J. Math. Pures Appl.,
76, 1997, *p.* 353-375

APPROXIMATE CONTROLLABILITY OF THE STOKES SYSTEM ON CYLINDERS BY EXTERNAL UNIDIRECTIONAL FORCES

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ABSTRACT. – We give some negative and positive results on the approximate controllability of the Stokes system formulated on a cylinder $\Omega = G \times \mathbb{R}$ of $\mathbb{R}^3$ when the control is a density of external unidirectional forces. We distinguish the case where the direction of the controls $\mathbf{e}$ is parallel to the cylinder generatrix ($\mathbf{e} = \mathbf{e}_3$) from the one where $\mathbf{e}$ is orthogonal to this generatrix ($\mathbf{e} = \mathbf{e}_1$). A negative result in the case of $\mathbf{e} = \mathbf{e}_3$ is proved for periodic boundary conditions on x_3 and homogeneous Dirichlet conditions on $\partial G \times \mathbb{R}$ where G is a general set of $\mathbb{R}^2$. In contrast to that, the approximate controllability is proved for homogeneous Dirichlet conditions on $\partial\Omega$ (*i.e.* zero on $\partial G \times \mathbb{R}$ and solutions in $\left(L^2(G \times \mathbb{R})\right)^3$ for any t), when G is a rectangle and $\mathbf{e} = \mathbf{e}_1$ is orthogonal to the cylinder generatrix.

1. Introduction

This paper is devoted to the study of the approximate controllability of the Stokes system

$$(\mathcal{P}) \quad \begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + \nabla p = u\chi_\omega \mathbf{e}, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \mathbf{y} = 0, & \text{in } (0,T) \times \Omega, \\ \mathbf{y}(0,\cdot) = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{B}\mathbf{y} = 0, & \text{on } (0,T) \times \partial\Omega, \end{cases}$$

where Ω is an infinity cylinder, $\Omega = G \times \mathbb{R} \subset \mathbb{R}^3$ with G an open bounded set of $\mathbb{R}^2$, $\mathbf{e}$ is a given vector of $\mathbb{R}^3$, χ_ω denotes the characteristic function of an open subdomain ω of Ω (*i.e.* $\chi_\omega = 1$ over ω and $\chi_\omega = 0$ over $\mathbb{R}^3 \setminus \omega$) and $\mathbf{B}$ is a boundary operator which denotes either Dirichlet or periodic boundary conditions or a combination of both. The flow velocity and the pressure of the flow are denoted by $\mathbf{y} = (y_1, y_2, y_3)$ and p respectively. The control is a scalar function u and the term $u\chi_\omega\mathbf{e}$ represents the density of some external forces spatially concentrated on ω and acting merely in the direction $\mathbf{e}$ and during a given period of time T. We note that the zero initial condition does not represent any restriction because of the linearity of the problem.

One formulates the approximate controllability problem as follows: Let $\mathcal{X}$ be a functional space with norm $\|\cdot\|$, and let ϵ be a small fixed positive number. Find a control u such that the solution $\mathbf{y}$ to the associated problem $(\mathcal{P})$ approximates at time T a saught velocity

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES. – 0021-7824/1997/04/\$ 7.00

$\hat{\mathbf{y}}$ in $\mathcal{X}$ as

$$\|\hat{\mathbf{y}}(\cdot) - \mathbf{y}(T, \cdot)\| \leq \epsilon. \tag{1}$$

We shall mainly work with the $(L^2(\Omega))^3$-norm, and shall take as $\mathcal{X}$ a certain closed subspace of $(L^2(\Omega))^3$. This is so because we seek *admisible velocities* $\hat{\mathbf{y}}(x)$ that must satisfy div $\hat{\mathbf{y}} = 0$ in Ω together with suitable boundary conditions (at least in a weak sense).

If we replace the right hand side of the equation of $(\mathcal{P})$ by a general vectorial control $\mathbf{u}\chi_\omega$, it is easy to prove the approximate controllability assumed $\mathbf{u} \in (L^2(\omega \times (0, T)))^3$. This can be obtained by the (already classical) arguments of J.-L. Lions, (*i.e.* the Hahn-Banach theorem and the Mizohata unique continuation theorem; see, *e.g.* Lions [5], [6] for details). However, many interesting questions arise if the controls are subject to some *complementary constraints*. Some of them have been solved in Fursikov and Imanuvilov [1], [2]. Here we consider the case of directional constraints on the control, more precisely, the case when the control is a one-directional vector field of the kind,

$$\mathbf{u} = u\mathbf{e}, \tag{2}$$

where $\mathbf{e}$ is a fixed vector from $\mathbb{R}^3$. The case of two-directional control of the form

$$\mathbf{u} = (u_1, u_2, 0) \tag{3}$$

was considered by J.-L. Lions in [6, p.78], [7]; in these publications he also attracted the attention of specialists to the case (2).

