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Équations aux dérivées partielles/*Partial Differential Equations*

Propagation properties for scalar conservation laws

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Abstract. We study the propagation of an initial disturbance $u_0(x)$ of an equilibrium state $\mathrm{s} \in \mathbb{R}$ for the scalar conservation law

$$u_t + \varphi(u)_x = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}.$$

We give a necessary and sufficient condition on φ for the following propagation property: *if support of* $(u_0(\,.\,) - \mathrm{s})$ *is compact then the support of* $(u_0(t, \,.\,) - \mathrm{s})$ *is also compact for* $t \in [0, T_0)$, *for some* $T_0 \in (0, +\infty]$. The proofs are based on the study of suitable associated Riemann problems.

Propriétés de propagation pour les lois de conservation scalaires

Résumé. *On étudie la propagation d'une perturbation initiale* $u_0(x)$ *d'un état d'équilibre* $\mathrm{s} \in \mathbb{R}$ *de la loi de conservation scalaire*

$$u_t + \varphi(u)_x = 0 \quad \text{dans } (0, +\infty) \times \mathbb{R}.$$

On donne une condition nécessaire et suffisante sur φ *pour que la propriété suivante de propagation soit satisfaite :* si le support de $(u_0(\,.\,) - \mathrm{s})$ est compact alors le support de $(u_0(t, \,.\,) - \mathrm{s})$ est aussi compact pour tout $t \in [0, T_0)$, pour un certain $T_0 \in (0, +\infty]$. *Les démonstration sont basées sur l'étude de quelques problèmes de Riemann associés.*

Version française abrégée

On considère la *propriété* suivante *de propagation localisée* (PLP) par rapport à l'état d'équilibre $\mathrm{s} \in \mathbb{R}$ pour la solution d'entropie généralisée (g.e.s.) du problème de Cauchy associé à la loi de conservation scalaire (1) où φ est une fonction continue et $u_0 \in \mathrm{L}^\infty(\mathbb{R})$: *si le support de* $(u_0(\,.\,) - \mathrm{s})$ *est compact, alors le support de* $(u(t, \,.\,) - \mathrm{s})$ *est aussi compact pour tout* $t \in [0, T_0)$, *pour une valeur* $T_0 \in (0, +\infty]$. Notre résultat principal (théorème 1) donne un critère pour la PLP en termes d'une condition locale sur $\varphi(u)$ en $u = \mathrm{s}$: la condition (5). La technique des déformations repose sur la considération des g.e.s. associées à des données initiales constantes par morceaux et

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qui sont utilisées comme des *fonctions barrières*. Ces g.e.s. sont construites par l'intermédiaire de certains problèmes de Riemann, et par un résultat dû à Gelfand [3] (*voir* aussi [7]) elles peuvent être caractérisées en remplaçant φ par ses enveloppes convexe et concave: si $\varphi \in C[a, b]$, on désigne par $\varphi^{\text{c-x}}_{a,b}(u)$ (respectivement $\varphi^{\text{c-v}}_{a,b}(u)$) l'enveloppe convexe (respectivement concave) de φ sur l'intervalle $[a, b]$. Pour montrer que (5) est suffisante on établit d'abord un lemme qui suppose $\max\{N(m), N(\mathrm{s}), N(M)\} < +\infty$ pour certains $m, M \in \mathbb{R}$ (qui, après, sont choisis comme $m < m_0 := \operatorname{ess\,inf} u_0$ et $M > M_0 = \operatorname{ess\,sup} u_0$).

On montre que (5) est nécessaire dans le sens suivant : si $N(\mathrm{s}) = \infty$, pour tout $u_0(x)$ satisfaisant $u_0(x) \geq \mathrm{s}$ et $u_0(x) \geq c > \mathrm{s}$ sur un intervalle $[a, b]$ alors la g.e.s. $u(t, x)$ est telle que $\operatorname{supp}(u(t, .) - \mathrm{s})$ est non borné pour tout $t \in (0, T_0]$, pour quelque $T_0 > 0$. En effet: si $u^b_c(t, x)$ désigne la solution du problème de Riemann au point $(0, b)$ alors $\operatorname{supp}(u^b_c(t, .) - s)$ est non borné et la conclusion est obtenue par comparaison grâce au fait que si $(\varphi^{\text{c-x}}_{\mathrm{s},c})(u))'_{u=c-0} < +\infty$ alors le problème de Riemann au point $(0, a)$ peut être résolu par le moyen de $\varphi^{\text{c-x}}_{\mathrm{s},c}(u)$. Si $(\varphi^{\text{c-x}}_{\mathrm{s},c}(u))'_{u=c-0} = +\infty$, on remplace la constante c par $c^* \in (\mathrm{s}, c)$, telle que $\varphi^{\text{c-x}}_{\mathrm{s},c}(c^*) = \varphi(c^*)$.

