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# SYMMETRIZATION IN A PARABOLIC-ELLIPTIC SYSTEM RELATED TO CHEMOTAXIS

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**Abstract.** We study symmetrization in a nonlinear parabolic-elliptic system related to chemotaxis by using the decreasing rearrangement, and establish comparison results for solutions of an initial-boundary value problem to such a system. As an application of the comparison results, the large time behavior of the solutions is obtained.

## 1. Introduction

This paper is devoted to the study of the initial-boundary value problem to the nonlinear parabolic–elliptic system, which is denoted by $\mathrm{P_D}(\Omega)$,

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) \quad \text{in } Q_T = \Omega \times (0,\ T], \tag{1.1}$$

$$0 = \Delta v - \gamma v + \alpha u \quad \text{in } Q_T, \tag{1.2}$$

$$u = 0, \quad \mathcal{B}v = 0 \quad \text{on } \Sigma_T = \partial\Omega \times (0,\ T], \tag{1.3}$$

$$u(\cdot, 0) = u_0 \quad \text{on } \Omega, \tag{1.4}$$

where $\Omega$ is a bounded domain in $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$, $\mathcal{B}$ is a boundary operator such that

$$\mathcal{B}v = v|_{\partial\Omega} \text{ (Dirichlet condition)} \quad \text{or} \quad \mathcal{B}v = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} \text{ (Neumann condition)},$$

$\chi$ and $\alpha$ are positive numbers, and $\gamma$ is a non-negative number such that

$$\gamma \geq 0 \text{ if } \mathcal{B}v = v|_{\partial\Omega}, \quad \gamma > 0 \text{ if } \mathcal{B}v = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega}.$$

The non-trivial initial function is assumed to satisfy

$$u_0 \geq 0 \text{ on } \Omega \quad \text{and} \quad u_0 \in W_0^{1,p}(\Omega)\ (p > N).$$

The system of this type arises in the mathematical modelling of chemotaxis ( aggregation of organisms sensitive to a gradient of a chemical substance ), and is a simplified version of one which appeared in [14]. It is conjectured in [6, 7, 18] that there exists a solution $(u, v)$ such that $u$ blows up in finite time when $N \geq 2$. Under homogeneous Neumann boundary conditions on $u$ and $v$, the global existence in time and the existence of blow-up solutions have been studied by [13, 17] in radially symmetric situations. As concerns the boundedness, [17] shows that $(u, v)$ is bounded on $\Omega \times [0, \infty)$ under the condition $\alpha\chi \int_\Omega u_0(x)dx < 8\pi$ for the radially symmetric initial function $u_0$ on a ball $\Omega$ in $R^2$. Under homogeneous Dirichlet boundary conditions on $u$, rearrangement techniques are very effective in studying the boundedness of solutions $(u, v)$ to $\mathbf{P}_{\mathrm{D}}(\Omega)$ without radially symmetric assumptions on $u_0$. Such techniques will give us the exponential decay of $(u, v)$ to $(0, 0)$ as $t \to \infty$ under the condition $\alpha\chi \int_\Omega u_0(x)dx < 8\pi$ in two dimensions.

For a solution of $\mathbf{P}_{\mathrm{D}}(\Omega)$ on $Q_T$ we mean a function $(u, v)$ on $Q_T$ such that

(i) $u \in C([0, T]; W^{1,p}(\Omega)) \cap C^1((0, T]; L^p(\Omega))$, $\quad u(\cdot, t) \in W^{2,p}(\Omega) \quad$ for $0 < t \leq T$,

(ii) $v \in C((0, T]; W^{2,p}(\Omega))$,

(iii) $(u, v)$ satisfies (1.1)–(1.4).

A function $(u, v)$ on $Q = \Omega \times (0, \infty)$ is said to be a global solution of $\mathbf{P}_{\mathrm{D}}(\Omega)$ if $(u, v)$ is a solution of $\mathbf{P}_{\mathrm{D}}(\Omega)$ on $Q_T$ for any $T > 0$.

In Sect.2 it will be shown that there exists uniquely a non-negative solution $(u, v)$ of $\mathbf{P}_{\mathrm{D}}(\Omega)$ on $Q_T$ for some $T > 0$, which becomes a classical solution on $Q_T$. By using the strong maximum principle, we see that

$$u(x, t) > 0 \quad \text{and} \quad v(x, t) > 0 \text{ on } Q_T.$$

In Sect.3 we shall compare the solution $(u, v)$ of $\mathbf{P}_{\mathrm{D}}(\Omega)$ with the solution $(U, V)$ of the following symmetrized problem, which is denoted by $\mathbf{SP}_{\mathrm{D}}(\tilde{\Omega})$,

$$\frac{\partial U}{\partial t} = \nabla \cdot (\nabla U - \chi U \nabla V) \quad \text{in } \tilde{Q}_T = \tilde{\Omega} \times (0, T], \tag{1.5}$$

$$0 = \Delta V + \alpha U \quad \text{in } \tilde{Q}_T, \tag{1.6}$$

$$U = V = 0 \quad \text{on } \partial\tilde{\Omega} \times (0, T], \tag{1.7}$$

$$U(\cdot, 0) = \tilde{u}_0 \quad \text{on } \tilde{\Omega}, \tag{1.8}$$

where $\tilde{\Omega}$ is the ball in $R^N$ centered at the origin with the same measure as $\Omega$, $\partial\tilde{\Omega}$ is the boundary of $\tilde{\Omega}$ and $\tilde{u}_0$ is the symmetric rearrangement of $u_0$ defined in Sect. 3. By the condition on $u_0$,

$$\tilde{u}_0 \geq 0 \text{ on } \tilde{\Omega} \quad \text{and} \quad \tilde{u}_0 \in W_0^{1,p}(\tilde{\Omega}).$$

A function $(U, V)$ on $\tilde{Q}_T$ is called a solution of $\mathbf{SP}_{\mathrm{D}}(\tilde{\Omega})$ if $(U, V)$ satisfies the following :



(i) $U \in C([0, T]; W^{1,p}(\tilde{\Omega})) \cap C^1((0, T]; L^p(\tilde{\Omega}))$, $\quad U(\cdot, t) \in W^{2,p}(\tilde{\Omega}) \quad$ for $0 < t \leq T$,

(ii) $V \in C((0, T]; W^{2,p}(\tilde{\Omega}))$,

(iii) $(U, V)$ satisfies (1.5)–(1.8).

There exists a unique solution $(U, V)$ of $\mathbf{SP}_{\mathrm{D}}(\tilde{\Omega})$ on $\tilde{Q}_T$ for some $T > 0$, which satisfies

(i) $U(x, t) \geq 0$ and $V(x, t) \geq 0$ on $\tilde{Q}_T$,

(ii) $U$ and $V$ are radially symmetric in $x \in \tilde{\Omega}$ and decrease along the radii.

The purpose in Sect.3 is to show the integral comparison

$$\int_0^s u^*(\sigma, t)d\sigma \leq \int_0^s U^*(\sigma, t)d\sigma \quad \text{for } s \in [0, |\Omega|]. \tag{1.9}$$

Here $u^*(\cdot, t)$ ( resp. $U^*(\cdot, t)$ ) is the decreasing rearrangement of $u(\cdot, t)$ ( resp. $U(\cdot, t)$ ) with respect to $x$ defined in Sect. 3. Then the comparison of $L^r$-norms $(1 \leq r \leq \infty)$ of $u$ and $U$

$$||u(\cdot, t)||_{L^r(\Omega)} \leq ||U(\cdot, t)||_{L^r(\tilde{\Omega})} \tag{1.10}$$

can be derived from (1.9). The integral comparison was first proved by [4] for strong solutions to linear parabolic equations and by [16] for weak solutions. For nonlinear parabolic equations we refer to [2, 3, 5, 9, 10, 23].

In Sect.4 it is shown that there is no stationary solution to $\mathbf{P}_{\mathrm{D}}(\Omega)$ except a trivial stationary solution. This means that the structure of stationary solutions to (1.1)–(1.2) under Dirichlet boundary conditions on $u$ is quite different from that under Neumann boundary conditions on $u$ (see [19]). As an application of (1.10), we shall show the global existence of non-negative solution $(u, v)$ to the problem $\mathbf{P}_{\mathrm{D}}(\Omega)$ and the exponential decay of $(u, v)$ to $(0, 0)$ as $t \to \infty$ under each of the following conditions.