So, the main goal of this article is to study the approximate controllability of the Stokes system when the control is unidirectional as in the formulation of $(\mathcal{P})$. As was said already, we restrict ourselves to the cases in which Ω is a cylindrical domain, *i.e.*

$$\Omega = G \times \mathbb{R}. \tag{4}$$

where $G \subset \mathbb{R}^2$ is an open bounded set. Furthermore, we shall distinguish the cases when the control direction $\mathbf{e}$ is parallel to the cylinder generatrix and when $\mathbf{e}$ is orthogonal to the generatrix. First we prove that this problem is not approximate controllable in the case of the boundary condition $\mathbf{y} = \mathbf{0}$ on $\partial G \times \mathbb{R}$ in the class of periodic solutions with respect to the last spatial variable (*i.e.* $\mathbf{y}(t, x_1, x_2, x_3 + L_3) = \mathbf{y}(t, x_1, x_2, x_3)$ for some $L_3 > 0$ and almost any $t \in (0, T), (x_1, x_2) \in G$ and $x_3 \in \mathbb{R}$). This is proved for L_3-periodic external unidirectional parallel controls with $\mathbf{e} = \mathbf{e}_3$, where $\mathbf{e}_3$ is the third element of the orthonormal basis in $\mathbb{R}^3$. The result holds even for $\omega = \Omega$.

We prove also a positive assertion on the approximate controllability of the Stokes problem $(\mathcal{P})$ in the case of Dirichlet boundary conditions $\mathbf{y} = 0$ on $(0, T) \times \partial G \times \mathbb{R}$ and $\mathbf{y}(t, \cdot) \in (L^2(G \times \mathbb{R}))^3$. This is obtained in the following case:

$$\begin{cases} \mathbf{e} \text{ is orthogonal to the generatrix,} \\ \qquad\qquad \text{and } G \text{ is a rectangle.} \end{cases} \tag{5}$$

The organization of this paper is as follows. The uncontrollability of L_3-periodic solutions is proved in Section 2. The rest of the paper is devoted to give some positive answer on the approximate controllability problem: Firstly we reduce the problem to the investigation of the *unique continuation property* for an overdeterminated boundary value problem for the linear heat equation having an harmonic function as right-hand side. This is presented in Section 3. In Section 4 we use the Fourier transform and apply complex analysis in order to prove this property, *i.e.* we show that any solution of the above overdetermined problem must be identically zero. Thanks to the Hahn-Banach theorem this proves the approximate controllability of the corresponding Stokes systems.

After the completion of a first version of this article the authors become aware of a recent result by Lions and Zuazua [9] where problem $(\mathcal{P})$ is considered for controls paralelly directed to the cylinder generatrix ($\mathrm{e} = \mathrm{e}_3$). They show that the approximate controllability is a *generic* property with respect to the domain G (the answer is positive if the eigenvalues are simple and the problem becomes uncontrollable if G is a ball).

2. Uncontrollability of L$_3$-periodic solutions of the Stokes system.

Let $L_3 > 0$ and G be a regular open bounded domain of $\mathbb{R}^2$. Let ω_0 be an open set of $G \times (0, L_3)$ and define ω as the periodic extension of ω_0 to the set $G \times \mathbb{R}$ ($i.e.(x_1, x_2, x_3) \in \omega$ iff there exists $n \in \mathbf{Z}$ such that $(x_1, x_2, x_3 + nL_3) \in \omega_0$). We consider the L_3-periodic Stokes problem

$$(\mathcal{P}_{\text{per}}) \quad \begin{cases} \mathbf{y}_t - \Delta \mathbf{y} + \nabla p = u(t,x)\chi_\omega \mathbf{e}, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \mathbf{y} = 0, & \text{in } (0,T) \times \Omega, \\ \mathbf{y}(0,\cdot) = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{y}(t, x_1, x_2, x_3) = \mathbf{0}, & t \in (0,T), (x_1, x_2) \in \partial G, \ x_3 \in \mathbb{R}, \\ \mathbf{y}(t, x_1, x_2, x_3 + L_3) = \mathbf{y}(t, x_1, x_2, x_3), & t \in (0,T), (x_1, x_2) \in G, \ x_3 \in \mathbb{R}\ . \end{cases}$$

We introduce the functional spaces:

$$(6) \quad L^2_{\text{per}}(\Omega) = \{\tilde{v} : \Omega \to \mathbb{R},\ \tilde{v} \in L^2(G \times (0, L_3))\},\ \tilde{v}(x_1, x_2, x_3 + L_3) = \tilde{v}(x_1, x_2, x_3)\},$$

$$(7) \qquad \mathrm{H}^1_{\text{per}}(\Omega) = \left\{ v \in L^2_{\text{per}}(\Omega), \nabla v \in \left(L^2_{\text{per}}(\Omega)\right)^3 \right\},$$

