) was located far from the initial plane $t = 0$ and the lateral boundary of Q.

5) Let us present an example of the direct estimating of total energy in terms of the input data. Consider the Dirichlet problem

$$u = G(x,t) \text{ on the parabolic boundary of } Q\,. \tag{4.2}$$

Set in (1.5)

$$\xi(x,\theta) = \xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m\big(u(x,s) - G(x,s)\big)\,\mathrm{d}s\,,$$

where ξ_k, T_m are determined in (2.1), and $G(x,t)$ is a function continuing the initial and boundary data into Q. Proceeding now like in section 2, we get the estimate

$$\begin{aligned} D(u) &:= b(T,\Omega) + \int_Q \big(|\nabla u|^p + |u|^{\lambda+1}\big)\,\mathrm{d}x\,\mathrm{d}t \\ &\le K\left[\sup_{t\le T}\int_\Omega |G(x,t)|\,\mathrm{d}x + \int_Q \big(|\nabla G|^p + \right. \\ &\quad \left. + |G_t|^{(1+\lambda)/(1+\lambda-\alpha)} + |G|^{+\lambda} + |f|^{(1+\lambda)/\lambda}\big)\,\mathrm{d}x\,\mathrm{d}t\right]\,, \end{aligned}$$

where $K \equiv K(\alpha,\lambda,p,M_1 - M_4)$, and, additionally to (1.3),

$$\forall\ (x,t,s) \in \Omega\times\mathbb{R}^+\times\mathbb{R} \quad |\varphi(x,t,s)| \le M_4|s|^\lambda\,,$$

and $\lambda \le \alpha \le 1+\alpha$.

Note that in the case of model equation (1.6) the latter restriction on the parameters of nonlinearity has the form

$$\gamma \le 1 \le 1+m$$

and, so, it is always fulfilled.

In the Cauchy problem

$$u(x,0) = u_0(x) \quad \text{as } x\in\Omega\equiv\mathbb{R}^N$$

for equation (1.1), those solutions which vanish as $|x|\to\infty$, possess the estimate

$$D(u) \le K\left[\int_\Omega |u_0(x)|^{1+\alpha}\,\mathrm{d}x + \int_Q |f(x,t)|^{(1+\lambda)/\lambda}\,\mathrm{d}x\,\mathrm{d}t\right]$$

with $K \equiv K(\alpha,\lambda,p,M_1,M_3)$.

Bibliography

[1] ALT (H. W.) and LUCKHAUS (S.) .— *Quasilinear elliptic-parabolic differential equations*, Math. Z. **183** (1983), pp. 311-341.

[2] ANTONTSEV (S. N.) .— *On the localization of solutions of nonlinear degenerate elliptic and parabolic equations*, Dokl. Akad. Nauk. SSSR, **260**, No 6 (1981), pp. 1289-1293 (in Russian; English translation in Soviet Math. Dokl., **24**, No 2 (1981), pp. 420-424).

[3] ANTOTSEV (S. N.), DÍAZ (J. I.) and SHMAREV (S. I.) .— *New applications of energy methods to parabolic free boundary problems*, Uspekhi Mat. Nauk, **46**, issue 6 (282) (1991), pp. 181-182 (in Russian; English translation: in Russian Mathematical Surveys, **46**, No 6 (1991), pp. 193-194).

[4] ANTONTSEV (S. N.), DÍAZ (J. I.) and SHMAREV (S. I.) .— *New results of the character of localization of solutions for elliptic and parabolic equations*, Birkhäuser-Verlag, International Series of Numerical Mathematics **106** (1992), pp. 59-66.

[5] ANTONTSEV (S. N.) and SHMAREV (S. I.) .— *Localization of solutions of nonlinear parabolic equations with linear sources of general type*, Dinamica Sploshnoi Sredy, Novosibirsk **89** (1989), pp. 28-42.

[6] ANTONTSEV (S. N.) and SHMAREV (S. I.) .— *Local energy method and vanishing of weak solutions of nonlinear parabolic equations*, Dokl. Akad, Nauk SSSR, **318**, No 4 (1991), pp. 777-781 (in Russian; English translation: in Soviet Math. Doklady **43** (1991)).

