

## On the Interfaces in a Nonlocal Quasilinear Degenerate Equation Arising in Population Dynamics

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We study regularity and propagation properties of interfaces separating regions where nonnegative weak solutions of the Cauchy problem for the equation

$$u_t = (u^m)_{xx} + \left[ u \left( \int_{-\infty}^x u(y, t) dy - \int_x^{\infty} u(y, t) dy \right) \right]_x, \quad m > 1,$$

are strictly positive or equal to zero. It is shown that under suitable conditions on the initial data the interfaces are (not necessarily monotone)  $C^\infty$ -curves and they do not lose this regularity at their turning points. The study of the interface regularity is performed via Lagrangian coordinates. We show that the initial behavior of interfaces is determined by the character of growth of the initial datum near the endpoints of the initial support. Estimates (from below and from above) on the width of the positivity set of solutions are also obtained.

*Key words:* interface, nonlocal equation, degenerate parabolic equation, Lagrangian coordinate

### 1. Introduction

In this paper, we study some properties of solutions of the Cauchy problem for the quasilinear nonlocal degenerate parabolic equation

$$u_t = (u^m)_{xx} + \left[ u \left( \int_{-\infty}^x u(y, t) dy - \int_x^{\infty} u(y, t) dy \right) \right]_x \quad \text{in } S = \mathbf{R} \times (0, T) \quad (1)$$

assuming  $m > 1$ .

This equation presents a simplified model of the process of diffusion of a biological population when the nonlocal interaction of individuals is taken into account. The reader is referred to the papers [21, 22, 23] for further information on the genesis of this problem and the biological background of the formulation. In this

model  $u(x, t)$  represents the density of the biological population and is assumed to be nonnegative in  $\bar{S}$ . The initial distribution of the population density

$$u(x, 0) = u_0(x) \quad \text{in } \mathbf{R} \tag{2}$$

is subject to the following conditions:

A<sub>1</sub>)  $u_0(x) > 0$  in  $(0, a)$ ,  $(a < \infty)$ ;  $u_0(x) \equiv 0$  for  $x \in \mathbf{R} \setminus (0, a)$ ;

A<sub>2</sub>)  $\left\| (u_0^{m-1})' \right\|_{L^\infty(0,a)} \leq M$ .

In what follows, we use the notion of weak solutions to (1), (2) proposed in [21].

DEFINITION 1. A function  $u(x, t)$  is said to be a weak solution to (1), (2) if it is nonnegative and bounded in  $\bar{S}$  and satisfies the following conditions:

- (i)  $u \in C(S) \cap L^\infty_{loc}(0, T; L^1(\mathbf{R}))$ ,
- (ii)  $(u^m)_x \in L^\infty(\mathbf{R} \times (\tau, T))$  for all  $0 < \tau < T$ ,
- (iii)

$$\int_{-\infty}^{\infty} u(y, t) dy \in C([0, T]),$$

$$\lim_{t \rightarrow 0} \int_{-\infty}^x u(y, t) dy = \int_{-\infty}^x u_0(y) dy \quad \text{for any } x \in (-\infty, \infty];$$

- (iv) for each  $f \in C^1(S)$ , compactly supported in  $S$ , there holds the identity

$$\int_S \left\{ u f_t - \left[ (u^m)_x + \left( \int_{-\infty}^x u(y, t) dy - \int_x^{\infty} u(y, t) dy \right) \right] f_x \right\} dx dt = 0.$$

Let us begin with a brief overview of the results already known. Consider the Cauchy problem associated to the equation

$$u_t = (u^m)_{xx} + \left[ u\Phi \left( \int_{-\infty}^x u(y, t) dy \right) \right]_x \tag{1'}$$

generalizing (1), and assume

N<sub>1</sub>)  $\Phi \in C^3$ ;

N<sub>2</sub>)  $u_0$  is nonnegative, bounded and integrable on  $\mathbf{R}$ .

Under these assumptions it is shown [21] that there exists a unique weak solution (in what follows simply termed “a solution”)  $u(x, t)$  of (1'), (2) which possesses the same regularity properties which are known to be the best possible for the “unperturbed” equation (1') with  $\Phi \equiv 0$ , usually called the porous medium equation

$$u_t = (u^m)_{xx}, \quad m > 1. \tag{PME}$$

In particular, this solution satisfies the following conservation property

$$\int_{\mathbf{R}} u(x, t) dx = \int_{\mathbf{R}} u_0(x) dx \quad \text{for any } t \in (0, T] \tag{3}$$

which is of a specific importance for our further purposes.

The paper [22] deals with asymptotic properties of solutions of (1'), (2). Assuming

N<sub>3</sub>)  $\Phi' > 0$ ,

it is shown that solutions of (1'), (2) converge as  $t \rightarrow \infty$  to the travelling wave solution of this problem  $U(s)$ ,  $s = x - kt$  ( $k = \text{const}$ ), defined by the conditions

$$\begin{cases} kU' + \left[ (U^m)' + U\Phi \left( \int_{-\infty}^s U(\eta) d\eta \right) \right]' = 0 & \text{in } \mathbf{R}, \\ U(-\infty) = U(\infty) = 0, \quad \int_{\mathbf{R}} U(x) dx = \int_{\mathbf{R}} u_0(x) dx = b, \\ \int_{\mathbf{R}} \int_{-\infty}^x [u_0(\xi) - U(\xi)] d\xi dx = 0. \end{cases}$$

Here

$$k = -\frac{\varphi(b) - \varphi(0)}{b}, \quad \Phi = \varphi'.$$

This allows one to prove that the solutions of (1'), (2) are localized in space: there exist two constants  $\alpha_1, \alpha_2$  such that for any  $t \in [0, T]$ ,

$$u(x, t) \equiv 0 \quad \text{for } x \in (-\infty, \alpha_1 + kt) \cup (\alpha_2 + kt, \infty).$$

In the particular case of equation (1) we have, using (3), that

$$\int_{-\infty}^x u(y, t) dy - \int_x^{\infty} u(y, t) dy = 2 \int_{-\infty}^x u(y, t) dy - b$$

and, consequently,

$$\Phi(s) = 2s - b; \quad \varphi(s) = s^2 - bs; \quad k = 0.$$

In this case the travelling wave solution changes into a stationary one.

The localization result shows that there exist two interfaces

$$\eta(t) = \sup\{x \in \mathbf{R} : u(x, t) > 0\}, \quad \zeta(t) = \inf\{x \in \mathbf{R} : u(x, t) > 0\},$$

i.e., the curves separating regions where the solution  $u(x, t)$  of (1'), (2) is strictly positive or equals zero. The investigation of the interface behavior was undertaken in [23], where problem (1), (2) was considered under assumptions A<sub>1</sub>), A<sub>2</sub>), and

N<sub>4</sub>)  $u_0$  is piecewise monotone in  $\mathbf{R}$ .

It is shown, in particular, that

- (i) the functions  $\eta(t), \zeta(t)$  are globally Lipschitz-continuous on  $[0, T]$ ;
- (ii) the positivity set  $P$  of the solution  $u(x, t)$  is given by

$$P = \{(x, t) \in S : \zeta(t) < x < \eta(t)\},$$

and the magnitude

$$(u^{m-1})_x(\eta(t), t) \equiv \lim_{P \ni (x,t), x \rightarrow \eta(t)-0} (u^{m-1})_x(x, t)$$

exists for all  $t \in (0, T]$  and belongs to  $L^\infty(0, T)$  (the corresponding results hold for  $\zeta(t)$ );

- (iii) the interfaces follow the equations

$$\eta'(t) = -\frac{m}{m-1} (u^{m-1})_x(\eta(t), t) - b, \quad \zeta'(t) = -\frac{m}{m-1} (u^{m-1})_x(\zeta(t), t) + b,$$

which are understood in the sense of distributions and hold a.e. in  $(0, T)$ ;

- (iv)  $\lim_{t \rightarrow \infty} \eta(t) = \delta, \quad \lim_{t \rightarrow \infty} \zeta(t) = \gamma$ , where the constants  $\delta, \gamma$  correspond to the boundary points of the support of the stationary solution.

In the present paper, we study further regularity properties of solutions to problem (1), (2). Due to the symmetry of equation (1) we concentrate our study on the right interface  $\eta(t)$ . The change of variables  $x' = a - x$  allows us to apply this study to the case of the left interface  $\zeta(t)$ .

Let us assume, additionally to  $A_1), A_2)$ , the following condition:

$A_3) \quad (u_0^{m-1})' \in C[0, a] \text{ and } \lim_{x \rightarrow a-0} (u_0^{m-1})'(x) \leq -q, \quad q = \text{const} > 0.$

**THEOREM 1.** *Let conditions  $A_1)$ – $A_3)$  hold and let  $u(x, t)$  be the weak solution of (1), (2). Then*

$$\eta \in C_{\text{loc}}^{1, (m-1)/(3m-1)}[\tau, T] \text{ for any } \tau > 0.$$

**THEOREM 2.** *Under the above assumptions*

$$\eta \in C_{\text{loc}}^\infty[\tau, T] \text{ for any } \tau > 0.$$

**REMARK 1.** We consciously separate these two assertions because of the difference in the technical tools used for their proofs.

**REMARK 2.** As a matter-of-fact, assumption  $A_3)$  is not a serious limitation. Indeed, as we mentioned earlier, for all  $t > 0$  the weak solution  $u(x, t)$  is compactly supported in  $P$  and the function  $(u^{m-1})_x$  is smooth inside  $P$  and has finite limits at the lateral boundaries of  $P$ . Moreover, as follows from Theorem 4 below, one may merely skip  $A_3)$  since the function  $(u^{m-1})_x$  necessarily becomes separate from zero in a finite time at the lateral boundaries of  $P$ .

REMARK 3. The assertions of Theorems 1, 2 hold without any changes for the interfaces occurring in solutions of (1'), (2). In Sections 2-5 we concentrate on the case of equation (1) just to avoid some technical complications. In Section 6 we show how all these arguments may be carried out in the case of equation (1').