(C1) $N = 1$,

(C2) $N = 2$ and $\alpha\chi \int_\Omega u_0(x)dx < 8\pi$,

(C3) $N \geq 3$ and $\alpha\chi ||u_0||_{L^N(\Omega)} < N\kappa_N^{\frac{2}{N}} |\Omega|^{-\frac{1}{N}}$,

where $\kappa_N$ is the volume of the unit ball in $R^N$.

## 2. Local existence in time

Let $(u, v)$ be a solution of $\mathbf{P}_{\mathrm{D}}(\Omega)$ on $Q_T$. Since $p > N$,

$$W^{1,p}(\Omega) \subset C(\overline{\Omega}) \tag{2.1}$$

with continuous inclusion (see [11]). By $u \in C([0, T]; W^{1,p}(\Omega))$ and (2.1),

$$\sup_{0 \leq t \leq T} ||u(\cdot, t)||_{L^\infty(\Omega)} \leq \text{Const.} \sup_{0 \leq t \leq T} ||u(\cdot, t)||_{W^{1,p}(\Omega)} < \infty. \tag{2.2}$$

Using $L^r$-estimates ( $1 < r < \infty$ ) for elliptic equations (see [21]), we have

$$||v(\cdot,t)||_{W^{2,r}(\Omega)} \leq \text{Const.}||u(\cdot,t)||_{L^r(\Omega)} \quad \text{for } 0 < t \leq T,$$

which together with (2.1) implies that

$$\sup_{0<t\leq T}\left(||v(\cdot,t)||_{L^\infty(\Omega)} + ||\nabla v(\cdot,t)||_{L^\infty(\Omega)}\right) \leq \text{Const.} \sup_{0\leq t\leq T}||u(\cdot,t)||_{L^p(\Omega)} < \infty. \tag{2.3}$$

We first show the uniqueness of solutions to $\mathrm{P_D}(\Omega)$ on $Q_T$.

**Lemma 1.** *The uniqueness holds for the problem* $\mathrm{P_D}(\Omega)$ *on* $Q_T$.

*Proof.* Let $(u_1, v_1)$ and $(u_2, v_2)$ be the solutions of $\mathrm{P_D}(\Omega)$ on $Q_T$ with the same initial function $u_0 \in W_0^{1,p}(\Omega)$. Put $a = u_1 - u_2$ and $b = v_1 - v_2$, which satisfy

$$\begin{cases} \dfrac{\partial a}{\partial t} = \Delta a - \chi\nabla\cdot(a\nabla v_1 + u_2\nabla b) & \text{in } Q_T, \\ 0 = \Delta b - \gamma b + \alpha a & \text{in } Q_T, \end{cases} \tag{2.4}$$

and the initial-boundary conditions

$$a = 0, \quad \mathcal{B}b = 0 \text{ on } \Sigma_T \quad \text{and} \quad a(\cdot,0) = 0 \text{ on } \Omega.$$

Multiplying the first equation in (2.4) by $|a|^{p-2}a$ and integrating over $\Omega$, we have

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega |a|^p dx + (p-1)\int_\Omega |a|^{p-2}|\nabla a|^2 dx \\ &= \chi(p-1)\int_\Omega |a|^{p-2}a\nabla v_1\cdot\nabla a dx + \chi(p-1)\int_\Omega |a|^{p-2}u_2\nabla b\cdot\nabla a dx. \end{aligned} \tag{2.5}$$

By using (2.3), the first term in the right-hand side of (2.5) is estimated as

$$\begin{aligned} &\chi(p-1)\int_\Omega |a|^{p-2}a\nabla v_1\cdot\nabla a dx \\ &\leq \chi(p-1)||\nabla v_1||_{L^\infty(\Omega)}\left(\int_\Omega |a|^p dx\right)^{\frac{1}{2}}\left(\int_\Omega |a|^{p-2}|\nabla a|^2 dx\right)^{\frac{1}{2}} \\ &\leq \frac{p-1}{4}\int_\Omega |a|^{p-2}|\nabla a|^2 dx + \text{Const.}\int_\Omega |a|^p dx. \end{aligned}$$

The second term in the right-hand side of (2.5) is estimated as

$$\begin{aligned} &\chi(p-1)\int_\Omega |a|^{p-2}u_2\nabla b\cdot\nabla a dx \\ &\leq \chi(p-1)||u_2||_{L^\infty(\Omega)}\left(\int_\Omega |a|^{p-2}|\nabla a|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |a|^{p-2}|\nabla b|^2 dx\right)^{\frac{1}{2}} \\ &\leq \frac{p-1}{4}\int_\Omega |a|^{p-2}|\nabla a|^2 dx + \text{Const.}\int_\Omega |a|^{p-2}|\nabla b|^2 dx. \end{aligned}$$



By Hölder's inequality and $||\nabla b||_{L^p(\Omega)} \leq \text{Const.}||a||_{L^p(\Omega)}$,

$$\int_\Omega |a|^{p-2}|\nabla b|^2 dx \leq \left(\int_\Omega |a|^p dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |\nabla b|^p dx\right)^{\frac{2}{p}} \leq \text{Const.}\int_\Omega |a|^p dx.$$

Combining (2.5) with these inequalities obtained above, we obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |a|^p dx + \frac{p-1}{2}\int_\Omega |a|^{p-2}|\nabla a|^2 dx \leq \text{Const.}\int_\Omega |a|^p dx \quad \text{for } 0 < t \leq T. \tag{2.6}$$

Since $a(x,0) = 0$ on $\Omega$, (2.6) yields that

$$\int_\Omega |a|^p(x,t)dx = 0 \quad \text{for } 0 \leq t \leq T.$$

Hence, we have $a(x,t) = 0$ on $Q_T$ and then $b(x,t) = 0$ on $Q_T$, which implies the uniqueness of solutions to $\mathrm{P_D}(\Omega)$ on $Q_T$. □

**Lemma 2.** *Let* $(u,v)$ *be a solution of* $\mathrm{P_D}(\Omega)$ *on* $Q_T$ *with the non-negative initial function* $u_0$. *Then* $u(x,t) \geq 0$ *and* $v(x,t) \geq 0$ *on* $Q_T$.

*Proof.* We note that $u_-(\cdot,t) \in W_0^{1,p}(\Omega)$, where $u_-(x,t) = -\min\{u(x,t),0\}$. Multiply (1.1) by $(u_-)^{p-2}u_-$ and integrate over $\Omega$ to get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (u_-)^p dx + (p-1)\int_\Omega (u_-)^{p-2}|\nabla u_-|^2 dx = \chi(p-1)\int_\Omega (u_-)^{p-1}\nabla u_-\cdot\nabla v dx.$$

The right-hand side of this relation is estimated as

$$\begin{aligned} &\chi(p-1)\int_\Omega (u_-)^{p-1}\nabla u_-\cdot\nabla v dx \\ &\leq \chi(p-1)||\nabla v||_{L^\infty(\Omega)}\left\{\int_\Omega (u_-)^p dx\right\}^{\frac{1}{2}}\left\{\int_\Omega (u_-)^{p-2}|\nabla u_-|^2 dx\right\}^{\frac{1}{2}} \\ &\leq \frac{p-1}{2}\int_\Omega (u_-)^{p-2}|\nabla u_-|^2 dx + \text{Const.}\int_\Omega (u_-)^p dx. \end{aligned}$$

Hence,

$$\frac{d}{dt}\int_\Omega (u_-)^p dx \leq \text{Const.}\int_\Omega (u_-)^p dx \quad \text{for } 0 < t \leq T. \tag{2.7}$$

Since $u_-(x,0) = 0$ on $\Omega$, it follows from (2.7) that

$$\int_\Omega (u_-)^p(x,t)dx = 0 \quad \text{for } 0 < t \leq T,$$

which implies that $u_-(x,t) = 0$ on $Q_T$. Therefore, $u(x,t) \geq 0$ on $Q_T$, and then $v(x,t) \geq 0$ on $Q_T$ by the maximum principle for $-\Delta v + \gamma v = \alpha u$ in $\Omega$ with $\mathcal{B}v = 0$ on $\partial\Omega$. □

In order to show the local existence of solutions to $\mathrm{P_D}(\Omega)$, we apply abstract results in [12] to Cauchy problem for semilinear equations in a Banach space.