$$(8) \qquad \begin{cases} \mathrm{H}^1_{0,\text{per}}(\Omega) = \left\{ v \in \mathrm{H}^1_{\text{per}}(\Omega), v = 0 \quad \text{on } \partial G \times \mathbb{R} \right\}, \\ \mathbf{H}^1_{\text{per}}(\Omega) = \left\{ \mathbf{w} \in (L^2_{\text{per}}(\Omega))^3, \nabla \mathbf{w} \in (L^2_{\text{per}}(\Omega))^9 \right\}, \end{cases}$$

$$\text{(9)} \qquad \mathbf{V}_{\text{per}} = \{\mathbf{w} \in (\mathcal{C}^{\infty}(\Omega))^3,\ \mathbf{w} = \mathbf{0} \text{ in a neighbourhood of } \partial\Omega,$$
$$\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \text{ and } \mathbf{w}(x_1, x_2, x_3) = \mathbf{w}(x_1, x_2, x_3 + L_3)\},$$

$$\text{(10)} \qquad \mathcal{H}^0_{\text{per}} = \text{closure of } \mathbf{V}_{\text{per}} \text{ in } \left(L^2_{\text{per}}(\Omega)\right)^3,$$

$$\text{(11)} \qquad \mathcal{H}^1_{\text{per}} = \text{closure of } \mathbf{V}_{\text{per}} \text{ in } \mathbf{H}^1_{\text{per}}(\Omega).$$

Sets $\left(L^2_{\text{per}}(\Omega)\right)^3$ and $\mathbf{H}^1_{\text{per}}(\Omega)$ are Hilbert spaces with respect to scalar products:

$$(\mathbf{w}, \mathbf{v})_{\left(L^2_{\text{per}}(\Omega)\right)^3} = \int_{G\times(0,L_3)} \mathbf{w}\cdot\mathbf{v}\,dx$$
$$(\mathbf{w}, \mathbf{v})_{\mathbf{H}^1_{\text{per}}(\Omega)} = \int_{G\times(0,L_3)} \mathbf{w}\cdot\mathbf{v}\,dx + \int_{G\times(0,L_3)} \nabla\mathbf{w}\cdot\nabla\mathbf{v}\,dx.$$

$\mathcal{H}^0_{\text{per}}$ and $\mathcal{H}^1_{\text{per}}$ are closed subspaces of $\left(L^2_{\text{per}}(\Omega)\right)^3$ and $\mathbf{H}^1_{\text{per}}(\Omega)$ respectively, so they are also Hilbert spaces. We shall use the notation

$$\|\mathbf{w}\|_{\mathcal{H}^0_{\text{per}}} = \|\mathbf{w}\|_{\left(L^2_{\text{per}}(\Omega)\right)^3}, \quad \mathbf{w} \in \mathcal{H}^0_{\text{per}},$$
$$\|\mathbf{w}\|_{\mathcal{H}^1_{\text{per}}} = \|\mathbf{w}\|_{\mathbf{H}^1_{\text{per}}(\Omega)}, \quad \mathbf{w} \in \mathcal{H}^1_{\text{per}}.$$

We also introduce

$$\text{(12)} \qquad \mathrm{H}^{-1}_{\text{per}}(\Omega) = \text{dual space of } \mathrm{H}^1_{0,\text{per}}(\Omega) \text{ with respect to the duality generated by the scalar product of } L^2_{\text{per}}(\Omega).$$

$$\text{(13)} \qquad \mathcal{H}^{-1}_{\text{per}} = \text{dual space of } \mathcal{H}^1_{\text{per}} \text{ with respect to the duality generated by the scalar product of } \left(L^2_{\text{per}}(\Omega)\right)^3.$$

Finally, let

$$\text{(14)} \qquad \mathbf{Y}_{\text{per}} = \left\{\mathbf{y} \in L^2(0,T;\mathcal{H}^1_{\text{per}}) \text{ such that } \frac{\partial \mathbf{y}}{\partial t} \in L^2(0,T;\mathcal{H}^{-1}_{\text{per}})\right\}.$$

The existence and uniqueness of a solution $(\mathbf{y}, p)$ of $(\mathcal{P}_{\text{per}})$ with $\mathbf{y} \in \mathbf{Y}_{\text{per}}$ and $\nabla p \in L^2(0, T : \mathrm{H}^{-1}_{\text{per}}(\Omega))$, assumed $u \in L^2(0, T; L^2_{\text{per}}(\Omega))$, can be easily proved by standard methods (see Ladyzenskaya [4] and Temam [12]). We also remark that if we eliminate the control constraint (by replacing u in $(\mathcal{P}_{\text{per}})$ by a general vectorial control $\mathbf{u} \in \mathcal{H}^0_{\text{per}}$) then we can apply the arguments of Lions [5], [6] and Fursikov and Imanuvilov [1], [2] showing the $\mathcal{H}^0_{\text{per}}$-approximate controllability of $(\mathcal{P}_{\text{per}})$. As we shall see, the situation is radically different for unidirectional controls. We assume that $\mathbf{e}$ is parallel to the cylinder generatrix *i.e.*,