Our next concern is the study of the initial behavior of the interfaces. The interfaces give boundaries between the populated region and the unpopulated one. It is of interest to know whether, for given initial data, the populated region spreads out or shrinks for a short time. In [1, 2, 10] the initial behavior of interfaces for the Cauchy problem to the diffusion-convection equation

$$u_t = (u^m)_{xx} + c(u^\lambda)_x, \quad c > 0, \quad \lambda > 0 \tag{4}$$

was studied. In [2], it was discussed whether the right interface is, initially, a progressing front or a reversing one according to some mass growth conditions on  $u_0$ . (We postpone the explanation of the relevance of the term "mass" until Section 2.) Returning to problem (1), (2), we have the following theorem for  $\eta(t)$ .

THEOREM 3. *Let us denote  $\ell = m/(m - 1)$  and  $\ell(b) = \ell^{-\ell} b^{1/(m-1)}$ , where  $b = \int_{\mathbb{R}} u_0(x) dx$ .*

1) *Assume that*

$$\limsup_{x \rightarrow a-0} (a - x)^{-\ell} \int_x^\infty u_0(y) dy < \ell(b).$$

*Then there exist  $c > 0$  and  $t_0 > 0$  such that*

$$\eta(t) \leq a - ct \quad \text{for } 0 \leq t \leq t_0.$$

2) *Assume that*

$$\liminf_{x \rightarrow a-0} (a - x)^{-\ell} \int_x^\infty u_0(y) dy > \ell(b).$$

*Then there exist  $c > 0$  and  $t_0 > 0$  such that*

$$\eta(t) \geq a + ct \quad \text{for } 0 \leq t \leq t_0.$$

Notice that if the function  $(u^{m-1})_x(\cdot, t)$  is continuous at the interface  $\eta(t)$  for  $t \in (0, t_0]$ , (i.e.,  $(u^{m-1})_x(\eta(t), t) = 0$ ), then the interface equation leads to the relation

$$\eta(t) = a - bt \quad \text{for } 0 \leq t \leq t_0.$$

In consequence, the support of the solution shrinks initially. For the interfaces of (PME), a phenomenon of a similar nature corresponds to the existence of a

positive time  $t_0$ , called *the waiting time*, such that the interface remains stationary for  $t \in [0, t_0]$  ([3, 17, 28]). For equation (4) we refer to [1, 2]. Let us put

$$t^* = \sup\{t_0 : \eta(t) = a - bt \text{ for } 0 \leq t \leq t_0\}.$$

The following theorem gives necessary and sufficient conditions on  $u_0$ , implying the positivity of  $t^*$  and presents some estimates of the maximal and minimal distance between the endpoints of the sets  $P \cap \{t = \text{const}\}$  on the time interval  $[0, t^*]$ .

THEOREM 4. *Let*

$$\alpha = \inf_{t \geq 0} \zeta(t) (> -\infty), \quad K = \left\{ \frac{b^2}{4} + \max_{[0, a]} u_0^m \right\}^{1/m}.$$

1) *If  $t^* > 0$ , then*

$$t^* \leq \frac{1}{b} \left( a - \alpha - \frac{b}{K} \right).$$

2)  *$t^*$  is positive if and only if*

$$\limsup_{x \rightarrow a-0} (a - x)^{-(m+1)/(m-1)} \int_x^\infty u_0(y) dy < +\infty. \tag{5}$$

3) *For all  $t \in [0, t^*]$*

$$\frac{b}{K} \leq \eta(t) - \zeta(t) \leq a - bt - \alpha.$$

Let us give a short comment on the results just formulated. First, the interfaces occurring in solutions of equation (1) need not be monotone (Theorem 3). Though the reversing fronts are admitted here, Theorem 4 asserts that these fronts never meet. Moreover, the distance between these fronts is measured in terms of initial data. On the other hand, due to the results on the asymptotic convergence of solutions of (1), (2) to the stationary solution, it is clear that each reversing front necessarily has a turning point. Moreover, these results hold simultaneously with the assertions of Theorems 1, 2. In particular, the interfaces pass their eventual turning points without losing their smoothness. In Section 8 we show numerical experiments on the initial behavior of the interfaces to supplement our conclusions of Theorems 3, 4.

**2. Lagrangian Coordinates: Statement of Problem (L)**

Regardless of the context in which problem (1), (2) arises, let us view it as the mathematical model of some fluid flow. Assuming that a fluid has the density  $\rho = u(x, t)$  and the velocity

$$V(x, t) = -\frac{m}{m-1} (u^{m-1})_x - \left\{ \int_{-\infty}^x u(y, t) dy - \int_x^\infty u(y, t) dy \right\},$$

one may review equation (1) in the form of the mass balance law of the fluid motion, i.e.,

$$\rho_t + (\rho V)_x = 0. \tag{6}$$

The initial density and velocity of the flow are derived from condition (2). Equality (6) is the mathematical expression of the mass conservation law in the Euler coordinates system, which is fixed in space. There exists, however, an alternative approach to the mathematical modeling of fluid motions. It is based on introducing Lagrangian (or material) coordinates defined by the initial state of the substance. Following the generally accepted scheme of Lagrangian coordinates [9, 24], let us define the quantity

$$\xi = \int_{-\infty}^x u_0(s) ds : [0, a] \mapsto [0, b]$$

to be a new independent variable. The value of  $\xi$  equals the total mass of the substance contained in the interval  $[0, x]$  at the instant  $t = 0$ . Due to (3), for each instant of time the fluid volume contains the same number of particles. Thus, the total mass of the fluid volume is constant and the variable  $\xi$  (usually called *the mass Lagrangian coordinate*) at each  $t \in [0, T]$  runs the interval  $[0, b]$ . This is very convenient for our further purposes since the problem reformulated in the variables of Lagrange will be posed automatically in a fixed domain and, therefore, the interfaces will give the lateral boundaries of the new problem domain. Remark also that under the above assumptions  $b$  is always finite.

As the new unknowns we choose  $x = X(\xi, t)$  – the trajectory of the fluid particle labelled by its position  $\xi$  at the instant  $t = 0$ , and the density  $v(\xi, t) = u(X(\xi, t), t)$ . The flow is governed by the following relations: the trajectory equation

$$\begin{aligned} X_t(\xi, t) &= V(X(\xi, t), t), \quad t > 0, \\ X(\xi, 0) &\in [0, a] \end{aligned} \tag{7}$$

and the mass balance law which now takes the form

$$v(\xi, t) X_\xi(\xi, t) = 1 \quad \text{in } Q := \{(\xi, t) : \xi \in (0, b), t \in (0, T)\}. \tag{8}$$

Having assumed all the functions we are dealing with to be smooth enough, let us perform several transformations. First, relying on (8) and assuming that the trajectories of the fluid particles do not intersect, we have

$$\int_{-\infty}^x u(y, t) dy \equiv \int_{X(0, t)}^{X(\xi, t)} u(y, t) dy = \int_0^\xi u(X(\eta, t), t) dX(\eta, t) = \xi,$$

whence, with the use of (3),

$$W(\xi, t) \equiv V(X(\xi, t), t) = -(v^m)_\xi - 2\xi + b. \tag{9}$$



Next, differentiating (7) in  $\xi$  and (8) in  $t$ , we get the equation for  $v(\xi, t)$ :

$$\left(\frac{1}{v}\right)_t + (v^m)_{\xi\xi} + 2 = 0 \quad \text{in } Q. \quad (10)$$

As already mentioned, the lateral boundaries of  $Q$  correspond to the unknown (free) boundaries of the fluid volume, where the density  $v(\xi, t)$  vanishes. The constraint therefore is to pose the boundary conditions

$$v(0, t) = v(b, t) = 0 \quad \text{as } t \in [0, T]. \quad (11)$$

The initial conditions for  $v(\xi, t)$  are provided by (2):

$$v(\xi, 0) = v_0(\xi) \equiv u_0(X(\xi, 0)) \quad \text{in } \xi \in [0, b]. \quad (12)$$

Here the function  $X(\xi, t)$  is to be defined from the relation introducing  $\xi$

$$\xi = \int_0^{X(\xi, 0)} u_0(s) ds \quad (13)$$

which is always invertible due to  $A_1$ ).

This analogy with fluid mechanics justifies the use of the notation "mass". From now on we term the problem (10)–(12) "The problem (L)".

The conditions on the initial function  $v_0$  are generated by  $A_1$ – $A_3$ ):

- B<sub>1</sub>)  $v_0 > 0$  in  $(0, b)$ ;  $v_0(0) = v_0(b) = 0$ ;
- B<sub>2</sub>)  $\|(v_0^m)'\|_{L^\infty(0, b)} \leq M_1$ ;
- B<sub>3</sub>) there exists  $\varepsilon_0 > 0$  such that for each  $\varepsilon \in (0, \varepsilon_0)$  the set  $\{\xi \in (0, b) : v_0(\xi) > \varepsilon\}$  is an interval;
- B<sub>4</sub>) there exists a constant  $C > 0$  such that  $v_0^m \geq C(b - \xi)$  in some neighborhood of the point  $\xi = b$ .

Problem (L) is the object of the further discussion in the part connected with the interface regularity. Having problem (L) formulated, we eliminate the free boundary from explicit consideration. As we show later on, the function  $\eta(t)$  gets the representation

$$\eta(t) = a - bt - \lim_{\xi \rightarrow b} \int_0^t (v^m)_\xi(\xi, \theta) d\theta, \quad t > 0. \quad (14)$$

This representation shows that the questions concerning the interface behavior and its regularity properties are no longer independent. They are reduced to the study of the boundary smoothness of solutions to problem (L).

To legitimize this change of variables we treat (L) as a separate mathematical object (independently of its interpretation as a description of a fluid motion) and

show that under assumptions B<sub>1</sub>) and B<sub>2</sub>) it has a unique classical solution. Condition B<sub>3</sub>) is a byproduct of A<sub>1</sub>) and A<sub>3</sub>) and is outlined into a separate line only for convenience of further reference; B<sub>4</sub>) is a byproduct of A<sub>3</sub>). Thereafter we present explicit formulas restoring a weak solution of (1), (2) by the constructed solution of (L). Due to the uniqueness of weak solutions to (1), (2), this solution necessarily coincides with the one obtained in [21, 22, 23].

