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On a Nonlocal Stationary Free-Boundary Problem Arising in the Confinement of a Plasma in a Stellarator Geometry

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Abstract

We prove the existence and some qualitative properties of the solution to a two-dimensional free-boundary problem modeling the magnetic confinement of a plasma in a Stellarator configuration. The nonlinear elliptic partial differential equation on the plasma region was obtained from the three-dimensional magnetohydrodynamic system by HENDER & CARRERAS in 1984 by using averaging arguments and a suitable system of coordinates (Boozer's vacuum coordinates). The free boundary represents the separation between the plasma and vacuum regions, and the model is described by an inverse-type problem (some nonlinear terms of the equation are unknown). Using the zero net current condition for the Stellarator configurations, we reformulate the problem with the help of the notion of relative rearrangement, leading to a new problem involving nonlocal terms in the equation. We use an iterative algorithm and establish some new properties on the relative rearrangement in order to prove the convergence of the algorithm and then the existence of a solution.

Contents

1. Introduction. Statement of the main result

This paper deals with the mathematical treatment of a two-dimensional free-boundary problem modeling the magnetic confinement of a plasma in a Stellarator. The derivation of the model from the ideal magnetohydrodynamics static system is presented in Section 2. In contrast with Tokamak devices, Stellarator geometries are not axisymmetric. The magnetohydrodynamic equilibrium state depends on the toroidal angle, and thus the nested magnetic surfaces are very complex because of the three-dimensional character of these configurations. Nevertheless, certain types of Stellarator geometries lead to bidimensional problems by the method of averaging. The model under consideration is obtained through the averaging results by HENDER & CARRERAS [HC]. They used a special *inverse* coordinate system (ρ, θ, ϕ) (the Boozer vacuum coordinates system; see BOOZER [Bo]) where $\rho = \rho(x, y, z)$ is a function which is constant on each nested toroid ($\rho > 0$ except for the magnetic axis where $\rho = 0$), $\theta = \theta(x, y, z)$ is the poloidal angle (i.e., θ is constant on any toroidal loop) and $\phi = \phi(x, y, z)$ is the toroidal angle (i.e., constant on any poloidal circuit). Boozer's coordinates are constructed so that the vacuum magnetic field lines are straight in the (θ, ϕ)-plane. By averaging in ϕ and adding the free-boundary formulation to the Grad-Shafranov type equation obtained by HENDER & CARRERAS, DÍAZ [D1] formulated the problem in the following terms: Let $\Omega = \{(\rho, \theta)\colon 0 < \rho < R, \theta \in (0, 2\pi)\}$ and define $\partial\Omega = \Gamma_R \cup \Gamma_p \cup \Gamma_0$ by

$$\Gamma_R = \{(R, \theta)\colon \theta \in (0, 2\pi)\},\ \Gamma_p = \{(\rho, 0) \text{ or } (\rho, 2\pi)\colon \rho \in (0, R)\},\ \Gamma_0 = \{(0, \theta)\colon \theta \in (0, 2\pi)\}.$$

Given $\lambda > 0$, $F_v > 0$, $a, b \in L^\infty(\Omega)$ with $b > 0$ in Ω and $\gamma \in \mathbb{R}$, find

$$u\colon \Omega \to \mathbb{R}, \quad F\colon \mathbb{R} \to \mathbb{R}_+$$

such that $F \in W^{1,\infty}(\mathbb{R})$, $F(s) = F_v$ for all $s \leqq 0$ and (u, F) satisfy

$$-\mathscr{L}u = a(\rho, \theta)F(u) + F(u)F'(u) + \lambda b(\rho, \theta)u_+ \quad \text{in } \Omega, \tag{1}$$

$$(\mathscr{P}_1) \qquad u|_{\Gamma_R} = \gamma, \quad u(\rho, 0) = u(\rho, 2\pi) \ \text{for } \rho \in (0, R), \quad \frac{\partial u}{\partial \theta} = 0 \ \text{on } \Gamma_0, \tag{2}$$

$$\int_{\{u \geqq t\}} [F(u)F'(u) + \lambda b(\rho, \theta)u_+]\,\rho\, d\rho\, d\theta = 0 \ \ \forall t \in [\inf_\Omega u, \sup_\Omega u], \tag{3}$$

where $\mathscr{L}$ is a suitable elliptic second-order differential operator with coefficients depending on ρ and θ and where $u_+ := \max(u, 0)$.

First of all, let us mention the main differences between $(\mathscr{P}_1)$ and the model considered in the mathematical literature on the study of the confinement of a plasma of Tokamak devices (see e.g. TEMAM [T1, T2], BERESTYCKI & BREZIS [BB], BLUM [B], FRIEDMAN [Fri], MOSSINO & TEMAM [MT], RAKOTOSON [R3] and their references). Due to the axisymmetry of the geometry, the unknown u (the magnetic flux) in the Tokamak case may be written as a direct function of the standard cylindrical coordinates system and so the operator $\mathscr{L}$ is the usual Laplacian operator ($\mathscr{L} = \Delta$). A more important factor seems to be the difference between the additional condition (3) (expressing the Stellarator condition of zero net current within each flux magnetic surface) and the Tokamak condition of positive total current

$$\int_{\Omega} [F(u)F'(u) + \lambda b u_+] dx = I$$

for a prescribed $I > 0$. Due to this fact, in the Tokamak case it does not seem possible to determine the function F unless we have some extra information, such as the value of the normal derivative of u at $\partial\Omega$ (see Beretta & Vogelius [BV] and its references). Then the mathematical model for Tokamaks assumes an equation of state for F similar to the equation of state for the pressure. It is usually assumed that $F(u)F'(u) + \lambda b(x)u_+$ can be written as $\mu c(x)u_+$ for some $\mu \in \mathbb{R}_+$ and $c \in L^\infty(\Omega)$ with $c > 0$ in Ω. The coefficient a in (1) is intrinsic to Stellarator configurations, and so it does not appear ($a \equiv 0$) in the Tokamak model. We point out that terms of the form $aF(u)$ appear very often in models of the Stellarator case (even if (ρ, θ) are taken in different ways; see, e.g., Green & Johnson [GJ]). In conclusion, Stellarators lead to inverse-type models, such as $(\mathscr{P}_1)$, with F unknown, while in the Tokamak case, the final model corresponds formally to $F \equiv 0$. Finally, due to the choice of the inverse coordinates (ρ, θ), the constant γ appearing in (2) can be assumed to be known, which is not the case for the Tokamak model.

The main aim of this article is to study the existence of solutions to problem $(\mathscr{P}_1)$. A special statement of our existence result is:

Theorem 1. *Suppose that* $\operatorname{ess\,inf}_\Omega |a| > 0$, $\gamma \leqq 0$. *Then there exists* $\Lambda > 0$ *such that if* $\lambda |b|_\infty < \Lambda$, *then there is a pair* (u, F) *satisfying* $(\mathscr{P}_1)$ *and*

(i) $u \in W^{1,\infty}(\Omega)$ *with* $\operatorname{meas}\{(\rho, \theta) \in \Omega : \frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial \theta} = 0\} = 0$,

(ii) F *is entirely determined by* u, $F(t) > 0$ *for* $t \in [\hat{m} := \inf_\Omega u, M := \sup_\Omega u]$ *and* $F \in W^{1,\infty}(]\hat{m}, M[)$.

One of the key ideas of our approach is to reformulate problem $(\mathscr{P}_1)$ in terms of a different problem $(\mathscr{P}_2)$ of a nonlocal nature which eliminates the unknown F. The rough idea of this new model (see Díaz [D2]) is to use condition (3). Differentiating with respect to t and using the notation $p(t) = \frac{1}{2}\lambda t_+$, we find that

$$F(t)F'(t) = \frac{-p'(t)\displaystyle\int_{\{u=t\}} \frac{b(\rho, \theta)}{|\nabla u|} \rho \, d\rho \, d\theta}{\displaystyle\int_{\{u=t\}} \frac{\rho \, d\rho \, d\theta}{|\nabla u|}} \quad \text{for } t \in [0, \sup u]. \tag{4}$$

We point out that functions of a similar nature appear very often in the study of plasma problems (see, e.g., Grad, Hu & Stevens [GHS]). After some change of variables it is possible to express (4) in terms of the notion of the *relative rearrangement* (see Díaz [D2]) introduced first by Mossino & Temam [MT] to deal with a different nonlocal problem in plasma physics and later applied in many different contexts (see Díaz & Mossino [DM], Mossino [M], Rakotoson [R1, R2, R3], Rakotoson & Temam [RT], etc.). In our case there are some technical difficulties due to the presence of the Jacobian weight ρ in the integrals.

After recalling the modeling of the problem, in Section 2, we introduce some auxiliary mathematical tools. Since the operator $\mathscr{L}$ becomes degenerate near the boundary Γ_0, it is convenient to work with suitable weighted function spaces, which is done in Section 3. As we stated before, we start by reformulating problem $(\mathscr{P}_1)$ into an equivalent problem $(\mathscr{P}_2)$, eliminating the unknown F by using the notions of monotone and relative rearrangements. These steps are carried out in Sections 4 and 5 (some technical details are presented independently in Section 8). This new problem $(\mathscr{P}_2)$ will be solved by using an iterative method. The chief difficulty comes from the lack of continuity of the relative rearrangement mapping. To overcome this difficulty, we obtain a new expression for the relative rearrangement in terms of a quotient of derivatives of two weighted monotone rearrangements. Thanks to a slight modification of a result of ALMGREM & LIEB [AL] concerning the continuity of the first derivative of the monotone rearrangement mapping, we can take limit in the iteration process. Finally we complete our study with some qualitative properties on the solution of problem $(\mathscr{P}_2)$, such as a sufficient condition assuring the existence of the free boundary, and an estimate from below on the size of the associated plasma region.

A summary of part of the results of this article, concerning a special formulation of problem $(\mathscr{P}_1)$ was presented in DÍAZ & RAKOTOSON [DR].

2. Modeling

The Stellarators are a class of toroidal plasma-confinement devices alternative to the Tokamaks. The currents producing poloidal magnetic fields in Stellarators flow in external conductors, allowing a range of magnetic configurations wider than those found in Tokamaks. The geometry of these magnetic configurations is very important since it is directly related to the stability of the plasma.

Ideal magnetohydrodynamics is the most basic single-fluid model for determining the macroscopic properties of a plasma. Magnetohydrodynamic equilibrium is determined by the system

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{5}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{J}, \tag{6}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{7}$$

where p is the pressure, $\boldsymbol{B}$ the magnetic field and $\boldsymbol{J}$ the current density. From (5) it follows that

$$\boldsymbol{B} \cdot \nabla p = 0, \tag{8}$$

$$\boldsymbol{J} \cdot \nabla p = 0. \tag{9}$$

Thus the pressure is constant on each magnetic surface (i.e., a surface made up of magnetic field lines; by (9) they are also current surfaces). If such a surface lies in a bounded volume of space and has no edges and if neither $\boldsymbol{B}$ nor $\boldsymbol{J}$ vanish

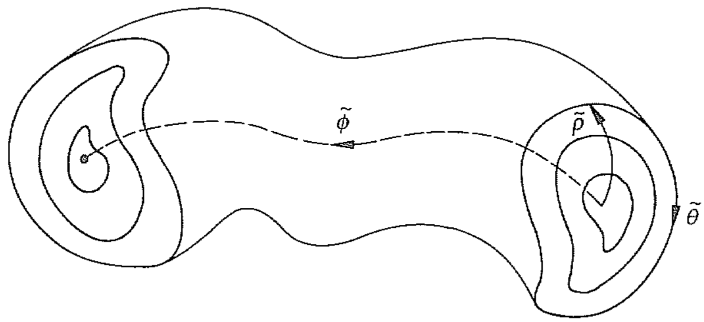

Figure 1

anywhere on it, then by a well-known theory due to ALEXANDROFF & HOPF it must be a toroid (i.e., a topological torus). Since the magnetic field lines are in toroidal nested surfaces (see Figure 1), it is useful to introduce a set of new toroidal coordinates $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$, such that $\tilde{\rho} = \tilde{\rho}(x, y, z)$ is an arbitrary function constant on each nested toroid and $\tilde{\theta} = \tilde{\theta}(x, y, z)$ is the poloidal angle which is constant on any toroidal circuit but changes by 2π over a poloidal circuit. (Here by a *toroidal circuit*, we mean any closed loop that encircles the axis of the torus once, and by a *poloidal circuit* a closed loop that encircles the minor axis once.) The toroidal angle $\tilde{\phi}$ is defined analogously, by interchanging *poloidal* and *toroidal*.

Notice that since the toroidal nested surfaces are not necessarily symmetric, the coordinate system $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ does not coincides, in general, with the "standard" toroidal coordinates (ρ, θ, ϕ) associated with a family of symmetric toroidal nested surfaces.

There are several special choices of $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ which are relevant for different purposes. Here we use the Boozer vacuum coordinate system (BOOZER [Bo]) which is very useful for Stellarators since magnetic field lines become "straight" in the $(\tilde{\theta}, \tilde{\phi})$-plane. In what follows, for the sake of simplicity in the notation, we denote this set of coordinates by (ρ, θ, ϕ).

For a vacuum configuration (i.e., one without any plasma), the magnetic field $\boldsymbol{B}_v$ may be written in contravariant form as

$$\boldsymbol{B}_v = B_0 \rho \nabla \rho \times \nabla(\theta - t_v(\rho)\phi)$$

where $t_v(\rho)$ is the so called *vacuum rotational transform* and B_0 is a positive constant. The covariant form of $\boldsymbol{B}_v$ is

$$B_v = F_v \nabla \phi \tag{10}$$

where F_v is a constant (which is customarily taken to be positive). In practice, the quasi-cylindrical-like Boozer set of coordinates $(\rho, \rho\theta, \phi)$ which have the usual near-axis behaviour of the field components is commonly used.

In contrast to Tokamaks, the Stellarators-type configurations are very complicated due to the fully three-dimensional nature of the device. To simplify the model to a two-dimensional problem, different averaging methods have been used; see GREENE & JOHNSON [GJ] and HENDER & CARRERAS [HC]. Following the last reference, we may decompose the magnetic field in terms of its toroidally averaged and rapidly varying parts. For a general function f, this decomposition takes the form

$$f = \langle f \rangle + \tilde{f}$$

where

$$\langle f \rangle := \frac{1}{2\pi} \int_0^{2\pi} f \, \mathrm{d}\phi.$$

In our case, motivated by the set of coordinates $(\rho, \rho\theta, \phi)$, the natural way of doing that is to set

$$\frac{B^i}{D} = \left\langle \frac{B^i}{D} \right\rangle + \left(\frac{\tilde{B}^i}{D} \right)$$

where B^i are the contravariant components of the vacuum magnetic field, $i = \rho, \theta, \phi$, and D is the Jacobian

$$D = (\nabla\rho \times \rho\nabla\theta) \cdot \nabla\phi.$$

Using a suitable assumption (the Stellarator expansion hypothesis) HENDER & CARRERAS [HC] show that (7) leads to

$$\frac{\partial}{\partial\rho}\left(\rho\left\langle \frac{B^\rho}{D} \right\rangle\right) + \frac{\partial}{\partial\theta}\left(\left\langle \frac{B^\theta}{D} \right\rangle\right) = 0,$$

and thus to the existence of the *averaged poloidal flux function* $\psi = \psi(\rho, \theta)$ defined by

$$\left\langle \frac{B^\rho}{D} \right\rangle = \frac{1}{\rho}\frac{\partial\psi}{\partial\theta}, \quad \left\langle \frac{B^\theta}{D} \right\rangle = -\frac{\partial\psi}{\partial\rho}. \tag{11}$$

They also show that $\langle B_\phi \rangle$ is a function ψ alone, as is $\langle p \rangle$ (recall (8)). By introducing the usual notation

$$F(\psi) := \langle B_\phi \rangle, \quad p(\psi) := \langle p \rangle \tag{12}$$

HENDER & CARRERAS [HC] obtain a Grad-Shafranov type equation for ψ:

$$-\mathscr{L}\psi = a(\rho, \theta)F(\psi) + F(\psi)F'(\psi) + b(\rho, \theta)p'(\psi) \tag{13}$$

where

$$\mathscr{L}\psi := \frac{1}{\rho}\left\{\frac{\partial}{\partial\rho}\left(a_{\rho\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\rho}\left(a_{\rho\theta}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\theta}\left(a_{\theta\rho}\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\theta}\left(a_{\theta\theta}\frac{\partial\psi}{\partial\theta}\right)\right\}$$

with

$$a_{\rho\rho}(\rho, \theta) := \rho\langle g^{\rho\rho}\rangle(\rho, \theta),$$
$$a_{\rho\theta}(\rho, \theta) = a_{\theta\rho}(\rho, \theta) := \langle g^{\rho\theta}\rangle(\rho, \theta),$$
$$a_{\theta\theta}(\rho, \theta) := \frac{1}{\rho}\langle g^{\theta\theta}\rangle(\rho, \theta),$$

and where $\langle g^{i,j} \rangle$, $i, j = \rho, \theta$ are the averaged components of the Riemannian metric associated with the vacuum coordinate system (all those coefficients are 2π-periodic functions in θ). The rest of the coefficients in (13) are given by

$$a(\rho, \theta) := \frac{B_0}{\rho F_v}\left[\frac{\partial}{\partial \rho}(\rho^2 t(\rho)\langle g^{\rho\rho}\rangle) + \frac{\partial}{\partial \theta}(\rho t(\rho)\langle g^{\rho\theta}\rangle)\right],$$

$$b(\rho, \theta) := \frac{F_v}{B_0}\left\langle \frac{1}{D} \right\rangle(\rho, \theta).$$

We remark that $b > 0$ and that the function a usually does not have any singularity.