$$\text{(15)} \qquad \mathbf{e} = \mathbf{e}_3,$$

where $\mathbf{e}_3$ is the third element of the orthonormal basis of $\mathbb{R}^3$. Roughly speaking, we shall show that

$$\int_0^{L_3} \mathbf{y}(T, x_1, x_2, x_3) dx_3$$

has the direction $\mathbf{e}_3$ and so the approximate controllability can not hold over $\mathcal{H}^0_{\text{per}}$. More precisely, we define the subspace $\mathbf{H}$ of $\mathcal{H}^0_{\text{per}}$ by the formula:

$$\mathbf{H} = \{v(x) = (v_1(x), v_2(x), v_3(x)) \in \mathcal{H}^0_{\text{per}} : v_3 \equiv 0 \text{ and } \\ v_j(x_1, x_2, x_3),\ j = 1, 2, \text{ do not depend on } x_3\}$$

THEOREM 1. – *Assume* (15). *Let* $u \in L^2(0, T : L^2_{\text{per}}(\Omega))$ *and let* $(\mathbf{y}, p)$ *be the solution to* $(\mathcal{P}_{\text{per}})$. *Then* $\mathbf{y}(T, \cdot)$ *is orthogonal to the subspace* $\mathbf{H}$, *i.e.*

$$\int_{G\times(0,L_3)} \mathbf{g}(x_1, x_2) \cdot \mathbf{y}(T, x_1, x_2, x_3) dx_1 dx_2 dx_3 = 0, \quad \textit{for any } \mathbf{g} \in \mathbf{H}. \tag{16}$$

In particular, problem $(\mathcal{P}_{\text{per}})$ *is not* $\mathcal{H}^0_{\text{per}}$*-approximately controllable.*

Proof. – Let $\mathbf{g} \in \mathbf{H}$ and consider the adjoint problem:

$$\begin{cases} -\mathbf{a}_t - \Delta\mathbf{a} + \nabla q = 0, & \text{in } (0, T) \times \Omega, \\ \text{div } \mathbf{a} = 0, & \text{in } (0, T) \times \Omega, \\ \mathbf{a}(t, x_1, x_2, x_3) = \mathbf{0}, & t \in (0, T), (x_1, x_2) \in \partial G, x_3 \in \mathbb{R}, \\ \mathbf{a}(t, x_1, x_2, x_3 + L_3) = \mathbf{a}(t, x_1, x_2, x_3), & t \in (0, T), (x_1, x_2) \in G, x_3 \in \mathbb{R}, \\ \mathbf{a}(T, x_1, x_2, x_3) = \mathbf{g}(x_1, x_2), & (x_1, x_2, x_3) \in \Omega. \end{cases} \tag{17}$$

The existence and uniqueness of such solution $(\mathbf{a}, \nabla q) \in \mathbf{Y}_{\text{per}} \times L^2(0, T; H^{-1}_{\text{per}}(\Omega))$ is again standard. It is easy to see that

$$\mathbf{a}(t, \cdot) \in \mathbf{H} \quad \text{for } t \in (0, T). \tag{18}$$

Indeed, by assumption $\mathbf{g}(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2), 0)$ for some $g_i \in L^2(G)$, $i = 1, 2$ and if we consider the two-dimensional Stokes problem

$$\begin{cases} -\mathbf{b}_t - \Delta\mathbf{b} + \nabla\ell = 0, & \text{in } (0, T) \times G, \\ \text{div } \mathbf{b} = 0, & \text{in } (0, T) \times G, \\ \mathbf{b} = \mathbf{0}, & \text{on } (0, T) \times \partial G, \\ \mathbf{b}(T, x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2)), & (x_1, x_2) \in G, \end{cases} \tag{19}$$

we know that problem (19) has a unique solution $(\mathbf{b}(t, x_1, x_2), \nabla\ell(t, x_1, x_2))$. Then it is clear that the pair $((\mathbf{b}, 0), \nabla\ell)$ satisfies problem (17) and so, by uniqueness, it must coincide with $(\mathbf{a}, \nabla q)$ which proves (18). Multiplying the equation of $(\mathcal{P}_{\text{per}})$ by $\mathbf{a}$, using (18) and the relation

$$\int_{G\times(0,L_3)} \nabla q \cdot \mathbf{w} dx = 0 \quad \text{for any } \mathbf{w} \in \mathcal{H}^0_{\text{per}}$$

and integrating by parts we get:

$$0 = \int_{(0,T)\times G\times(0,L_3)} u\chi_\omega \mathbf{a}\cdot\mathbf{e} dx dt = \int_{(0,T)\times G\times(0,L_3)} \mathbf{a}\cdot(\mathbf{y}_t - \Delta\mathbf{y} + \nabla p) dx dt$$
$$= \int_{G\times(0,L_3)} \mathbf{g}(x)\cdot\mathbf{y}(T,x)dx + \int_{(0,T)\times G\times(0,L_3)} (-\mathbf{a}_t - \Delta\mathbf{a} + \nabla q)\cdot\mathbf{y} dx dt$$
$$= \int_{G\times(0,L_3)} \mathbf{g}(x)\cdot\mathbf{y}(T,x)dx,$$

where we have identified $\mathbf{g}(x)$ with $\mathbf{g}(x_1, x_2)$ for any $x = (x_1, x_2, x_3) \in G \times \mathbb{R}$. □