The idea of passing to Lagrangian coordinates in order to study properties of interfaces occurring in an evolution equation of divergence form comes from the papers [5, 16, 20]. Later on this method was used in a number of works: see [8, 11, 15, 25, 26] and references therein. In particular, in [15] it was used for the study of smoothness properties of interfaces arising in solutions of the Cauchy problem for the diffusion-convection equation. Here we will follow the techniques of the papers [11, 25] in the part concerning the proof of correctness of problem (L) and restoring a solution to the original Cauchy problem (1), (2). The proof of  $C^{1,\alpha}$ -regularity of interfaces will follow the scheme of arguments proposed in the paper [26], dealing with the same questions for a related degenerate parabolic equation. The proof of  $C^\infty$ -regularity of interfaces will be reduced to the same problem but for interfaces occurring in solutions of the Cauchy problem for the porous medium equation (PME), which is already solved in [4].

### 3. Solvability of Problem (L)

#### 3.1. Regularization: statement of problems (L<sub>n</sub>)

A solution of (L) will be obtained as the pointwise limit of a sequence of solutions of some regularized non-degenerate problems. Let {w<sub>n</sub>} be a sequence of solutions of the problems

$$\mathcal{L}_n w_n \equiv w_{nt} - mw_n^{(m+1)/m} w_{n\xi\xi} - 2mw_n^{(m+1)/m} = 0 \quad \text{in } Q, \tag{15}$$

$$w_n(0, t) = w_n(b, t) = \frac{1}{n}, \quad t \in [0, t], \tag{16}$$

$$w_n(\xi, 0) \equiv w_{0n}(\xi) \quad \text{in } (0, b), \quad (n = 1, 2, 3, \dots), \tag{17}$$

which we term “Problems (L<sub>n</sub>)”. Equation (15) follows from (10) by the formal replacement of  $v^m(\xi, t)$  by  $w_n(\xi, t)$ . The approximating sequence {w<sub>0n</sub>} will be chosen so as to satisfy the following conditions:

- C<sub>1</sub>)  $w_{0n} \in C^\infty[0, b]$ ,  $w_{0n} \searrow w_0 \equiv v_0^m$  as  $n \rightarrow \infty$ ;
- C<sub>2</sub>)  $\frac{1}{n} \leq w_{0n} \leq M_0 + 1$ ;  $\|w'_{0n}\|_{L^\infty(0,b)} \leq M_1 + 2$ ;
- C<sub>3</sub>) there exists  $\varepsilon'_0 > 0$  such that for each  $\varepsilon \in (0, \varepsilon'_0)$  the sets  $P_n(\varepsilon) = \{\xi \in (0, b) : w_{0n}(\xi) > \varepsilon\}$  are intervals;
- C<sub>4</sub>)  $w_{0n}(0) = w_{0n}(b) = \frac{1}{n}$ ;  $\mathcal{L}_n w_{0n} = 0$  as  $\xi = 0, \xi = b$ ;

C<sub>5</sub>) there exists a constant  $C' > 0$  such that

$$w_{0n}(\xi) \geq \frac{1}{n} + C'(b - \xi) \quad \text{for } \frac{b}{2} \leq \xi \leq b.$$

Here  $M_0 = \max_R |w_0(\xi)|$ , and the constant  $M_1$  is taken from B<sub>2</sub>). Such a sequence  $\{w_{0n}\}$  may be constructed by the given function  $w_0(\xi)$  via a standard regularizing procedure as follows. For  $\varepsilon > 0$  let us put

$$\omega_\varepsilon(\xi) = \frac{1}{\varepsilon} \omega\left(\frac{\xi}{\varepsilon}\right) \quad \text{on } \mathbf{R},$$

where  $\omega(\xi)$  is an even smooth function such that

$$\omega(\xi) > 0 \quad \text{for } |\xi| < 1, \quad \omega(\xi) = 0 \quad \text{for } |\xi| \geq 1 \quad \text{and} \quad \int_{\mathbf{R}} \omega(\xi) d\xi = 1.$$

For each integer  $n$  large enough we consider the function  $W_n(\xi)$  satisfying

$$W_n''(\xi) + 2 = 0 \quad \text{on } \mathbf{R}, \quad W_n(0) = \gamma_n, \quad W_n'(0) = M_1 + 1,$$

where

$$\gamma_n = \int_{\mathbf{R}} \omega_{1/n^3}(\xi) \xi^2 d\xi.$$

The function  $W_n$  is given by  $W_n(\xi) = -\xi^2 + (M_1 + 1)\xi + \gamma_n$ . Along with  $W_n$ , define the function  $V_n(\xi)$  by  $V_n(\xi) = W_n(b - \xi)$ , which satisfies

$$V_n''(\xi) + 2 = 0 \quad \text{on } \mathbf{R}, \quad V_n(b) = \gamma_n, \quad V_n'(b) = -(M_1 + 1).$$

Let  $\mu_n$  and  $\nu_n$  be such points that

$$\begin{aligned} \mu_n &= \min \left\{ \xi \in [0, b] : w_0(\xi) = W_n\left(\frac{1}{nM_1}\right) \right\}, \\ \nu_n &= \max \left\{ \xi \in [0, b] : w_0(\xi) = V_n\left(b - \frac{1}{nM_1}\right) \right\}. \end{aligned}$$

We note that  $1/(nM_1) < \mu_n < \nu_n < b - 1/(nM_1)$ , since  $W_n(1/(nM_1)) > w_0(\xi)$  for  $0 \leq \xi \leq 1/(nM_1)$  and  $V_n(b - 1/(nM_1)) > w_0(\xi)$  for  $b - 1/(nM_1) \leq \xi \leq b$ . Now set

$$Z_n(\xi) = \begin{cases} W_n(\xi) & \text{if } \xi < \frac{1}{nM_1}, \\ W_n\left(\frac{1}{nM_1}\right) & \text{if } \frac{1}{nM_1} \leq \xi \leq \mu_n, \\ w_0(\xi) & \text{if } \mu_n < \xi < \nu_n, \\ V_n\left(b - \frac{1}{nM_1}\right) & \text{if } \nu_n \leq \xi \leq b - \frac{1}{nM_1}, \\ V_n(\xi) & \text{if } b - \frac{1}{nM_1} < \xi. \end{cases}$$

The desired function  $w_{0n}(\xi)$  is then defined by

$$w_{0n}(\xi) = \int_{\mathbf{R}} Z_n(\xi - \zeta) \omega_{1/n^3}(\zeta) d\zeta + \frac{1}{n}.$$

**3.2. Solvability of problems  $(L_n)$**

Given an integer  $n$  fixed, consider the corresponding problem  $(L_n)$ . If  $(L_n)$  has a classical solution  $z_n(\xi, t)$  then

$$z_n \geq \frac{1}{n} \quad \text{on } \overline{Q}, \tag{18}$$

since  $\mathcal{L}_n(1/n) = -2mn^{-(m+1)/m} < 0$  and  $z_n \geq 1/n$  on the parabolic boundary of  $Q$  [12, Ch.2, Th.16]. Instead of dealing with  $z_n$ , let us consider the function

$$S_n = z_n + \xi^2 - b\xi,$$

which satisfies the conditions

$$S_{nt} - mz_n^{(m+1)/m} S_n \xi \xi = 0 \quad \text{in } Q, \tag{19}$$

$$S_n(0, t) = S_n(b, t) = \frac{1}{n} \quad \text{as } t \in [0, T], \tag{20}$$

$$S_n(\xi, 0) = w_{0n}(\xi) + \xi^2 - b\xi \quad \text{on } [0, b]. \tag{21}$$

By the maximum principle,  $S_n \leq \max_{[0,b]} S_n(\xi, 0)$  on  $\overline{Q}$  providing the upper estimate for  $z_n$  on  $\overline{Q}$ :

$$z_n \leq \frac{b^2}{4} + \max_{[0,b]} w_{0n}. \tag{22}$$

It follows from (18) and (22) that equation (19) is uniformly parabolic and in this range of values of  $z_n$  problem (19)–(21) always has a unique classical solution  $S_n \in C^\infty(Q)$ . The function  $S_n$  generates a solution  $w_n$  to  $(L_n)$  which must coincide with  $z_n$ .

Moreover, the choice of the approximating sequence  $\{w_n\}$  provides the monotonicity of the sequence  $\{w_n\}$  at each point of  $\overline{Q}$ :  $w_n \geq w_{n+1}$ ,  $n = 1, 2, 3, \dots$ . Thus, there exists a function

$$w(\xi, t) = \lim_{n \rightarrow \infty} w_n(\xi, t), \quad (\xi, t) \in \overline{Q}$$

which, as we shall see, is the searched solution to the degenerate problem (L) and satisfies the upper estimate

$$w \leq \frac{b^2}{4} + \max_{[0,b]} v_0'' \quad \text{on } \overline{Q}. \tag{23}$$

3.3. *A priori estimates*

LEMMA 1. *There exists a positive constant  $M_2$  not depending on  $n$  such that*

$$|w_{n\xi}| \leq M_2 \quad \text{on } \overline{Q}. \tag{24}$$

*Proof.* It will be convenient to deal with the functions  $S_n = w_n + \xi^2 - b\xi$  instead of  $w_n$ . By (19)

$$\frac{\partial}{\partial t}(S_{n\xi}) - m \frac{\partial}{\partial \xi} \left[ w_n^{(m+1)/m} \frac{\partial}{\partial \xi}(S_{n\xi}) \right] = 0.$$

Hence,  $S_{n\xi}$  assumes both the positive maximas and negative minimas at the parabolic boundary of  $Q$ . Due to  $C_2$ ),  $S_{n\xi}$  is bounded as  $t = 0$ . Let us control it at the lateral boundaries of  $Q$ . As a classical solution of (19)–(21),  $S_n$  admits the upper and lower barriers

$$\frac{1}{n} - C_1(b - \xi) \leq S_n \leq \frac{1}{n} + C_1(b - \xi), \quad \frac{1}{n} - C_0\xi \leq S_n \leq \frac{1}{n} + C_0\xi \quad \text{in } \overline{Q}$$

provided  $C_0, C_1$  are chosen sufficiently large. Dividing these inequalities by  $b - \xi$  or  $\xi$  respectively and letting then  $\xi \rightarrow b$ ,  $\xi \rightarrow 0$ , we get

$$-C_0 \leq S_{n\xi}(0, t) \leq C_0, \quad -C_1 \leq S_{n\xi}(b, t) \leq C_1 \quad \text{as } t \in [0, T].$$

REMARK 4. The uniform Lipschitz-continuity of the sequence  $\{w_n\}$  in  $\xi$  implies its uniform Hölder-continuity in  $t$  [13, 18]. Therefore, the limit function  $w$  belongs to  $C(\overline{Q})$ , at least.