Equation (13) holds only on the (averaged) region occupied by the plasma. In order to get a global formulation as a free-boundary problem, we remark that $\nabla p = 0$ in the vacuum region, and so, using (10), we employ a simpler analysis than before to obtain

$$-\mathscr{L}\psi_v = a(\rho, \theta) F_v.$$

Besides, it is clear that the free boundary (separating the plasma and vacuum regions) is a (toroidal) magnetic surface and, since $p = p(\psi)$, by normalizing, we can identify the free boundary as the level line $\{\psi = 0\}$, the plasma region as $\{\psi > 0\}$ (and thus $\{p > 0\}$) and the vacuum region by $\{\psi < 0\}$ and $\{p = 0\}$). It is also well-known that the pressure cannot be obtained from the magnetohydrodynamic system, and some constitutive law must be assumed. Here, for simplicity, we assume a quadratic law (see, e.g., TEMAM [T1]):

$$p = \frac{\lambda}{2}[\psi_+]^2, \quad \psi_+ = \max\{\psi, 0\}, \tag{14}$$

which is compatible with the above normalization. In order to extend the *unknown* $F(\psi)$ for negative values of ψ we again use (10), and so we must find $\psi(\rho, \theta)$ and $F : \mathbb{R} \to \mathbb{R}_+$ such that $F(s) = F_v$ for any $s \leqq 0$, satisfying

$$-\mathscr{L}\psi = a(\rho, \theta) F(\psi) + F(\psi) F'(\psi) + \lambda b(\rho, \theta)\psi_+ \tag{15}$$

on any bidimensional open set (in the variables (ρ, θ) associated with a physical three-dimensional domain Ω^3 (i.e., in the original Cartesian variables (X, Y, Z)) containing in its interior the plasma region. If we take as Ω^3 the interior of a vacuum magnetic surface, the construction of the Boozer coordinates implies that the associated open set in the (ρ, θ) variables becomes

$$\Omega = \{(\rho, \theta) : \rho \in (0, R), \theta \in (0, 2\pi)\}.$$

The boundary of Ω^3 is assumed to be a *perfectly conducting wall*, and thus $\boldsymbol{B} \cdot \boldsymbol{n}^3 = 0$ over Ω^3, where $\boldsymbol{n}^3$ denotes the outer normal vector to $\partial\Omega^3$. The averaging process implies that over the associated part of $\partial\Omega^3$, i.e., on

$$\Gamma_R = \{(R, \theta) : \theta \in (0, 2\pi)\},$$

we must have

$$\langle \boldsymbol{B} \rangle \cdot \boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ is the outer normal to Γ_R. From (11) we obtain that

$$\frac{\partial \psi}{\partial \tau} = 0 \quad \text{on } \Gamma_R$$

with τ the unit tangent vector to Γ_R. In other words,

$$\frac{\partial \psi}{\partial \theta}(R, \theta) = 0,$$

which shows that

$$\psi = \gamma \quad \text{on } \Gamma_R \tag{16}$$

for some (negative) constant γ. Since the variable θ has been constructed as an angle, we know that

$$\psi(\rho, 0) = \psi(\rho, 2\pi) \quad \text{for } \rho \in (0, R), \tag{17}$$

which also gives a boundary condition on

$$\Gamma_p = \{(\rho, 0) \text{ or } (\rho, 2\pi) : \rho \in (0, R)\}.$$

Finally, the remaining part of $\partial\Omega$ is

$$\Gamma_0 = \{(0, \theta) : \theta \in (0, 2\pi)\},$$

and the required boundary condition is

$$\psi = \text{constant } \forall\, \theta \in (0, 2\pi) \text{ or equivalently } \frac{\partial \psi}{\partial \theta} = 0 \text{ on } \Gamma_0. \tag{18}$$

This comes from the fact that the three-dimensional problem does not have any singularity at $\rho = 0$.

We point out that if we understand (ρ, θ) as the polar coordinates associated with a Cartesian bidimensional space in the variables (x, y), then the set Ω is transformed into the ball

$$\hat{\Omega} := \{(x, y) : x^2 + y^2 < R^2\} \tag{19}$$

and that if we define the identification

$$\hat{\psi}(x, y) := \psi(\rho, \theta), \tag{20}$$

then the boundary conditions (16)–(18) become

$$\hat{\psi} = \gamma \quad \text{on } \partial\hat{\Omega}. \tag{21}$$

This approach allows us to simplify many technical details, but since the coefficients of the operator $\mathscr{L}$ are given in the (ρ, θ) variables, we shall not follow this way. We also notice that, in general, the variables (x, y) do not coincide with the two first components of the physical three-dimensional variables (X, Y, Z).

In contrast with Tokamak devices, it is not restrictive to assume γ given a priori, since in the vacuum region $\psi(\rho, \theta) = \psi_v(\rho)$, we have the relation

$$\psi_v'(\rho) = C\rho t(\rho)$$

with $t(\rho)$ and C known. If, for instance, the Stellarator possesses a *limiter*, then the location of the free boundary is well-determinated and so are the rest of the vacuum levels.

To complete the formulation of the problem under consideration, we must add the Stellarator condition imposing a zero net current within each flux magnetic surface. According to the averaging method of HENDER & CARRERAS [HC], this condition can be expressed (DÍAZ [D1]) as

$$\int_{\{\psi \geqq t\}} [F(\psi)F'(\psi) + \lambda b\psi_+]\rho\, d\rho\, d\theta = 0 \quad \text{for any } t \in [\inf \psi, \sup \psi]. \tag{22}$$

Notice that in the Stellarators, this condition comes from the design of the external conductors. This contrasts with the usual condition of positive total current due to the inner toroidal current in the plasma for such configurations (see, e.g., TEMAM [T1] and BLUM [B]).

3. Weighted function spaces and the operator $\mathscr{L}$

Throughout this paper, Ω denotes the rectangle $\{(\rho, \theta): 0 < \rho < R, \theta \in (0, 2\pi)\}$. Nevertheless, many of our results remain true if Ω is merely a connected open set included in $\{(\rho, \theta) \in \mathbb{R}^2: \rho > 0\}$. We define the following weighted spaces: For $1 \leqq r < +\infty$,

$$L^r(\Omega, \rho) = \left\{u: \Omega \to \mathbb{R} \text{ is Lebesgue measurable with } \int_\Omega |u(\rho, \theta)|^r \rho\, d\rho\, d\theta < \infty\right\}.$$

It is a Banach space endowed with the natural norm $|u|_{r,\rho}^r = \int_\Omega |u(\rho, \theta)|^r \rho\, d\rho\, d\theta$. We also define

$$\tilde{H}_\rho^1(\Omega) = \left\{u \in L^2(\Omega, \rho), \frac{\partial u}{\partial \rho} \in L^2(\Omega, \rho), \frac{1}{\rho}\frac{\partial u}{\partial \theta} \in L^2(\Omega, \rho)\right\},$$

$$W^{1,2}(\Omega, \rho) = \left\{u \in L^2(\Omega, \rho), \nabla u = \left(\frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \theta}\right) \in L^2(\Omega, \rho)^2\right\}.$$

If we denote by $(f, g)_\rho = \int_\Omega f(\rho, \theta) g(\rho, \theta)\rho\, d\rho\, d\theta$ the scalar product on $L^2(\Omega, \rho)$, then we can define the scalar product on $\tilde{H}_\rho^1(\Omega)$:

$$(f, g)_{\tilde{H}_\rho^1(\Omega)} = (f, g)_\rho + \left(\frac{\partial f}{\partial \rho}, \frac{\partial g}{\partial \rho}\right)_\rho + \left(\frac{1}{\rho}\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial g}{\partial \theta}\right)_\rho$$

(which makes this space a Hilbert space). $W^{1,2}(\Omega, \rho)$ when endowed with the natural scalar product is also a Hilbert space. We easily have the following continuous injection:

Lemma 1.

$$\tilde{H}^1_\rho(\Omega) \subsetneq W^{1,2}(\Omega, \rho).$$

Motivated by the boundary conditions (2), we introduce the subspace $\tilde{H}^1_{\rho,0}(\Omega)$ as the closure in $\tilde{H}^1_\rho(\Omega)$ of the set

$$\left\{ u \in C^\infty(\bar{\Omega}),\ u = 0 \text{ on } \Gamma_R,\ \frac{\partial u}{\partial \theta} = 0 \text{ on } \Gamma_0 \text{ and } u(\rho, 0) = u(\rho, 2\pi) \text{ for any } \rho \in (0, R) \right\}.$$

Remark 1. As indicated in [D3], the identification (19), (20) mentioned in Section 3 converts the spaces $\tilde{H}^1_\rho(\Omega)$ and $\tilde{H}^1_{\rho,0}(\Omega)$ to the usual Sobolev space $H^1(\hat{\Omega})$ and $H^1_0(\hat{\Omega})$. The associated Sobolev–Poincaré type inequalities on $W^{1,2}(\Omega, \rho)$ and $\tilde{H}^1_{\rho,0}$ are given in [RS] and [S].

Another useful result is

Lemma 2. *If* $1 \leqq q < 6$, *then the imbedding*

$$W^{1,2}(\Omega, \rho) \subsetneq L^q(\Omega, \rho)$$

is continuous and compact. Furthermore, for each $q \in [1, 6[$, *there exists a constant* $c > 0$ *such that*

$$|u|_{q,\rho} \leqq c \left[\left| \frac{\partial u}{\partial \rho} \right|^2_{2,\rho} + \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2_{2,\rho} \right]^{1/2} \quad \textit{for all } u \in \tilde{H}^1_{\rho,0}(\Omega).$$

In particular, the norm $|u|^2_{\tilde{H}^1_\rho(\Omega)}$ *on* $\tilde{H}^1_{\rho,0}(\Omega)$, *is equivalent to the norm*

$$\left| \frac{\partial u}{\partial \rho} \right|^2_{2,\rho} + \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2_{2,\rho}.$$

For $(\varphi, \psi) \in \tilde{H}^1_\rho(\Omega) \times \tilde{H}^1_\rho(\Omega)$ we define the bilinear form $\mathfrak{a}(\varphi, \psi)$ by

$$\mathfrak{a}(\varphi, \psi) = \int_\Omega \left[\langle g^{\rho\rho} \rangle \frac{\partial \varphi}{\partial \rho} \frac{\partial \psi}{\partial \rho} + \langle g^{\theta\rho} \rangle \frac{1}{\rho} \frac{\partial \varphi}{\partial \rho} \frac{\partial \psi}{\partial \theta} + \langle g^{\rho\theta} \rangle \frac{1}{\rho} \frac{\partial \varphi}{\partial \theta} \frac{\partial \psi}{\partial \rho} + \langle g^{\theta\theta} \rangle \frac{1}{\rho^2} \frac{\partial \varphi}{\partial \theta} \frac{\partial \psi}{\partial \theta} \right] \rho \, d\rho \, d\theta.$$

We have

Lemma 3. (i) *The bilinear form* $\mathfrak{a}$ *is continuous, coercive and symmetric on* $\tilde{H}^1_{\rho,0}(\Omega) \times \tilde{H}^1_{\rho,0}(\Omega)$.

(ii) *For each* ψ *in* $\tilde{H}^1_{\rho,0}(\Omega)$ *such that* $\dfrac{\partial \psi}{\partial \theta} = 0$ *on* Γ_0, *there is a unique element* $-\mathscr{L}\psi$ *in the dual of* $\tilde{H}^1_{\rho,0}(\Omega)$ *such that*

$$\mathfrak{a}(\varphi, \psi) = \langle \varphi, -\mathscr{L}\psi \rangle \quad \textit{for all } \varphi \in \tilde{H}^1_{\rho,0}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ *denotes the duality product between the dual space* $\tilde{H}^1_{\rho,0}(\Omega)'$ *and* $\tilde{H}^1_{\rho,0}(\Omega)$.

(iii) *For each* $f \in L^2(\Omega, \rho)$, *there exists an unique element* ψ *of* $\tilde{H}^1_{\rho,0}(\Omega)$ *satisfying* $-\mathscr{L}\psi = f$ *and* $\dfrac{\partial \psi}{\partial \theta} = 0$ *on* Γ_0.

Proof. The first point (i) was proved in [D1] by using the structure of the coefficients of $\mathfrak{a}$. Statement (ii) follows directly from (i). The third statement comes from a direct application of the Lax-Milgram theorem. □

Remark 2. From the coercivity condition on the coefficients, we deduce that for any relatively compact subset $\omega \subset\subset \Omega$, there exists a constant $c_\omega > 0$ such that for all $(z_1, z_2) \in \mathbb{R}^2$,

$$a_{\rho\rho} z_1^2 + 2a_{\theta\rho} z_1 z_2 + a_{\theta\theta} z_2^2 \geqq c_w [z_1^2 + z_2^2] \quad \text{a.e. in } \omega.$$

That means that $\mathscr{L}$ behaves "locally" like a non-degenerate operator. So, using the Agmon-Douglis-Nirenberg regularity theory, we derive the following result: If $f \in L^s(\Omega, \rho)$, $1 < s < +\infty$, then the solution $\psi \in \tilde{H}^1_{\rho,0}(\Omega)$ of $-\mathscr{L}\psi = f$ belongs to $W^{2,s}_{\text{loc}}(\Omega)$. As a matter of fact, this regularity result also holds globally since the transformation (19), (20) leads to a regular Dirichlet problem on the ball $\hat{\Omega}$ to which we can apply standard results (see, e.g., GILBARG & TRUDINGER [GT]). In particular, if $f \in L^2(\Omega, \rho)$, then the solution $\psi \in \tilde{H}^1_{\rho,0}(\Omega)$ of $-\mathscr{L}\psi = f$ belongs to $L^\infty(\Omega)$ and there exists an universal constant $Q_0 > 0$ (independent of ψ) such that

$$|\psi|_{L^\infty(\Omega)} \leqq Q_0 |\mathscr{L}\psi|_{2,\rho}.$$

If in addition $f \in L^s(\Omega, \rho)$ for some $s \geqq p^* > 2$, then $\psi \in W^{1,\infty}(\Omega)$ and there exists another universal constant $Q_1 > 0$ (independent of ψ) such that

$$|\psi|_{W^{1,\infty}(\Omega)} \leqq Q_1 |\mathscr{L}\psi|_{s,\rho}.$$

4. Decreasing and relative rearrangement on weighted spaces

We start by introducing the weighted decreasing rearrangement.

Definition 1. A function $\sigma: \Omega \to \mathbb{R}_+$ is a *weight function* on Ω if (i) $\sigma(\rho, \theta) > 0$ for almost every $(\rho, \theta) \in \Omega$ and (ii) $\sigma \in L^\infty(\Omega)$ (for simplicity).

If $\sigma(\rho, \theta) = \rho$, we simply denote this weight by ρ. If E is a measurable subset of Ω (in the sense of Lebesgue), we denote $|E|_\sigma = \int_E \sigma(\rho, \theta)\, d\rho\, d\theta$.

Definition 2. Let σ be a weight function on Ω and let $u: \Omega \to \mathbb{R}$ a Lebesgue measurable function. Then the *distribution function of* u *relative to the weight* σ is defined by

$$m_u^\sigma(t) = |u > t|_\sigma = \int_{\{(\rho,\theta) \in \Omega : u(\rho,\theta) > t\}} \sigma(\rho, \theta)\, d\rho\, d\theta$$

for $t \in \mathbb{R}$. (If no confusion is to be feared, we simply set $m^\sigma(t) = m_u^\sigma(t)$.) The generalized inverse of m_u^σ is called the *weighted decreasing rearrangement of u*. That is the function u_*^σ: $]0, |\Omega|_\sigma[\to \bar{\mathbb{R}}$ such that for $s \in]0, |\Omega|_\sigma[$, $u_*^\sigma(s) = \inf\{t \in \mathbb{R}: m^\sigma(t) \leqq s\}$ and $u_*^\sigma(|\Omega|_\sigma) = \operatorname{ess\,inf}_\Omega u$.

Remark 3. u_*^σ has the same properties as the usual rearrangement (see [CR, S, M]). In particular, u and u_*^σ are equimeasurable, that is, $|u_*^\sigma > t|_\sigma = |u > t|_\sigma$ for all $t \in \mathbb{R}$. We set $\Omega_*^\sigma =]0, |\Omega|_\sigma[$. Later we shall need some results on the regularity of the first derivative of u_*^σ. For this, we need some additional assumptions on σ.

Definition 3. Let σ be a weight function on Ω. Given $1 \leqq p \leqq +\infty$, we define the space

$$W^{1,p}(\Omega, \sigma) = \{u \in W_{\text{loc}}^{1,1}(\Omega) \text{ such that } (u, \nabla u) \in L^p(\Omega, \sigma)^3\}.$$

Given $q > 1$, we say that σ *belongs to the class* $Q(\Omega, q)$ if there exists a constant $c > 0$ such that

$$\inf_{t \in \mathbb{R}} (|u - t|_{q,\sigma}) \leqq c|\nabla u|_{1,\sigma} \quad \text{for all } u \in W^{1,1}(\Omega, \sigma).$$

Here, $|v|_{q,\sigma}$ is the natural norm of $L^q(\Omega, \sigma)$. The following results are shown in [RS] (see also [S]).

Lemma 4. (a) *If* $\sigma(\rho, \theta) = \rho$, *then* $\sigma \in Q(\Omega, \frac{3}{2})$. (b) *If* σ_1 *is a weight such that* ess $\inf_\Omega \sigma_1 > 0$, *then* $\sigma_1 \in Q(\Omega, 2)$. (c) *If* $\sigma \in Q(\Omega, q)$ *and* σ_1 *is as in* (b), *then* $\sigma_1 \sigma \in Q(\Omega, q)$.

Definition 4. A weight σ is *admissible* if either $\sigma \in Q(\Omega, q)$ for some $q > 1$ or there exists $\sigma_1 \in Q(\Omega, q)$ such that $\sigma_1 \leqq \sigma$ a.e. in Ω.