Let $A_D$ and $A_B$ be closed operators in $L^p(\Omega)$ defined by

$$A_D u = -\Delta u \qquad \text{for } u \in D(A_D) = \{u \in W^{2,p}(\Omega); u = 0 \text{ on } \partial\Omega\},$$
$$A_B u = -\Delta u \qquad \text{for } u \in D(A_B) = \{u \in W^{2,p}(\Omega); \mathcal{B}u = 0 \text{ on } \partial\Omega\}.$$

$A_D$ and $A_B$ are sectorial operators in $L^p(\Omega)$, and

$$\sigma(A_D) \subset \{z \in \mathbf{C}; Re(z) > 0\}, \quad \sigma(A_B) \subset \{z \in \mathbf{C}; Re(z) \geq 0\},$$

where $\sigma(A_D)$ and $\sigma(A_B)$ stand for the spectrumes of $A_D$ and $A_B$, respectively (see [21]). The fractional powers $A_D^\gamma$ of $A_D$ is defined, and in [12, 22] $D(A_D^{1/2})$ is characterized by

$$D(A_D^{1/2}) = W_0^{1,p}(\Omega).$$

Since $v = \alpha(\gamma + A_B)^{-1}u$ for a solution $(u, v)$ to $\mathrm{P_D}(\Omega)$, we rewrite $\mathrm{P_D}(\Omega)$ on $Q_T$ as the following Cauchy problem in $L^p(\Omega)$

$$\text{(CP)} \qquad \begin{cases} \dfrac{du}{dt}(t) + A_D u(t) = f(u(t)), \quad 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

where

$$f(u) = -\alpha\chi\left\{\nabla(\gamma + A_B)^{-1}u \cdot \nabla u + \gamma u(\gamma + A_B)^{-1}u - u^2\right\}.$$

To deal with the nonlinear term $f(u)$ we need the following lemma (see [11]).

**Lemma 3.** *Let $1 < q < \infty$. For $0 \leq \gamma \leq 1$,*

$$D(A_D^\gamma) \subset W^{1,q^*}(\Omega) \qquad \text{when } 1 - N/q^* < 2\gamma - N/q, \ q^* \geq q,$$
$$D(A_D^\gamma) \subset C^\nu(\overline{\Omega}) \qquad \text{when } 0 \leq \nu < 2\gamma - N/q,$$

*where each imbedding is continuous.*

We note $p > N$ and take $\gamma$ such that

$$\frac{N}{4p} + \frac{1}{2} < \gamma < 1.$$

Since $1 - N/(2p) < 2\gamma - N/p$, Lemma 3 implies that

$$||u||_{W^{1,2p}(\Omega)} \leq \text{Const.}||A_D^\gamma u||_{L^p(\Omega)} \quad \text{for } u \in D(A_D^\gamma).$$

By using Hölder's inequality and the inequality

$$||(\gamma + A_B)^{-1}u||_{W^{2,2p}(\Omega)} \leq \text{Const.}||u||_{L^{2p}(\Omega)} \quad \text{for } u \in L^{2p}(\Omega),$$

we have

$$||f(v) - f(w)||_{L^p(\Omega)} \leq \text{Const.}\left(||v||_{W^{1,2p}(\Omega)} + ||w||_{W^{1,2p}(\Omega)}\right)||v - w||_{W^{1,2p}(\Omega)}.$$



Hence,

$$||f(v) - f(w)||_{L^p(\Omega)} \leq \text{Const.}\left(||A_D^\gamma v||_{L^p(\Omega)} + ||A_D^\gamma w||_{L^p(\Omega)}\right)||A_D^\gamma(v - w)||_{L^p(\Omega)}$$

for $v, w \in D(A_D^\gamma)$.

**Theorem 1.** (i). *For an initial function $u_0 \in W_0^{1,p}(\Omega)$ there exists a positive number $T$ such that $\mathrm{P_D}(\Omega)$ has a unique solution $(u, v)$ on $Q_T$, which becomes a classical solution.*

(ii). *If the non-trivial initial function $u_0$ is non-negative on $\Omega$, then $u(x,t) > 0$ and $v(x,t) > 0$ on $Q_T$.*

(iii). *Let $T_{max}$ be a maximal existence time of $(u, v)$. If $||u(\cdot,t)||_{L^p(\Omega)} \leq$ Const. on $(0, T_{max})$, then $T_{max} = \infty$ and $||u(\cdot,t)||_{L^\infty(\Omega)} \leq$ Const. on $(0, \infty)$.*

*Proof.* We apply Theorems 1 and 2 in [12] to get the local existence of solution to (CP). For $u_0 \in D(A_D^{1/2}) = W_0^{1,p}(\Omega)$ there exists a positive number $T$ such that (CP) has a solution $u$ on $[0, T]$ satisfying

$$u \in C([0,T]; D(A_D^{1/2})) \cap C^1((0,T]; L^p(\Omega)), \quad u(t) \in D(A_D) \text{ for } 0 < t \leq T.$$

By putting $v = \alpha(\gamma + A_B)^{-1}u$, $(u, v)$ becomes the unique solution of $\mathrm{P_D}(\Omega)$ on $Q_T$. As concerns classical solution of $\mathrm{P_D}(\Omega)$, by Lemma 3 and the regularity theory for elliptic equations, similar arguments in [11, 12] yield that $u \in C(\overline{\Omega}\times[0,T]) \cap C^{2,1}(\overline{\Omega}\times(0,T])$, $v(\cdot,t) \in C^2(\overline{\Omega})$ and $(u, v)$ is a classical solution of $\mathrm{P_D}(\Omega)$ on $Q_T$.

Note that $|\nabla v|$ and $\Delta v$ are bounded on $Q_T$ by (2.2), (2.3) and (1.2). If $u_0 \geq 0$ on $Q_T$, then $u(x,t) \geq 0$ and $v(x,t) \geq 0$ on $Q_T$ by Lemma 2. Since $u$ is a classical solution of the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi\nabla v \cdot \nabla u - \chi(\Delta v)u \quad \text{in } Q_T, \\ u = 0 \quad \text{on } \Sigma_T, \quad u(x,0) = u_0(x) \quad \text{on } \Omega, \end{cases}$$

the strong maximum principle yields that

$$u(x,t) > 0 \quad \text{and} \quad v(x,t) > 0 \quad \text{on } Q_T.$$

Let us prove the assertion (iii). By $||u(\cdot,t)||_{L^p(\Omega)} \leq$ Const. on $(0, T_{max})$ and (2.3),

$$||\nabla v(\cdot,t)||_{L^\infty(\Omega)} \leq \text{Const.} \quad \text{on } (0, T_{max}).$$

Hence, by Proposition A.2 we get

$$(2.8) \qquad ||u(\cdot,t)||_{L^\infty(\Omega)} \leq \text{Const.} \quad \text{on } (0, T_{max}).$$

Assume $T_{max} < \infty$. For fixed $t_0 \in (0, T_{max})$, $u(t_0) \in D(A_D^\gamma)$. For $t \in (t_0, T_{max})$, $u(t)$ is rewritten as

$$u(t) = e^{-(t-t_0)A_D}u(t_0) + \int_{t_0}^t e^{-(t-s)A_D}f(u(s))ds,$$

where $\{e^{-tA_D}\}_{t\geq 0}$ is an analytic semi-group generated by $-A_D$ in $L^p(\Omega)$. Using (2.3) and (2.8), we see that

$$||f(u(t))||_{L^p(\Omega)} \leq \text{Const.}\left(||A_D^\gamma u(t)||_{L^p(\Omega)} + 1\right) \quad \text{on } [t_0,\ T_{max}).$$

By a similar way in the proof of Corollary 3.3.5 in [11], we have

$$||A_D^\gamma u(t)||_{L^p(\Omega)} \leq \text{Const.} \quad \text{on } [t_0,\ T_{max}),$$

which implies that

$$||f(u(t))||_{L^p(\Omega)} \leq \text{Const.} \quad \text{on } [t_0,\ T_{max}).$$

A similar argument in Theorem 3.3.4 in [11] gives that $\lim_{t\to T_{max}} u(t)$ exists in $D(A_D^\gamma)$. Hence, the solution $u(t)$ of (CP) can be extended beyond time $T_{max}$, which contradicts maximality of $T_{max}$. □

We remark that the same results as in Theorem 1 hold to the problem $\mathbf{SP}_\mathrm{D}(\tilde\Omega)$ by using similar arguments. The solution $(U,V)$ of $\mathbf{SP}_\mathrm{D}(\tilde\Omega)$ is radially symmetric in $x\in\tilde\Omega$ by the uniqueness of the solution and the symmetry of the problem.