Remark 1. – We point out that Theorem 1 holds even for the case $\omega = \Omega$. □

3. Approximate controllability: an overdeterminated auxiliary problem

This section and the following one are devoted to prove the approximate controllability of the Stokes system with Dirichlet boundary conditions by means of unidirectional controls. Again, we assume $\Omega \doteq G \times \mathbb{R}$ with $G \subset \mathbb{R}^2$ a bounded open set with Lipschitz boundary ∂G and $\omega \subset \Omega$. Moreover, we shall assume that

$$G = \{(x_1, x_2) \in \mathbb{R}^2 : \ |x_1| < \frac{\pi}{2,} |x_2| < 1\}. \tag{20}$$

The lenghts π and 2 of the rectangle sides are not essential and can be changed to be arbitrary ones. Let consider the Dirichlet problem for the Stokes system

$$(\mathcal{P}_D) \quad \begin{cases} \mathbf{y}_t - \Delta\mathbf{y} + \nabla p = u\chi_\omega\mathbf{e}, & \text{in } (0,T)\times\Omega, \\ \text{div } \mathbf{y} = 0, & \text{in } (0,T)\times\Omega, \\ \mathbf{y}(0,\cdot) = \mathbf{0}, & \text{in } \Omega, \\ \mathbf{y} = \mathbf{0}, & \text{on } (0,T)\times\partial\Omega. \end{cases}$$

We will use the well-known functional spaces:

$$\mathbf{V} = \{\mathbf{w} \in (\mathcal{C}_0^\infty(\Omega))^3;\ \text{div } \mathbf{w} = 0 \text{ in } \Omega\}, \tag{21}$$

$$\mathcal{H}^0 = \text{ closure of } \mathbf{V} \text{ in } (L^2(\Omega))^3, \tag{22}$$

$$\mathcal{H}^1 = \text{ closure of } \mathbf{V} \text{ in } (H^1(\Omega)))^3, \tag{23}$$

where $\mathrm{H}^1(\Omega)$ is the usual Sobolev space $\mathrm{H}^1(\Omega) = \{v \in L^2(\Omega), \nabla v \in (L^2(\Omega))^3\}$,

$$\begin{aligned} \mathrm{H}^{-1}(\Omega) = {} & \text{dual space to } \mathrm{H}_0^1(\Omega) \text{ with respect to the duality} \\ & \text{generated by the scalar product of } L^2(\Omega) \\ & \text{where } \mathrm{H}_0^1(\Omega) = \{v \in \mathrm{H}^1(\Omega), v(x) = 0, x \in \partial\Omega\}, \end{aligned} \tag{24}$$

$$\begin{aligned} \mathcal{H}^{-1} = {} & \text{dual space to } \mathcal{H}^1 \text{ with respect to the duality} \\ & \text{generated by the scalar product of } (L^2(\Omega))^3, \end{aligned} \tag{25}$$

$$\mathcal{Z} = \left\{ \mathbf{y} \in L^2(0,T : \mathcal{H}^1) : \frac{\partial \mathbf{y}}{\partial t} \in L^2(0,T : \mathcal{H}^{-1}) \right\}. \tag{26}$$

Given $u \in L^2(0,T : L^2(\Omega))$ the existence and uniqueness of a solution $(\mathbf{y}, \nabla p) \in \mathcal{Z} \times L^2(0,T : \mathrm{H}^{-1}(\Omega))$ of $(\mathcal{P}_D)$ can be found, for instance, in Ladyzenskaya [4] or Temam [12].

Our main goal is to study the $\mathcal{H}^0$-approximate controllability property when the vector $\mathbf{e}$ is given by,

$$\mathbf{e} = \mathbf{e}_1 \tag{27}$$

where $\mathbf{e}_1$ is the first element of the orthonormal basis of $\mathbb{R}^3$.