Lemma 5. *Assume that σ is admissible. Then there is a constant $r_\sigma > 1$ such that for each $r \in [1, r_\sigma[$, there exists a $c_r > 0$ such that for all $u \in W^{1,\infty}(\Omega)$,*

$$u_*^\sigma \in W^{1,r}(\Omega_*^\sigma), \qquad \left|\frac{du_*^\sigma}{ds}\right|_{L^r(\Omega_*^\sigma)} \leqq c_r |\nabla u|_{L^\infty(\Omega)}.$$

Remark 4. The proof of Lemma 5 uses the Nirenberg translation method as in RAKOTOSON & TEMAM [RT] and [R2]. A slight difficulty comes from the Sobolev-Poincaré inequality, $\inf_t [u - t|_{q,\sigma} \leqq c|\nabla u|_{1,\sigma}$.

To conclude our iterative method we need some strong convergence of the sequence of the first derivatives du_{j*}^σ/ds. To do this, we borrow some notions introduced by ALMGREN & LIEB [AL]: Let $u \in W_{\text{loc}}^{1,1}(\Omega)$. For $t \in \mathbb{R}$ we set

$$m_0^\sigma(t) = m_{u,0}^\sigma(t) = |\{(\rho, \theta) \in \Omega, u(\rho, \theta) > t, \nabla u(\rho, \theta) = 0\}|_\sigma,$$

$$m_1^\sigma(t) = m_{u,1}^\sigma(t) = m_u^\sigma(t) - m_{u,0}^\sigma(t) = m^\sigma(t) - m_1^\sigma(t).$$

Notice that m^σ, m_1^σ, m_0^σ belong to $BV(\mathbb{R})$.

Definition 5. u is a *co-area regular function* if the Radon measure $(m_{u,0}^{\sigma})'$ is singular with respect to the Lebesgue measure on $\mathbb{R}$.

Remark 5. This definition means that in the Radon-Nikodym-Lebesgue decomposition $(m_{u,0}^{\sigma})' = -h \wedge dt - \nu$, the absolutely continous part is $h \equiv 0$.

As we shall see in Section 8, using a Federer-Sard theorem and a Lusin-type approximation as in [R3], we can prove

Theorem 2. *If* $u \in W_{\mathrm{loc}}^{2,p}(\Omega)$ *for some* $p > 1$, *then* u *is a co-area regular function (and so is the same for any weight* σ, $\Omega \subset \mathbb{R}^2$).

The proof is sketched in Section 8. Another simple condition to get a co-area regular function is given by

Lemma 6. *Let* $u \in W_{\mathrm{loc}}^{1,1}(\Omega)$. *If* $\mathrm{meas}\{(\rho, \theta) \in \Omega, \nabla u(\rho, \theta) = 0\} = 0$, *then* u *is a co-area regular function.*

Proof. Indeed, in that case, $m_{u,0}^{\sigma} \equiv 0$. □

The relation between the notion of co-area regular function and the convergence of the derivative of a sequence u_{j*}^{σ} is given by

Theorem 3. *Let* σ *be an admissible weight and let* u *be a co-area regular function in* $W^{1,\infty}(\Omega)$. *If* u_j *is a bounded sequence in* $W^{1,\infty}(\Omega)$ *converging to* u *in* $W^{1,1}(\Omega)$, *then*

$$\frac{du_{j*}^{\sigma}}{ds} \quad \textit{converge to} \; \frac{du_{*}^{\sigma}}{ds} \quad \textit{in } L^q(\Omega_*^{\sigma})$$

for $1 \leqq q < r_\sigma$ (r_σ *is as in Lemma 5*).

Remark 6. The assumptions on the regularity of u and u_j in Theorem 3 can be weakened. These assumptions are essentially made for the applications. A sketch of the proof is given in Section 8. It follows essentially the same ideas as in ALMGREN & LIEB [AL].

Now we introduce the *relative rearrangement on a weighted measure space.* Given a measurable function $u: \Omega \to \mathbb{R}$, we denote $P(u) = \{(\rho, \theta) \in \Omega: |u = u(\rho, \theta)| > 0\}$. Here, $|u = u(\rho, \theta)|$ is the Lebesgue measure of $\{u = u(\rho, \theta)\}$. It is easy to see that if σ is a weight function, then $P(u) = \{(\rho, \theta): |u = u(\rho, \theta)|_\sigma > 0\}$ is at most countable and does not depend on σ. If u_*^{σ} is the generalized inverse of $m^{\sigma} = m_u^{\sigma}$, then we define $D = \{t \in \mathbb{R}, |u_*^{\sigma} = t| > 0\}$. If $s \in \overline{\Omega_*^{\sigma}} = [0, |\Omega|_\sigma]$ is such that $u_*^{\sigma}(s) \in D$, then we set

$$P_s(u) = P_s = \{(\rho, \theta) \in \Omega, u(\rho, \theta) = u_*^{\sigma}(s)\}.$$

Consider $u \in L^1(\Omega, \sigma)$ and $v \in L^p(\Omega, \sigma)$ for some $1 \leqq p \leqq +\infty$. For $s \in \overline{\Omega_*^\sigma}$, we define the function

$$w^\sigma(s) = \int_{u > u_*^\sigma(s)} v(\rho, \theta)\sigma(\rho, \theta)\,\mathrm{d}\rho\,\mathrm{d}\theta + \int_0^{s - |u > u_*^\sigma(s)|_\sigma} (v|_{P_s})_*^\sigma(\tau)\,d\tau.$$

Here $v|_{P_s}$ is the restriction of v to $P_s = \{u = u_*^\sigma(s)\}$ and $(v_{|P_s})_*^\sigma$ is the decreasing rearrangement of $v_{|P_s}$ with respect to the measure $\sigma\,d\rho\,d\theta$. The following theorem is proved in [S] (see also [MT, RS, M, MR]).

Theorem 4. *For* $\lambda > 0$, *let* $w_\lambda^\sigma(s) = \int_0^s \frac{(u + \lambda v)_*^\sigma - u_*^\sigma}{\lambda}(\tau)d\tau$ *for* $s \in \overline{\Omega_*^\sigma}$. *If* $\lambda \to 0$, *then*

$$\frac{dw_\lambda^\sigma}{ds} \rightharpoonup \frac{dw^\sigma}{ds} \quad \text{in} \begin{cases} L^p(\Omega_*^\sigma) \ \text{weak if } 1 < p < +\infty, \\ L^\infty(\Omega_*^\sigma) \ \text{weak-star if } p = +\infty, \\ \sigma(L^1(\Omega_*^\sigma),\ L^\infty(\Omega_*^\sigma)) \ \text{if } p = 1. \end{cases}$$

Furthermore,

$$\left|\frac{dw^\sigma}{ds}\right|_{L^p(\Omega_*^\sigma)} \leqq |v|_{L^p(\Omega,\sigma)}.$$

Definition 6. The function $\frac{dw^\sigma}{ds}$ is called the *σ-relative rearrangement of v with respect to u and is denoted by*

$$v_{*u}^\sigma = \frac{dw^\sigma}{ds}.$$

This function possesses many properties (see, for instance, [S, MR, MT, RS]):

Lemma 7. *Let* u, v_1, v_2 *be elements of* $L^1(\Omega, \sigma)$. (i) *If* $v_1 \leqq v_2$ *a.e. in* Ω, *then* $v_{1*u}^\sigma \leqq v_{2*u}^\sigma$ *a.e. in* $\Omega_*^\sigma =]0, |\Omega|_\sigma[$. (ii) *If* $k \in \mathbb{R}$, *then* $(v_1 + k)_{*u}^\sigma = v_{1*u}^\sigma + k$. (iii) *If* $F_0: \mathbb{R} \to \mathbb{R}$ *is a Borel function such that* $F_0(u) \in L^1(\Omega, \sigma)$, *then* $F_0(u)_{*u}^\sigma = F_0(u_*^\sigma)$.

Now we define the mean operators or integral transformations.

Definition 7. Let $u \in L^1(\Omega, \sigma)$, and let g: $\Omega_*^\sigma \to \mathbb{R}$ be an integrable function. We define the *(first-category) mean operator* $M_u(g)$: $\Omega \to \mathbb{R}$ by

$$M_u(g)(x) = \begin{cases} g(m^\sigma(u(x))) & \text{if } x \in \Omega \backslash P(u), \\ \dfrac{1}{|u = u(x)|_\sigma} \displaystyle\int_{|u > u(x)|_\sigma}^{|u \geqq u(x)|_\sigma} g(\tau)d\tau & \text{otherwise.} \end{cases}$$

(Here $x = (\rho, \theta) \in \Omega$.)

Lemma 8. *$M_u(g)$ is well defined. Moreover, $\int_\Omega M_u(g)(\rho,\theta)\sigma(\rho,\theta)d\rho\, d\theta = \int_{\Omega_*^\sigma} g(\tau)\, d\tau$, and M_u: $L^p(\Omega_*^\sigma) \to L^p(\Omega,\sigma)$ is a linear and continuous functional with norm $\|M_u\| = 1$.*

Definition 8. Let u, v be two functions of $L^1(\Omega,\sigma)$ and let g: $\Omega_*^\sigma \to \mathbb{R}$ be an integrable function. We write

$$P(u) = \bigcup_{t_i \in D} P_i(u), \quad P_i(u) = \{(\rho,\theta): u(\rho,\theta) = t_i\}$$

and $v_i = v_{|P_i(u)}$ = restriction of v to $P_i(u)$. We define (*the second-category*) *mean operator* $M_{u,v}(g)$: $\Omega \to \mathbb{R}$ by

$$M_{u,v}(g)(x) = \begin{cases} g(m^\sigma(u(x))) & \text{if } x \in \Omega \backslash P(u), \\ M_{v_i}(h_i)(x) & \text{if } x \in P_i(u) \end{cases}$$

where h_i: $(0, |P_i(u)|_\sigma) \to \mathbb{R}$, $h_i(s) = g(s + |u > t_i|_\sigma)$ and M_{v_i} is the first-category mean operator associated with v_i.

Lemma 9. *The operator $M_{u,v}(g)$ is well defined and*
(i) $\int_\Omega M_{u,v}(g)(\rho,\theta)\cdot\sigma(\rho,\theta)d\rho\, d\theta = \int_{\Omega_*^\sigma} g(\tau)d\tau$,
(ii) *$M_{u,v}$: $L^p(\Omega_*^\sigma) \to L^p(\Omega,\sigma)$ is linear and continuous of norm 1 for any $p \in [1, +\infty]$.*

The link between the σ-relative rearrangement and those operators is summarized in

Lemma 10. *Let $u \in L^1(\Omega,\sigma)$ and $v \in L^p(\Omega,\sigma)$, $1 \leqq p \leqq +\infty$. Then*

$$\int_{\Omega_*^\sigma} v_{*u}^\sigma(s) g(s) ds = \int_\Omega [M_{u,v}(g)v](\rho,\theta)\sigma(\rho,\theta)\, d\rho\, d\theta \quad \textit{for any } g \in L^{p'}(\Omega_*^\sigma), \frac{1}{p} + \frac{1}{p'} = 1.$$

Corollary 1. *If the assumptions of Lemma 10 hold and if furthermore* meas $(P(u)) = 0$, *then*

$$\int_{\Omega_*^\sigma} v_{*u}^\sigma(s) g(s)\, ds = \int_\Omega g(m^\sigma(u(\rho,\theta))v(\rho,\theta)\sigma(\rho,\theta)\, d\rho\, d\theta.$$

Let us introduce some applications of the mean operators which will be useful later.

Lemma 11. *Let $u \in L^1(\Omega,\sigma)$ be such that $u_*^\sigma \in C(\Omega_*^\sigma)$ and* meas $(P(u)) = 0$. *Let F_0: $\mathbb{R} \to \mathbb{R}$ be a Borel function such that $F_0(u) \in L^1(\Omega,\sigma)$. If $b \in L^\infty(\Omega)$, then*

$$[F_0(u)b]_{*u}^\sigma = F_0(u_*^\sigma) b_{*u}^\sigma.$$

Proof. Let $g \in C(\overline{\Omega_*^\sigma})$. Then from Corollary 1 *we have*

$$\begin{aligned} \int_{\Omega_*^\sigma} g(s)[F_0(u)b]_{*u}^\sigma(s) ds &= \int_\Omega g(m_u^\sigma(u(\rho,\theta)))[F_0(u)b](\rho,\theta)\sigma(\rho,\theta)d\rho\, d\theta \\ &= \int_\Omega (gF_0(u_*^\sigma))(m_u^\sigma(u(\rho,\theta)))b(\rho,\theta)\sigma(\rho,\theta)d\rho\, d\theta. \end{aligned}$$

(since u_*^σ being continuous on Ω_*^σ, $u_*^\sigma(m_u^\sigma(t)) = t$ for $t \in (\text{ess inf}_\Omega u, \text{ess sup}_\Omega u)$). Again by Corollary 1, we have

$$\int_{\Omega_*} g(s)[F_0(u)b]_{*u}^\sigma(s)\,ds = \int_{\Omega_*} g(s)F_0(u_*^\sigma(s))b_{*u}^\sigma(s)\,ds \quad \forall g \in C(\overline{\Omega_*})$$

if $[F_0(u)b]_{*u}^\sigma$ and $F_0(u_*^\sigma)b_{*u}^\sigma$ are in $L^1(\Omega_*^\sigma)$ and the conclusion holds. □

Remark 7. (a) The assumption that $\text{meas}(P(u)) = 0$ can be removed by using a different proof. (b) It is shown in [S] that if $u \in C(\overline{\Omega})$, then u_*^σ is in $C(\overline{\Omega_*^\sigma})$ (for Ω connected) and that if $u_*^\sigma \in W_{\text{loc}}^{1,1}(\Omega_*^\sigma)$, then $u_*^\sigma \in C(\Omega_*^\sigma)$.

Theorem 5. *Let u_n, u be in $L^1(\Omega, \sigma)$ and assume that u_n converges to u in $L^1(\Omega, \sigma)$. Then for all $v \in L^p(\Omega, \sigma)$, for some $1 < p \leqq +\infty$ we have*

$$(v\chi_{\Omega\setminus P(u)})_{*u_n}^\sigma \rightharpoonup (u\chi_{\Omega\setminus P(u)})_{*u}^\sigma$$

weakly in $L^p(\Omega_*^\rho)$ if $p < +\infty$ and weak star in $L^\infty(\Omega_*^\sigma)$ if $p = +\infty$. Here, χ_E denotes the characteristic function of the set E.

Proof. Let $g \in C(\overline{\Omega_*^\sigma})$ and set $w = v\chi_{\Omega\setminus P(u)}$. From the mean-value operator property (Lemma 10), we have

$$\int_{\Omega_*^\sigma} g(s)w_{*u_n}^\sigma(s)\,ds = \int_{\Omega\setminus P(u)} [M_{u_n,w}(g)v\sigma](\rho,\theta)\,d\rho\,d\theta. \tag{23}$$

One can check easily that

$$\lim_{n\to+\infty} M_{u_n,w}(g)(\rho,\theta) = g(m_u^\sigma(u(\rho,\theta))) = M_{u,w}(g)(\rho,\theta) \quad \text{for } (\rho,\theta) \in \Omega\setminus P(u).$$

So, if we take the limit in relation (23), we get

$$\lim_{n\to+\infty} \int_{\Omega_*^\sigma} g(s)w_{*u_n}^\sigma(s)\,ds = \int_\Omega [M_{u,w}(g)w](\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta$$
$$= \int_{\Omega_*} g(s)w_{*u}^\sigma(s)\,ds \tag{24}$$

(again by Lemma 10). Since $|w_{*u_n}^\sigma|_{L^p(\Omega_*^\sigma)} \leqq |v|_{L^p(\Omega,\sigma)}$, relation (24) leads to the result. □

5. An equivalent problem involving the relative rearrangement

As we announced in the Introduction, we show that for the given smooth function u satisfying the relation

$$\int_{u>t} [F(u)F'(u) + \lambda u_+ b](\rho,\theta)\rho\,d\rho\,d\theta = 0 \quad \text{for all } t \in [\inf_\Omega u, \sup_\Omega u], \tag{25}$$

we can determine F explicitly in terms of u (in fact, the expression for F is obtained so that relation (25) holds). In this way, we reduce problem $(\mathscr{P}_1)$ to a problem of the form (with only one unknown)

$$(\mathscr{P}_2)\qquad \begin{aligned} &-\mathscr{L}u = a(\rho,\theta)F_u(\rho,\theta) + \lambda u_+[b(\rho,\theta) - b^\rho_{*u}(m^\rho(u(\rho,\theta)))] \quad \text{in } \Omega,\\ &u-\gamma\in \tilde{H}^1_{\rho,0}(\Omega)\cap W^{1,\infty}(\Omega), \quad \frac{\partial u}{\partial\theta} = 0 \quad \text{on } \Gamma_0, \end{aligned}$$

where F_u will be specified later. For simplicity, we use the notation $p(t) = \frac{1}{2}\lambda t_+^2$. Given $(\rho,\theta)\in\Omega$, we set

$$F_u(\rho,\theta) = \left[F_v^2 - 2\int_{m^\rho(0)}^{m^\rho(u_+(\rho,\theta))} [p(u^\rho_*)]'(s)b^\rho_{*u}(s)\,ds\right]_+^{1/2},$$

and for $t\in[\hat{m}, M]$, we introduce the function

$$F(t) = \left[F_v^2 - 2\int_0^{t_+} p'(\tau)b^\rho_{*u}(m^\rho(\tau))\,d\tau\right]_+^{1/2}.$$

Here, $\hat{m} = \inf_\Omega u$ and $M = \sup_\Omega u$. The weight $\sigma(\rho,\theta) = \rho$ is simply denoted by ρ, m^ρ is the distribution function of u with respect to the weight ρ and u^ρ_* is the generalized inverse of m^ρ.