## 3. Rearrangement and integral comparison

For a measurable set $E$ in $R^N$, we denote $|E|$ its Lebesgue measure. Let $f$ be a measurable function on $\Omega$. For simplicity we denote a subset $\{x\in\Omega; f(x)>s\}$ in $R^N$ by $\{f>s\}$. Let us put $\Omega^* = (0,\ |\Omega|)$. The decreasing rearrangement $f^*$ of $f$ is the function from $\overline{\Omega^*}$ into $[-\infty,\infty]$ defined by

$$f^*(s) = \begin{cases} \inf\{\tau; |\{f>\tau\}|\leq s\} & \text{if } 0\leq s<|\Omega|, \\ \operatorname{ess\,inf}\{f(x); x\in\Omega\} & \text{if } s=|\Omega|. \end{cases}$$

$f^*$ is non-increasing and right-continuous, and satisfies

$$f^*(0) = \operatorname{ess\,sup}\{f(x); x\in\Omega\}.$$

The symmetric rearrangement $\tilde f$ of $f$ on $\tilde\Omega$ is defined by

$$\tilde f(x) = f^*(\kappa_N|x|^N) \quad (x\in\tilde\Omega).$$

Some basic facts about rearrangement are as follows (see [4, 8, 15]):

(i) For every Borel measurable function $F$ from $R$ to $R^+$,

$$\int_\Omega F(f)dx = \int_{\tilde\Omega} F(\tilde f)dx = \int_{\Omega^*} F(f^*)ds.$$

In particular, If $f\in L^r(\Omega)$ $(1\leq r\leq\infty)$, then

$$||f||_{L^r(\Omega)} = ||\tilde f||_{L^r(\tilde\Omega)} = ||f^*||_{L^r(\Omega^*)}.$$



(ii) For $f\in L^r(\Omega)$ and $g\in L^q(\Omega)(1\leq r\leq\infty,\ 1/r+1/q=1)$,

$$\int_\Omega fg\,dx \leq \int_{\Omega^*} f^*g^*ds.$$

(iii) If $f\in W_0^{1,r}(\Omega)(1\leq r\leq\infty)$ and $f\geq 0$ on $\Omega$, then $\tilde f\in W_0^{1,r}(\tilde\Omega)$ and

$$||\nabla\tilde f||_{L^r(\tilde\Omega)} \leq ||\nabla f||_{L^r(\Omega)}.$$

We remark that if $f\in W_0^{1,r}(\Omega)(1\leq r\leq\infty)$ and $f\geq 0$ on $\Omega$, then $f^*\in W^{1,r}(\delta,|\Omega|)$ for every $\delta$ and $f^*\in C(\Omega^*)$.

Let $(u,v)$ be a non-negative solution of $\mathbf{P}_\mathrm{D}(\Omega)$ on $Q_T$. We indicate with $\mu$ the distribution function of $u$ with respect to $x$ defined by

$$\mu(s) = |\{u(\cdot,t)>s\}|,$$

which is a function of $s$ and $t$. We consider the decreasing rearrangement of $u(\cdot,t)$ with respect to $x$ as

$$u^*(s,t) = u(\cdot,t)^*(s) = \inf\{\tau; \mu(\tau)\leq s\}$$

and the symmetric rearrangement $\tilde u(\cdot,t)$ of $u(\cdot,t)$ by

$$\tilde u(x,t) = u^*(\kappa_N|x|^N,t) \quad (x\in\tilde\Omega).$$

Since $u^*(\cdot,t)\in C(\Omega^*)$, we see that

$$u^*(\mu(s),t) = s \quad \text{for } 0<s<u^*(0,t).$$

In [16] it is shown that $u\in H^1(0,T;L^p(\Omega))$ implies that $u^*\in H^1(0,T;L^p(\Omega^*))$,

$$||\frac{\partial u^*}{\partial t}(\cdot,t)||_{L^p(\Omega^*)} \leq ||\frac{\partial u}{\partial t}(\cdot,t)||_{L^p(\Omega)} \quad \text{a.a. } t,$$

$$\int_{\{u>s\}} \frac{\partial u}{\partial t}dx = \frac{\partial k}{\partial t}(\mu(s),t) \quad \text{a.a. } (s,t),$$

where

$$k(s,t) = \int_0^s u^*(\sigma,t)d\sigma \quad \text{for } s\in\overline{\Omega^*} \quad \text{and} \quad t\in[0,\ T].$$

As for the regularity of $k$, we have

$$k\in L^\infty(Q_T^*)\cap H^1(0,T;W^{1,p}(\Omega^*))\cap\bigcap_{\delta>0} L^2(0,T;W^{2,p}(\delta,|\Omega|)),$$

where $Q_T^* = \Omega^*\times(0,\ T)$.

**Lemma 4.** *$k$ satisfies the following partial differential inequality and initial-boundary conditions*

$$\begin{cases} \dfrac{\partial k}{\partial t} - d(s)\dfrac{\partial^2 k}{\partial s^2} - \alpha\chi k\dfrac{\partial k}{\partial s} \leq 0 \ \textit{a.e. in } Q_T^*, \\ k(0,t)=0, \quad \dfrac{\partial k}{\partial s}(|\Omega|,t)=0 \quad \textit{for any} \quad t\in[0,\ T], \\ k(s,0) = \displaystyle\int_0^s u_0^*(\sigma)d\sigma \quad \textit{for any} \quad s\in\Omega^*, \end{cases}$$

*where* $d(s) = N^2\kappa_N^{2/N}s^{2(N-1)/N}$.

*Proof.* For $\tau \in (0,\ u^*(0,t))$ and $h > 0$, define the function $T_{\tau,h}$ on $(-\infty,\ \infty)$ by

$$T_{\tau,h}(s) = \begin{cases} 0 & \text{if } s \le \tau, \\ s-\tau & \text{if } \tau < s \le \tau + h, \\ h & \text{if } s > \tau + h. \end{cases}$$

$T_{\tau,h}(u(\cdot,t)) \in W_0^{1,p}(\Omega)$ since $u(\cdot,t) \in W_0^{1,p}(\Omega)$ and $T_{\tau,h}$ is Lipschitz continuous on $(-\infty,\infty)$. Multiply (1.1) by $T_{\tau,h}(u)$ and integrate over $\Omega$. Integrating by parts gives

$$\int_\Omega \frac{\partial u}{\partial t}T_{\tau,h}(u)dx + \int_\Omega \nabla u\cdot\nabla T_{\tau,h}(u)dx - \chi\int_\Omega u\nabla v\cdot\nabla T_{\tau,h}(u)dx = 0.$$

From the definition of $T_{\tau,h}$ and $|\{u=\tau\}| = 0$ a.a. $\tau > 0$ for each $t \in (0,\ T)$ it follows that

$$\lim_{h\to 0}\frac{1}{h}\int_\Omega \frac{\partial u}{\partial t}T_{\tau,h}(u)dx = \int_{\{u>\tau\}}\frac{\partial u}{\partial t}dx = \frac{\partial k}{\partial t}(\mu(\tau),t).$$

We next see that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}\int_\Omega \nabla u\cdot\nabla T_{\tau,h}(u)dx &= \lim_{h\to 0}\frac{1}{h}\left(\int_{\{u>\tau\}}|\nabla u|^2dx - \int_{\{u>\tau+h\}}|\nabla u|^2dx\right) \\ &= -\frac{\partial}{\partial\tau}\int_{\{u>\tau\}}|\nabla u|^2dx.\end{aligned}$$

Let us define the function $\Phi_{\tau,h}$ on $(-\infty,\ \infty)$ by

$$\Phi_{\tau,h}(s) = \int_0^s \sigma\frac{d}{d\sigma}T_{\tau,h}(\sigma)d\sigma = \begin{cases} 0 & \text{if } s\le\tau, \\ \frac{1}{2}(s^2-\tau^2) & \text{if } \tau < s \le \tau+h, \\ h(\tau+\frac{h}{2}) & \text{if } s > \tau+h. \end{cases}$$