LEMMA 2. *There exists a positive constant  $C$  not depending on  $n$  such that*

$$\frac{w_{nt}}{w_n} \geq -\frac{C}{t} \quad \text{in } \overline{Q}. \tag{25}$$

*Proof.* Dropping subindex, we derive from (15) the following equation for the function  $\sigma(\xi, t) = w_t(\xi, t)/w(\xi, t)$

$$\mathcal{B}\sigma \equiv \sigma_t - mw^{(m+1)/m}\sigma_{\xi\xi} - 2mw^{1/m}w_{\xi}\sigma_{\xi} + 2mw^{1/m}\sigma - \frac{m+1}{m}\sigma^2 = 0 \quad \text{in } Q.$$

Consider the function  $\omega(t) = -CT/t$ ,  $C = \text{const} > 0$ . By (16),  $\sigma > \omega$  at the parabolic boundary of  $Q$ . Next,

$$\mathcal{B}\omega = \frac{CT}{t^2} \left( 1 - 2mw^{1/m}t - \frac{m+1}{m}CT \right) < 0 \quad \text{in } Q,$$

provided

$$C > \frac{m}{(m+1)T},$$

and we have  $\sigma > \omega$  in  $\bar{Q}$  by [12, Ch.2, Th.16].

REMARK 5. Letting  $n \rightarrow \infty$  in (25) we get

$$w_t \geq -\frac{C}{t}w \quad \text{in } \mathcal{D}'(Q). \tag{26}$$

LEMMA 3.  $w(\xi, t)$  is strictly positive on  $(0, b) \times [0, T]$ .

*Proof.* Fix an arbitrary  $\xi_0 \in (0, b)$ . By continuity of  $w$  in  $\bar{Q}$  there always exists  $t_0 > 0$  such that  $w(\xi_0, t) > 0$  on  $[0, t_0]$ . Let us rewrite estimate (25) in the form  $(t^C w_n)_t \geq 0$  and then integrate over the interval  $(t_0, t)$ . Then

$$w_n(\xi_0, t) \geq \left(\frac{t_0}{t}\right)^C w_n(\xi_0, t_0)$$

and making  $n \rightarrow \infty$ , we complete the proof.

### 3.4. Solvability of (L) and the inverse coordinate transformation

LEMMA 4. The function  $v(\xi, t) = w^{1/m}(\xi, t)$  is the unique classical solution of problem (L). Moreover,  $w \in C(\bar{Q}) \cap C^\infty(Q)$ .

*Proof. Existence.* The proof repeats the arguments used for the proof of the same propositions in [11, 25]. Multiplying (15) by an arbitrary smooth  $f(\xi, t)$ , compactly supported in  $Q$ , and integrating by parts in  $Q$ , we obtain the integral identity for  $w_n$ . Estimates (22), (26), strict positivity of  $w$  in  $Q$ , and pointwise convergence of  $w_n$  to  $w$  allow one to pass to the limit as  $n \rightarrow \infty$  which leads to the integral identity for  $w$ . Thus,  $w$  is a weak solution to (15) in  $Q$ . Due to strict positivity of  $w$  in  $Q$ , in each subdomain  $Q'$  of  $Q$  separated from the parabolic boundary of  $Q$ ,  $w$  may be treated as a solution of a uniformly parabolic equation and its smoothness may be improved then by the “bootstrap” argument [14].

*Uniqueness.* Uniqueness of a strictly positive solution of problem (L) follows from [6], where it was stated for strictly positive solutions of problem (L) but with equation (10) replaced by

$$\left(\frac{1}{v}\right)_t + (v^m)_{\xi\xi} = 0.$$

Hence, the proof is complete.

To complete the justification of the passage to Lagrangian coordinates, we must still check that the solution obtained for (L) generates a solution of the original problem (1), (2).

Let  $X(\xi, t)$  be defined by (13). Introduce the function

$$X(\xi, t) = X(\xi, 0) - \int_0^t (w + \xi^2 - b\xi)_\xi d\theta \quad \text{on } Q \tag{27}$$

and set

$$\Omega(t) = \{x \in \mathbf{R} : x = X(\xi, t), \xi \in (0, b)\}.$$

Define

$$u(x, t) = \begin{cases} v(\xi, t) & \text{if } x \in \Omega(t), \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

It is easy to verify that  $u(x, t)$  satisfies all properties listed in items 1)–3) of Definition 1. As for item 4), it may be checked as follows. Let  $f(x, t)$  be an admissible test function from Definition 1. Let  $F(\xi, t) \equiv f(X(\xi, t), t)$ . Using (8), (9), (27) and Lemma 4, we have

$$\begin{aligned} 0 &= \int_Q \frac{d}{dt} F(\xi, t) d\xi dt = \int_Q (f_t + f_x X_t) d\xi dt = \int_Q (f_t + f_x W) d\xi dt \\ &= \int_S \left[ f_t + f_x \left( -\frac{m}{m-1} (u^{m-1})_x - \int_{-\infty}^x u(y, t) dy + \int_x^{\infty} u(y, t) dy \right) \right] u dx dt. \end{aligned}$$

We also see that by (23), (28) and Lemma 4,  $u$  is estimated as

$$u(x, t) \leq \left\{ \frac{b^2}{4} + \max_{[0, a]} u_0^m \right\}^{1/m} = K \quad \text{for } x \in \mathbf{R}, t \geq 0. \tag{29}$$

**3.5. A representation formula for the interface**

To prove (14) it suffices to show that  $X(\xi, t)$ , given by (27), is monotone in  $\xi$  for each  $t \in [0, T]$  fixed. Differentiating (27) in  $\xi$  and using (10), (12), (13), we have at once

$$X_\xi = \frac{1}{v(\xi, t)} - \int_0^t (v^m + \xi^2 - b\xi)_{\xi\xi} d\theta = \frac{1}{v(\xi, 0)} > 0.$$

*Global Lipschitz-continuity of the interface.* First, we show that the function  $W(\xi, t)$  for each  $t \in (0, T]$  has a finite limit as  $\xi \rightarrow b$ . By virtue of (26) this may be done by the standard arguments [11, 17]. By (24) we also have that  $|W(b, t)| \leq M_2$ . Hence, for all  $t, t + \Delta t \in (0, T]$

$$|\eta(t + \Delta t) - \eta(t)| = \left| \int_t^{t+\Delta t} W(b, \theta) d\theta \right| \leq M_2 |\Delta t|.$$

*The interface equation.* It follows from (27) that

$$X_t = -(v^m + \xi^2 - b\xi)_\xi \quad \text{in } Q.$$

Due to global Lipschitz-continuity, the right-hand side of this equation has a finite limit as  $\xi \rightarrow b$  for each  $t \in (0, T]$  fixed and so is  $X(\xi, t)$ . It follows that

$$\eta'(t) + \lim_{\xi \rightarrow b} (v^m + \xi^2 - b\xi)_\xi = 0 \quad \text{in } \mathcal{D}'(0, T).$$

4. Estimating the Time Derivative  $w_t$

4.1. A lower estimate for the spatial derivative

LEMMA 5. Assume  $A_1$ - $A_3$ ). Then there exists a constant  $L > 0$  such that

$$(v^m)_\xi(b, t) \leq -L \quad \text{for any } t \in (0, T]. \tag{30}$$

*Proof.* Let

$$\mu = \inf_{(0, T)} w \left( \frac{b}{2}, t \right) > 0.$$

Take a constant  $L > 0$  so small that  $1/n + Lb/2 < \mu$  (for  $n$  large enough) and

$$\frac{1}{n} + L(b - \xi) \leq w_{0n}(\xi) \quad \text{for any } \xi \in \left( \frac{b}{2}, b \right).$$

The function  $\omega_n(\xi) = 1/n + L(b - \xi)$  is a lower barrier for  $w_n$  in the closure of the rectangle  $D = Q \cap \{\xi > b/2\}$ . Hence,  $w_n \geq \omega_n$  in  $\bar{D}$ . Letting  $n \rightarrow \infty$  we get the inequality

$$w(\xi, t) \geq L(b - \xi) \quad \text{on } \bar{D},$$

whence (30) by using (11).

4.2. Preliminary estimate of the time derivative

The same arguments for the proof of Lemma 5 show that for all integer  $n$  large enough

$$w_{n\xi}(b, t) \leq -L \quad \text{for } t \in (0, T]$$

with the same constant  $L$ . These inequalities coupled with (24) show that in a neighborhood of the right-side lateral boundary of  $Q$   $w_n$  and  $w$  may be chosen as the new independent space variables.