Proposition 1. *Let $u\in W^{1,\infty}(\Omega)$ be such that*

$$\operatorname{meas}\left\{(\rho,\theta)\in\Omega: \frac{\partial u}{\partial\rho}(\rho,\theta) = \frac{\partial u}{\partial\theta}(\rho,\theta) = 0\right\} = 0.$$

Then,

$$F_u(\rho,\theta) = F(u(\rho,\theta)) \quad \textit{for all } (\rho,\theta)\in\bar{\Omega}.$$

Proof. Since meas $\{(\rho,\theta): \nabla u(\rho,\theta) = 0\} = 0$, the Federer formula (see [F]) ensures that the map $t\in[\hat{m}, M]\to m^\rho(t) = \int_{u>t}\rho\,d\rho\,d\theta$ is absolutely continuous. Furthermore, since $u\in W^{1,\infty}(\Omega)$, the inverse of m^ρ (that is u^ρ_*) is then in $W^{1,1}(\Omega^\sigma_*)$. Introducing the integral $J = \int_0^t p'(\tau)b^\rho_{*u}(m^\rho(\tau))d\tau$, we can make the change of variable $\tau = u^\rho_*$ to derive

$$J = \int_{m^\rho(0)}^{m^\rho(t_+)} p'(u^\rho_*(s))\frac{du^\rho_*}{ds}(s)\,b^\rho_{*u}(s)\,ds = \int_{m^\rho(0)}^{m^\rho(t_+)} \frac{d}{ds}\,p(u^\rho_*(s))b^\rho_{*u}(s)\,ds.$$

This gives the result. $\square$

In order to prove the equivalence of $(\mathscr{P}_1)$ and $(\mathscr{P}_2)$, we need a few lemmas. Throughout this section, $u\in W^{1,\infty}(\Omega)$ and satisfies

$$\operatorname{meas}\{(\rho,\theta): \nabla u(\rho,\theta) = 0\} = 0.$$

$\hat{m} = \inf_\Omega u$, $M = \sup_\Omega u$, and F and F_u are defined as before.

Lemma 12. *Assume that* $\min\{F(t), t \in [\hat{m}, M]\} > 0$. *Then*

(i) $$F \in W^{1,\infty}(]\hat{m}, M[),$$

(ii) $$F(t)F'(t) + p'(t)b^{\rho}_{*u}(m^{\rho}(t)) = 0 \quad \textit{for almost every } t \in]\hat{m}, M[. \tag{26}$$

Proof. The function b^{ρ}_{*u} is in $L^{\infty}(\Omega^{\rho}_{*})$. Thus the map

$$t \to \int_0^{t_+} p'(\tau)b^{\rho}_{*u}(m^{\rho}(\tau))d\tau$$

is in $W^{1,\infty}(\hat{m}, M)$. From the assumption that $\min\{F(t), t \in [\hat{m}, M]\} > 0$ and from the previous results we deduce that $F \in W^{1,\infty}(\hat{m}, M)$. Then $F^2 \in W^{1,\infty}(\hat{m}, M)$ and $\frac{d}{dt}F^2 = 2FF'$ in $\mathscr{D}'(\hat{m}, M)$ and almost everywhere. If we differentiate

$$F^2(t) = F_v^2 - 2\int_0^{t_+} p'(\tau)b^{\rho}_{*u}(m^{\rho}(\tau))d\tau,$$

we easily find that

$$2F(t)F('t) = \frac{d}{dt}F^2(t) = -2p'(t)b^{\rho}_{*u}(m^{\rho}(t)) \quad \text{a.e. in } [\hat{m}, M]$$

(notice that $p'(t) = 0 = F^2(t)'$ if $t \leqq 0$). □

Lemma 13. *If N is a null set of $[\hat{m}, M]$, then the sets*

$$\{(\rho, \theta) \in \Omega,\ u(\rho, \theta) \in N\}, \quad \{\rho \in \Omega^{\rho}_{*}:\ u^{\rho}_{*}(\rho) \in N\}$$

are also of measure zero.

Proof. By equimeasurability, we have

$$|\{(\rho, \theta) \in \Omega:\ u(\rho, \theta) \in N\}|_{\rho} = |\{s \in \Omega^{\rho}_{*}:\ u^{\rho}_{*}(s) \in N\}|.$$

Since m^{ρ} is absolutely continuous (remember that the set $\{(\rho, \theta) \in \Omega, \nabla u(\rho, \theta) = 0\}$ is assumed to be of measure zero), $\text{meas}\{m^{\rho}(N)\} = 0$. Observing that $\{s \in \Omega^{\rho}_{*}: u^{\rho}_{*}(\rho) \in N\} \subset m^{\rho}(N)$, we get the conclusion. □

Corollary 2. *Under the assumption of Lemma 12,*

(i) $F(u(\rho, \theta))F'(u(\rho, \theta)) + p'(u(\rho, \theta))b^{\rho}_{*u}(m^{\rho}(u(\rho, \theta))) = 0$ *for almost every* $(\rho, \theta) \in \Omega$,

(ii) $F(u^{\rho}_{*}(s))F'(u^{\rho}_{*}(s)) + p'(u^{\rho}_{*}(s))b^{\rho}_{*}(s) = 0$ *for almost every* $s \in \Omega^{\rho}_{*}$.

Proof. Consider the set $N := \{t \in [\hat{m}, M]$ such that (26) does not hold$\}$. Then $\text{meas}(N) = 0$, and from Lemma 13, the set $N_0 = \{(\rho, \theta) \in \Omega:\ u(\rho, \theta) \in N\}$ is also of measure zero. If we take $(\rho, \theta) \in \Omega \backslash N_0$ and replace t by $u(\rho, \theta)$ in (26), we get (i). The same argument holds for $N_0^* = \{\rho \in \Omega^{\rho}_{*}:\ u^{\rho}_{*}(\rho) \in N\}$. Since $m^{\rho}(u^{\rho}_{*}(s)) = s$, putting $t = u^{\rho}_{*}(s)$ in (26) with $s \in \Omega^{\rho}_{*} \backslash N_0^*$, we get (ii). □

Lemma 14. *Assume that* $\min\{F(t), t\in[\hat{m}, M]\} > 0$. *Then*

$$\int_{u>t} [F(u)F'(u) + p'(u)b](\rho,\theta)\rho\,d\rho\,d\theta = 0 \quad \forall\, t\in[\hat{m}, M].$$

Proof. Consider the function

$$w_\rho(s) = \int_{u>u_*^\rho(s)} [F(u)F'(u) + p'(u)b](\rho,\theta)\,\rho\,d\rho\,d\theta, \quad s\in\overline{\Omega_*^\rho},$$

and set $v(\rho,\theta) = [F(u)F'(u) + p'(u)b](\rho,\theta)$. We can see that $v\in L^\infty(\Omega)$. Since u "has no plateau" (i.e., $\text{meas}(P(u)) = 0$), by Theorem 4, we know that $w_\rho\in W^{1,\infty}(\Omega_*^\rho)$ and $\frac{d}{ds}w_\rho(s) = v_{*u}^\rho$. By linearity, we can write

$$\frac{d}{ds}w_\rho(s) = [F(u)F'(u)]_{*u}^\rho(s) + [p'(u)b]_{*u}^\rho(s) \quad \text{a.e. in } \Omega_*^\rho.$$

From Lemma 7, $[F(u)F'(u)]_{*u}^\rho = F(u_*^\rho)F'(u_*^\rho)$, and since $u_*^\rho\in C(\overline{\Omega_*^\rho})$, Lemma 11 implies that

$$[p'(u)b]_{*u}^\rho = p'(u_*^\rho)b_{*u}^\rho.$$

The last three relations combined together imply that

$$\frac{d}{ds}w_\rho(s) = F(u_*^\rho)F'(u_*^\rho)(s) + p'(u_*^\rho(s))b_{*u}^\rho(s) = 0,$$

where we used Corollary 2. Since $w_\rho\in W^{1,\infty}(\Omega_*^\rho)$, we have $w_\rho(s) = \text{constant} = w_\rho(0) = 0$ for all $s\in\overline{\Omega_*^\rho}$, which means that

$$\int_{u>t} v(\rho,\theta)\rho\,d\rho\,d\theta = 0 \quad \forall\, t\in[\hat{m}, M]. \quad \square$$

Conversely, we have

Lemma 15. *Let* $u\in W^{1,\infty}(\Omega)$ *be such that* $\text{meas}\{(\rho,\theta):\nabla u(\rho,\theta) = 0\} = 0$ *and assume* $\hat{m} = \inf_\Omega u \leqq 0$. *If* $F\in W^{1,\infty}(\hat{m}, M)$ *is a function satisfying* $F\colon [\hat{m}, M]\to \mathbb{R}_+$, $F(t) = F_v$ *for* $t\leqq 0$, *and*

$$\int_{u>t} [F(u)F'(u) + p'(u)b](\rho,\theta)\,\rho\,d\rho\,d\theta = 0, \quad \forall\, t\in[\hat{m}, M],$$

then, necessarily,

$$F(t) = \left[F_v^2 - 2\int_0^{t_+} p'(\tau)b_{*u}^\rho(m^\rho(\tau))d\tau\right]_+^{1/2} \quad \forall\, t\in[\hat{m}, M].$$

Proof. From the relation $\int_{u>t}[F(u)F'(u) + p'(u)b]\rho\,d\rho\,d\theta = 0$, we deduce that

$$0 = w_\rho(s) = \int_{u>u_*^\rho(s)} [F(u)F'(u) + p'(u)b]\,\rho\,d\rho\,d\theta \quad \forall\, s\in\overline{\Omega_*^\rho}.$$

Thus, $\frac{d}{ds} w_\rho(s) = 0$. Using the proof of Lemma 14, we have

$$F(u_*^\rho(s))F'(u_*^\rho(s)) + p'(u_*^\rho(s))b_{*u}^\rho(m^\rho(u_*^\rho(s))) = 0 \tag{27}$$

for almost every $s \in \Omega_*^\rho$ (we have used the fact that $s = m^\rho(u_*^\rho(s))$). If we set $N_{1*} = \{s \in \Omega_*^\rho$ such that s does not satisfy relation (27)$\}$, then meas $(N_{1*}) = 0$. Since $u_*^\rho \in W^{1,1}(\Omega_*^\rho)$ (see Lemmas 4 and 5), meas $(u_*^\rho(N_{1*})) = 0$, which implies that the set $\{t \in [\hat{m}, M], t \in u_*^\rho(N_{1*})\}$ is also a null set. Setting $t = u_*^\rho(s)$ in relation (27), we then have

$$F(t)F'(t) + p'(t)b_{*u}^\rho(m^\rho(t)) = 0 \tag{28}$$

for almost all $t \in [\hat{m}, M]$. Since F is in $W^{1,\infty}(\hat{m}, M)$, it follows that F^2 is in $W^{1,\infty}(\hat{m}, M)$ and

$$\frac{d}{dt} F^2(t) = 2F(t)F'(t) = -2p'(t)b_{*u}^\rho(m^\rho(t)). \tag{29}$$

Integrating relation (29) from 0 to $t_+ = \max(t, 0)$, we obtain

$$F^2(t_+) - F^2(0) = -2 \int_0^{t_+} p'(\tau)b_{*u}^\rho(m^\rho(\tau))\,d\tau \tag{30}$$

(notice that $\inf_\Omega u \leqq 0$ implies that $[0, t_+] \subset [\hat{m}, M]$). But $F^2(0) = F_v^2$ and $F^2(t_+) = F^2(t)$ by the assumptions of F). Thus relation (30) is equivalent to

$$F(t) = \left[F_v^2 - 2 \int_0^{t_+} p'(\tau)b_{*u}^\rho(m^\rho(\tau))\,d\tau\right]_+^{1/2}. \quad \square$$

We now arrive at the main result of this section:

Theorem 6. *Let* $u \in W^{1,\infty}(\Omega)$ *be such that* meas $\{(\rho, \theta): \nabla u(\rho, \theta) = 0\} = 0$. *Assume that* $\hat{m} = \inf_\Omega u \leqq 0$ *and* $F_u(\rho, \theta) > 0$ *a.e. in* Ω. *If* (u, F) *is a solution of* $(\mathscr{P}_1)$ *such that* $F: \mathbb{R} \to \mathbb{R}_+$, $F \in W^{1,\infty}(\hat{m}, M)$ *and* $F(t) = F_v$ *for* $t \leqq 0$, *then* u *is a solution of* $(\mathscr{P}_2)$ *and*

$$F(t) = \left[F_v^2 - 2 \int_0^{t_+} p'(\tau)b_{*u}^\rho(m(\tau))\,d\tau\right]_+^{1/2}. \tag{31}$$

Conversely, if u *is a solution of* $(\mathscr{P}_2)$ *and* F *is given by relation* (31), *then the pair* (u, F) *satisfies* $(\mathscr{P}_1)$, *and* $F \in W^{1,\infty}(\hat{m}, M)$.

Proof. Suppose that (u, F) satisfies $(\mathscr{P}_1)$. Applying Lemma 15, we derive the expression of F in terms of u. Proposition 1 shows that $F_u(\rho, \theta) = F(u(\rho, \theta))$. Thus the assumptions that $F_u(\rho, \theta) > 0$ and $u \in C(\bar{\Omega})$ imply that $\min\{F(t): t \in [\hat{m}, M]\} > 0$. The conditions of Corollary 2 are then fulfilled, allowing us to get

$$F(u(\rho, \theta))F'(u(\rho, \theta)) = -p'(u(\rho, \theta))b_{*u}^\rho(m^\rho(u(\rho, \theta))). \tag{32}$$

We thus obtain

$$a(\rho,\theta)F(u)+F(u)F'(u)+p'(u)b=a(\rho,\theta)F_u(\rho,\theta)+p'(u)[b(\rho,\theta)-b^{\rho}_{*u}(m^{\rho}(u))], \tag{33}$$

and u satisfies the first equation of $(\mathscr{P}_2)$. Conversely if u is a solution of $(\mathscr{P}_2)$ with F given by relation (31), then $F(u(\rho,\theta))=F_u(\rho,\theta)>0$, which implies that $\min\{F(t), t\in[\hat{m}, M]\}>0$. We then apply Lemma 14 to derive relation (3) of $(\mathscr{P}_1)$. On the other hand, Corollary 2 gives the relation (33), and thus the first equation of $(\mathscr{P}_1)$ holds. □

6. Solving problem $(\mathscr{P}_2)$

The main difficulty in solving problem $(\mathscr{P}_2)$ is that the terms b^{ρ}_{*u} and $b^{\rho}_{*u}(m^{\rho}(u(\rho,\theta)))$ do not possess any good continuity property with respect to u. So our first approach consists in finding a suitable approximation of these quantities. This is the purpose of the next lemmas of this section. We assume that $b\rho$ is an admissible weight. (If, for instance, ess $\inf_{\Omega} b>0$, then using Lemma 4 we deduce that $b\rho\in Q(\Omega,\frac{3}{2})$.)

Lemma 16. *Let* $u\in W^{1,\infty}(\Omega)$ *be such that* meas $\{(\rho,\theta)\in\Omega\colon \nabla u(\rho,\theta)=0\}=0$. *Then*

$$b^{\rho}_{*u}(|u>u(\rho,\theta)|_{\rho})=\frac{(u^{\rho}_{*})'(|u>u(\rho,\theta)|_{\rho})}{(u^{b\rho}_{*})'(|u>u(\rho,\theta)|_{b\rho})}$$

for almost every $(\rho,\theta)\in\Omega$. *Furthermore,* $(u^{b\rho}_{*})'(|u>u(\rho,\theta)|_{b\rho})<0$ *a.e. in* Ω.

Proof. Let us denote by σ the weight $b\rho$ or ρ. We set $\Omega_0=\{(\rho,\theta)\in\Omega\colon \nabla u(\rho,\theta)=0\}$. Using the Federer formula, we deduce the following facts:
(i) For almost every $t\in[\hat{m}, M]$, $H_1(u^{-1}(t)\cap\Omega_0)=0$ (H_1 is the Lebesgue measure on $\mathbb{R}$ which is also the 1-dimensional Hausdorff measure).