Dans le cas où $u_0(x) \geq \mathrm{s}$ on $\mathbb{R}$ et $\operatorname{supp}(u_0(\,.\,) - \mathrm{s}) \subset [a, +\infty)$ on montre (proposition 1) que si on a (6) alors $\operatorname{supp}(u(t, .) - \mathrm{s}) \subset [a + p_{\mathrm{s}} t, +\infty)$ pour tout $t > 0$ ou $p_{\mathrm{s}} := (\varphi^{\text{c-x}}_{a,b}(u))'_{u=\mathrm{s}+0} < +\infty$. Le théorème 2 montre la double estimation $[a + p_{\mathrm{s}} t, +\infty) \supset \operatorname{supp}(u(t, .) - \mathrm{s}) \supset [b + q_d t, +\infty)$ pour des données initiales telles que $u_0(x) - \mathrm{s} \geq (d - \mathrm{s})\chi_{[a,b]}(x)$ *p.p.* $x \in \mathbb{R}$, quand on suppose (6), (7) et (8). De nouveau l'étude des problèmes de Riemann associés conduisent à la démonstration.

Le reste de la Note est consacré à l'étude du cas $\mathrm{s} = 0$ et φ strictement concave, $\varphi'(+0) = +\infty$ (*voir* (11)). On définit les transformées de Legendre : $q = \varphi'(r)$,

$$\text{(LT)} \qquad F(r) := \varphi(r) - r\varphi'(r), \qquad G(q) := F(H(q)) \qquad \text{et} \qquad H(\varphi'(r)) \equiv r.$$

On commence par caractériser la g.e.s. associée a $u^M_0(x) = M\chi_{[a,b]}(x)$ où $M \in (0, M_0]$ (*voir* lemme 2). La solution correspondante $u^M(t, x)$ est obtenue explicitement en utilisant l'onde de choc

$$(\mathrm{E}_\sigma) \qquad \sigma(t) = \begin{cases} a + \dfrac{\varphi(M)}{M} t & \text{pour} \quad \leq t \leq t_0, \\ b + tG^{-1}\left(\dfrac{M(b-a)}{t}\right) & \text{pour} \quad t \geq t_0. \end{cases}$$

Grâce à ce résultat technique on peut améliorer la conclusion du théorème 2 : dans le théorème 3 on montre que $u(t, x) \leq u^{M_0}(t, x)$ ce qui implique $\operatorname{supp} u(t, \cdot) \subset \left[a + \frac{\varphi(M_0)}{M_0} t, +\infty\right)$ si $t \in [0, t_0]$ et $\operatorname{supp} u(t, \cdot) \subset \left[b + tG^{-1}\left(\frac{M_0(b-a)}{t}\right), +\infty\right)$ si $t \in [t_0, +\infty)$ avec t_0 donné par (12). Dans le cas de $u_0(x) \geq m$ sur $[\alpha, \beta] \subset [a, b]$ avec $m \in (0, M_0]$ on obtient des estimations différentes (*voir* théorème 4).

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1. Introduction and statement of the main results

This Note is devoted to the study of the perturbation of equilibrium states in the class of the *generalized entropy solutions* of the Cauchy problem (CP):

$$(1) \qquad u_t + \varphi(u)_x = 0 \quad \text{in } (0, T) \times \mathbb{R}, \qquad T \leq \infty,$$

$$(2) \qquad u(0, \cdot) = u_0(\cdot) \quad \text{in } \mathbb{R},$$

where φ is a continuous real function and $u_0 \in L^\infty(\mathbb{R})$ (*see e.g.* [5] and [6]). We assume that for *a.e.* $x \in \mathbb{R}$

$$-\infty < m_0 := \text{ess inf } u_0 \leq u_0(x) \leq \text{ess sup } u_0 := M_0 < +\infty. \tag{3}$$

DEFINITION. – A bounded measurable function $u(t, x)$ on $(0, T) \times \mathbb{R}$ is called a *generalized entropy solution* (in short *g.e.s.*) of (1), (2) if $|u - k|_t + \operatorname{sign}(u - k)(\varphi(u) - \varphi(k))]_x \leq 0$ in the sense of distributions, for any $k \in \mathbb{R}$, and $u(t, x) \to u_0(x)$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \to 0$ essentially.