Multiplying (1.2) by $\Phi_{\tau,h}(u(\cdot,t)) \in W_0^{1,p}(\Omega)$ and integrating over $\Omega$ gives

$$J = \int_\Omega u\nabla v\cdot\nabla T_{\tau,h}(u)dx = \int_\Omega \nabla v\cdot\nabla\Phi_{\tau,h}(u)dx = \int_\Omega(\alpha u - \gamma v)\Phi_{\tau,h}(u)dx.$$

It follows from the definition of $\Phi_{\tau,h}$ that

$$\begin{aligned}\frac{J}{h} &= \frac{1}{2h}\int_{\{\tau<u\le\tau+h\}}(\alpha u-\gamma v)(u^2-\tau^2)dx + \int_{\{u>\tau+h\}}(\alpha u-\gamma v)\left(\tau+\frac{h}{2}\right)dx \\ &\longrightarrow \tau\int_{\{u>\tau\}}(\alpha u-\gamma v)dx \quad \text{as} \quad h\longrightarrow 0.\end{aligned}$$



By $u^*(\mu(\tau),t) = \tau$ and

$$\int_{\{u>\tau\}}udx = \int_0^{\mu(\tau)}u^*(\sigma,t)d\sigma,$$

we have

$$\begin{aligned}\lim_{h\to 0}\frac{J}{h} &= u^*(\mu(\tau),t)\left\{\alpha\int_0^{\mu(\tau)}u^*(\sigma,t)d\sigma - \gamma\int_{\{u>\tau\}}vdx\right\} \\ &= \frac{\partial k}{\partial s}(\mu(\tau),t)\left\{\alpha k(\mu(\tau),t) - \gamma\int_{\{u>\tau\}}vdx\right\}.\end{aligned}$$

Hence,

$$\frac{\partial k}{\partial t}(\mu(\tau),t) - \frac{\partial}{\partial\tau}\int_{\{u>\tau\}}|\nabla u|^2dx - \chi\frac{\partial k}{\partial s}(\mu(\tau),t)\left\{\alpha k(\mu(\tau),t) - \gamma\int_{\{u>\tau\}}vdx\right\} = 0,$$

which together with $u^* \ge 0$ and $v \ge 0$ yields that

$$-\frac{\partial}{\partial\tau}\int_{\{u>\tau\}}|\nabla u|^2dx \le -\frac{\partial k}{\partial t}(\mu(\tau),t) + \alpha\chi k(\mu(\tau),t)\frac{\partial k}{\partial s}(\mu(\tau),t) \tag{3.1}$$

To estimate further we need the following inequality (see [8, 15, 20]):

$$N\kappa_N^{1/N}\mu(\tau)^{(N-1)/N} \le (-\mu'(\tau))^{1/2}\left(-\frac{\partial}{\partial\tau}\int_{\{u>\tau\}}|\nabla u|^2dx\right)^{1/2}.$$

From this inequality and (3.1) it follows that

$$N^2\kappa_N^{2/N}\mu(\tau)^{2(N-1)/N} \le -\mu'(\tau)\left\{-\frac{\partial k}{\partial t}(\mu(\tau),t) + \alpha\chi k(\mu(\tau),t)\frac{\partial k}{\partial s}(\mu(\tau),t)\right\},$$

which implies that for almost all $\tau \in (0,\ u^*(0,t))$,

$$1 \le N^{-2}\kappa_N^{-2/N}\mu(\tau)^{2(1-N)/N}(-\mu'(\tau))\left\{-\frac{\partial k}{\partial t}(\mu(\tau),t) + \alpha\chi k(\mu(\tau),t)\frac{\partial k}{\partial s}(\mu(\tau),t)\right\}. \tag{3.2}$$

The function $G(s,t) = -\partial k/\partial t(s,t) + \alpha\chi k(s,t)\partial k/\partial s(s,t)$ is continuous on $[0,\ |\Omega|]$ for fixed $t$. By using $|\{u(\cdot,t)>0\}| = |\Omega|$ and a similar way in Lemma 2 in [10], we see that $G(s,t) \ge 0$ on $[0,\ |\Omega|]$ for fixed $t$. Integrate (3.2) over $(\tau_1,\ \tau_2) \subset [0,\ |\Omega|]$. By a similar way in Lemma 2 in [10], we get

$$\tau_1 - \tau_2 \le N^{-2}\kappa_N^{-2/N}\int_{\mu(\tau_1)}^{\mu(\tau_2)}s^{2(1-N)/N}\left\{-\frac{\partial k}{\partial t}(s,t) + \alpha\chi k(s,t)\frac{\partial k}{\partial s}(s,t)\right\}ds,$$

and then, for almost all $s$ in $\Omega^*$

$$0 \le -\frac{\partial^2 k}{\partial s^2}(s,t) = -\frac{\partial u^*}{\partial s}(s,t) \le N^{-2}\kappa_N^{-2/N}s^{2(1-N)/N}\left\{-\frac{\partial k}{\partial t}(s,t) + \alpha\chi k(s,t)\frac{\partial k}{\partial s}(s,t)\right\},$$

which implies the desired partial differential inequality. □

Let $(U, V)$ be a solution of the symmetrized problem $\mathbf{SP}_{\mathrm{D}}(\tilde{\Omega})$ on $\tilde{Q}_T$. We have seen in Sect. 2 that $(U, V)$ is a classical solution and that $U$ and $V$ are non-negative and radially symmetric in $x$.

**Lemma 5.** *$U$ and $V$ decrease along the radii. Hence, $U = \tilde{U}$ and $V = \tilde{V}$.*

*Proof.* We rewrite (1.5) and (1.6) in terms of $r = |x|$ as

$$\left\{\begin{aligned} &\frac{\partial U}{\partial t} = r^{1-N}\frac{\partial}{\partial r}\left\{r^{N-1}\left(\frac{\partial U}{\partial r} - \chi U\frac{\partial V}{\partial r}\right)\right\}, \\ &0 = r^{1-N}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial V}{\partial r}\right) + \alpha U. \end{aligned}\right. \tag{3.3}$$

It is easy to get

$$r^{N-1}\frac{\partial V}{\partial r} = -\alpha\int_0^r U\rho^{N-1}d\rho,$$

from which together with $U \geq 0$ it follows that $\partial V/\partial r \leq 0$. Hence, $V$ decreases along the radii.

By (3.3),

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial r^2} + \frac{N-1}{r}\frac{\partial U}{\partial r} - \chi\left(\frac{\partial V}{\partial r}\frac{\partial U}{\partial r} - \alpha U^2\right).$$

Differentiating this relation with respect to $r$ and putting $w = \partial U/\partial r$, we get

$$\frac{\partial w}{\partial t} = \Delta w - \chi\nabla V\cdot\nabla w + C(x,t)w \quad \text{in } \tilde{Q}_T,$$

where

$$C(x,t) = -\frac{N-1}{|x|^2} - \chi\frac{\partial^2 V}{\partial r^2}(|x|,t) + 2\alpha\chi U.$$

We note that $C(x,t)$ is bounded above on $\tilde{Q}_T$ by $U \in L^\infty(Q_T)$ and the second equation in (3.3). Hence, by $w \leq 0$ on $\partial\tilde{\Omega}\times(0,\ T)$ and $w(x,0) \leq 0$ on $\Omega$, the maximum principle yields that $w \leq 0$ on $\tilde{Q}_T$. □

Define the function $K$ on $\overline{Q_T^*}$ by

$$K(s,t) = \int_0^s U^*(\sigma,t)d\sigma. \tag{3.4}$$

We then have the following lemma.

**Lemma 6.** *$K$ and $V^*$ satisfy the following:*

(i) $K \in L^\infty(Q_T^*) \cap H^1(0,T;W^{1,p}(\Omega^*)) \cap \bigcap_{\delta>0} L^2(0,T;W^{2,p}(\delta,|\Omega|))$,

(ii) $V^*(\cdot,t) \in C^1(\Omega^*)$ *for* $t > 0$,



(iii) $d(s)\dfrac{\partial V^*}{\partial s} + \alpha K = 0$ *in* $Q^*$,

(iv) $\dfrac{\partial K}{\partial t} - d(s)\dfrac{\partial^2 K}{\partial s^2} - \alpha\chi K\dfrac{\partial K}{\partial s} = 0$ *in* $Q^*$,

(v) $K(s,0) = k(s,0)$ *on* $\Omega^*$ *and* $K(0,t) = 0$, $\dfrac{\partial K}{\partial s}(|\Omega|,t) = 0$ *for any* $t \in [0,\ T]$.