(ii) The map $\tau\to\Delta^{\sigma}_u(\tau)=\displaystyle\int_{\{(\rho,\theta)\colon u(\rho,\theta)=\tau\}}\frac{\sigma(\rho,\theta)dH_1(\rho,\theta)}{|\nabla u(\rho,\theta)|}$ belongs to $L^1(\hat{m}, M)$, and

$$\int_{\hat{m}}^{M}\Delta^{\sigma}_u(\tau)d\tau=\int_{\{(\rho,\theta)\colon\nabla u(\rho,\theta)\neq 0\}}\sigma(\rho,\theta)\,d\rho\,d\theta.$$

In particular, m^{σ} is absolutely continuous on $(\hat{m}, M)$ and thus $\Delta^{b\rho}_u(u^{\sigma}_{*}(s))$ and $\Delta^{\rho}_u(u^{\sigma}_{*}(s))$ are finite for almost every $s\in\Omega^{\sigma}_{*}$. Since $u\in W^{1,\infty}(\Omega)$, $u^{\sigma}_{*}\in W^{1,1}(\Omega^{\sigma}_{*})$. This regularity allows us to apply the chain rule and to get the identity

$$\frac{d}{ds}\int_{u^{\sigma}_{*}(s)}^{M}\Delta^{b\rho}_u(\tau)d\tau=-\frac{du^{\sigma}_{*}}{ds}(s)\int_{u=u^{\sigma}_{*}(s)}\frac{b(\rho,\theta)\rho\,dH_1(\rho,\theta)}{|\nabla u(\rho,\theta)|} \tag{34}$$

for almost every $s \in \Omega_*^\sigma$. Again applying Federer's formula and the definition of $u_*^{b\rho}$, we have

$$\int_{u_*^{b\rho}(s)}^{M} \Delta_u^{b\rho}(\tau)\,d\tau = \int_{u > u_*^{b\rho}(s)} b(\rho,\theta)\rho\,d\rho\,d\theta = s$$

for any $s \in \Omega_*^{b\rho}$. Taking $\sigma = b\rho$ in relation (34) and differentiating this last relation, we easily obtain

$$-\frac{du_*^{b\rho}}{ds}(s) = \frac{1}{\displaystyle\int_{u = u_*^{b\rho}(s)} \frac{b(\rho,\theta)\rho\,dH_1(\rho,\theta)}{|\nabla u(\rho,\theta)|}} \in]0, +\infty[\tag{35}$$

for almost every $s \in \Omega_*^{b\rho}$. From relation (35), we want to deduce that

$$\frac{du_*^{b\rho}}{ds}(|u > u(\rho,\theta)|_{b\rho}) = -\frac{1}{\displaystyle\int_{\{(z,z'):u(z,z') = u(\rho,\theta)\}} \frac{b(z,z')z\,dH_1(z,z)}{|\nabla u(z,z')|}} \tag{36}$$

for almost all $(\rho,\theta) \in \Omega$. We can take $s = |u > u(\rho,\theta)|_{b\rho}$ in relation (35), but we have to check that the relation we get is true for almost all $(\rho,\theta) \in \Omega$. That is, we need to prove that if $N^* \subset \Omega_*^{b\rho}$ is a null set, then $\{(\rho,\theta) \in \Omega\colon |u > u(\rho,\theta)|_{b\rho} \in N^*\}$ is also of measure zero. Indeed,

$$\{(\rho,\theta) \in \Omega\colon |u > u(\rho,\theta)|_{b\rho} \in N^*\} \subset \{(\rho,\theta)\colon u(\rho,\theta) \in u_*^{b\rho}(N^*)\},$$

but $\operatorname{meas}\{u_*^{b\rho}(N^*)\} = 0$. We can apply Lemma 13 to deduce that $\operatorname{meas}\{(\rho,\theta) u(\rho,\theta) \in u_*^{b\rho}(N^*) = 0$. Thus $\{(\rho,\theta) \in \Omega\colon\ |u > u(\rho,\theta)|_{b\rho} \in N^*\}$ is of measure zero. Taking $N^* = \{s \in \Omega_*^{b\rho}$ such that relation (35) does not hold$\}$, we find that $\operatorname{meas}(N^*) = 0$. Thus relation (36) holds for $(\rho,\theta) \in \Omega \backslash \{(z,z')\colon |u > u(z,z')|_{b\rho} \in N^*\}$, that is, almost everywhere on Ω. A similar argument justifies the following computation: Taking $\sigma = \rho$ in relation (34) and replacing relation (26) by

$$\int_{u_*^\rho(s)}^{M} \Delta_u^{b\rho}(\tau)\,d\tau = \int_{u > u_*^\rho(s)} b(\rho,\theta)\rho\,d\rho\,d\theta \quad \forall\, s \in \Omega_*^\rho$$

we get from the definition of b_{*u}^ρ that

$$b_{*u}^\rho(s) = -\frac{du_*^\rho}{ds}(s) \int_{u = u_*^\rho(s)} \frac{b(\rho,\theta)\rho\,dH_1(\rho,\theta)}{|\nabla u(\rho,\theta)|} \tag{37}$$

for almost all $s \in \Omega_*^\rho$. Taking $s = |u > u(\rho,\theta)|_\rho$ in relation (37) with $(\rho,\theta) \in \Omega \backslash \{(z,z')\colon\ |u > u(z,z')|_\rho \in \bar{N}^*\}$ and $\bar{N}^* = \{s \in \Omega_*^\rho$ such that relation (37) does not hold$\}$, we deduce that

$$b_*^\rho(|u > u(\rho,\theta)|_\rho) = -\frac{du_*^\rho}{ds}(|u > u(\rho,\theta)|_\rho) \int_{\{(z,z'):u(z,z') = u(\rho,\theta)\}} \frac{b(z,z')z\,dH_1(z,z')}{|\nabla u(z,z')|} \tag{38}$$

for almost all $(\rho, \theta) \in \Omega$. Combining relations (36) and (38), we have

$$b^{\rho}_{*u}(|u > u(\rho, \theta)|_{\rho}) = \frac{(u^{\rho}_{*})'(|u > u(\rho, \theta)|_{\rho})}{(u^{b\rho}_{*})'(|u > u(\rho, \theta)|_{b\rho})}.$$

Corollary 3. *Let u be in $W^{2,s}_{\text{loc}}(\Omega) \cap W^{1,\infty}(\Omega)$ for some $s > 1$. Then*

$$-(\text{ess}\inf_{\Omega} b)\frac{d^+u^{b\rho}_{*}}{ds}(|u > u(\rho, \theta)|_{b\rho}) \leqq -\frac{d^+u^{\rho}_{*}}{ds}(|u > u(\rho, \theta)|_{\rho})$$

$$\leqq -|b|_{\infty}\frac{d^+u^{b\rho}_{*}}{ds}(|u > u(\rho, \theta)|_{b\rho})$$

for almost all $(\rho, \theta) \in \Omega$. Here $\frac{d^+u^{\rho}_{*}}{ds}$ and $\frac{d^+u^{b\rho}_{*}}{ds}$ denote the right derivatives of u^{ρ}_{*} and $u^{b\rho}_{*}$.

Proof. Consider u_n, a sequence of $W^{1,\infty}(\Omega)$ such that u_n converges to u in $W^{1,1}(\Omega)$ and such that meas $\{(\rho, \theta) \in \Omega, \nabla u_n(\rho, \theta) = 0\} = 0$. (This holds if, for instance, u_n is analytic in Ω.) Since ess $\inf_{\Omega} b \leqq b^{\rho}_{*u_n} \leqq$ ess $\sup_{\Omega} b$, Lemma 16 applied to u_n implies that for almost all (ρ, θ),

$$-b_i\frac{d^+u^{b\rho}_{n*}}{ds}(|u_n > u_n(\rho, \theta)|_{b\rho})\chi_{\Omega\backslash P(u)}(\rho, \theta) \leqq -\frac{d^+u^{\rho}_{n*}}{ds}(|u_n > u_n(\rho, \theta)|_{\rho})\chi_{\Omega\backslash P(u)}(\rho, \theta),$$

$$-\frac{d^+u^{\rho}_{n*}}{ds}(|u_n > u_n(\rho, \theta)|_{\rho})\chi_{\Omega\backslash P(u)}(\rho, \theta) \leqq -|b|_{\infty}\frac{d^+u^{\rho}_{n*}}{ds}(|u_n > u_n(\rho, \theta)|_{b\rho})$$

$$\times \chi_{\Omega\backslash P(u)}(\rho, \theta), \tag{39}$$

where $b_i =$ ess $\inf_{\Omega} b$.

Before finishing the proof we need the following fundamental convergence result:

Lemma 17. *Let σ be equal to ρ or $b\rho$. Then, there exists a number $r_{\sigma} > 1$ (depending on σ) such that*

$$-\frac{d^+u^{\sigma}_{n*}}{ds}(|u_n > u_n(\rho, \theta)|_{\sigma})\chi_{\Omega\backslash P(u)}(\rho, \theta) \rightharpoonup -\frac{d^+u^{\sigma}_{*}}{ds}(|u > u(\rho, \theta)|_{\sigma})\chi_{\Omega\backslash P(u)}(\rho, \theta)$$

weakly in $L^r(\Omega, \sigma)$ for any $r \in [1, r_{\sigma}[$.

Proof of Lemma 17. Since u_n and u are in $W^{1,\infty}(\Omega)$, we know from Lemma 5 that for $\sigma = \rho$ or $\sigma = b\rho$ there exists a number $r_{\sigma} > 1$ such that for each $r \in [1, r_{\sigma}[$, there exists a $c_r > 0$ satisfying

$$\left|\frac{du^{\sigma}_{n*}}{ds}\right|_{L^r(\Omega^{\sigma}_{*})} \leqq C_r|\nabla u_n|_{L^{\infty}(\Omega)}. \tag{40}$$

The same relation holds for u. But u is a co-area regular function (because $u \in W^{2,s}_{\mathrm{loc}}$ and because of Theorem 2), so we deduce from Theorem 3 that u^σ_{n*} converges strongly to u^σ_* in $W^{1,1}(\Omega^\sigma_*)$. Now, consider $v \in L^\infty(\Omega)$ and set $w = v\chi_{\Omega\setminus P(u)}$. Applying the mean-value operator property (see Lemma 10), we obtain

$$\int_\Omega \frac{d^+u^\sigma_{n*}}{ds}(|u_n > u_n(\rho,\theta)|_\sigma) w(\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta$$

$$= \int_\Omega M_{u_n,w}\left(\frac{d^+u^\sigma_{n*}}{ds}\right) w(\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta = \int_{\Omega^\sigma_*} \frac{d^+u^\sigma_{n*}}{ds}(s) w^\sigma_{*u_n}(s)\,ds.$$

Since $\dfrac{d^+u^\sigma_{n*}}{ds}$ converges to $\dfrac{d^+u^\sigma_*}{ds}$ strongly in $L^1(\Omega^\sigma_*)$ and since $w^\sigma_{*u_n}$ converges to w^σ_{*u} in $L^\infty(\Omega^\sigma_*)$-weak star, by Theorem 5 we deduce that

$$\lim_n \int_{\Omega^\sigma_*} \frac{d^+u^\sigma_{n*}}{ds}(s) w^\sigma_{*u_n}(s)\,ds = \int_{\Omega^\sigma_*} \frac{d^+u^\sigma_*}{ds}(s) w^\sigma_{*u}(s)\,ds = \tilde{J}.$$

Again, using the mean-value operator, we have

$$\tilde{J} = \int_\Omega M_{u,v}\left(\frac{d^+u^\sigma_*}{ds}\right) w(\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta = \int_\Omega \frac{d^+u^\sigma_*}{ds}(|u > u(\rho,\theta)|_\sigma) w\sigma\,d\rho\,d\theta.$$

Thus, we have shown that

$$\lim_n \int_\Omega \frac{d^+u^\sigma_{n*}}{ds}(|u_n > u_n(\rho,\theta)|_\sigma)\chi_{\Omega\setminus P(u)}(\rho,\theta)v(\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta$$

$$= \int_\Omega \frac{d^+u^\sigma_*}{ds}(|u > u(\rho,\theta)|_\sigma)\chi_{\Omega\setminus P(u)}(\rho,\theta)v(\rho,\theta)\sigma(\rho,\theta)\,d\rho\,d\theta \tag{41}$$

for any $v \in L^\infty(\Omega)$. Lemma 17 will be proved if we show that the sequence $\dfrac{d^+u^\sigma_{n*}}{ds}(|u_n > u_n(\rho,\theta)|_\sigma)\chi_{\Omega\setminus P(u)}(\rho,\theta)$ remains in a bounded set of $L^r(\Omega,\sigma)$. Indeed, using the equimeasurability property, we deduce that

$$\left|\frac{d^+u^\sigma_{n*}}{ds}(|u_n > u_n(\rho,\theta)|_\sigma)\chi_{\Omega\setminus P(u)}(\rho,\theta)\right|_{L^r(\Omega,\sigma)}$$

$$\leqq \left|\frac{d^+u^\sigma_{n*}}{ds}\right|_{L^r(\Omega^\sigma_*)} \leqq c_r|\nabla u_n|_{L^\infty(\Omega)} \leqq \text{constant}. \tag{42}$$

The same argument shows that

$$\frac{d^+u^\sigma_*}{ds}(|u > u(\rho,\theta)|_\sigma)\chi_{\Omega\setminus P(u)}(\rho,\theta) \text{ belongs to } L^r(\Omega,\sigma),\ r \in [1, r_\sigma[. \quad \square$$

To end the proof of Corollary 3 we need a simple result:

Lemma 18. *If f_n, g_n are measurable functions on Ω such that* (i) $0 \leqq f_n(x) \leqq g_n(x)$ *for almost every $x \in \Omega$,* (ii) *f_n converges weakly to f in $L^r(\Omega, \sigma)$, and g_n converges weakly to g in $L^r(\Omega, b\rho)$, $r > 1$. Then $0 \leqq f(x) \leqq g(x)$ for almost every $x \in \Omega$.*

The proof is easy so we omit it.

Completion of the proof of Corollary 3. We apply Lemma 18 with

$$f_n(\rho, \theta) = -\frac{d^+ u_{n*}^{\sigma}}{ds}(|u_n > u_n(\rho, \theta)|_{\sigma})\,\chi_{\Omega \setminus P(u)}(\rho, \theta) b_i,$$

$$g_n(\rho, \theta) = -\frac{d^+ u_{n*}^{b\sigma}}{ds}(|u_n > u_n(\rho, \theta)|_{b\sigma})\,\chi_{\Omega \setminus P(u)}(\rho, \theta)\,|b|_{\infty}.$$

Relation (39) implies that

$$0 \leqq f_n(\rho, \theta) \leqq g_n(\rho, \theta) \quad \text{for almost every } (\rho, \theta) \in \Omega. \tag{43}$$

Lemma 17 implies that f_n converges weakly to f in $L^r(\Omega, \rho)$ and g_n converges weakly to g in $L^r(\Omega, b\rho)$. Here, we set

$$f(\rho, \theta) = -b_i \frac{d^+ u_*^{\rho}}{ds}(|u > u(\rho, \theta)|_{b\sigma})\chi_{\Omega \setminus P(u)}(\rho, \theta),$$

$$g(\rho, \theta) = -|b|_{\infty} \frac{d^+ u_*^{b\rho}}{ds}(|u > u(\rho, \theta)|_{b\sigma})\chi_{\Omega \setminus P(u)}(\rho, \theta).$$

We then have $0 \leqq f(\rho, \theta) \leqq g(\rho, \theta)$ for almost every $(\rho, \theta) \in \Omega$. This inequality shows that inequalities of Corollary 3 hold for almost all $(\rho, \theta) \in \Omega \setminus P(u)$. Since $\frac{d^+ u_*^{\sigma}}{ds}(|u > u(\rho, \theta)|_{\sigma}) = 0$ a.e. ($\sigma = \rho$ or $\sigma = b\rho$) for $(\rho, \theta) \in P(u)$, the inequalities of the statement remain true on $P(u)$. □

From now on, for u measurable on Ω and $(\rho, \theta) \in \Omega$ we define

$$\mathfrak{b}_u(\rho, \theta) = \begin{cases} \dfrac{(u_*^{\rho})'(|u > u(\rho, \theta)|_{\rho})}{(u_*^{b\rho})'(u > u(\rho, \theta)|_{b\rho})} & \text{if } (u_*^{b\rho})'(|u > u(\rho, \theta)|_{b\rho}) \neq 0, \\ b(\rho, \theta) & \text{otherwise.} \end{cases} \tag{44}$$

We easily have

Proposition 2. *Let $u \in W^{1,\infty}(\Omega) \cap W^{2,s}_{\text{loc}}(\Omega)$ for some $s > 1$. For almost all $(\rho, \theta) \in \Omega$,*

(i) $\operatorname{ess\,inf}_{\Omega} b \leqq \mathfrak{b}_u(\rho, \theta) \leqq \operatorname{ess\,sub}_{\Omega} b$ *and* $\mathfrak{b}_{(u+\gamma)}(\rho, \theta) = \mathfrak{b}_u(\rho, \theta)\ \forall \gamma \in \mathbb{R}$.

(ii) *If furthermore* $\operatorname{meas}\{(\rho, \theta) \in \Omega: \nabla u(\rho, \theta) = 0\} = 0$, *then* $\mathfrak{b}_u(\rho, \theta) = b^{\rho}_{*u}(|u > u(\rho, \theta)|_{\rho})$.

The proof is a consequence of Lemma 16 and Corollary 3.

The idea of the existence proof relies on an iterative process. We first derive a key theorem concerning the weak solution to the problem $(\mathscr{P}_2)$.

Let us introduce the first eigenvalue λ_1 for $-\mathscr{L}$ on $\tilde{H}^1_{\rho,0}(\Omega)$. Let ψ_1 be the associated eigenfunction, i.e., $-\mathscr{L}\psi_1 = \lambda_1 \psi_1$, $\psi_1 \in \tilde{H}^1_{\rho,0}(\Omega)$ and $\lambda_1 = \inf\{\mathfrak{a}(\psi, \psi): \int_\Omega \psi^2(\rho, \theta)\, \rho\, d\rho\, d\theta = 1, \psi \in \tilde{H}^1_{\rho,0}(\Omega)\}$. We remark that in fact $\psi_1 > 0$ on Ω since the transformation (19), (20) reduces the study to the standard Dirichlet problem on the ball $\hat{\Omega}$. We set $p(t) = \frac{1}{2}\lambda t_+^2$ for $t \in \mathbb{R}$ and $\operatorname{osc}_\Omega b := \operatorname{ess\,sup}_\Omega b - \operatorname{ess\,inf}_\Omega b$.

Theorem 7. *Let* $\lambda \operatorname{osc}_\Omega b < \lambda_1$. *Then there exists a function* $u \in W^{2,p}_{\text{loc}}(\Omega) \cap W^{1,\infty}(\Omega)$ *for all* $p \in [1, +\infty[$ *and two functions* $\hat{b}_{\rho,u} \in L^\infty(\Omega^\rho_*)$, $\tilde{b}_{\rho,u} \in L^\infty(\Omega)$ *satisfying*

$$(\mathscr{P}_3)\quad \begin{aligned} -\mathscr{L}u = a(\rho, \theta)&\left[F_v^2 - 2 \int_{m^\rho(0)}^{m^\rho(u_+(\rho,\theta))} [p(u^\rho_*)]'(s)\hat{b}_{\rho,u}(s)\,ds\right]_+^{1/2} \\ &+ p'(u(\rho, \theta))[b(\rho, \theta) - \tilde{b}_{\rho,u}(\rho, \theta)] \\ u - \gamma \in \tilde{H}^1_{\rho,0}(\Omega),\ &\frac{\partial u}{\partial \theta} = \quad \text{on } \Gamma_0. \end{aligned}$$

Furthermore, $\hat{b}_{\rho,u}$ *and* $\tilde{b}_{\rho,u}$ *satisfy*

$$\text{(i)}\qquad \hat{b}_{\rho,u}(s) \in \left[(b\chi_{\Omega\setminus P(u)})^\rho_{*u}(s), (b\chi_{\Omega\setminus P(u)})^\rho_{*u}(s) + \operatorname*{ess\,sup}_{P(u)} b\right]$$

for almost every $s \in \Omega^\rho_*$ *(with the convention that* $\operatorname{ess\,sup}_\Omega b = 0$ *whenever* $\operatorname{meas}(P(u)) = 0$),

(ii)

$$\tilde{b}_{\rho,u}(\rho, \theta) = \mathfrak{b}_u(\rho, \theta) \textit{ for almost all } (\rho, \theta) \in \Omega_{b\rho} := \left\{x \in \Omega: \frac{d^+ u^{b\rho}_*}{ds}(|u > u(x)|_{b\rho}) \neq 0\right\}.$$

Definition 9: A function u satisfying $(\mathscr{P}_3)$ with (i) and (ii) is said to be a *weak solution* of $(\mathscr{P}_2)$.