If we assume that the $\mathcal{H}^0$-approximate controllability does not hold then it must exist $\mathbf{f} \in \mathcal{H}^0$, $f \neq 0$, such that

$$\int_\Omega \mathbf{y}(T,x) \cdot \mathbf{f}(x) dx = 0, \tag{28}$$

for the solution $\mathbf{y}$ of $(\mathcal{P}_D)$ corresponding to an arbitrary $u \in L^2(0,T : L^2(\Omega))$, $\operatorname{supp} u \subset (0,T) \times \omega$. To get a contradiction, we consider the adjoint problem:

$$\begin{cases} -\mathbf{a}_t - \Delta \mathbf{a} + \nabla q = 0, & \text{in } (0,T) \times \Omega, \\ \operatorname{div} \mathbf{a} = 0, & \text{in } (0,T) \times \Omega, \\ \mathbf{a} = \mathbf{0}, & \text{on } (0,T) \times \partial\Omega, \\ \mathbf{a}(T,\cdot) = \mathbf{f}(\cdot), & \text{in } \Omega. \end{cases} \tag{29}$$

Below we need to consider problem (29) not only on $(0,T) \times \Omega$ but on the set $(-\infty, T) \times \Omega$ also. As $\mathbf{f} \in \mathcal{H}^0$, the proof of the existence and uniqueness of the solution $(\mathbf{a}, q)$ of (29) on $(0,T) \times \Omega$ as well as on the set $(-\infty, T) \times \Omega$ is standard. The special structure of the controls defined in (27) leads to the following result.

LEMMA 1. – *Let* $\mathbf{f} \in \mathcal{H}^0$ *satisfying* (28) *and let* $(\mathbf{a}, \nabla q)$ *be the solution of* (29). *Then if* $\mathbf{a} = (a_1, a_2, a_3)$ *we have*

$$a_1(t,x) \equiv 0 \textit{ for } (t,x) \in (-\infty, T) \times \Omega. \tag{30}$$

In particular $f_3 = 0$ *on* Ω. *Moreover, for a.e.* $t \in (-\infty, T)$:

$$\begin{cases} q(t,\cdot) \text{ is an harmonic function and} \\ \text{does not depend on the } x_1 \text{ variable.} \end{cases} \tag{31}$$

Proof. – Multiplying the first equation in (29) by $\mathbf{y}$ that is solution of $(\mathcal{P}_D)$ with arbitrary $u \in L^2(0, T : L^2(\omega))$, $\operatorname{supp} u \in (0,T) \times \omega$, and using (28) we get:

$$0 = \int_{(0,T)\times\Omega} \left(-\frac{\partial \mathbf{a}}{\partial t} - \Delta \mathbf{a} + \nabla q \right) \cdot \mathbf{y} dx dt = \int_{(0,T)\times\Omega} \mathbf{a} \cdot \left(\frac{\partial \mathbf{y}}{\partial t} - \Delta \mathbf{y} \right) dx dt$$
$$= \int_{(0,T)\times\Omega} \mathbf{a} \cdot (u \chi_\omega \mathbf{e}_1 - \nabla p) dx dt = \int_{(0,T)\times\omega} u a_1 dx dt.$$

As u is arbitrary we conclude that

$$a_1(t,x) = 0 \quad a.e.\ (t,x) \in (0,T) \times \omega \tag{32}$$

and therefore (using the first equation in (29))

$$\frac{\partial q}{\partial x_1}(t,x) = 0 \quad a.e.\ (t,x) \in (0,T) \times \omega. \tag{33}$$

Applying the divergence operator to the first equation in (29) and using that div $\mathbf{a} = 0$ and div $(\Delta \mathbf{a}) = \Delta(\text{div } \mathbf{a})$ we get that

$$\Delta q = 0 \quad \text{in } (0,T) \times \Omega. \tag{34}$$

This also shows that $\dfrac{\partial q}{\partial x_1}(t,\cdot)$ is an harmonic function. By well-known results, property (33) implies that $\dfrac{\partial q}{\partial x_1}(t,\cdot) \equiv 0$ on Ω which proves (31). Using this information in (29) and (27) we obtain that:

$$\begin{cases} -\dfrac{\partial a_1}{\partial t} - \Delta a_1 = 0, & \text{in } (0,T) \times \Omega, \\ a_1 = 0, & \text{on } (0,T) \times \partial\Omega, \\ a_1(T,\cdot) = f_1(\cdot), & \text{on } \Omega. \end{cases}$$

From (32) and the Mizohata unique continuation theorem (see Mizohata [10]) we obtain (30) for $(t,x) \in (0,T) \times \Omega$, that yields the identity $f_1 = 0$ on Ω.

We consider the boundary value problem

$$\begin{cases} \dfrac{\partial a_2}{\partial t}(t,x_1,x_2,x_3) - \Delta a_2 + \dfrac{\partial p}{\partial x_2}(t,x_1,x_2,x_3) = 0, & \text{in } (-\infty,T) \times \Omega, \\ \dfrac{\partial a_3}{\partial t}(t,x_1,x_2,x_3) - \Delta a_3 + \dfrac{\partial p}{\partial x_3}(t,x_1,x_2,x_3) = 0, & \text{in } (-\infty,T) \times \Omega, \\ \dfrac{\partial a_2}{\partial x_2} + \dfrac{\partial a_3}{\partial x_3} = 0, & \text{in} (-\infty,T) \times \Omega, \\ a_2(T,x) = f_2(x),\ a_3(T,x) = f_3(x), & \text{in } \Omega, \\ a_2 = a_3 = 0, & \text{on } (-\infty,T) \times \partial\Omega, \end{cases}$$