For convenience for our techniques of proofs we extend φ outside $[m_0, M_0]$ by assuming $\varphi(u) \equiv \varphi(m_0)$ for $u < m_0$ and $\varphi(u) \equiv \varphi(M_0)$ for $u > M_0$.

One of main characterizations of the *propagation of an initial perturbation* $u_0(x)$ *from an equilibrium state* s (here assumed to be a constant such that $\mathrm{s} \in [m_0, M_0]$) is the evolution of the support of the function $u(t, .) - \mathrm{s}$. The aim of this Note is the study of such an evolution. The technique of our proofs is based on the consideration of special g.e.s. associated to piecewise constant initial functions which are used as *barrier functions*. These special g.e.s. are defined through appropriated Riemann problems constructed with the help of convex and concave envelopes of flux function: for given $\varphi(u) \in C[a, b]$, we shall denote by $\varphi^{\text{c-x}}_{a,b}(u)$ (respectively $\varphi^{\text{c-v}}_{a,b}(u)$) the convex (respectively concave) envelope of $\varphi(u)$ on the interval $[a, b]$. We introduce the following special

PROPERTY OF LOCALIZED PROPAGATION. – *We say that problem* (1), (2) *satisfies the property of localized propagation* (PLP *in the following*) *if given* u_0 *such that* $\operatorname{supp}(u_0(\,.\,) - \mathrm{s})$ *is compact then* $\operatorname{supp}(u(t, .) - \mathrm{s})$ *is also compact for any* $t \in [0, T_0)$, *for some* $T_0 \in (0, +\infty]$.

It is well-known (*see* [5]) that in the case of Lipschitz continuous flux functions φ on $[m_0, M_0]$ such property holds, with $T_0 = +\infty$, in view of the characteristic cone estimate

$$\int_{S_t} |u(t, x) - \mathrm{s}|dx \leq \int_{S_0} |u_0(x) - \mathrm{s}|dx,$$

with $S_t := \{x : |x - x_0| \leq \mathrm{R} - \mathrm{L}t\}$, x_0 and R arbitrary and L the Lipschitz constant of φ.

Our first and main result (theorem 1) gives a criterion for PLP as a local condition on $\varphi(u)$ near $u = \mathrm{s}$; given $w \in \mathbb{R}$ define

$$N(w) = \limsup_{r \to w} \frac{|\varphi(r) - \varphi(w)|}{|r - w|}.$$

LEMMA 1 (three points lemma). – *Assume that there exist* m, s, $M \in \mathbb{R}$ *for which*

$$\max\{N(m), N(\mathrm{s}), N(M)\} < +\infty. \tag{4}$$

Let u *be the g.e.s. with* $u_0(x)$ *satisfying* (3) *and* $m \leq m_0 \leq \mathrm{s} \leq M_0 \leq M$. *Then the* PLP *holds.*

THEOREM 1. – *Given* s, $\mathrm{s} \in [m_0, M_0]$, *the sufficient and necessary condition for* PLP *is*

$$N(\mathrm{s}) < \infty. \tag{5}$$

The necessity part is understood in the following sense: if $N(\mathrm{s}) = \infty$, *for any* $u_0(x)$ *satisfying* $u_0(x) \geq \mathrm{s}$ *and* $u_0(x) \geq c > \mathrm{s}$ *on some interval* $[a, b]$ *then the g.e.s.* $u(t, x)$ *is such that* $\operatorname{supp}(u(t, .) - \mathrm{s})$ *is unbounded for any* $t \in (0, T_0]$, *for some* $T_0 > 0$.

Our next result concerns the behavior of the boundary of $\operatorname{supp}(u(t, .) - \mathrm{s})$. We start by assuming

$$\limsup_{r \to \mathrm{s}+0} \frac{\varphi(r) - \varphi(\mathrm{s})}{r - \mathrm{s}} > -\infty. \tag{6}$$

Notice that $p_s := (\varphi_{a,b}^{c\text{-}x}(u))'_{u=s+0} < +\infty$ in view of (6).