Corollary 4. (i) *If the solution u found in Theorem 7 satisfies*

$$\operatorname{meas}\{(\rho, \theta) \in \Omega: \nabla u(\rho, \theta) = 0\} = 0, \tag{45}$$

then u satisfies $(\mathscr{P}_2)$, *that is,*

$$\hat{b}_{\rho,u}(s) = b^\rho_{*u}(s) \qquad \text{a.e. in } \Omega^\rho_*,$$

$$\tilde{b}_{\rho,u}(\rho, \theta) = b^\rho_{*u}(|u > u(\rho, \theta)|_\rho) \quad \text{a.e. in } \Omega.$$

(ii) *If* $\operatorname{ess\,inf}_\Omega |a| > 0$, *then there is a constant* $\Lambda > 0$ *such that if* $\lambda |b|_\infty < \Lambda$, *then condition* (45) *is fulfulled.*

It is clear that Corollary 4 implies Theorem 1.

Proof of Theorem 7. *Step* 1. Consider a sequence $(w^j)_{j \geqq 0}$ of $\tilde{H}^1_{\rho,0}(\Omega)$ defined by (i) $w^0 = -\gamma$ and $w^{j+1} \in \tilde{H}^1_{\rho,0}(\Omega) \cap W^{1,\infty}(\Omega) \cap W^{2,s}_{\text{loc}}(\Omega)$ for any $s > 1$ satisfying

$$(\mathscr{P}^j) \qquad \mathfrak{a}(w^{j+1}, \varphi) = (aG^j + J^j, \varphi)_\rho \quad \forall \varphi \in \tilde{H}^1_{\rho,0}(\Omega)$$

where

$$G^j(\rho, \theta) = \left[F_v^2 - 2 \int\limits_{|w^j + \gamma > 0|_\rho}^{|w^j + \gamma > (w^j + \gamma)_+(\rho, \theta)|_\rho} [p[(w^j + \gamma)^\rho_*]]'(s) b^\rho_{*w^j}(s)\, ds \right]_+^{1/2},$$

$$J^j(\rho, \theta) = p'(w^j + \gamma)(\rho, \theta)[(b(\rho, \theta) - \mathfrak{b}_{w^j}(\rho, \theta))].$$

We need some a priori estimates:

Lemma 19. *Assume that* $\lambda \operatorname{osc}_\Omega b < \lambda_1$. *Then there exists a unique solution* w^{j+1} *of problem* $(\mathscr{P}^j)$. *Furthermore, the sequence* $(w^j)_{j \geqq 0}$ *remains in a bounded set of* $W^{1,\infty}(\Omega) \cap W^{2,p}_{\text{loc}}(\Omega)$ *for all* $p \in [1, +\infty[$, *and*

$$|w^j + \gamma|_{2,\rho} \leqq \frac{(|a|_\infty F_v + \lambda_1 |\gamma|) |\Omega|_\rho^{1/2}}{\lambda_1 - \lambda \operatorname{osc}_\Omega b} := \mu. \tag{46}$$

Proof of Lemma 19. We use an induction argument of Lemma 3. Assume that $w^j \in W^{2,s}_{\text{loc}}(\Omega) \cap W^{1,\infty}(\Omega)$ for all $s \in [1, +\infty[$ is constructed. It suffices to show that $f^j := aG^j + J^j$ is in $L^2(\Omega, \rho)$ and then to apply Lemma 3. Indeed,

$$0 \leqq |a| G^j(\rho, \theta) \leqq |a|_\infty F_v, \tag{47}$$

$$|J^j(\rho, \theta)| \leqq \lambda \left(\operatorname{osc}_\Omega b \right) |(w^j + \gamma)(\rho, \theta)|. \tag{48}$$

Relations (47) and (48) show that $f^j \in L^2(\Omega, \rho)$. Thus there exists a unique $w^{j+1} \in \tilde{H}^1_{\rho,0}(\Omega)$ such that

$$\mathfrak{a}(w^{j+1}, \varphi) = (f^j, \varphi)_\rho \quad \forall \varphi \in \tilde{H}^1_{\rho,0}(\Omega). \tag{49}$$

We take $\varphi = w^{j+1}$ in (49). With the help of relations (47) and (48), we deduce that

$$\lambda_1 \int\limits_\Omega |w^{j+1}(\rho, \theta)|^2 \rho\, d\rho\, d\theta \leqq |a|_\infty F_v \int\limits_\Omega |w^{j+1}(\rho, \theta)| \rho\, d\rho\, d\theta$$
$$+ \lambda \left(\operatorname{osc}_\Omega b \right) |w^{j+1}|_{2,\rho} |w^j + \gamma|_{2,\rho}. \tag{50}$$

Using the Cauchy-Schwarz inequality, we find that (50) implies that

$$\lambda_1 |w^{j+1}|_{2,\rho} \leqq |a|_\infty F_v |\Omega|_\rho^{1/2} + \lambda \left(\operatorname{osc}_\Omega b \right) |w^j + \gamma|_{2,\rho}. \tag{51}$$

On the other hand, we have that

$$|w^{j+1}+\gamma|_{2,\rho} \leqq |w^{j+1}|_{2,\rho} + |\gamma||\Omega|_\rho^{1/2}. \tag{52}$$

Relations (51) and (52) lead to

$$\lambda_1|w^{j+1}+\gamma|_{2,\rho} \leqq \lambda\left(\underset{\Omega}{\operatorname{osc}} b\right)|w^j+\gamma|_{2,\rho} + \lambda_1|\gamma||\Omega|_\rho^{1/2} + |a|_\infty F_v|\Omega|_\rho^{1/2}. \tag{53}$$

We set $\varepsilon = \dfrac{\lambda(\operatorname{osc} b)}{\lambda_1}$, $\delta = \dfrac{(|a|_\infty F_v + \lambda_1|\gamma|)|\Omega|_\rho^{1/2}}{\lambda_1}$ and $a_j = |w^j+\gamma|_{2,\rho}$, so that inequality (53) can be written as

$$a_{j+1} \leqq \varepsilon a_j + \delta, \quad j \geqq 0, a_0 = 0. \tag{54}$$

Then $a_{j+1} \leqq \delta/(1-\varepsilon)$ for all $j \geqq 0$. Replacing δ and ε by their values, we get $|w^j+\gamma|_{2,\rho} \leqq \mu$. Again using the estimates (47) and (48), we deduce that

$$|f^j|_{2,\rho} \leqq |a|_\infty F_v|\Omega|_\rho^{1/2} + \lambda\left(\underset{\Omega}{\operatorname{osc}} b\right)|w^j+\gamma|_{2,\rho} \leqq \tilde{\mu} \tag{55}$$

where $\tilde{\mu} := |a|_\infty F_v|\Omega|_\rho^{1/2} + \lambda(\operatorname{osc}_\Omega b)\mu$. Next, we show that w^j remains in a bounded set of $W^{1,\infty}(\Omega)$. Using the coercivity condition on $\mathfrak{a}$, we derive

$$\alpha|w^{j+1}|^2_{\tilde{H}^1_{\rho,0}(\Omega)} \leqq \mathfrak{a}(w^{j+1}, w^{j+1}) = (f^j, w^{j+1})_\rho \leqq |f^j|_{2,\rho}|w^{j+1}|_{2,\rho}. \tag{56}$$

With relation (55), relation (56) leads to

$$|w^{j+1}|^2_{\tilde{H}^1_\rho(\Omega)} \leqq \frac{\tilde{\mu}}{\alpha}(\mu + |\gamma||\Omega|_\rho^{1/2}) = \mu_1. \tag{57}$$

It follows from the Poincaré-Sobolev inequality given in Lemma 2 that for all $p \in [1, 6[$, there exists $c_p > 0$ such that

$$|w^{j+1}|_{p,\rho} \leqq c_p \quad \text{for all } j \geqq 0. \tag{58}$$

Relations (47) and (48) then imply that $f^j = aG^j + J^j = -\mathscr{L}w^{j+1}$ remains in a bounded set of $L^p(\Omega, \rho)$ for $1 \leqq p \leqq 6$. From Remark 2 there exists a constant Q_1 such that if $p > 2$, then

$$|w^{j+1}|_{W^{1,\infty}(\Omega)} \leqq Q_1|\mathscr{L}w^{j+1}|_{p,\rho} \leqq Q_1^* \quad \forall j \geqq 0. \tag{59}$$

Again by relations (47) and (48), we deduce that f^j remains in a bounded set of $L^\infty(\Omega)$. Since $-\mathscr{L}$ is coercive in any open set relatively compact in Ω, the Agmon-Douglis-Nirenberg regularity theory can be applied to w^{j+1}, and thus it remains in a bounded set of $W^{2,s}_{\text{loc}}(\Omega)$ for all $s \in [1, +\infty[$. □

Step 2. Passing to the limit. The above uniform boundedness allows us to derive the existence of a function $w \in W^{1,\infty}(\Omega) \cap \tilde{H}^1_{\rho,0}(\Omega) \cap W^{2,s}_{\text{loc}}(\Omega)$ for $s \in [1 + \infty[$ such that

$$w^j \to w \quad \text{weakly-* in } W^{1,\infty}(\Omega), \tag{60}$$

$$w^j \to w \quad \text{strongly in } W^{1,s}(\Omega) \text{ for all } s \in [1, +\infty[, \tag{61}$$

$$w^j \to w \quad \text{in } C^{0,\alpha}(\bar{\Omega}) \cap C^{1,\alpha}_{\text{loc}}(\Omega) \text{ for some } 0 \leqq \alpha < 1, \tag{62}$$

$$w^j \to w \quad \text{strongly in } \tilde{H}^1_{\rho,0}(\Omega). \tag{63}$$

The convergences (60), (61) and (62) come from standard results, while the convergence (63) comes from the fact that the sequence $(w^{j+1})_{j \geqq 0}$ is a Cauchy-sequence in $\tilde{H}^1_{\rho,0}(\Omega)$. Indeed, since $\mathfrak{a}$ is coercive on $\tilde{H}^1_{\rho,0}(\Omega)$ and $|f_j|_{L^\infty(\Omega)} \leqq$ constant, we have

$$\mathfrak{a}(w^{j+1} - w^{n+1}, w^{j+1} - w^{n+1}) \leqq C\,|w^{j+1} - w^{n+1}|_{L^1(\Omega)} \to 0.$$

From now on, we set

$$u^j = w^j + \gamma, \quad j \geqq 0, \; u = w + \gamma. \tag{64}$$

We notice that

$$b^\rho_{*w^j} = b^\rho_{*u^j}, \quad \mathfrak{b}_{w^j} = \mathfrak{b}_{u^j}. \tag{65}$$

Step 3. Convergence of $b^\rho_{*u^j}$. Since $|b^\rho_{*w^j}|_\infty \leqq |b|_\infty$, we can assume that $b^\rho_{*w^j}$ converges to a function $\hat{b}_{\rho,u}$ in $L^\infty(\Omega^\rho_*)$-weak-star. From Lemma 7, we deduce that

$$\operatorname*{ess\,inf}_\Omega b \leqq b \leqq \operatorname*{ess\,sup}_\Omega b \quad \text{implies that} \quad \operatorname*{ess\,inf}_\Omega b \leqq b^\rho_{*w^j} \leqq \operatorname*{ess\,sup}_\Omega b.$$

Thus

$$\operatorname*{ess\,inf}_\Omega b \leqq \hat{b}_{\rho,u}(s) \leqq \operatorname*{ess\,sup}_\Omega b. \tag{66}$$

We need a more precise result:

Lemma 20. *For almost all* $s \in \Omega^\rho_*$,

$$\hat{b}_{\rho,u}(s) \in \left[(b\chi_{\Omega\setminus P(u)})^\rho_*(s), (b\chi_{\Omega\setminus P(u)})^\rho_{*u}(s) + \operatorname*{ess\,sup}_{P(u)} b \right]$$

with the convention that $\operatorname{ess\,sup}_{P(u)} b = 0$ *if* $\operatorname{meas}(P(u)) = 0$. *(Recall that* $P(u) = \{(\rho, \theta) \in \Omega := |u = u(\rho, \theta)| > 0\}$ *and that* $\chi_{\Omega\setminus P(u)}$ *is the characteristic function of the set* $\Omega\setminus P(u)$.)

Proof of Lemma 20. Almost everywhere in Ω,

$$b\chi_{\Omega\setminus P(u)}(\rho, \theta) \leqq b(\rho, \theta) \leqq b\chi_{\Omega\setminus P(u)}(\rho, \theta) + \operatorname*{ess\,sup}_{P(u)} b.$$

From Lemma 7 we know that

$$(b\chi_{\Omega\setminus P(u)})^\rho_{*u^j} \leqq b^\rho_{*u^j} \leqq (b\chi_{\Omega\setminus P(u)})^\rho_{*u^j} + \operatorname*{ess\,sup}_{P(u)} b \quad \text{for almost all } s \in \Omega^\rho_*. \tag{67}$$

Applying Theorem 5, we find that

$$\lim_j \, (b\chi_{\Omega\setminus P(u)})^\rho_{*u^j} = (b\chi_{\Omega\setminus P(u)})^\rho_{*u} \quad \text{in } L^\infty(\Omega^\rho_*) \text{ weak-}*.$$

Thus inequality (67) implies that

$$(b\chi_{\Omega\setminus P(u)})^{\rho}_{*u} \leqq \hat{b}_{\rho,u} \leqq (b\chi_{\Omega\setminus P(u)})^{\rho}_{*u} + \operatorname*{ess\,sup}_{P(u)} b \quad \text{in } \mathscr{D}'(\Omega^{\rho}_{*})$$

and thus also almost everywhere. □

Step 4. Convergence of $a_j(\rho, \theta) := 2\int_{|u^j>0|_\rho}^{|u^j>u^j_+(\rho,\theta)|_\rho} b^{\rho}_{*u^j}(s)[p[u^j]^{\rho}_{*}]'(s)\,ds$. We need

Lemma 21. *For any admissible weight* σ,

$$\frac{d}{ds}(u^j)^{\sigma}_{*} = \frac{d}{ds}(w^j)^{\sigma}_{*} \text{ converges to } \frac{du^{\sigma}_{*}}{ds} = \frac{dw^{\sigma}_{*}}{ds}$$

strongly in $L^1(\Omega^{\sigma}_{*})$.

Proof of Lemma 21. Since $u \in W^{2,s}_{\text{loc}}(\Omega)$ for some $s > 1$, Theorem 2 implies that u is a co-area regular function. But u^j is a bounded sequence in $W^{1,\infty}(\Omega)$ converging to u in $W^{1,1}(\Omega)$. Thus Theorem 3 allows us to conclude that

$$\frac{d}{ds}(u^j)^{\sigma}_{*} \to \frac{du^{\sigma}_{*}}{ds} \quad \text{strongly in } L^1(\Omega^{\sigma}_{*}).$$

(Note that $u^{\sigma}_{*} = w^{\sigma}_{*} + \gamma$ and $(u^j)^{\sigma}_{*} = (w^j)^{\sigma}_{*} + \gamma$.) □

From the estimate of Lemma 5, the sequence $(u^j)^{\sigma}_{*}$ remains in a bounded set of $W^{1,r}(\Omega^{\sigma}_{*})$ for some $r \in [1, r_\sigma] r_\sigma > 1$. Then, from Lemma 21, we easily have

Lemma 22. *For any admissible weight* σ, *there exists a number* $r_\sigma > 1$ *such that* $\dfrac{d}{ds}(u^j)^{\sigma}_{*}$ *converges to* $\dfrac{du^{\sigma}_{*}}{ds}$ *strongly in* $L^r(\Omega^{\sigma}_{*})$ *for any* $r \in [1, r_\sigma]$.