where $\frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} = 0$. We want to establish the solvability and uniqueness of the solution of this problem. To do this we introduce the spaces which are analogous to spaces (21)-(26):

$$\underline{V} = \{w = (w_2, w_3) \in (C_0^\infty(\Omega))^2 : \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} = 0\},$$
$$\underline{\mathcal{H}}^0 = \text{closure } \underline{V} \text{ in } (L^2(\Omega))^2,$$
$$\underline{\mathcal{H}}^1 = \text{closure } \underline{V} \text{ in } (H^1(\Omega))^2,$$
$$\underline{\mathcal{H}}^{-1} = \text{dual space } to\ \underline{\mathcal{H}}^1,$$
$$\underline{Z} = \{\mathbf{y} \in L^2(-\infty, T; \underline{\mathcal{H}}^1) : \frac{\partial \mathbf{y}}{\partial t} \in L^2(-\infty, T; \underline{\mathcal{H}}^{-1}).$$

Using these spaces and repeating the proof of existence and uniqueness of solutions for the Stokes boundary value problem (see Temam [12], Ladyzhenskaya [4]) we get the desired result. Let (a_2, a_3, p) be the obtained solution of the above mentioned boundary value problem. By means of a straightforward verification we establish that quadraplet $(a_1, a_2, a_3, p) \equiv (\mathbf{a}, q)$ with $a_1 \equiv 0$ satisfies the boundary value problem (29) considered on $(-\infty, T) \times \Omega$. This fact and the uniqueness of the solutions for the Stokes boundary value problem (29) yield assertions (30), (31) for $(t, x) \in (-\infty, T) \times \Omega$. □

Given $x_1^0 \in \mathbb{R}$, we denote by $P(x_1^0)$ the plane in $\mathbb{R}^3$ which is orthogonal to the axis x_1 and intersects this axis at a point $(x_1^0, 0, 0)$. We shall use the notation:

$$\Omega(x_1^0) = \Omega \cap P(x_1^0). \tag{35}$$

Notice that if $\Omega(x_1^0)$ is not void then $\Omega(x_1^0)$ is of the form $\{x_1^0\} \times I \times \mathbb{R}$ with $I = \{x_2 \in \mathbb{R} : |x_2| < 1\}$ (recall (20)).

Using (30) and div $\mathbf{a} = 0$ we have that:

$$\frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = 0 \quad \text{in } (-\infty, T) \times \Omega. \tag{36}$$

Moreover from (31) we deduce that a_2 and a_3 are regular functions and so, for any $(x_1^0, x_2, x_3) \in \Omega(x_1^0)$ and $t \in (-\infty, T)$, we can define the *potential function*:

$$V(t, x_1^0, x_2, x_3) = \int_\gamma (a_3(t, \cdot)dx_2 - a_2(t, \cdot)dx_3), \tag{37}$$

where γ is any curve in $\overline{\Omega}(x_1^0)$ joining (x_1^0, x_2, x_3) with a fixed point $(x_1^0, x_2^0, x_3^0) \in \partial\Omega(x_1^0)$. The definition of V is, in fact, independent on γ because the field $(a_3, -a_2)$ is conservative with respect to (x_2, x_3) (due to (36)) and $\Omega(x_1^0)$ is a simply connected set (due to (20)). Function V is then well defined on $\overline{\Omega}$.

LEMMA 2. – *Assume that G is the rectangle* (20). *Then the potential function V given in* (37) *satisfies the equation*

$$\frac{\partial V}{\partial t}(t, x) + \Delta V(t, x) = A(t, x_2, x_3) + B(t, x_1) \text{ in } (-\infty, T) \times \Omega \tag{38}$$

and the two boundary conditions

$$(39)\qquad \begin{cases} V = 0 \ on \ (-\infty, T) \times \partial\Omega, \\ \dfrac{\partial V}{\partial x_2}(t, x_1, x_2, x_3)\big|_{|x_2|=1} = 0 \ if \ (t, x_1, x_2, x_3) \in (-\infty, T) \times (\partial\Omega \cap \partial\Omega(x_1)). \end{cases}$$