[7] ANTONTSEV (S. N.) and SHMAREV (S. I.) .— *Local energy method and vanishing properties of weak solutions of quasilinear parabolic equations. In Free boundary problems involving solids*, J. Chadam and H. Rasmussen eds., Pitman Res. Notes in Math. Ser., **281**, Longman (1993), pp. 2-5.

[8] BANDLE (C.) and STAKGOLD (I.) .— *The formation of the dead core in parabolic reaction-diffusion problems*, Trans. Amer. Math. Soc. **286** (1984), pp. 275-293.

[9] BERNIS (F.) .— *Existence results for double nonlinear higher order parabolic equations on unbounded domains*, Math Annalen **279** (1988), pp. 373-394.

[10] DÍAZ (J. I.) and HERNÁNDEZ (J.) .— *Some results on the existence of free boundaries for parabolic reaction-diffusion systems*, In Trends in theory and pratice of nonlinear differential equations, V. Lakshmiklantham ed., Marcel Dekker (1984), pp. 149-156.

[11] DÍAZ (J. I.) and HERNÁNDEZ (J.) .— *Qualitative properties of free boundaries for some nonlinear degerenate parabolic equations. In Nonlinear parabolic equations: qualitative properties of solutions*, L. Boccardo and A. Tesei eds., Pitman 149 (1987), pp. 85-93.

[12] DÍAZ (J. I.) and DE THÉLIN (F.) .— *On a nonlinear parabolic problem arising in some models related to turbulent flows*, SIAM J. Math. 25, No 4 (1994), pp. 1085-1111.

[13] DÍAZ (J. I.) and VERON (L.) .— *Local vanishing properties of solutions of elliptic and parabolic quasilinear equations*, Trans. Amer. Math. Soc. 290, No 2 (1985), pp. 787-814.

[14] EVANS (L. C.) and KNERR (B.) .— *Instantaneous shrinking of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities*, Illinois J. Math., 23 (1979), pp. 153-166.

[15] GALAKTIONOV (V. A.) and VAZQUEZ (J. L.) .— *Extinction for a quasilinear heat equation with absorption II, A dynamical systems approach*, to appear

[16] GILDING (B.) and KERSNER (R.) .— *The characterization of reaction-convection-diffusion processes by travelling waves*, Faculty of Applied Mathematics, University of Twente, Memorandum 1111 (1993).

[17] KAČUR (J.) .— *On a solution of degenerate elliptic-parabolic systems in Orlicz-Sobolev spacesII*, Math. Z. 203 (1990), pp. 569-579.

[18] KALASHNIKOV (A. S.) .— *Some questions of the qualitative theory of nonlinear degenerate second-order parabolic equations*, Russian Math Surveys 42, No 2 (1987), pp. 169-222.

[19] KERSNER (R.) .— *On the behavior of temperature fronts in nonlinear heat-conductive media with absorption*, Vestnik MGU, Ser. 1, Mathematics and Mechanics, 5 (1978), pp. 44-51.

[20] KERSNER (R.) .— *Nonlinear heat conduction with absorption: space localization and extinction in finite time*, SIAM J. Appl. Math. 43 (1983), pp. 1274-1285.

[21] PALIMSKII (I.) .— *Some qualitative properties of solutions of equations of nonlinear heat conductivity with absorption*, Chislennye Metody Mekhaniki Sploshnoi Sredy, Novosibirsk 16, No 1 (1985), pp. 136-145 (in Russian).

[22] SAMARSKY (A. A.), GALAKTIONOV (V. A), KURDUMOV (S. P.) and MIKHAYLOV (A. M.) .— *Blowup in Problems for Quasilinear Parabolic Problems*, Nauka, Moscow (1987) (in Russian; English translation: Walter de Gruyter, Berlin, to appear).

[23] SHMAREV (S. I.) .— *Local properties of solutions of nonlinear equation of heat conduction with absorption*, Dinamika Sploshnoi Sredy, Novosibirsk 95 (1990), pp. 28-42 (in Russian).

[24] SHMAREV (S. I.) .— *The effect of support shrinking for solutions of a degenerate parabolic equation*, Dinamica Sploshnoi Sredy 97 (1990), pp. 194-200 (in Russian).