In order to study the convergence of $a_j(\rho, \theta)$, we introduce the following intervals:

$$I^j(\rho, \theta) = [|u^j > u^j_+(\rho, \theta)|_\rho, |u^j > 0|_\rho],$$

$$J_0(\rho, \theta) = [|u \geqq u_+(\rho, \theta)|_\rho, |u > 0|_\rho],$$

$$J_1(\rho, \theta) = [|u > u_+(\rho, \theta)|_\rho, |u \geqq 0|_\rho],$$

$$J_2(\rho, \theta) = [|u > u_+(\rho, \theta)|_\rho, |u > 0|_\rho].$$

We again denote by χ_A the characteristic function of a set A. Since

$$|u > 0|_\rho \leqq \liminf_j |u^j > 0|_\rho \leqq \limsup_j |u^j > 0|_\rho \leqq |u \geqq 0|_\rho, \tag{68}$$

$$|u > u_+(\rho, \theta)|_0 \leqq \liminf_j |u^j > u^j_+(\rho, \theta)|_\rho \leqq \limsup_j |u^j > u^j_+(\rho, \theta)|_\rho$$
$$\leqq |u \geqq u_+(\rho, \theta)|_\rho,$$

we deduce that

$$\chi_{J_0(\rho,\theta)}(s) \leqq \liminf_j \chi_{I^j(\rho,\theta)}(s) \leqq \limsup_j \chi_{I^j(\rho,\theta)}(s) \leqq \chi_{J_1(\rho,\theta)}(s) \quad \text{for all } s \in \Omega_*^\rho. \tag{69}$$

But for $i = 1, 2$ and for almost every $s \in \Omega_*^\rho$, we have

$$[p(u_*^\rho)]'(s)\chi_{J(\rho,\theta)}(s) = [p(u_*^\rho)]'(s)\chi_{J_i(\rho,\theta)}(s)), \tag{70}$$

$$\lim_j [p[u^j]_*^\rho]' = [p(u_*^\rho)]' \quad \text{strongly in } L^1(\Omega_*^\rho). \tag{71}$$

Thus, from (68)–(71), we deduce that

$$[p[u^j]_*^\rho]'\chi_{I^j(\rho,\theta)} \text{ converges to } [p(u_*^\rho)]'\chi_{J_2(\rho,\theta)} \quad \text{strongly in } L^1(\Omega_*^\rho) \tag{72}$$

when j goes to infinity. Finally, defining

$$a_j(\rho, \theta) := 2 \int_{\Omega_*^\rho} [p[u^j]_*^\rho]'(s)\chi_{I^j(\rho,\theta)}(s) b_{*u^j}^\rho(s)\, ds,$$

we see that the strong convergence of relation (72) and the weak-star convergence of b_{*uj}^ρ to $\hat{b}_{\rho,u}$ imply that

$$a_j(\rho, \theta) \underset{j\to+\infty}{\longrightarrow} 2\int_{\Omega_*^\rho} [p(u_*^\rho)]'(s)\chi_{J_2(\rho,\theta)}(s)\hat{b}_{\rho,u}(s)\, ds \tag{73}$$

for almost every $(\rho, \theta) \in \Omega$. But the expression for $G^j(\rho, \theta)$ can be written as

$$G^j(\rho, \theta) = a(\rho, \theta)[F_v^2 - a_j(\rho, \theta)]_+^{1/2}. \tag{74}$$

Thus, the convergence (73) shows us that if $j \to +\infty$, then

$$G^j(\rho, \theta) \to a(\rho, \theta)[F_v^2 - 2 \int_{|u>0|_\rho}^{|u>u_+(\rho,\theta)|_\rho} [p(u_*^\rho)]'(s)\hat{b}_{\rho,u}(s)\, ds]_+^{1/2} = (aG)(\rho, \theta). \tag{75}$$

The estimate (47) (i.e., $|G^j(\rho, \theta)| \leqq |a|_\infty F_v$) implies that

$$G^j \text{ converges to } G \text{ in } L^s(\Omega) \text{ for all } s \in [1, +\infty[. \tag{76}$$

Moreover, from Proposition 2, we know that

$$\operatorname*{ess\,inf}_\Omega b \leqq \mathrm{b}_{u^j}(\rho, \theta) \leqq \operatorname*{ess\,sup}_\Omega b$$

for almost every $(\rho, \theta) \in \Omega$. Thus we can assume that b_{u^j} converges to a function $\tilde{b}_{\rho,u}$ in $L^\infty(\Omega)$-weak*. By the continuity of p', we get that

$$J^j = p'(u^j)[b - \mathrm{b}_{u^j}] \to J = p'(u)[b - \tilde{b}_{\rho,u}] \quad \text{in } L^\infty(\Omega)\text{-weak-*}. \tag{77}$$

Step 5. Convergence of the rest of the terms of the equation. From the convergences (76) and (77), we easily have that $u \in W_{\text{loc}}^{2,p}(\Omega) \cap W^{1,\infty}(\Omega)$ is a solution of

$$(\mathscr{P}_3) \qquad -\mathscr{L}u = aG + J \text{ in } \tilde{H}_{\rho,0}^1(\Omega)', \quad u - \gamma \in \tilde{H}_{\rho,0}^1(\Omega), \quad \frac{\partial u}{\partial \theta} = 0 \text{ on } \Gamma_0$$

with G given by relation (75) and $J = p'(u)[b - \tilde{b}_{\rho u}]$. It remains to give some information about the function $\tilde{b}_{\rho,u}$. For this, we have the following strong convergence result, which completes the proof of Theorem 7.

Lemma 23. *Let $\sigma = b\rho$ or $\sigma = \rho$. Then there is a number $r_\sigma > 1$ such that*

$$g_j(\rho, \theta) = \frac{d^+(u^j)_*^\sigma}{ds}(|u^j > u^j(\rho, \theta)|_\sigma)\chi_{\Omega\backslash P(u)}(\rho, \theta)$$

converges strongly to

$$g(\rho, \theta) = \frac{d^+u_*^\sigma}{ds}(|u > u(\rho, \theta)|_\sigma)\chi_{\Omega\backslash P(u)}(\rho, \theta)$$

in $L^r(\Omega, \sigma)$ for any $r \in [1, r_\sigma[$.

Proof of Lemma 23. From Lemma 17, we know that g_j converges to g weakly in $L^r(\Omega, \sigma)$ for any $r \in [1, r_\sigma[$. By equimeasurability, we deduce that

$$|g_j|_{L^r(\Omega,\sigma)} \leqq \left|\frac{d^+(u^j)_*^\sigma}{ds}\right|_{L^r(\Omega_*^\sigma)}. \tag{78}$$

From the strong convergence of Lemma 22, we have

$$\lim_j \left|\frac{d^+(u^j)_*^\sigma}{ds}\right|_{L^r(\Omega_*^\sigma)} = \left|\frac{d^+(u_*^\sigma)}{ds}\right|_{L^r(\Omega_*^\sigma)}. \tag{79}$$

Again by equimeasurability, we also have that

$$|g|_{L^r(\Omega,\sigma)} = \left|\frac{d^+u_*^\sigma}{ds}\right|_{L^r(\Omega_*^\sigma)}. \tag{80}$$

From relations (78)–(80), we conclude that

$$\lim_j |g_j|_{r,\sigma} \leqq |g|_{r,\sigma}. \tag{81}$$

But if $r \in]1, r_\sigma[$, then $L^r(\Omega, \sigma)$ is uniformly convex, so that the weak convergence of g_j to g and relation (81) imply that $|g_j - g|_{L^r(\Omega,\sigma)}$ converges to zero. For a subsequence (still denoted by g_j), we can assume that $g_j(\rho, \theta)$ converges to $g(\rho, \theta)$ a.e. in Ω. We set $\Omega^\sharp := \{(\rho, \theta) \in \Omega$: $g_j(\rho, \theta)$ converges to $g(\rho, \theta)$ for $\sigma = b\rho$ and $\sigma = \rho\}$. If $(\rho, \theta) \in \Omega^\sharp$ is such that $\frac{d^+u_*^{b\rho}}{ds}(|u > u(\rho, \theta)|_{b\rho}) \neq 0$, then we deduce from the expression for b_{u^j} that

$$\mathrm{b}_{u^j}(\rho, \theta) \xrightarrow[j\to+\infty]{} \frac{(u_*^\rho)'(|u > u(\rho, \theta))|_\rho)}{(u_*^{b\rho})'(|u > u(\rho, \theta)|_{b\rho})} = \bar{g}(\rho, \theta).$$

Since b_{u^j} converges to $\tilde{b}_{\rho,u}$ in $L^\infty(\Omega)$ weak-star, we obtain that $\tilde{b}_{\rho,u}(\rho, \theta) = \bar{g}(\rho, \theta)$ a.e. on $\Omega_{b\rho} = \{(\rho, \theta) \in \Omega$: $\frac{d^+u_*^{b\rho}}{ds}(|u > u(\rho, \theta)|_{b\rho}) < 0\}$. □

Proof of Corollary 4. Assume that the function found previously as the solution of $(\mathscr{P}_3)$ satisfies the condition (45), i.e.,

$$\text{meas}\{(\rho, \theta) \in \Omega: \nabla u(\rho, \theta) = 0\} = 0.$$

Then the conditions for the application of Lemma 16 are fulfilled, and thus $\Omega_{b_\rho} = \Omega$ (modulo a null set) and $\tilde{b}_{\rho,u}(\rho, \theta) = b^{\rho}_{*u}(|u > u(\rho, \theta)|_\rho)$ for almost every $(\rho, \theta) \in \Omega$. Furthermore, condition (45) implies that meas $(P(u)) = 0$, and thus, from Lemma 20, we deduce that

$$b^{\rho}_{*}(s) = \hat{b}_{\rho,u}(s) \quad \text{a.e. on } \Omega^{\rho}_{*}.$$

To show part (ii) we must give a condition ensuring (45). To do this, we need an a priori bound for $\max_\Omega |u(\rho, \theta)|$. We set

$$Q_\infty(\Omega) = \sup\left\{\frac{|w|_\infty}{|f|_{2,\rho}}: -\mathscr{L}w = f,\ w \in \tilde{H}^1_{\rho,0}(\Omega) \text{ and } f \in L^2(\Omega, \rho)\right\}. \tag{82}$$

Due to the properties of $\mathscr{L}$, $Q_\infty(\Omega)$ is finite. Let μ be given by (46) and assume that

$$\lambda \operatorname*{osc}_{\Omega} b < \lambda_1 .$$

From now on, we consider u to be a solution of $(\mathscr{P}_3)$ found by the preceding iteration method.

Lemma 24.

$$\max_\Omega |u(x)| \leqq |\gamma| + \left[\lambda\left(\operatorname*{osc}_{\Omega} b\right)\mu + |a|_\infty F_v |\Omega|_\rho^{1/2}\right] Q_\infty(\Omega) := M_w.$$

Proof of Lemma 24. From relation (46) of Lemma 19, we deduce that

$$|u|_{2,\rho} \leqq \mu. \tag{83}$$

From relation (47) and (48), we derive that

$$|aG + J|_{2,\rho} \leqq |a|_\infty F_v |\Omega|_\rho^{1/2} + \lambda(\operatorname*{osc}_{\Omega} b)|u|_{2,\rho}. \tag{84}$$

Then, from (83), (84) and the definition of $Q_\infty(\Omega)$ we obtain that

$$\max_\Omega |u(x) - \gamma| \leqq Q_\infty(\Omega)|aG + J|_{2,\rho} \leqq Q_\infty(\Omega)\left[|a|_\infty F_v |\Omega|_\rho^{1/2} + \lambda\left(\operatorname*{osc}_{\Omega} b\right)\mu\right], \tag{85}$$

and thus we get the result. □

Lemma 25. *Let* $v := \dfrac{\lambda M_w^2 |b|_\infty}{F_v^2}$. *If* $v < 1$, *then* $G(\rho, \theta) \geqq (1 - v)^{1/2} F_v > 0$.

Proof of Lemma 25. Since $|\hat{b}_{\rho,u}(s)| \leqq |b|_\infty$, we have

$$2 \int_{|u>0|_\rho}^{|u>u_+(\rho,\theta)|_\rho} [p(u_*^\rho)]'(s)\hat{b}_{\rho,u}(s)ds \leqq \lambda|b|_\infty u_+^2(\rho,\theta) \leqq \lambda|b|_\infty M_w^2 = \nu F_v^2. \tag{86}$$

Thus, we get

$$G(\rho,\theta) = \left[F_v^2 - 2 \int_{|u>0|_\rho}^{|u>u_+(\rho,\theta)|_\rho} [p(u_*^\rho)]'(s)\hat{b}_{\rho,u}(s)ds\right]_+^{1/2} \geqq (1-\nu)^{1/2}F_v. \quad \square \tag{87}$$

Lemma 26. *Assume that* $a^{2m} :=$ ess $\inf_\Omega a^2 > 0$, $\dfrac{\lambda M_\omega^2 |b|_\infty}{F_v^2} = \nu < 1$ *and* $\lambda|b|_\infty < \dfrac{a^{2m}(1-\nu)}{\nu}$. *Then,* meas $\{(\rho,\theta)\in\Omega\colon \nabla u(\rho,\theta) = 0\} = 0$.

Proof of Lemma 26. Suppose that meas$\{(\rho,\theta)\in\Omega\colon \nabla u(\rho,\theta) = 0\} > 0$. Since $u \in W_{\text{loc}}^{2;p}(\Omega)$ satisfies $-\mathscr{L}u = aG + J$ in Ω, we deduce from Stampacchia's theorem that $\mathscr{L}u = 0$ a.e. on $S_0 = \{(\rho,\theta)\in\Omega\colon \nabla u(\rho,\theta) = 0\}$. Thus

$$-\lambda u_+(\rho,\theta)[b(\rho,\theta) - \tilde{b}_{\rho u}(\rho,\theta)] = a(\rho,\theta)G(\rho,\theta) \quad \text{a.e. on } S_0. \tag{88}$$

We have

$$|\lambda u_+(\rho,\theta)[b(\rho,\theta) - \tilde{b}_{\rho u}(\rho,\theta)]| \leqq \lambda M_\omega |b|_\infty, \tag{89}$$

and from Lemma 25,

$$a(\rho,\theta)^2 G(\rho,\theta)^2 \geqq a^{2m}F_v^2(1-v). \tag{90}$$

From relations (89) and (90), we then deduce that

$$\lambda^2 M_\omega^2 |b|_\infty^2 \geqq a^{2m}F_v^2(1-\nu). \tag{91}$$

Using the definition of ν, we find that $\lambda|b|_\infty \nu \geqq a^{2m}(1-\nu)$, which contradicts the choice of $\lambda|b|_\infty$. $\square$

Proof of Theorem 1. Under the assumptions of Lemma 26, we conclude from Corollary 4 that u satisfies the problem $(\mathscr{P}_2)$, that is, $u \in W^{1,\infty}(\Omega) \cap W_{\text{loc}}^{2,r}(\Omega)$ for all $r \in [1, +\infty[$ and

$$-\mathscr{L}u = a(\rho,\theta)[F_v^2 - 2\int_{|u>0|_\rho}^{|u>u_+(\rho,\theta)|_\rho} [p(u_*^\rho)]'(s)\cdot \hat{b}_{*u}^\rho(s)ds]_+^{1/2}$$
$$+ p'(u(\rho,\theta))[b(\rho,\theta) - b_{*u}^\rho(|u > u(\rho,\theta)|_\rho)],$$
$$u - \gamma \in \tilde{H}_{\rho,0}^1(\Omega), \quad \frac{\partial u}{\partial\theta} = 0 \text{ on } \Gamma_0.$$

Furthermore, $F_u(\rho,\theta) = F(u(\rho,\theta)) > 0$. Moreover, if $\gamma \leqq 0$, then $\hat{m} = \inf_\Omega u \leqq 0$. Finally, since the conditions of Theorem 7 are fulfilled, we conclude that a solution u of $(\mathscr{P}_2)$ is also a solution of $(\mathscr{P}_1)$. $\square$

Remark 8. This proof based on an iterative process can be replaced by a standard Galerkin approach as is done by DIAZ, PADIAL & RAKOTOSON ("Mathematical treatment of the magnetic confinement in a current carrying Stellarator", submitted) and by RAKOTOSON ("Galerkin approximation, strong continuity of the relative rearrangement map and application to Plasma Physic Equations", submitted). This Galerkin approach presents many advantages: For instance, the proof of existence now becomes shorter and the numerical implementation seems to the much easier than in other approaches.

7. Some qualitative results

We start by establishing a condition for the existence of a free boundary. Let ψ_1 be the first eigenfunction associated with λ_1 for the operator $\mathscr{L}$ with a Dirichlet condition, i.e., $\psi_1 \in \tilde{H}^1_{0,\rho}(\Omega)$ and $-\mathscr{L}\psi_1 = \lambda_1\psi_1$ on Ω. Thanks to the identification (19), (20), we know that $\psi_1 > 0$ on Ω. Moreover we can renormalize ψ_1 such that $\lambda_1 \int_\Omega \psi_1(\rho, \theta)\,\rho\,d\rho\,d\theta = 1$.

Theorem 8. *Assume that*

$$-\gamma < F_v \int_\Omega a(\rho, \theta)\psi_1(\rho, \theta)\rho\,d\rho\,d\theta := -\gamma_0.$$

Then any weak solution of $(\mathscr{P}_3)$ *satisfies* $u_+ \not\equiv 0$.