*Proof.* For $\hat{\psi} \in C_0^\infty(\Omega^*)$ let us define

$$\psi(s) = \int_s^{|\Omega|}\hat{\psi}(\sigma)d\sigma \quad \text{and} \quad \varphi(x) = \psi(\kappa_N|x|^N).$$

Multiply (1.6) by $\varphi \in C_0^1(\tilde{\Omega})$ and integrate over $\tilde{\Omega}$. Integrating by parts and using $U^* = \partial K/\partial s$ gives

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}}(\nabla V\cdot\nabla\varphi - \alpha U\varphi)dx = \int_{\Omega^*}\left(d(s)\frac{\partial V^*}{\partial s}\frac{d\psi}{ds} - \alpha U^*\psi\right)ds \\ &= -\int_{\Omega^*}\left(d(s)\frac{\partial V^*}{\partial s} + \alpha K\right)\hat{\psi}ds, \end{aligned}$$

which implies (iii). Next, multiply (1.5) by $\varphi$ and integrate over $\tilde{\Omega}$. Using integration by parts, we obtain

$$\begin{aligned} 0 &= \int_{\tilde{\Omega}}\left\{\frac{\partial U}{\partial t}\varphi + (\nabla U - \chi U\nabla V)\cdot\nabla\varphi\right\}dx \\ &= \int_{\Omega^*}\left\{\frac{\partial U^*}{\partial t}\psi + d(s)\left(\frac{\partial U^*}{\partial s} - \chi U\frac{\partial V^*}{\partial s}\right)\frac{d\psi}{ds}\right\}ds \\ &= -\int_{\Omega^*}\left\{-\frac{\partial K}{\partial t} + d(s)\left(\frac{\partial^2 K}{\partial s^2} - \chi\frac{\partial K}{\partial s}\frac{\partial V^*}{\partial s}\right)\right\}\hat{\psi}ds, \end{aligned}$$

from which together with (iii) we get (iv). □

We are now in a position to give the integral comparison.

**Theorem 2.** *The following inequalities hold:*

(i) $\displaystyle\int_0^s u^*(\sigma,t)d\sigma \leq \int_0^s U^*(\sigma,t)d\sigma$ *for* $(s,t) \in \overline{\Omega^*}\times[0,\ T]$.

(ii) $\|u(\cdot,t)\|_{L^r(\Omega)} \leq \|U(\cdot,t)\|_{L^r(\tilde{\Omega})}$ *for* $t \in [0,\ T]$ *and* $r \in [1,\ \infty]$.

*Proof.* (i) implies (ii) (see [4, 8]). From $u, U \in L^\infty(Q_T)$ the condition (ii) in Proposition A1 follows. Hence, by applying Proposition A1, (i) is obtained. □

**Remark.** Under the homogeneous Dirichlet boundary condition on $v$ in $\mathrm{P}_{\mathrm{D}}(\Omega)$, we establish the mass comparison for $v^*$ and $V^*$

$$\int_0^s v^*(\sigma,t)d\sigma \leq \int_0^s V^*(\sigma,t)d\sigma$$

by using the inequality on $u^*$ and $U^*$ in Theorem 2 and the standard approach ( see for example, Theorem 1.26 and Remark 1.16 in [8], Corollary 1.4 in [15]). As a matter of fact, we can prove the stronger comparison $v^* \le V^*$.

## 4. Global existence and exponential decay

A function $(a(x), b(x))$ on $\overline{\Omega}$ is called a stationary solution to the problem $\mathbf{P}_{\mathrm{D}}(\Omega)$ if $(a, b)$ satisfies the following

$$\begin{cases} a, b \in L^\infty(\Omega) \cap W^{2,2}(\Omega), \\ \nabla \cdot (\nabla a - \chi a \nabla b) = 0, \quad \Delta b - \gamma b + \alpha a = 0 \quad \text{in } \Omega, \\ a = 0, \quad \mathcal{B} b = 0 \quad \text{on } \partial\Omega. \end{cases}$$

**Proposition 1.** *There is no stationary solution to* $\mathbf{P}_{\mathrm{D}}(\Omega)$ *except* $(a, b) = (0, 0)$.

*Proof.* By the following relation

$$e^{\chi b} \nabla (a e^{-\chi b}) = \nabla a - \chi a \nabla b,$$

we get

$$\nabla \cdot \left\{ e^{\chi b} \nabla (a e^{-\chi b}) \right\} = \nabla \cdot (\nabla a - \chi a \nabla b) = 0. \tag{4.1}$$

Multiply (4.1) by $a e^{-\chi b}$and integrate over $\Omega$. Integrating by parts, we have

$$\int_\Omega e^{\chi b} \left| \nabla (a e^{-\chi b}) \right|^2 dx = - \int_\Omega \nabla \cdot \left\{ e^{\chi b} \nabla (a e^{-\chi b}) \right\} a e^{-\chi b} dx = 0,$$

which implies that

$$\nabla (a e^{-\chi b}) = 0 \quad \text{on } \Omega.$$

Since $a = 0$ on $\partial\Omega$, we get $a \equiv 0$ on $\Omega$. The maximum principle for $0 = \Delta b - \gamma b$ in $\Omega$ with $\mathcal{B} b = 0$ on $\partial\Omega$ gives $b \equiv 0$ on $\Omega$. □

Let $(u, v)$ be a non-negative solution of $\mathbf{P}_{\mathrm{D}}(\Omega)$. We will give an application of Theorem 2, which gives the global existence of the solution to $\mathbf{P}_{\mathrm{D}}(\Omega)$ and the exponential decay of $(u, v)$ to $(0, 0)$ as $t \to \infty$ through the estimation of the solution $(U, V)$ to the symmetrized problem $\mathbf{SP}_{\mathrm{D}}(\tilde{\Omega})$. We denote a maximal existence time of $(U, V)$ by $T^*_{max}$. For $U$, let $K(s, t)$ be the one defined in (3.4).

**Lemma 7.** *If* $K(s, t) \le \mathrm{Const.} \sqrt{d(s)}$ *on* $\Omega^* \times (0,\ T^*_{max})$, *then*

$$\|V(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.}, \quad \|\nabla V(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} \quad \text{on } (0,\ T^*_{max}).$$

*Proof.* By Lemma 6,

$$0 = d(s) \frac{\partial V^*}{\partial s} + \alpha K \quad \text{in } \Omega^* \times (0,\ T^*_{max}).$$



Hence,

$$\frac{\partial V}{\partial r} = \sqrt{d(s)} \frac{\partial V^*}{\partial s} = - \frac{\alpha K}{\sqrt{d(s)}} \quad (r = |x|,\ s = \kappa_N r^N),$$

which implies

$$\|\nabla V(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} \quad \text{on } (0,\ T^*_{max}).$$

By $V(|\Omega|, t) = 0$ and the estimate of $\nabla V$, we get the desired estimate about $V$. □

**Lemma 8.** *Assume that*

$$\|\nabla V(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} \quad on\ (0,\ T^*_{max}).$$

*If there exists a constant* $\lambda \ge 0$ *such that*

$$\|U(\cdot, t)\|_{L^1(\tilde{\Omega})} \le \mathrm{Const.} e^{-\lambda t} \quad for\ t \in (0,\ T^*_{max}),$$

*then* $T^*_{max} = \infty$ *and*

$$\|U(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} e^{-\lambda t} \quad for\ t > 0.$$

*Proof.* Define the function $f$ by

$$f(x, t) = e^{\lambda t} U(x, t) \quad \text{on } \tilde{\Omega} \times [0,\ T^*_{max}),$$

which satisfies

$$\|f(\cdot, t)\|_{L^1(\tilde{\Omega})} \le \mathrm{Const.} \quad \text{on } [0,\ T^*_{max}).$$

We see that $f$ satisfies

(i) $f \in L^\infty(\tilde{Q}_T) \cap L^2(0, T; W_0^{1,2}(\tilde{\Omega})) \cap H^1(0, T; L^2(\tilde{\Omega}))$ for any $T \in (0,\ T^*_{max})$,

(ii) for any $\varphi \in W_0^{1,2}(\tilde{\Omega})$ and $t \in (0,\ T^*_{max})$,

$$\int_{\tilde{\Omega}} \left\{ \frac{\partial f}{\partial t} \varphi + (\nabla f - \chi f \nabla V) \cdot \nabla \varphi - \lambda f \varphi \right\} dx = 0,$$

(iii) $f(\cdot, 0) = \tilde{u}_0 \in L^\infty(\tilde{\Omega})$.