Fix an integer  $n$  large enough. Following [7], let us introduce the new independent space variables  $\eta$  and the new searched function  $y_n(\eta, t)$  by the relations

$$\begin{cases} \eta + \frac{1}{n} = w_n(b - y_n(\eta, t), t), \\ y_n(\eta, t) = b - \xi. \end{cases} \tag{31}$$

By the same formulas, letting  $n \rightarrow \infty$ , we introduce the space variable  $\eta$  related to  $w$ . By (31)

$$w_{nt} = w_{n\xi}y_{nt}, \quad w_{n\xi} = -\frac{1}{y_{n\eta}}.$$

Substituting these expressions into (15), we deduce the equation for the function  $y_n$ :

$$y_{nt} = m \frac{\left(\eta + \frac{1}{n}\right)^{(m+1)/m}}{(y_{n\eta})^2} y_{n\eta\eta} - 2m y_{n\eta} \left(\eta + \frac{1}{n}\right)^{(m+1)/m}. \tag{32}$$



The same equation holds as  $n \rightarrow \infty$  for  $y(\eta, t)$ . We will consider equation (32) in the domain  $H \equiv (0, \eta_0) \times (t_0, t_0 + 1]$  with an arbitrary  $t_0 \in (0, T - 1)$  and  $\eta_0$  chosen by the unique condition:

$$\frac{1}{M_2} \leq y_{nn}(\eta, t) \leq \frac{2}{L} \quad \text{in } \overline{H}. \tag{33}$$

The first estimate on the functions  $w_{nt}$  or, equivalently,  $y_{nt}$  follows by a simple adaptation of the estimation procedure proposed in [4] in the treatment of the porous medium equation. As follows from (33), the rate of decreasing (or vanishing in the limit case) of the coefficient of the major derivative is already estimated from below and from above. Having derived the equation for  $y_{nn}$  from (32), it is possible to perform a local rescaling transformation in  $\eta$  and  $t$  to render this equation nondegenerate parabolic and refer to the results of the standard parabolic theory [19, Ch. 5, Th. 3.1], which gives  $y_{nn\eta} \sim \eta^{-1}$  and

$$|y_{nt}| \leq C' \eta^{1/m} \quad \text{in } H$$

with a constant  $C'$  depending only on  $m, M_2, L$ .

This estimate may be improved by means of the following result:

LEMMA 6. *For each  $t \in (t_0, t_0 + 1]$*

$$\eta^{-1/m} y_t(\eta, t) \rightarrow 0 \quad \text{as } \eta \rightarrow 0.$$

*Proof.* Fix some  $n$  and put  $S = y_{nt}$ . Differentiating (32) in  $t$ , we deduce the following equation for  $S$ :

$$S_t - m \left( \eta + \frac{1}{n} \right)^{(m+1)/m} \frac{\partial}{\partial \eta} \left[ \frac{1}{(y_{nn})^2} \frac{\partial S}{\partial \eta} - 2S \right] = 0 \quad \text{in } H. \tag{34}$$

Since  $y_{nt}(0, t) = 0$ , it follows from (34) that

$$\frac{\partial}{\partial \eta} \left[ \frac{1}{(y_{nn})^2} \frac{\partial S}{\partial \eta} - 2S \right] = 0 \quad \text{as } \eta = 0. \tag{35}$$

Consider the function

$$\rho = \frac{S_\eta}{(y_{nn})^2} - 2S.$$

With the use of the identity

$$\left( \frac{S_\eta}{(y_{nn})^2} \right)_t = \frac{S_{\eta t}}{(y_{nn})^2} - 2y_{nn} \left( \frac{S_\eta}{(y_{nn})^2} \right)^2$$

we obtain the following equation for  $\rho(\eta, t)$ : in  $H$

$$\begin{aligned} \mathcal{M}_n \rho \equiv \rho_t - \frac{m}{(y_{nn})^2} \frac{\partial}{\partial \eta} \left[ \left( \eta + \frac{1}{n} \right)^{(m+1)/m} \frac{\partial \rho}{\partial \eta} \right] + 2m \left( \eta + \frac{1}{n} \right)^{(m+1)/m} \frac{\partial \rho}{\partial \eta} \\ + 8y_{nn} \rho S + 2y_{nn} \rho^2 = -8y_{nn} S^2 < 0. \end{aligned}$$

Now take the function

$$\varphi(t) = \frac{K}{t - t_0}, \quad K = \text{const} > 0.$$

$K$  may be always chosen so large that  $\varphi(t) > \rho(\eta, t)$  at the down-face and the right-side lateral boundary of  $H$ . For the left-side one, it follows from (35) that  $(\varphi - \rho)_\eta = 0$  as  $\eta = 0$ . Using (33), we calculate

$$\mathcal{M}_n \varphi = \frac{8y_{n\eta}KS}{t - t_0} - \frac{K}{t - t_0} + 2y_{n\eta} \frac{K^2}{(t - t_0)^2} \geq \frac{K}{(t - t_0)^2} \left( \frac{K}{M_2} - 1 \right) > 0$$

in  $H$ , provided  $K > M_2$ . By [12, Ch.2, Th.16] we conclude that  $\varphi(t) > \rho(\eta, t)$  in  $H$ . Thus,

$$\frac{\partial}{\partial \eta}(y_{nt})(\eta, t) < \frac{K'}{t - t_0} \quad \text{on } H$$

for sufficiently large  $K'$  independent of  $n$ , and as  $n \rightarrow \infty$ ,

$$\frac{\partial}{\partial \eta}(y_t)(\eta, t) \leq \frac{K'}{t - t_0} \quad \text{in } H. \tag{36}$$

One may always assume that the constant  $K'$  in (36) is not less than the constant  $C$  in (26). Observe that in our choice of  $K'$  the function

$$y_t - \frac{K'\eta}{t - t_0}$$

is nonpositive and non-increasing in  $\eta$  for each  $t \in (t_0, t_0 + 1]$  fixed. We then have the following upper estimate

$$\eta^{-1/m} y_t(\eta, t) \leq \frac{K'}{t - t_0} \eta^{1-1/m} \quad \text{in } H. \tag{37}$$

Let us derive a lower estimate for  $y_t$  in  $H$ . First, let us remember that by the Cauchy criterion

$$\frac{1}{y_\eta(2s, t)} - \frac{1}{y_\eta(s, t)} \rightarrow 0 \quad \text{as } s \rightarrow 0.$$

Take an arbitrary  $s \in (0, \eta_0/2)$ . By virtue of (32) the following chain of relations holds:

$$\begin{aligned} & - \left[ \frac{1}{y_\eta(2s, t)} - \frac{1}{y_\eta(s, t)} \right] = \int_s^{2s} \frac{\partial}{\partial \eta} \left[ -\frac{1}{y_\eta(\eta, t)} \right] d\eta \\ & = \int_s^{2s} \frac{y_t}{m\eta^{(m+1)/m}} d\eta + 2 \int_s^{2s} y_\eta d\eta \\ & = \int_s^{2s} \frac{1}{m\eta^{(m+1)/m}} \left( y_t - \frac{K'\eta}{t - t_0} \right) d\eta + \frac{K'}{m(t - t_0)} \int_s^{2s} \frac{1}{\eta^{1/m}} d\eta + 2 \int_s^{2s} y_\eta d\eta \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

The function in the curved braces in the integrand of  $I_1$  is nonpositive and non-increasing on  $[s, 2s]$ . Therefore we may estimate  $I_1$  as follows:

$$\begin{aligned}
 I_1 &\leq \left( y_t(s, t) - \frac{K's}{t - t_0} \right) \int_s^{2s} \frac{1}{m\eta^{(m+1)/m}} d\eta = \left( y_t(s, t) - \frac{K's}{t - t_0} \right) \frac{2^{1/m} - 1}{(2s)^{1/m}} \\
 &< \frac{2^{1/m} - 1}{2^{1/m}} s^{-1/m} y_t(s, t).
 \end{aligned}$$

Also for  $I_2$  and  $I_3$ , they may be evaluated directly. Hence, we get

$$\frac{2^{1/m} - 1}{2^{1/m}} s^{-1/m} y_t(s, t) > - \left[ \frac{1}{y_\eta(2s, t)} - \frac{1}{y_\eta(s, t)} \right] - I_2 - I_3 \rightarrow 0 \quad \text{as } s \rightarrow 0$$

for each  $t \in (t_0, t_0 + 1]$ .

**4.3. An improved estimate for the time derivative**

The estimate of Lemma 6 allows us to adopt (in the new variables) the barrier construction proposed in a similar situation in [4].

LEMMA 7. *Given  $t_0 \in (0, T)$ , there exist positive constants  $C'$  and  $\eta' \in (0, \eta_0)$  such that*

$$|y_t(\eta, t)| \leq C' \frac{\eta}{t - t_0} \quad \text{on } [0, \eta'] \times (t_0, t_0 + 1]. \tag{38}$$

*Proof.* As above, we denote  $S = y_t$  on  $\bar{H}$ . Let us transform equation (34) for  $S$  to the following form:

$$\mathcal{N}S \equiv S_t - m\eta^{(m+1)/m} \frac{1}{(y_\eta)^2} S_{\eta\eta} + 6m\eta^{(m+1)/m} S_\eta + \frac{2}{y_\eta} S S_\eta = 0.$$

Following [4], let us construct for  $S$  a local lower barrier of the form

$$z(\eta, t) = -\gamma\eta^{1/m} - \varepsilon \frac{\eta}{\delta(t - t_0) + \eta^{(m-1)/m}},$$

where positive constants  $\gamma, \varepsilon$  and  $\delta$  are to be defined. Calculating  $\mathcal{N}z$ , we obtain

$$\begin{aligned}
 \mathcal{N}z &\leq \gamma\eta^{-1+2/m} \left\{ -\frac{m-1}{m} \frac{1}{(y_\eta)^2} + \frac{2\gamma}{m} \frac{1}{y_\eta} \right\} \\
 &\quad + \frac{\eta}{\{\delta(t - t_0) + \eta^{(m-1)/m}\}^2} \left\{ \varepsilon\delta + \varepsilon^2 \frac{2}{y_\eta} - \varepsilon \frac{m-1}{m} \frac{1}{(y_\eta)^2} \right\}.
 \end{aligned}$$

The first guess in the choice of the parameters  $\gamma, \varepsilon, \delta$  is to take them so small that  $\mathcal{N}z < 0$  in  $H$ . Next, for  $t = t_0$ ,  $\eta$  small enough, and with  $\varepsilon$  already fixed

$$z(\eta, t_0) = -(\gamma + \varepsilon)\eta^{1/m} \leq -\varepsilon\eta^{1/m} < y_t(\eta, t_0)$$

due to Lemma 6. We must still control the difference  $z - y_t$  at the lateral boundaries of  $(0, \eta') \times (t_0, t_0 + 1]$ . By Lemma 6 we conclude that for each  $\gamma > 0$  fixed  $z - y_t < 0$  as  $\eta \rightarrow 0$ . For  $\eta = \eta'$  we need the inequality

$$y_t(\eta', t) \geq -\gamma(\eta')^{1/m} - \varepsilon \frac{\eta'}{\delta(t - t_0) + (\eta')^{(m-1)/m}} \quad \text{for } t \in [t_0, t_0 + 1].$$

Set  $\delta = (\eta')^{(m-1)/m}$ . Now we make the second step of choosing  $\eta'$  taking it so that

$$-\frac{\varepsilon}{2}(\eta')^{1/m} \leq y_t(\eta', t) \quad \left( = o(\eta')^{1/m} \right).$$

Applying [12, Ch.2, Th.16] we conclude that  $z < y_t$  on  $[0, \eta'] \times [t_0, t_0 + 1]$  for all  $\gamma > 0$  small enough. Letting  $\gamma \rightarrow 0$ , we have

$$y_t(\eta, t) \geq -\frac{\varepsilon}{\delta} \frac{\eta}{t - t_0}.$$

Hence, we have the lower estimate. The upper estimate for  $y_t$  is already stated by (37).