Proof. The proof relies of the identity

$$\lambda_1 \int_\Omega u\psi_1\,\rho\,d\rho\,d\theta - \gamma = \int_\Omega aG\psi_1\rho\,d\rho\,d\theta + \lambda \int_\Omega u_+\psi_1[b(\rho, \theta) - \tilde{b}_{\rho,u}(\rho, \theta)]\rho\,d\rho\,d\theta. \tag{92}$$

Recall that $p'(u) = \lambda u_+$, $J = p'(u)[b - \tilde{b}_{\rho,u}]$ and that u satisfies $-\mathscr{L}u = aG + J$, i.e., $\mathfrak{a}(u - \gamma, v) = (aG + J, v)_\rho$ for all $v \in \tilde{H}^1_{\rho,0}(\Omega)$. We choose $v = \psi_1$. Since $\mathfrak{a}$ is symmetric, we then have

$$\mathfrak{a}(\psi_1, u - \gamma) = (aG + J, \psi_1)_\rho. \tag{93}$$

By the definition of $\mathfrak{a}$, we have $\mathfrak{a}(\psi_1, u - \gamma) = \lambda_1 \int_\Omega (\psi_1(u - \gamma)\rho\,d\rho\,d\theta$, which reduces to

$$\mathfrak{a}(\psi_1, u - \gamma) = \lambda_1 \int_\Omega \psi_1 u\rho\,d\rho\,d\theta - \gamma. \tag{94}$$

Combining relations (93) and (94), we get identity (92). We complete the proof of Theorem 8 by arguing by contradiction. Assume that $u_+ \equiv 0$. Thus, relation (92) is reduced to

$$\lambda_1 \int_\Omega u\psi_1\,\rho\,d\rho\,d\theta = \gamma + F_v \int_\Omega a\psi_1\,\rho\,d\rho\,d\theta. \tag{95}$$

In this case, $\lambda_1 \int_\Omega u\psi_1\,\rho\,d\rho\,d\theta \leqq 0$, so that relation (95) implies that $-\gamma \geqq F_v \int_\Omega a\psi_1\,\rho\,d\rho\,d\theta$. This relation contradicts the choice of γ. □

As a second qualitative property we estimate the quantity

$$|u>0|_\rho = \int_{u>0} \rho \, d\rho \, d\theta.$$

One way to estimate this from below relies on the simple inequality

$$\int_\Omega u_+(\rho,\theta)\,\rho\,d\rho\,d\theta \leqq |u>0|_\rho \max_{\bar\Omega} u_+. \tag{96}$$

But we already have an estimate for $\max_{\bar\Omega} u_+ \leqq M_w$ (see Lemma 24). So (96) gives

$$|u>0|_\rho \geqq \frac{1}{M_w}\int_\Omega u_+(\rho,\theta)\,\rho\,d\rho\,d\theta.$$

Now we estimate the L^1_ρ-norm of u_+. We use identity (92). First, we write $u = u_+ - u_-$, so that relation (92) becomes

$$\int_\Omega [aG + \lambda_1 u_-]\psi_1\,\rho\,d\rho\,d\theta + \gamma = \int_\Omega u_+\psi_1[\lambda_1 - \lambda(b - \tilde{b}_{\rho,u})]\,\rho\,d\rho\,d\theta, \tag{97}$$

from which we derive

$$\gamma + F_v \int_\Omega a\psi_1\,\rho\,d\rho\,d\theta + \int_\Omega a[G - F_v]\psi_1\,\rho\,d\rho\,d\theta$$
$$\leqq [\lambda_1 + \lambda|b|_\infty] \max_{\bar\Omega}\psi_1 \int_\Omega u_+\,\rho\,d\rho\,d\theta. \tag{98}$$

That is,

$$\gamma + F_v\int_\Omega a\psi \leqq (\lambda_1 + \lambda|b|_\infty)(\max_{\bar\Omega}\psi_1)\int_\Omega u_+ + |a\psi_1|_\infty \int_\Omega (F_v - G). \tag{99}$$

We have

$$F_v - G(\rho,\theta) \leqq \left[\lambda \int_{|u>0|_\rho}^{|u>u_+(\rho,\theta)|_\rho} [(u^\rho_{*+})^2]'(s)\hat{b}_{\rho,u}(\rho)d\rho\right]^{1/2},$$

i.e.,

$$F_v - G(\rho,\theta) \leqq \sqrt{\lambda|b|_\infty}\,u_+(\rho,\theta) \quad \text{for } (\rho,\theta)\in\Omega. \tag{100}$$

If we set

$$L(\lambda) := (\lambda_1 + \lambda|b|_\infty)\max_{\bar\Omega}\psi_1 + \sqrt{\lambda|b|_\infty}\,|a\psi_1|_\infty, \tag{101}$$

$$\gamma_0 = -F_v\int_\Omega a\psi_1,$$

then a combination of relations (99) and (100) implies that

$$\frac{\gamma-\gamma_0}{L(\lambda)} \leqq \int_\Omega u_+(\rho,\theta)\rho\,d\rho\,d\theta. \tag{102}$$

Thus we have demonstrated

Theorem 9. *Let u be a weak solution of $(\mathscr{P}_3)$, and let $M^+(\gamma)$ be an upper bound for* $\max_\Omega u_+ \leqq M^+(\gamma)$. *Then*

$$|u > 0|_\rho \geqq \frac{\gamma - \gamma_0}{M^+(\gamma) L(\lambda)} \tag{103}$$

where $L(\lambda)$, γ_0 are given by relation (101).

Another way of estimating $m^\rho(0) = |u > 0|_\rho$ from below is to use the Hölder inequality

$$\int_\Omega u_+ \rho \, d\rho \, d\theta \leqq [m^\rho(0)]^{1/p'} \left(\int_\Omega u_+^p \right)^{1/p} \tag{104}$$

with $1/p + 1/p' = 1$. Then one needs to estimate $\int_\Omega u_+^p$. If $p = 2$, we know from Lemma 19 that

$$\left(\int_\Omega u_+^2 \right)^{1/2} \leqq \mu = \frac{|a|_\infty F_v + \lambda_1 |\gamma|}{\lambda_1 - \lambda \operatorname{osc}_\Omega b} |\Omega|_\rho^{1/2}. \tag{105}$$

Thus we obtain

Theorem 10. *If $\mu^+(\gamma)$ is an upper bound for $\int_\Omega u_+^2$, then*

$$m^\rho(0) \geqq \frac{(\gamma - \gamma_0)^2}{L(\lambda)^2 \mu^+(\gamma)}$$

provided that $\gamma \geqq \gamma_0$. In particular,

$$m^\rho(0) \geqq \frac{(\gamma - \gamma_0)^2}{L(\lambda)^2 \mu^2}.$$

8. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let σ be a weight on $\Omega \subset \mathbb{R}^2$ and let $u \in W_{\text{loc}}^{2,p}(\Omega)$ for some $p > 1$. We want to show that $(m_{u,0}^\sigma)'$ is purely singular. We begin with the case when $u \in C^2(\bar{\Omega})$. The argument now is the same as that in Almgren and Lieb [AL]. For any set $U \subset \mathbb{R}$, we have

$$(m_u^\sigma)'(U) = \int_{\{(\rho,\theta) \in \Omega : u(\rho,\theta) \in U\} \cap \{(\rho,\theta) \in \Omega : \nabla u(\rho,\theta) = 0\}} \sigma(\rho, \theta) \, d\rho \, d\theta. \tag{106}$$

If we set $B = \{(\rho, \theta) \in \Omega \colon \nabla u(\rho, \theta) = 0\}$, then (106) shows us that the support of $(m_{u,0}^\sigma)'$ is included in $u(B)$. But by the Morse-Sard-Federer Theorem, we have meas $(u(B)) = 0$. So meas (support $(m_{u,0}^\sigma)'$) $= 0$, and this means that $(m_{u,0}^\sigma)'$ is purely singular. In the case where $u \in W_{\text{loc}}^{2,p}(\Omega)$ for some $p > 1$, we use a Lusin-type approximation given in [Z, page 159]. We conclude that u can be changed on a set of arbitrarily small measure to become a $C^2(\bar{\Omega})$ function on Ω. It follows readily from the $C^2(\bar{\Omega})$ case that $(m_{u,0}^\sigma)'$ has no absolutely continuous part. □

Proof of Theorem 3. Let $u \in W^{1,\infty}(\Omega)$ be a co-area regular function and u_j a bounded sequence $W^{1,\infty}(\Omega)$ such that

$$u_j \xrightarrow[j \to +\infty]{} u \quad \text{in } W^{1,1}(\Omega).$$

Without loss of generality, we may assume that $u \geqq 0$ and $u_j \geqq 0$. Indeed if $\alpha \in \mathbb{R}$ is such that $\inf_\Omega u \geqq \alpha$ and $\inf_\Omega u_j \geqq \alpha$ $(\forall j)$, then $u_j - \alpha \geqq 0$ converges to $u - \alpha$ in $W^{1,1}(\Omega)$ and $(u_j - \alpha)^\sigma_* = u^\sigma_{j*} - \alpha$, $u^\sigma_* - \alpha = (u - \alpha)^\sigma_*$, so the result on $u_j - \alpha$ and $u - \alpha$ implies the result for u_j and u. The proof now follows the same argument as in ALMGREN & LIEB [AL]. Let σ be an admissible weight and recall that

$$m^\sigma_u(t) = m^\sigma_{u,1}(t) + m^\sigma_{u,0}(t) = \int_{u>t} \sigma(\rho, \theta) d\rho \, d\theta,$$

$$m^\sigma_{u,0}(t) = \int_{\{(\rho,\theta) \in \Omega : u(\rho,\theta) > t, \nabla u(\rho,\theta) = 0\}} \sigma(\rho, \theta) d\rho \, d\theta.$$

Functions m^σ_u, $m^\sigma_{u,0}$ and $m^\sigma_{u,1}$ are in $BV(\mathbb{R})$. We write their Lebesgue-Radon-Nikodym decomposition as

$$(m^\sigma_u)' = (m^\sigma_u)'(t) \wedge dt - \nu, \; (m^\sigma_{u,1})' = (m^\sigma_{u,1})'(t) \wedge dt - \nu_1, \; (m^\sigma_{u,0})' = (m^\sigma_{u,0})'(t) \wedge dt - \nu_0.$$

Since u is co-area regular, the absolutely continuous part of $(m^\sigma_{u,0})'(t)$ is identically zero. The measures ν, ν_1, ν_0 (i.e., the singular parts) are then in $M(\mathbb{R})$ (the set of bounded Radon measures on $\mathbb{R}$). We now need some auxiliary results:

Lemma 27. *There exists a sequence still denoted by (u_j) such that for almost every $t \in \mathbb{R}$,*

$$\liminf_{j \to +\infty} |(m^\sigma_{u_j,1})'(t)| \geqq |(m^\sigma_{u,1})'(t)|.$$

This lemma is even true for u not necessarily a co-area regular function and for any weight σ. The proof closely follows Theorem 3.5 of [AL], so we omit it.

Lemma 28. *If u is a co-area regular function, then*

$$\liminf_j |(m^\sigma_{u_j})'(t)| \geqq |(m^\sigma_u)'(t)| \quad \textit{for almost every } t \in \mathbb{R}.$$

Proof of Lemma 28. If u is a co-area regular function, then the absolutely continuous parts of the measures $(m^\sigma_u)'$ and $(m^\sigma_{u,1})'$ are the same, i.e., for almost every $t \in \mathbb{R}$,

$$(m^\sigma_u)'(t) = (m^\sigma_{u,1})'(t).$$

From Lemma 26, we then have

$$|(m^\sigma_u)'(t)| = |(m^\sigma_{u,1})'(t)| \leqq \liminf_j |(m^\sigma_{u_j,1})'(t)|. \tag{107}$$

But

$$\liminf_j |(m^\sigma_{u_j,1})'(t)| \leqq \liminf_j |(m^\sigma_{u_j})'(t)|. \tag{108}$$

(Indeed, $|(m^\sigma_{u_j})'(t)| = -(m^\sigma_{u_j})'(t) = -(m^\sigma_{u_j,0})'(t) - (m^\sigma_{u_j,1})'(t) = |(m^\sigma_{u_j,0})'(t)| + |(m^\sigma_{u_j,1})'(t)|$ for almost every t.) Thus combining (107) and (108) we get the conclusion. □

One of the main ideas of ALMGREN & LIEB [AL] is the use of the notion of arc length associated with a nonincreasing function.

Definition 10. (see [AL]). Let $f\colon \mathbb{R} \to \mathbb{R}_+$ be a monotone nonincreasing function. Consider the Lebesgue-Radon-Nikodyn decomposition of the measure df:

$$df = f'(t)\,dt - dv$$

where $f'(t)$ is the a.e. derivative of f (the absolutely continuous part) and $dv \geqq 0$ the singular part. The *length of f over a bounded interval* $[\alpha, \beta]$ is defined as the number

$$L_{[\alpha,\beta]}(f) := \int_{[\alpha,\beta]} \sqrt{1 + f'(t)^2}\,dt + \int_{[\alpha,\beta]} dv.$$

The following result is the same as Theorem 8.3 given in [AL]:

Theorem 11. *Let $(f_j)_{j\geqq 0}$ be a sequence of nonincreasing functions from $\mathbb{R}$ into $\mathbb{R}_+$. Consider the Lebesgue-Radon-Nikodym decomposition*

$$df_j = f_j'(t)\,dt - dv_j, \quad j = 0, 1, \ldots.$$

Assume that

$$\liminf_j |f_j'(t)| \geqq |f_0'(t)| \quad \textit{for almost every } t \in \mathbb{R}, \tag{109}$$

$$|f_j - f_0| \to 0 \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}) \textit{ as } j \to \infty. \tag{110}$$

Then,

$$\lim_j \left(\int_{\mathbb{R}} \varphi \sqrt{1 + f_j'(t)^2}\,dt + \int_{\mathbb{R}} \varphi\,dv_j \right) = \int_{\mathbb{R}} \varphi \sqrt{1 + f_0'(t)^2}\,dt + \int_{\mathbb{R}} \varphi\,dv_0$$

for all $\varphi\colon \mathbb{R} \to \mathbb{R}$ continuous with compact support.

We shall apply Theorem 11 with

$$f_j(t) = m^\sigma_{u_j}(t), \quad f_0(t) = m^\sigma_u(t).$$

We have to check that the conditions (109) and (110) are fulfilled. First, Lemma 27 ensures that (109) is satisfied. For (110), we have

Lemma 29. (i) $\lim_j m^\sigma_{u_j}(t) = m^\sigma_u(t)$ *a.e. in* $\mathbb{R}$, *and* (ii) $|m^\sigma_{u_j} - m^\sigma_u| \xrightarrow[j\to+\infty]{} 0$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. For any $t \in \mathbb{R}$, we have

$$m_u^\sigma(t) = |u > t|_\sigma \leqq \liminf_j |u_j > t|_\sigma \leqq \limsup_j m_{u_j}^\sigma(t) \leqq |u \geqq t|_\sigma \tag{111}$$

(recall the notation $|u_j > t|_\sigma = m_{u_j}^\sigma(t)$). We know that the set $D = \{t \in \mathbb{R} : |u = t|_\sigma > 0\}$ is at most countable. So if $t \in \mathbb{R} \backslash D$, then $\lim_j m_{u_j}^\sigma(t) = m_u^\sigma(t)$. This shows (i). The statement (ii) is a consequence of the pointwise convergence (i) and the Lebesgue dominated convergence theorem. □

Corollary 5. *For any reals* (α, β), $\alpha < \beta$,

$$\lim_j \int_{[\alpha,\beta]} \sqrt{1 + (m_{u_j}^\sigma)'(t)^2}\, dt + \int_{[\alpha,\beta]} dv_j = \int_{[\alpha,\beta]} \sqrt{1 + (m_u^\sigma)'(t)^2}\, dt + \int_{[\alpha,\beta]} dv.$$

Here, $dm_{u_j}^\sigma = (m_{u_j}^\sigma)'(t)\, dt - dv_j$, $dm_u^\sigma = (m_u^\sigma)'(t)\, dt - dv$, $dv_j \geqq 0$ *and* $dv \geqq 0$.

Proof. This is a consequence of Theorem 11 applied to $m_{u_j}^\sigma$ and m_u^σ. □

The second important property of the arc length is its conservation under reflection. So if f has a generalized inverse, then the arc length of the generalized inverse is the same as the arc length of f. In particular, from Theorem 8.5 of ALMGREN & LIEB [AL], we can state

Lemma 30. *Let* $v \in L^\infty(\Omega)$. *Set* $m = \operatorname{ess\,inf}_\Omega v$ *and* $M = \operatorname{ess\,sup}_\Omega v$. *Let* $m_v^\sigma = |v > t|_\sigma = \int_{v>t} \sigma(\rho, \theta)\rho\, d\rho\, d\theta$, *the distributional function of* v *with respect to the measure* σ, *and let* v_*^σ *be its generalized inverse, i.e.,*

$$v_*^\sigma(s) = \begin{cases} \inf\{t \in \mathbb{R} m_v^\sigma(t) \leqq s\} & \text{if } s \in [0, |\Omega|_\sigma[, \\ M & \text{if } s = |\Omega|_\sigma. \end{cases}$$

Consider the Lebesgue decompositions of dm_v^σ *and* dv_*^σ:

$$dm_v^\sigma = (m_v^\sigma)'(t)\, dt - d\mu, \quad dv_*^\sigma = (v_*^\sigma)'(s)\, ds - dv.$$

Then

$$\int_0^{|\Omega|_\sigma} \sqrt{1 + (v_*^\sigma)'(s)^2}\, ds + \int_{[0,|\Omega|_\sigma]} dv = \int_m^M \sqrt{1 + (m_v^\sigma)'(t)^2}\, dt + \int_{[m,M]} d\mu.$$

In order to complete the proof of Theorem 3, we reproduce again Theorem 8.7 of ALMGREN & LIEB [AL]:

Theorem 12. *Assume that* $v, v_1, v_2, \ldots$ *are nonincreasing absolutely continuous functions mapping* $[0, b]$ *into an interval with*

$$\lim_{j \to +\infty} \int_0^b |v_j(s) - v(s)|\, ds = 0,$$

$$\lim_{j \to +\infty} \int_0^b \sqrt{1 + v_j'(s)^2}\, ds = \int_0^b \sqrt{1 + v'^2(s)^2}\, ds.$$

Then for any (α, β) *such that* $0 \leqq \alpha < \beta \leqq b$,

$$\lim_{j \to +\infty} \int_\alpha^\beta \sqrt{1 + v_j'(s)^2}\, ds = \int_\alpha^\beta \sqrt{1 + v'(s)^2}\, ds,$$

$$\lim_{j \to +\infty} \int_0^b |v_j'(s) - v'(s)|\, ds = 0.$$

Completion of the proof of Theorem 3. We apply Theorem 12 to $v_j = u_{j*}^\sigma$, $v = u_*^\sigma$, and $b = |\Omega|_\sigma$. If σ is an admissible weight, we know from Lemma 5 that u_{j*}^σ and u_*^σ are absolutely continuous on Ω_*^σ. Furthermore, since by assumption u_j converges to u in $L^1(\Omega)$ (and thus in $L^1(\Omega, \sigma)$), we deduce that

$$u_{j*}^\sigma \text{ converges to } u_*^\sigma \text{ in } L^1(\Omega_*^\sigma). \tag{112}$$

From Lemma 29 and the convergences of Corollary 5, we easily deduce that

$$\lim_{j \to +\infty} \int_0^{|\Omega|_\sigma} \sqrt{1 + (u_{j*}^\sigma)'(s)^2}\, ds = \int_0^{|\Omega|_\sigma} \sqrt{1 + (u_*^\sigma)'(s)^2}\, ds.$$

All the assumptions required for the application of Theorem 12 are fulfilled, and thus we have

$$\lim_{j \to +\infty} \int_0^{|\Omega|_\sigma} |(u_{j*}^\sigma)'(s) - (u_*^\sigma)'(s)|\, ds = 0.$$

With the help of the estimate of Lemma 5, we get the conclusion, that is,

$$\lim_{j \to +\infty} \int_0^{|\Omega|_\sigma} |(u_{j*}^\sigma)'(s) - (u_*^\sigma)'(s)|^q\, ds = 0 \; \forall q \in [1, r_\sigma[. \quad \square$$

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