By Proposition A2, we get

$$\|f(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} \quad \text{on } (0,\ T^*_{max}),$$

which implies $T^*_{max} = \infty$ and the desired inequality. □

**Proposition 2.** *Under each of* (C1)–(C3), $T^*_{max} = \infty$ *and there exists a positive constant* $\lambda$ *such that*

$$\|U(\cdot, t)\|_{L^\infty(\tilde{\Omega})} \le \mathrm{Const.} e^{-\lambda t} \quad for\ t > 0. \tag{4.2}$$

*Proof.* Let us consider under (C1). Define the function $Z$ by

$$Z(s, t) = e^{-\lambda t} w(s), \quad w(s) = p \frac{e^{\alpha \chi p s/2} - 1}{e^{\alpha \chi p s/2} + 1},$$

where $\lambda$ and $p$ are positive constants determined below. Take $p$ so large that

$$K(s,0) \leq w(s) \quad \text{on } \overline{\Omega^*}.$$

Choosing $\lambda$ such that

$$0 < \lambda \leq \alpha\chi w'(|\Omega|),$$

we get

$$\frac{\partial Z}{\partial t} - d(s)\frac{\partial^2 Z}{\partial s^2} - \alpha\chi Z\frac{\partial Z}{\partial s} = e^{-\lambda t}\left\{-\lambda w + \alpha\chi\left(2 - e^{-\lambda t}\right) ww'\right\}$$

$$> e^{-\lambda t} w\left(-\lambda + \alpha\chi w'(|\Omega|)\right) \geq 0.$$

Here we have used that $w'$ is decreasing. Noting that $Z(0,t) = 0$ and $\partial Z/\partial s(|\Omega|,t) > 0$, by Proposition A1 we get

$$K(s,t) \leq Z(s,t) \quad \text{on } \Omega^* \times [0,\ T^*_{max}),$$

which implies that

$$K(s,t) \leq \text{Const.}\sqrt{d(s)}. \quad \text{on } \Omega^* \times [0,\ T^*_{max})$$

and

$$||U(\cdot,t)||_{L^1(\tilde{\Omega})} = K(|\Omega|,t) \leq \text{Const.}e^{-\lambda t} \quad \text{for } t \in (0,\ T^*_{max}).$$

Therefore, by Lemmas 7 and 8, we have $T^*_{max} = \infty$ and (4.2).

Under (C2), define the function $Z$ by

$$Z(s,t) = e^{-\lambda t}\frac{pqs}{1+qs} \quad (s \in \overline{\Omega^*},\ t \geq 0).$$

Choose $p$ satisfying

$$K(|\Omega|,0) < p \quad \text{and} \quad \alpha\chi p < 8\pi.$$

Take $q$ so large that

$$K(s,0) \leq \frac{pqs}{1+qs} = Z(s,0) \quad \text{on } \overline{\Omega^*}.$$

We take $\lambda$ satisfying

$$0 < \lambda \leq \frac{q}{(1+q|\Omega|)^2}(8\pi - \alpha\chi p).$$

We then get

$$\frac{\partial Z}{\partial t} - d(s)\frac{\partial^2 Z}{\partial s^2} - \alpha\chi Z\frac{\partial Z}{\partial s} = \frac{pqs}{1+qs}\left\{-\lambda + 8\pi\frac{q}{(1+qs)^2} - e^{-\lambda t}\alpha\chi p\frac{q}{(1+qs)^2}\right\}$$

$$\geq \frac{pqs}{1+qs}\left\{-\lambda + \frac{q}{(1+qs)^2}(8\pi - \alpha\chi p)\right\} \geq 0.$$



Hence, $Z$ satisfies

$$\begin{cases} \dfrac{\partial Z}{\partial t} - d(s)\dfrac{\partial^2 Z}{\partial s^2} - \alpha\chi Z\dfrac{\partial Z}{\partial s} \geq 0 & \text{in } \Omega^* \times (0,\ \infty), \\ Z(0,t) = 0, \quad \dfrac{\partial Z}{\partial s}(|\Omega|,t) \geq 0 & \text{on } (0,\ \infty), \\ Z(s,0) \geq K(s,0) & \text{on } \overline{\Omega^*}. \end{cases}$$

By Proposition A1,

$$K(s,t) \leq Z(s,t) \quad \text{on } \overline{\Omega^*} \times [0,\ T^*_{max}),$$

which implies (4.2).

Under (C3) let us choose $\beta$ such that

$$\beta > \alpha\chi \quad \text{and} \quad \beta||u_0||_{L^N(\Omega)} \leq N\kappa_N^{\frac{2}{N}}|\Omega|^{-\frac{1}{N}}.$$

Define the function $Z$ by

$$Z(s,t) = e^{-\lambda t}w(s), \quad w(s) = qs^{1-\frac{1}{N}}, \quad q = \frac{N}{\beta}\kappa_N^{\frac{2}{N}}|\Omega|^{-\frac{1}{N}},$$

where $\lambda$ is a constant satisfying

$$0 < \lambda \leq (\beta - \alpha\chi)w'(|\Omega|).$$

Then,

$$\frac{\partial Z}{\partial t} - d(s)\frac{\partial^2 Z}{\partial s^2} - \alpha\chi Z\frac{\partial Z}{\partial s} \geq e^{-\lambda t}w\left\{-\lambda + (\beta - \alpha\chi)w'(|\Omega|)\right\} \geq 0.$$

Using Hölder's inequality and $||u_0^*||_{L^N(\Omega^*)} = ||u_0||_{L^N(\Omega)}$, we have

$$K(s,0) = \int_0^s u_0^* d\sigma \leq Z(s,0).$$

Noting that $Z(0,t) = 0$ and $\partial Z/\partial s(|\Omega|,t) > 0$, by Proposition A1 we get $K(s,t) \leq Z(s,t)$, which implies (4.2). □

From Theorem 2 and Proposition 2 the exponential decay of the non-negative solution $(u,v)$ of the problem $\mathrm{P_D}(\Omega)$ is obtained.

**Theorem 3.** *Under each of* (C1)–(C3), $T_{max} = \infty$ *and there exists a positive constant* $\lambda$ *such that*

$$||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{L^\infty(\Omega)} \leq \text{Const.}e^{-\lambda t} \quad \textit{for } t > 0.$$

*Proof.* By Theorem 2 and Proposition 2,

$$||u(\cdot,t)||_{L^\infty(\Omega)} \leq ||U(\cdot,t)||_{L^\infty(\Omega)} \leq \text{Const.}e^{-\lambda t} \quad \text{for } 0 < t < T_{max}. \tag{4.3}$$

Hence, $T_{max} = \infty$ by Theorem 1, and

$$||u(\cdot,t)||_{L^\infty(\Omega)} \leq \text{Const.}e^{-\lambda t} \quad \text{for } t > 0.$$

Since the function $v(\cdot,t)$ satisfies

$$\begin{cases} v(\cdot,t)\in L^\infty(\Omega)\cap W^{2,p}(\Omega), \\ (\Delta-\gamma)v(\cdot,t)=-\alpha u(\cdot,t) \quad \text{in } \Omega, \\ \mathcal{B}v=0 \quad \text{on } \partial\Omega, \end{cases}$$

$v(\cdot,t)$ is estimated as

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\le \text{Const.}\|u(\cdot,t)\|_{L^\infty(\Omega)},$$

which together with (4.3) establishes the conclusion of the theorem.

## Appendix

In what follows, $C_1(t)$ is a generic positive function on $(0,\ T)$ belonging to $L^2(0,T)$.

**Proposition A1.** *Let $f$ and $g$ be functions on $\overline{Q_T^*}$ satisfying the following:*

(i) $f,g\in L^\infty(Q_T^*)\cap H^1(0,T;L^2(\Omega^*))\cap\bigcap_{\delta>0}L^2(0,T;W^{2,2}(\delta,|\Omega|))$.