Since  $t_0$  was taken arbitrarily, we will use estimate (38) in the following formulation.

**COROLLARY 1.** *Given an arbitrary  $t' \in (0, T)$ , there exist constants  $C(t') > 0$  and  $\xi' \in (0, b)$  such that*

$$|w_t(\xi, t)| \leq C(t')w(\xi, t) \tag{39}$$

*in the closure of the rectangle  $E(t') = (\xi', b) \times (2t', T)$ .*

### 5. Regularity of the Interface

#### 5.1. $C^{1,\alpha}$ -regularity: proof of Theorem 1

Set  $S(\xi, t) = w(\xi, t) + \xi^2 - b\xi$ . Due to (14), it will be enough to prove that  $S_\xi(b, \cdot) \in C^\alpha(\tau, \tau + 1)$  for each  $\tau \in (0, T - 1]$ . Denote  $\sigma = S_\xi$ . For  $\sigma$  and  $S$  we have, respectively,

$$S_t - mw^{(m+1)/m}S_{\xi\xi} = 0, \quad \mathcal{P}\sigma \equiv \sigma_t - m \frac{\partial}{\partial \xi} \left( w^{(m+1)/m} \frac{\partial \sigma}{\partial \xi} \right) = 0 \quad \text{in } Q.$$

Given some  $\tau > 0$ , let us consider the domain  $E(\tau/2)$ . The results of the previous sections imply the following properties of  $\sigma$  in  $E(\tau/2)$ :

$$|\sigma(\xi, t) - \sigma(b, t)| \leq \int_\xi^b \frac{|S_t|}{mw^{(m+1)/m}} d\xi \leq N(b - \xi)^{(m-1)/m},$$

$$N = \text{const} > 0, \quad \text{use (39);}$$

$\sigma(\xi, t)$  is uniformly bounded by a constant  $L_0$  and  $\sigma_\xi(\xi, t) = O((b - \xi)^{-1/m})$  as  $\xi \rightarrow b$ .

All the constants appearing in these relations are finite and depend only on  $\tau$ . In the domain  $G(\rho) = (b - \rho, b) \times (\tau, \tau + 1]$ , let us consider the functions

$$U^\pm(\xi, t) = \pm\{\sigma(\xi, t) - \sigma(b, t_0)\} + L_1 \left\{ 4\rho^{(m-1)/m} - (b - \xi)^{(m-1)/m} \right\} + \frac{L_2}{\rho^2} \{L_3(t - t_0) + (b - \xi)^2\}$$

depending on positive constants  $L_1, L_2, L_3, \rho$ . Choosing  $L_1, L_2, L_3$  large enough and taking into account the above-listed properties of  $\sigma$ , one may easily check the following inequalities:

$$U^\pm(b - \rho, t) \geq -2L_0 + 3L_1\rho^{(m-1)/m} + L_2L_3\frac{t - t_0}{\rho^2} + L_2 > 0,$$

$$U^\pm(\xi, t) \geq -N(b - \xi)^{(m-1)/m} + 3L_1(b - \xi)^{(m-1)/m} + L_2\frac{(b - \xi)^2}{\rho^2} > 0,$$

$$\frac{\partial U^\pm}{\partial \xi} \geq \left(-N + \frac{m-1}{m}L_1\right)(b - \xi)^{-1/m} - 2L_2\frac{b - \xi}{\rho^2} > 0 \quad (\rightarrow \infty \text{ as } \xi \rightarrow b),$$

$$\begin{aligned} \mathcal{P}U^\pm &= \frac{L_2L_3}{\rho^2} - (m+1)w^{1/m}w_\xi \left\{ \frac{m-1}{m}L_1(b - \xi)^{-1/m} - 2L_2\frac{b - \xi}{\rho^2} \right\} \\ &\quad - w^{1/m}w_\xi \frac{m-1}{m}L_1(b - \xi)^{-(m+1)/m} - 2mw^{(m+1)/m}\frac{L_2}{\rho^2} > 0. \end{aligned}$$

These relations hold for an arbitrary  $\rho$  small enough. Referring to [12, Ch.2, Th.16] we conclude now that  $U^\pm(\xi, t) > 0$  in  $\overline{G(\rho)}$ . Letting  $\xi \rightarrow b$ , we then have

$$|\sigma(b, t_1) - \sigma(b, t_0)| \leq 4L_1\rho^{(m-1)/m} + L_2L_3\frac{t_1 - t_0}{\rho^2} \quad \text{for } t_1 > t_0. \tag{40}$$

Notice that: *inequality (40) holds regardless the concrete choice of  $\rho$* . Making use of this, we will choose  $\rho$  in a special way. Namely, assuming  $\Delta t = t_1 - t_0$  be small enough, we set

$$\rho = (\Delta t)^{m/(3m-1)}.$$

Then (40) takes the form

$$|\sigma(b, t_1) - \sigma(b, t_0)| \leq R(\Delta t)^{(m-1)/(3m-1)}, \quad R = \text{const} > 0.$$

Now recall that all constants in the previous proceeding did not depend on the concrete values of  $t_1, t_0$  and, correspondingly,  $\rho$  but are the same for all  $G(b - \xi')$ . That is why the previous arguments may be repeated in the case when the value of  $t_1$ , defining the top-face of the rectangle where (40) holds, is fixed and  $t_0$  is free.

**5.2.  $C^\infty$ -regularity: proof of Theorem 2**

Having proved  $C^{1,\alpha}$ -regularity of the interface, we may improve this result reducing the problem to those already solved for the porous medium equation.

Let us consider the Cauchy problem for (PME) with the initial function  $u_0$ , subject to conditions  $A_1$ – $A_3$ ). It is easy to see that the same arguments, i.e., the passage to Lagrangian coordinates, the proof of the unique solvability of the corresponding stable-boundary problem, the proof of equivalence between the original and Lagrangian statements, and the study of the interface regularity, are true. (See also [27] for the direct proof of this equivalence.) The interface in the Cauchy problem for (PME) has the representation formula

$$\eta(t) = a - \int_0^t (v^m)_\xi(b, \theta) d\theta,$$

where  $v$  is a solution of the equation

$$\left(\frac{1}{v}\right)_t + (v^m)_{\xi\xi} = 0 \quad \text{in } Q$$

satisfying the boundary and initial conditions (11), (12).

It is shown in [4] that in the case of (PME)  $\eta \in C^\infty(\tau, T)$ ,  $\tau > 0$ . This result holds for the monotone moving part of the interface, i.e., under  $A_3$ ), for the whole interface. The result of [4], being written down in the variables  $(\xi, t)$ , is the following: let

- (i)  $\frac{1}{C}(b - \xi) \leq v^m(\xi, t) \leq C(b - \xi)$ ,  $C = \text{const} > 0$ , in some rectangle  $E(\tau)$ ,  $\tau > 0$ ;
- (ii)  $|(v^m)_t(\xi, t)| \leq C_1 v^m(\xi, t)$  in  $\overline{E(\tau)}$ .

Then the functions  $D_t^s((v^m)_\xi)$ ,  $s = 1, 2, \dots$ , have finite limits as  $\xi \rightarrow b$ . This conclusion is made by the consideration of parabolic equations obtained for  $D_t^s((v^m)_\xi)$  from (PME). The structure of their coefficients and the estimates of (i) allow one to pick up a special rescaling transformation rendering (locally) these equations into uniform parabolic ones; thereafter one has to have recourse to the results of the classical parabolic theory [19, Ch. 5, Th. 3.1], which provide the requested estimates.

Let us now write down another Lagrangian analog of the Cauchy problem (1), (2). Eliminating  $v(\xi, t)$  from (7) and (8), we obtain the following conditions for  $X(\xi, t)$ :

$$X_t = m \frac{1}{X_\xi^{m+1}} X_{\xi\xi} - 2\xi + b \quad \text{in } Q.$$

A solution to this problem is given by the constructed solution to (L). Formally, the function  $\sigma = X_t$  satisfies the equation

$$\sigma_t = m \frac{1}{X_\xi^{m+1}} \sigma_{\xi\xi} - m(m + 1) \frac{X_{\xi\xi}}{X_\xi^{m+2}} \sigma_\xi.$$

The same equation with the coefficients possessing the same properties holds for the function  $\overline{X}_t$ , if we denote by  $\overline{X}$  the function standing for the trajectory of the particle in the Lagrangian “twin” of the Cauchy problem for (PME). This coincidence proves that the differential properties of the functions  $X$  and  $\overline{X}$  must be the same.