PROPOSITION 1. – *Assume* (6) *and let* $u_0(x)$ *satisfying* (3) *and* $u_0(x) \geq s$ *on* $\mathbb{R}$. *If* $\operatorname{supp}(u_0(\,.\,) - s) \subset [a, +\infty)$, *then* $\operatorname{supp}(u(t,\,.\,) - s) \subset [a, +p_s t, -\infty)$ *for any* $t > 0$.

To obtain a two-side estimate on $\operatorname{supp}(u(t,\,.\,) - s)$ we assume additionally that φ satisfies

$$\limsup_{r \to s+0} \frac{\varphi(r) - \varphi(s)}{r - s} = +\infty \tag{7}$$

and suppose, also, that there exists a number $d \in (s, M_0)$ for which

$$\limsup_{r \to d-0} \frac{|\varphi(r) - \varphi(d)|}{r - d} < +\infty. \tag{8}$$

Denote

$$p_d = (\varphi_{s,d}^{c\text{-}x}(u))'_{u=d-0} \qquad \text{and} \qquad q_d = (\varphi_{s,d}^{c\text{-}v}(u))'_{u=d-0}.$$

It is clear that both numbers are finite in view of (8) and that $p_d \geq q_d$.

THEOREM 2. – *Let* $u(t, x)$ *be the g.e.s. of the Cauchy problem* (CP) *with* u_0 *such that*

$$u_0(x) - s \geq (d - s)\chi_{[a,b]}(x) \quad a.e.\ x \in \mathbb{R}, \tag{9}$$

where $d \leq M_0$. *Then, if* $T_0 := (b - a)/(p_d - q_d)$ *and* $t \in [0, T_0)$

$$[a + p_s t, +\infty) \supset \operatorname{supp}(u(t, \cdot) - s) \supset [b + q_d t, +\infty). \tag{10}$$

The following results contain sharper estimates for the case of concave flux functions and $s = 0$. We introduce the condition

$$\varphi \in C^0([0, M_0]) \cap C^1((0, M_0]), \quad \varphi \text{ is strictly concave on } [0, M_0] \text{ and } \varphi'(+0) = +\infty. \tag{11}$$

We first define the auxiliary functions related to the Legendre transformation: $q = \varphi'(r)$,

$$F(r) := \varphi(r) - r\varphi'(r), \qquad G(q) := F(H(q)) \qquad \text{and} \qquad H(\varphi'(r)) \equiv r. \tag{LT}$$

Our next results are based on the following technical lemma.

LEMMA 2. – *Let* φ *satisfying* (11) *and assume* (3) *with* $m_0 = 0$. *Let* $u_0^M(x) = M\chi_{[a,b]}(x)$ *with* $M \in (0, M_0]$,

$$t_0 := \frac{M(b - a)}{F(M)} \qquad \text{and} \qquad x_0 := b + t_0\varphi'(M). \tag{12}$$

Then the g.e.s. $u^M(t, x)$ *of* (1), (2) *with initial datum* $u_0^M(x)$ *is given for* $t \geq 0$ *by* 0 *for* $x \leq \sigma(t)$, M *for* $\sigma(t) \leq x \leq b + t\varphi'(M)$, *and* $H\left(\frac{x-b}{t}\right)$ *for* $x \geq \max(\sigma(t), b + t\varphi'(M))$, *where the shock wave* $\sigma(t)$ *is defined by* (E_σ) *(see the french version).*

The following result improves the conclusion of theorem 2 when $s = 0$ and follows from lemma 2 via the comparison principle (*see* [6] and [1]).

THEOREM 3. – *Let* φ *satisfying* (11) *and let* $u_0(x)$ *satisfying* (3) *with* $m_0 = 0$ *and* $\operatorname{supp} u_0(\,.\,) \subset [a,\, b]$. *Let* $u(t,\, x)$ *be the g.e.s. of* (1), (2). *Then* $0 \le u(t,\, x) \le u^{M_0}(t,\, x)$ *for any* $t \ge 0,\ x \in \mathbb{R}$. *It follows that* $\operatorname{supp} u(t,\, .\,) \subset \left[a + \frac{\varphi(M_0)}{M_0} t,\, +\infty\right)$ *if* $t \in [0,\, t_0]$ *and* $\operatorname{supp} u(t,\, .\,) \subset \left[b + tG^{-1}\left(\frac{M_0(b-a)}{t}\right),\, +\infty\right)$ *if* $t \in [t_0,\, +\infty)$, *where* t_0 *is given by* (12).