(ii) $\left|\dfrac{\partial f}{\partial s}(s,t)\right|\le C_1(t)$ *and* $\left|\dfrac{\partial g}{\partial s}(s,t)\right|\le C_1(t)s^{-\ell}$ *on* $Q_T^*\cap\{0<s<1\}$,

*where $\ell$ is a constant satisfying $0\le\ell<1$.*

(iii) $\dfrac{\partial f}{\partial t}-d(s)\dfrac{\partial^2 f}{\partial s^2}-\alpha\chi f\dfrac{\partial f}{\partial s}\le\dfrac{\partial g}{\partial t}-d(s)\dfrac{\partial^2 g}{\partial s^2}-\alpha\chi g\dfrac{\partial g}{\partial s}$ *a.e. in* $Q_T^*$,

*where* $d(s)=N^2\kappa_N^{2/N}s^{2(N-1)/N}$.

(iv) $0=f(0,t)\le g(0,t)$ *and* $\dfrac{\partial f}{\partial s}(|\Omega|,t)\le\dfrac{\partial g}{\partial s}(|\Omega|,t)$ *for any* $t\in[0,\ T]$.

(v) $f(s,0)\le g(s,0)$ *on* $\Omega^*$ *and* $g(s,t)\ge 0$ *on* $Q_T^*$.

*Then*

$$f\le g \quad \text{on } Q_T^*.$$

*Proof.* Put $w=f-g$, which satisfies

$$\text{(A.1)}\qquad \begin{cases} \dfrac{\partial w}{\partial t}-d(s)\dfrac{\partial^2 w}{\partial s^2}-\alpha\chi\left(f\dfrac{\partial f}{\partial s}-g\dfrac{\partial g}{\partial s}\right)\le 0 \quad \text{a.e. in } Q_T^*, \\ w(0,t)\le 0,\quad \dfrac{\partial w}{\partial s}(|\Omega|,t)\le 0 \quad \text{for any} \quad t\in[0,\ T], \\ w(s,0)\le 0 \quad \text{for any} \quad s\in[0,\ |\Omega|]. \end{cases}$$



Multiplying the differential inequality in (A.1) by $s^{2(1-N)/N}w_+$, where $w_+=\max\{w,0\}$, we get

$$s^{2(1-N)/N}\frac{\partial w}{\partial t}w_+\le N^2\kappa_N^{2/N}\frac{\partial^2 w}{\partial s^2}w_+ +\alpha\chi s^{2(1-N)/N}\left(f\frac{\partial f}{\partial s}-g\frac{\partial g}{\partial s}\right)w_+ \quad \text{a.e. in } Q_T^*.$$

By $g\ge 0$, $f(0,t)=0$ and (ii), we have

$$\text{(A.2)}\qquad w_+(s,t)\le f(s,t)\le C_1(t)s \quad \text{on } \{s\in\Omega^*;w_+(s,t)>0\}\cap\{0<s<1\}.$$

From (A.2) it follows that $s^{2(1-N)/N}(w_+)^2$ belongs to $L^1(\Omega^*)$. Let us take $\delta$ satisfying $0<\delta<|\Omega|$. Using the integration by parts and $\partial w/\partial s(|\Omega|,t)\le 0$ yields that

$$\int_\delta^{|\Omega|}\frac{\partial^2 w}{\partial s^2}w_+ds\le-\frac{\partial w}{\partial s}(\delta,t)w_+(\delta,t)-\int_\delta^{|\Omega|}\left(\frac{\partial w_+}{\partial s}\right)^2ds.$$

We next have

$$\begin{aligned} &\alpha\chi\int_\delta^{|\Omega|}s^{2(1-N)/N}\left(f\frac{\partial f}{\partial s}-g\frac{\partial g}{\partial s}\right)w_+ds \\ &=\alpha\chi\int_\delta^{|\Omega|}s^{2(1-N)/N}(w_+)^2\frac{\partial f}{\partial s}ds+\alpha\chi\int_\delta^{|\Omega|}s^{2(1-N)/N}w_+\frac{\partial w}{\partial s}gds \\ &\le C_1(t)\int_{\Omega^*}s^{2(1-N)/N}(w_+)^2ds+\alpha\chi\int_\delta^{|\Omega|}s^{2(1-N)/N}w_+\left|\frac{\partial w_+}{\partial s}\right|fds. \end{aligned}$$

Using (A.2) and Hőlder's inequality, we obtain

$$\begin{aligned} &\alpha\chi\int_\delta^{|\Omega|}s^{2(1-N)/N}w_+\left|\frac{\partial w_+}{\partial s}\right|fds\le C_1(t)\int_\delta^{|\Omega|}s^{(1-N)/N}w_+\left|\frac{\partial w_+}{\partial s}\right|ds \\ &\le\frac{1}{2}N^2\kappa_N^{2/N}\int_\delta^{|\Omega|}\left(\frac{\partial w_+}{\partial s}\right)^2ds+\{C_1(t)\}^2\int_{\Omega^*}s^{2(1-N)/N}(w_+)^2ds. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{d}{dt}\int_\delta^{|\Omega|}s^{2(1-N)/N}w_+^2ds+N^2\kappa_N^{2/N}\int_\delta^{|\Omega|}\left(\frac{\partial w_+}{\partial s}\right)^2ds \\ &\le\{C_1(t)\}^2\int_{\Omega^*}s^{2(1-N)/N}w_+^2ds+\text{Const.}\left|\frac{\partial w}{\partial s}(\delta,t)\right|w_+(\delta,t) \quad \text{for} \quad t\in(0,\ T), \end{aligned}$$

from which together with $w_+(s,0)=0$ on $\Omega^*$ it follows that for $t\in(0,\ T)$,

$$\int_\delta^{|\Omega|}s^{2(1-N)/N}w_+(s,t)^2ds$$

$$\leq \int_0^t \{C_1(\tau)\}^2 \left(\int_{\Omega^*} s^{2(1-N)/N} w_+^2 ds\right) d\tau + \text{Const.} \int_0^t \left|\frac{\partial w}{\partial s}(\delta,\tau)\right| w_+(\delta,\tau) d\tau.$$

Note that $|(\partial w/\partial s)(\delta,t)| w_+(\delta,t) \leq \{C_1(t)\}^2 \delta^{1-\ell}$. Letting $\delta \to 0$, we get

$$\int_{\Omega^*} s^{2(1-N)/N} w_+(s,t)^2 ds \leq \int_0^t \{C_1(\tau)\}^2 \left(\int_{\Omega^*} s^{2(1-N)/N} w_+^2 ds\right) d\tau \quad \text{for} \quad t \in (0,\ T).$$

By Gronwall's inequality,

$$\int_{\Omega^*} s^{2(1-N)/N} w_+(s,t)^2 ds = 0 \quad \text{for} \quad t \in (0,\ T],$$

which implies $w_+ = 0$ in $\overline{Q_T^*}$. Hence, $f \leq g$. □

The following proposition is obtained by using Moser's technique (see Alikakos[1]).

**Proposition A2.** *Assume that there exists a positive constant $C$ independent of $T$ such that*

$$|a(x,t)| \leq C, \quad b(x,t) \leq C \quad \text{on } Q_T.$$

*Let $w$ be a non-negative function on $Q_T$ such that*

(i) $w \in L^\infty(Q_T) \cap L^2(0,T; W_0^{1,2}(\Omega)) \cap H^1(0,T; L^2(\Omega))$,

(ii) *for almost all $t \in (0,\ T)$ and all $\varphi \in W_0^{1,2}(\Omega)$ with $\varphi \geq 0$,*

$$\int_\Omega \left\{\frac{\partial w}{\partial t}\varphi + (\nabla w + wa)\cdot\nabla\varphi - bw\varphi\right\} dx \leq 0.$$

*Then there exists a positive constant $C$ independent of $T$ such that*

$$\sup_{0\leq t\leq T} ||w(\cdot,t)||_{L^\infty(\Omega)} \leq C \max\{1,\ ||w(\cdot,0)||_{L^\infty(\Omega)},\ \sup_{0\leq t\leq T} ||w(\cdot,t)||_{L^1(\Omega)}\}.$$

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