**6. Generalization to Equation (1')**

Here we point out how all the preceding analysis may be carried out in the case of general equation (1'). Following the above-described scheme of Lagrangian coordinate introduction we arrive at the equation

$$\left(\frac{1}{v}\right)_t + (v^m)_{\xi\xi} + \Phi'(\xi) = 0 \quad \text{in } Q. \tag{41}$$

So, the only difference between the above-considered case of linear function  $\Phi$  and the general case is that now one has to deal with equation (41), having the coefficient of the minor term not constant but bounded and positive (see  $N_3$ ). This provides the validity of all the steps of the former arguments, which rely only on the maximum principle. Passing to a sequence of solutions  $v_n$  of regularized problems, corresponding to (41), and denoting  $w_n = v_n^m$  (cf. (15)–(17)), we then introduce the functions

$$S_n = w_n + \int_0^\xi \Phi(s) ds,$$

which satisfy equations (19), the boundary conditions (20), and the initial conditions related to (21). It is easy to see that if  $\Phi$  is assumed to be smooth enough, then one may literally repeat the proofs of all propositions of Sections 1–5. As a conclusion we obtain

**THEOREM 5.** *Let conditions  $A_1$ )– $A_3$ ) hold. Assume that  $\Phi \in C^\infty$  and satisfies  $N_3$ ). Then*

$$\eta \in C_{loc}^\infty(0, T).$$

**7. On the Initial Behavior of the Interface**

Let  $u(x, t)$  be the solution of problem (1), (2). Define

$$z(x, t) = \int_x^\infty u(y, t) dy.$$

It is easy to see that  $z$  is a solution of the problem

$$\begin{cases} z_t = (|z_x|^{m-1} z_x)_x + \{(b - z)z\}_x & \text{in } S, \\ z(-\infty, t) = b, \quad z(+\infty, t) = 0 & \text{as } t \in [0, T], \\ z_x(x, t) \leq 0 & \text{in } S, \\ z(x, 0) = z_0(x) & \text{on } \mathbf{R}, \end{cases}$$

where

$$z_0(x) = \int_x^\infty u_0(y, t) dy \quad \text{and} \quad b = \int_{-\infty}^\infty u_0(y) dy.$$

The function  $z(x, t)$  also satisfies

$$z(x, t) = b \quad (x \leq \zeta(t)), \quad 0 < z(x, t) < b \quad (\zeta(t) < x < \eta(t)), \quad z(x, t) = 0 \quad (x \geq \eta(t))$$

for  $t \in [0, T]$ .

Define the function  $F(\omega; b)$  by

$$F(\omega; b) = \int_0^\omega \sigma^{-1/m} (b - \sigma)^{-1/m} d\sigma \quad \text{for} \quad 0 \leq \omega \leq b,$$

and put

$$L(b) = \int_0^b \sigma^{-1/m} (b - \sigma)^{-1/m} d\sigma.$$

Using the inverse function  $F^{-1}(x; b)$  of  $F(\omega; b)$ , we define the function  $\omega(x; b)$  on  $\mathbf{R}$  by

$$\omega(x; b) = \begin{cases} b & \text{if } x \leq a - L(b), \\ F^{-1}(a - x; b) & \text{if } a - L(b) < x < a, \\ 0 & \text{if } x \geq a. \end{cases}$$

The function  $\omega(x; b)$  is a solution of the problem

$$\begin{cases} 0 = (|\omega'|^{m-1} \omega')' + \{(b - \omega)\omega\}' & \text{in } \mathbf{R}, \\ \omega(-\infty) = b, \quad \omega(+\infty) = 0, \\ \omega'(x) \leq 0 & \text{on } \mathbf{R} \end{cases}$$

and is normalized by the condition  $\omega(a; b) = 0$ . It follows from the construction of  $\omega(x; b)$  that if  $b_1 < b_2$ , then

$$\omega(x; b_1) < \omega(x; b_2) \quad \text{for } x < a.$$

LEMMA 8.  $\lim_{x \rightarrow a-0} (a - x)^{-\ell} \omega(x; b) = \ell(b)$ .

*Proof.* Note that

$$\lim_{x \rightarrow a-0} \frac{\omega(x; b)}{(a - x)^\ell} = \lim_{\omega \rightarrow +0} \left\{ \frac{\omega^{1/\ell}}{F(\omega; b)} \right\}^\ell.$$

We then have

$$\lim_{\omega \rightarrow +0} \frac{\omega^{1/\ell}}{F(\omega; b)} = \frac{1}{\ell} \lim_{\omega \rightarrow +0} \frac{\omega^{-1/m}}{\omega^{-1/m} (b - \omega)^{-1/m}} = \frac{1}{\ell} b^{1/m},$$

which completes the proof.



### 7.1. Proof of Theorem 3

Assume that  $\limsup_{x \rightarrow a-0} (a-x)^{-\ell} z_0(x) < \ell(b)$ . By Lemma 8,

$$\limsup_{x \rightarrow a-0} \frac{z_0(x)}{\omega(x; b)} < 1,$$

which implies that

$$z_0(x) < K\omega(x; b) \quad (a_0 \leq x < a)$$

for some  $0 < K < 1$  and  $a_0 < a$ . Choosing  $b_1$  such that

$$0 < b_1 < b, \quad K\ell(b) < \ell(b_1),$$

we have

$$\lim_{x \rightarrow a-0} \frac{K\omega(x; b)}{\omega(x; b_1)} = \frac{K\ell(b)}{\ell(b_1)} < 1.$$

From this relation,

$$K\omega(x; b) < \omega(x; b_1) \quad (a_1 \leq x < a)$$

for some  $a_0 \leq a_1 < a$ . Hence,

$$z_0(x) < \omega(x; b_1) \quad (a_1 \leq x < a).$$

Having defined  $c = b - b_1 > 0$ , let us consider the function

$$q(x, t) = \omega(x + ct; b_1),$$

which satisfies

$$\begin{cases} q_t = (|q_x|^{m-1} q_x)_x + \{(b-q)q\}_x & \text{in } (a_1, \infty) \times (0, T], \\ z(\infty, t) = q(\infty, t) = 0 & \text{for } t \in [0, T], \\ z(x, 0) = z_0(x) \leq q(x, 0) & \text{on } [a_1, \infty). \end{cases}$$

By  $z(a_1, 0) < q(a_1, 0)$ , there exists  $t_0 \in (0, T]$  such that

$$z(a_1, t) < q(a_1, t) \quad (0 \leq t \leq t_0).$$

Hence, by the comparison theorem we obtain

$$z(x, t) \leq q(x, t) \quad (x \geq a_1, 0 \leq t \leq t_0),$$

which implies

$$\eta(t) \leq a - ct \quad (0 \leq t \leq t_0).$$

Under the condition  $\liminf_{x \rightarrow a-0} (a-x)^{-\ell} z_0(x) > \ell(b)$ , by using similar arguments  $b_1$  is taken as  $b_1 > b$  and it is shown that  $\eta(t) \geq a + ct$  ( $0 \leq t \leq t_0$ ), where  $c = b_1 - b > 0$ .

**7.2. Straight interface**

We first compare the interface  $\eta(t)$  to the right interface  $\xi(t)$  of the solution  $p(x, t)$  to the Cauchy problem for (PME) with  $p(\cdot, 0) = u_0$ . There holds that  $\xi(t) \geq a$  for  $t \geq 0$ .

LEMMA 9.  $\eta(t) \geq \xi(t) - bt$  for  $0 \leq t \leq T$ .

*Proof.* The function  $q(x, t) = \int_x^\infty p(y, t)dy$  satisfies

$$\begin{cases} q_t - (|q_x|^{m-1}q_x)_x = 0 & \text{in } S, \\ q(-\infty, t) = b, \quad q(\infty, t) = 0 & \text{for } t \in [0, T], \\ q(x, 0) = z_0(x) & \text{on } \mathbf{R}. \end{cases}$$

Let us define the function  $\varphi(x, t)$  by

$$\varphi(x, t) = z(x - bt, t),$$

which satisfies

$$\begin{cases} \varphi_t - (|\varphi_x|^{m-1}\varphi_x)_x = -2\varphi\varphi_x \geq 0 & \text{in } S, \\ \varphi(-\infty, t) = b, \quad \varphi(\infty, t) = 0 & \text{for } t \in [0, T], \\ \varphi(x, 0) = z_0(x) & \text{on } \mathbf{R}. \end{cases}$$

The comparison theorem concludes that

$$q(x, t) \leq \varphi(x, t) \quad \text{on } \bar{S}.$$

Noting that

$$q(x, t) > 0 \quad (x < \xi(t)), \quad q(x, t) = 0 \quad (x \geq \xi(t)),$$

we have

$$\xi(t) \leq \eta(t) + bt \quad (t \geq 0),$$

which completes the proof.

The next lemma shows that  $\eta(t) \leq a - bt$  for small time under condition (5).