Another application of lemma 2 is the following one

THEOREM 4. – *Suppose the conditions of theorem* 3 *and, in addition, that* $u_0(x) \ge m$ *on* $[\alpha,\, \beta] \subset [a,\, b]$, *for some constant* $m \in (0,\, M_0]$. *Define* $\tau_0 := \frac{m(\beta-\alpha)}{F(m)}$ *and* $y_0 := \beta + \tau_0 \varphi'(m)$. *Then*

$$u(t,\, x) \ge \begin{cases} H\left(\dfrac{x-\beta}{t}\right) > 0 & for \quad (t,\, x) \in \left\{x \ge \beta + t\varphi'(m),\ G\left(\dfrac{x-\beta}{t}\right) \ge \dfrac{m(\beta-\alpha)}{t}\right\}, \\ m & for \quad (t,\, x) \in \left\{0 \le t \le \tau_0,\ \alpha + \dfrac{\varphi(m)}{m} t \le x \le \beta + t\varphi'(m)\right\}. \end{cases}$$

Consequently

$$\operatorname{supp} u(t,\, \cdot) \supset \left[\alpha + \frac{\varphi(m)}{m} t,\, +\infty\right) \qquad if \quad 0 \le t \le \tau_0,$$
$$\operatorname{supp} u(t,\, \cdot) \supset \left[\beta + tG^{-1}\left(\frac{m(\beta-\alpha)}{t}\right),\, +\infty\right) \qquad if \quad t > \tau_0.$$

The proof of theorem 4 is a simple application of the comparison principle and the lemma 2 replacing a by α and b by β.

Remark 1. – Theorems 2, 3 and 4 improve proposition 6.3 of [2]. We notice that the estimates on the support of $u(t,\, x)$ given in theorem 4 show that if, for instance, $u_0(x) \ne 0,\ \forall x \in (a,\, b)$ and $u_0(x) \equiv 0$ *on* $\mathbb{R}\backslash(a,\, b)$, only one *interface* (separating the regions where $\{u = 0\}$ and $\{u \ne 0\}$), is formed under assumptions (7), (8). An explicit solution on the set $(0,\ \alpha/(1-\alpha)) \times \mathbb{R}$ for the special case of $\varphi(r) = |r|^\alpha/\alpha,\ \alpha \in (0,\, 1)$, was published in [6]. Notice also that theorems 3 and 4 give two side global estimates on the support of the solution. For other references on the propagation of the support of the g.e.s. of scalar conservation laws *see* [4], [9], [8] and the references therein.

2. Sketch of proofs

Proof of lemma 1. – Assume that $\operatorname{supp}(u_0(\,.\,) - \mathrm{s}) \subset [a,\, b]$. For any $c \in [m,\, M]$ let $u_c(t,\, x)$ be the g.e.s. of (1), (2) with initial datum $u_c(0,\, x) = c$ for $x \in [a,\, b]$, and $u_c(0,\, x) = \mathrm{s}$ for $x \in \mathbb{R}\backslash[a,\, b]$.

It is clear that, if $c = m$ and $c = M$, there is $t_c > 0$ such that the function $u_c(t,\, x)$ coincides, for $t \in [0,\, t_c]$, which the solution of the associated Riemann problem which is defined through the convex and concave envelopes of the flux function φ on the intervals $[m,\, \mathrm{s}]$ and $[\mathrm{s},\, M]$ (*see* [3], [7]). Notice that condition (4) implies that such envelopes are Lipschitz continuous functions and consequently, in particular, there exist some constants $\alpha_c,\, \beta_c$ such that $u_c(t,\, x) \equiv \mathrm{s}$ for $x < a + \alpha_c t$ and $x > b + \beta_c t$. From the comparison principle we deduce that $u_m(t,\, x) \le u(t,\, x) \le u_M(t,\, x)$ *on* $(0,\, +\infty) \times \mathbb{R}$. this proves PLP with $T_0 := \min(t_m,\, t_M)$. $\square$