LEMMA 10. *Assume condition (5). Then there exists  $t_0 > 0$  such that*

$$\eta(t) \leq a - bt \quad \text{for } 0 \leq t \leq t_0.$$

*Proof.* The function  $\varphi(x, t) = z(x - bt, t)$  satisfies

$$\begin{cases} \mathcal{L}\varphi \equiv \varphi_t - (|\varphi_x|^{m-1}\varphi_x)_x + (\varphi^2)_x = 0 & \text{in } S, \\ \varphi(-\infty, t) = b, \quad \varphi(\infty, t) = 0 & \text{for } t \in [0, T], \\ \varphi(x, 0) = z_0(x) & \text{on } \mathbf{R}. \end{cases}$$

Let us choose  $M > 0$  satisfying

$$\limsup_{x \rightarrow a-0} (a-x)^{-(m+1)/(m-1)} z_0(x) < M < +\infty.$$

Then there exists  $a_0 < a$  such that

$$\varphi(x, 0) < M(a-x)^{(m+1)/(m-1)} \quad \text{for } a_0 \leq x < a.$$

We define the function  $q(x, t)$  on  $[a_0, \infty) \times [0, \tau)$  by

$$q(x, t) = \begin{cases} M(a-x)^{(m+1)/(m-1)} \left(\frac{\tau}{\tau-t}\right)^{1/(m-1)} & \text{if } a_0 \leq x < a, \\ 0 & \text{if } x > a, \end{cases}$$

where a positive number  $\tau$  will be suitably chosen below. Let  $a_0 < x < a$ . A direct computation gives

$$\begin{aligned} \mathcal{L}q &= \frac{M}{m-1} (a-x)^{(m+1)/(m-1)} \left(\frac{\tau}{\tau-t}\right)^{m/(m-1)} \\ &\times \left\{ \frac{1}{\tau} - 2m \left(\frac{m+1}{m-1}\right)^m M^{m-1} - 2(m+1)M^2 (a-x)^{2/(m-1)} \left(\frac{\tau}{\tau-t}\right)^{(2-m)/(m-1)} \right\}. \end{aligned}$$

Note that

$$\left(\frac{\tau}{\tau-t}\right)^{(2-m)/(m-1)} \leq 2^{2/(m-1)} \quad \text{for } 0 \leq t \leq \frac{\tau}{2}.$$

Taking  $\tau$  and  $a - a_0$  small enough, we have

$$\mathcal{L}q > 0 \quad \text{in } (a_0, a) \times \left(0, \frac{\tau}{2}\right].$$

Hence,

$$\mathcal{L}q \geq 0 \quad \text{in } (a_0, \infty) \times \left(0, \frac{\tau}{2}\right].$$

By  $\varphi(a_0, 0) < q(a_0, 0)$ , there exists  $t_0 \in (0, \tau/2]$  such that

$$\varphi(a_0, t) < q(a_0, t) \quad \text{for } 0 \leq t \leq t_0.$$

Therefore,  $q(x, t)$  satisfies

$$\begin{cases} \mathcal{L}q \geq 0 & \text{in } (a_0, \infty) \times (0, t_0], \\ \varphi(a_0, t) < q(a_0, t), \quad \varphi(\infty, t) = q(\infty, t) = 0 & \text{for } t \in [0, t_0], \\ \varphi(x, 0) \leq q(x, 0) & \text{on } [a_0, \infty). \end{cases}$$

The comparison theorem concludes that

$$\varphi(x, t) \leq q(x, t) \quad \text{on } [a_0, \infty) \times [0, t_0],$$

which implies

$$\eta(t) + bt \leq a \quad \text{for } t \in [0, t_0].$$

Thus the proof is complete.

We are now in a position to prove Theorem 4.

*Proof of Theorem 4.* We begin with proving the assertions 1) and 3). By the definition of  $\alpha$ ,

$$\alpha \leq \zeta(t) \leq \eta(t) \quad \text{for } t \geq 0.$$

At the interval  $[0, t^*]$  this estimate may be specified: by the definition of  $t^*$

$$\alpha \leq \zeta(t) \leq \eta(t) = a - bt.$$

Also, from (3) and (29) we derive

$$b = \int_{\zeta(t)}^{\eta(t)} u(x, t) dx \leq K(\eta(t) - \zeta(t)) \leq K(\eta(t) - \alpha) \leq K(a - bt - \alpha) \quad (42)$$

for  $t \in [0, t^*]$ . It follows from (42) that the assertions 1) and 3) hold.

Let us prove the second assertion. For the first thing, we recall a result on the existence of the waiting-time effect for interfaces occurring in solutions of the porous medium equation ([28]). Let  $p(x, t)$  be a solution of the Cauchy problem for (PME) with  $p(\cdot, 0) = u_0$  and  $\xi(t)$  be the right interface. The following assertion is true: condition (5) is necessary and sufficient for the existence of  $t_1 > 0$  such that

$$\xi(t) = a \quad \text{for } t \in [0, t_1]. \quad (43)$$

Assume condition (5). Then it follows from Lemmas 9 and 10 and (43) that

$$a - bt \leq \eta(t) \leq a - bt \quad \text{for } 0 \leq t \leq \min\{t_0, t_1\},$$

which implies  $t^* > 0$ . We next assume

$$\limsup_{x \rightarrow a-0} (a - x)^{-(m+1)/(m-1)} \int_x^\infty u_0(y) dy = +\infty.$$

Then,  $\xi(t) > a$  for  $t > 0$ . Hence, by Lemma 9 we have

$$\eta(t) > a - bt \quad \text{for } t > 0,$$

which implies  $t^* = 0$ .

### 8. Numerical Experiments

We use a linearly implicit finite difference scheme to discretize the problem

$$\begin{cases} w_t = mw^{1+1/m}(w_{\xi\xi} + 2), & 0 < \xi < b, \ t > 0, \\ w(0, t) = w(b, t) = 0, & t \geq 0, \\ w(\xi, 0) = u_0^m(X(\xi, 0)), & 0 \leq \xi \leq b, \end{cases}$$

where  $X(\xi, 0)$  is given by the relation (13). This problem follows from the problem (10)-(12) by putting  $w(\xi, t) = v^m(\xi, t)$ . The numerical right interface is then obtained by discretizing the relation about the right interface  $\eta(t)$

$$\eta(t) = X(b, 0) - \int_0^t \{w_\xi(b, s) + b\} ds.$$

This approach was adopted by [6] to construct the numerical interfaces for the porous media equation, and the interfaces are tracked quite accurately.

To discuss the initial behavior of  $\eta(t)$  numerically, we take  $m = 2$  and the initial function  $u_0$  as

$$u_0(x) = \begin{cases} \frac{p+1}{\pi} \left(1 - \frac{2}{\pi} \left|x - \frac{\pi}{2}\right|\right)^p & \text{if } 0 \leq x \leq \pi, \\ 0 & \text{otherwise,} \end{cases} \tag{44}$$

where  $p > 0$ . In this case,

$$b \equiv \int_{-\infty}^{\infty} u_0(x) dx = 1, \quad X(b, 0) = \pi, \quad \ell = 2, \quad \ell(b) = \frac{1}{4}.$$

It is easy to see that

$$\lim_{x \rightarrow \pi-0} (\pi - x)^{-\ell} \int_x^{\infty} u_0(y) dy > \ell(b)$$

$$p = 1.9, 1.2, 0.6, 0.2$$

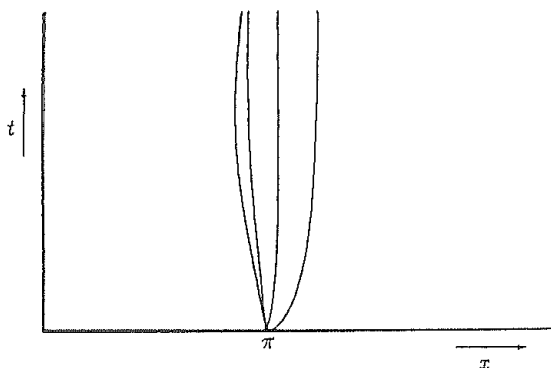


Fig. 1. The numerical right interfaces for  $p = 0.2, 0.6, 1.2, 1.9$ .

if and only if  $0 < p < 1$ , and that the reverse inequality holds if and only if  $p \geq 1$ . Figure 1 shows that the right interfaces  $\eta(t)$  for  $p = 0.2, 0.6$  (resp.  $p = 1.2, 1.9$ ) move to the right (resp. to the left) initially, and that the interfaces are not straight. This result is consistent with the conclusions of Theorem 3.

Concerning the positivity of  $t^*$ , it holds that  $t^* > 0$  if and only if  $p \geq 2$  (see Theorem 4). In Fig. 2(a) it is observed that the right interfaces  $\eta(t)$  for  $p = 3, 4, 5$  are straight initially with same slope  $b = 1$  from the definition of  $t^*$ .

Figure 2(b) shows the waiting time  $t_*$  for the right interface  $\xi(t)$  of the porous media equation. For  $t_*$  it holds that  $\xi(t) = \pi$  ( $0 \leq t \leq t_*$ ) and  $\xi(t) > \pi$  ( $t > t_*$ ). In Fig. 2 we observe a similar property between  $t^*$  and  $t_*$  such that both  $t^*$  and  $t_*$  decrease with increasing  $p$ . In the case of  $p > 2$ , we are unaware of any results for evaluating  $t_*$ , but it is shown in [28] that  $t_*$  is estimated as follows:

$$\frac{1}{36B} \leq t_* \leq \frac{1}{9B}, \quad B = \sup_{x < \pi} (\pi - x)^{-3} \int_x^\infty u_0(y) dy. \tag{45}$$

Denote  $B$  by  $B_p$  for  $p$  in (44). It is seen numerically that the relation

$$B_3 < B_4 < B_5 \tag{46}$$

holds. By taking into account (45), the relation (46) suggests that both  $t^*$  and  $t_*$  decrease with increasing  $p$ .

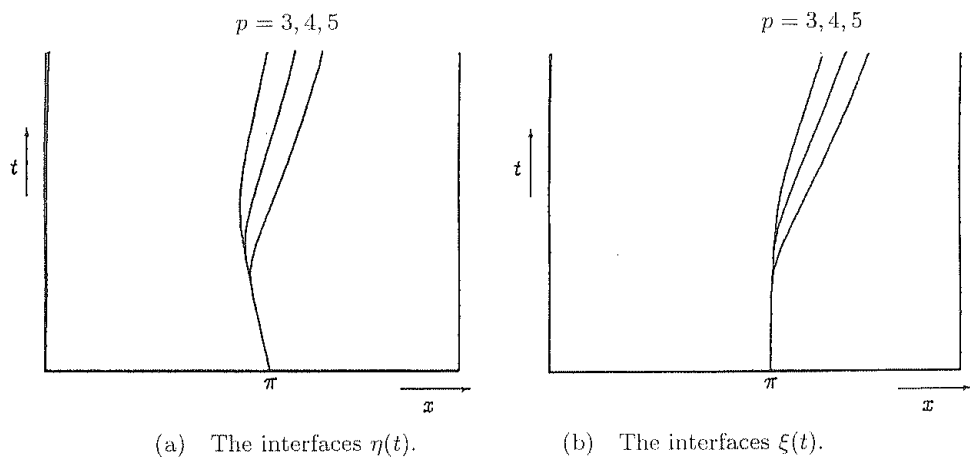


Fig. 2. The numerical right interfaces for  $p = 3, 4, 5$ .

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