Proof of theorem 1. – The proof of the sufficiency part is an immediate corollary of lemma 1. For the necessity part, suppose that $N(\mathrm{s}) = +\infty$. Without loss of generality we can assume (7). Let us show that for any $u_0(x)$ such that $u_0(x) \ge \mathrm{s}$ and $u_0(x) \ge c > \mathrm{s}$ on some interval $[a,\, b]$ then the g.e.s. $u(t,\, x)$ is such that $\operatorname{supp}(u(t,\, .\,) - \mathrm{s})$ is unbounded for any $t \in [0,\, T_0]$, for some $T_0 > 0$. Indeed, as in the proof of lemma 1 we introduce the g.e.s. $u_c(t,\, x)$. It is clear that if $u_c^b(t,\, x)$ denotes the solution of the Riemann problem in the point $(0,\, b)$ then $\operatorname{supp}(u_c^b(t,\, .\,) - \mathrm{s})$ is unbounded. Assume we have $(\varphi_{\mathrm{s},c}^{c\text{-x}})(u))'_{u=c-0} < +\infty$. Then the Riemann problem at the point $(0,\, a)$ is solved with the help of the Lipschitz-continuous envelope of $\varphi(u)$ and by applying the comparison principle we have $u(t, x) > u_c(t, x)$. In the case $(\varphi_{s,c}^{c-x})(u))'_{u=c-0} = +\infty$, we replace constant c by c' such that $c^* \in$ (s, c), $\varphi_{s,c}^{c-x})(c^*) = \varphi(c^*)$. □

Proof of proposition 1. – Let $w(t, x)$ be the g.e.s. of the Riemann problem with initial datum $w(0, x) \equiv M + (\mathrm{s} - M)\chi_{(-\infty,a]}(x)$. It is clear that $w(t, x) \equiv \mathrm{s}$ for $x < a + p_{\mathrm{s}}t$, $\forall\, t \geq 0$. Thus, the conclusion follows from the inequalities $\mathrm{s} \leq u(t, x) \leq w(t, x)$. □

Proof of theorem 2. – In view of (7) $(\varphi_{s,d}^{c-v})(u))'_u \to +\infty$ as $u \to \mathrm{s} + 0$. Thus the solution $u^b(t, x)$ of the Riemann problem for (1) with initial datum $u^b(0, x) := \mathrm{s} + (d - \mathrm{s})\chi_{(-\infty,b]}$ satisfies $u^b(t, x) \equiv d$ for $x < b + q_d t$, $t > 0$. Besides, there exist some sequences $\{s_\nu\} \uparrow \mathrm{s}$ and $\mathrm{N}_\nu \to +\infty$ as $\nu \to \infty$ such that $u^b(t, \mathrm{N}_\nu t) = \mathrm{s}_\nu$ for any $t > 0$. The right hand side of the inclusions (10) follows from the obvious inequality $u(t, x) \geq u_-(t, x)$, where $u_-(t, x)$ is the g.e.s. of (CP) (1), (2), with $u_-(0, x) = \mathrm{s} + (d - \mathrm{s})\chi_{[a,b]}$, if we remark $u_-(t, x) = u^b(t, x)$ for $x > b + q_d t$, $0 \leq t \leq \frac{d-a}{p_d - q_d} = T_0$ that (for $t \in (0, T_0]$ the function $u_-(t, x)$ may be constructed with the help of solutions of the Riemann problem at the points $(0, a)$ and $(0, b)$). The first inclusion in (10) follows from proposition 1. □

Proof of lemma 2. – It is based on the construction of $u^M(t, x)$. At the points $(0, a)$ and $(0, b)$ we have a shock wave and a rarefaction wave respectively. As a result of the interaction of these waves at the point (t_0, x_0) a new shock wave arises; along the corresponding discontinuity lines $x = \sigma(t)$, $t \geq t_0$ the Rankine-Hugoniot condition holds:

$$\text{(R-H)} \qquad \frac{dx}{dt} = \frac{\varphi(u)}{u}, \qquad u \equiv H\left(\frac{x - b}{t}\right).$$

Consequently $dx = \varphi'(u)dt + td\varphi'(u)$. Taking into account that $dF = -ud\varphi'(u)$ we obtain from (R-H) $-\frac{ud\varphi'(u)}{\varphi(u) - u\varphi'(u)} = \frac{dt}{t}$. Thus $-\frac{dF(u)}{F(u)} = \frac{dt}{t}$, which implies that $tF(u) = C$. The final formulae for $\sigma(t)$ follows from the initial condition $\sigma(t_0) = x_0$ and $G\left(\frac{\sigma(t) - b}{t}\right) = \frac{t_0}{t}F(M)$ for $t \geq t_0$. □

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