Function V also satisfies (in a weak sense) the final condition:

$$(40)\qquad V(T, x) = V_T(x) \doteq \int_\gamma (f_3 dx_2 - f_2 dx_3),$$

where γ is an arbitrary curve joining $x = (x_1, x_2, x_3) \in \Omega$ with a fixed point $(x_1, x_2^0, x_3^0) \in \partial\Omega(x_1)$. In addition, we have:

$$(41)\quad \text{ess sup}_{t\in(-\infty,T)} \left\{ \|V(t,\cdot)\|^2_{L^2(\Omega)} + \sum_{l=2,3} \left\| \frac{\partial V}{\partial x_l}(t,\cdot) \right\|^2_{L^2(\Omega)} + \int_{-\infty}^T \sum_{l_1,l_2=2,3} \left(\left\| \frac{\partial^2 V}{\partial x_{l_1} \partial x_{l_2}}(\tau,\cdot) \right\|^2_{L^2(\Omega)} + \left\| \frac{\partial^2 V}{\partial x_{l_1} \partial x_1}(\tau,\cdot) \right\|^2_{L^2(\Omega)} \right) d\tau \right\} \le C$$

for some positive constant C. Moreover, function $A(t, x_2, x_3)$ is harmonic with respect to x_2, x_3 and we have the a priori *estimate:*

$$(42)\qquad \int_{-\infty}^T \|A(t,\cdot)\|^2_{L^2(\Omega^1))} dt + \int_{-\infty}^T \|B(t,\cdot)\|_{L^2(-\frac{\pi}{2},\frac{\pi}{2})} dt \le C,$$

where $\Omega^1 = \{(x_2, x_3) : |x_2| < 1, x_3 \in \mathbb{R}\}$.

Proof. – Equality (37) implies:

$$(43)\qquad \frac{\partial V}{\partial x_2} = a_3 \qquad \text{and} \qquad \frac{\partial V}{\partial x_3} = -a_2.$$

Notice that function V does not depend on the choice of the point (x_1^0, x_2^0, x_3^0) in $\partial\Omega(x_1^0)$. From (20), G is a rectangle and we can take, for instance, $x_2^0 = -1$ in the definition (37) of V. But then, obviously, $V = 0$ on the parts of the boundary of the form $\{|x_1| < \pi/2, x_2 = -1, x_3 \in \mathbb{R}\}$ and $\{x_1 = \pm\pi/2, x_2 < 1, x_3 \in \mathbb{R}\}$ since $\Omega(x_1^0) = \{(x_1^0, x_2, x_3) : |x_2| < 1, x_3 \in \mathbb{R}\}$ and we can chose the path γ belonging to $\partial\Omega$ where $a_2 = a_3 = 0$. On the surface $\{|x_1| < \pi/2, x_2 = 1, x_3 \in \mathbb{R}\}$ the function $V(t, x_1, 1, x_3)$ does not depend on x_3 (i.e. $V(t, x_1, 1, x_3) = V(t, x_1, 1, 1)$). Let us show that $V(t, x_1, 1, 1) = 0$. The solution $\mathbf{a}$ of (29) satisfies the energy estimate:

$$(44)\qquad \|\mathbf{a}(t,\cdot)\|^2_{L^2(\Omega)} + \int_t^T \|\nabla\mathbf{a}(\tau,\cdot)\|^2_{L^2(\Omega)} d\tau \le \|\mathbf{f}\|^2_{L^2(\Omega)}, \ a.e. \ t \in (0, T).$$

Therefore, if $V(t^0, x_1^0, 1, 1) \neq 0$ for some (t^0, x_1^0) then by choosing the path in (37) as $\gamma = \gamma_1 \cup \gamma_2$ with

$$\gamma_1 = \{x_1 = x_1^0, x_2 = -1, x_3 \in (0, N)\},$$
$$\gamma_2 = \{x_1 = x_1^0, x_2 \in (-1, 1), x_3 = N\},$$

we deduce, from (29), that the integral along γ_1 vanishes for any arbitrary number N. From the interior regularity for problem (29) we know (*see* Ladyshenskaya [4]) that **a** is smooth in $(-\infty, T) \times \Omega$. Then, we can choose N so large that the integral in (37) will be as small as we want, and, for instance, will be less than $|V(t^0, x_1^0, 1, 1)|/2$. This is a contradiction which proves (39). To prove (38) we point out that using (43) we can rewrite the first equation of (29) in the following form

$$\frac{\partial}{\partial x_3}\left(\frac{\partial V}{\partial t} + \Delta V\right) = \frac{\partial q}{\partial x_2}, \tag{45}$$

$$\frac{\partial}{\partial x_2}\left(\frac{\partial V}{\partial t} + \Delta V\right) = \frac{\partial q}{\partial x_3}. \tag{46}$$

We can assume that $|x_1| < \frac{\pi}{2}$, so $(x_1, 0, 0) \in \Omega$. Integrating (45) and (46) along a path γ that joins the points $(x_1, 0, 0)$ and (x_1, x_2, x_3) we get

$$\begin{aligned}\left(\frac{\partial V}{\partial t} + \Delta V\right)(t, x_1, x_2, x_3) &= \left(\frac{\partial V}{\partial t} + \Delta V\right)(t, x_1, 0, 0)\\ &+ \int_\gamma \left\{\frac{\partial}{\partial x_2}\left(\frac{\partial V}{\partial t} + \Delta V\right)dx_2 + \frac{\partial}{\partial x_3}\left(\frac{\partial V}{\partial t} + \Delta V\right)dx_3\right\} = A(t, x_2, x_3)\\ &+ B(t, x_1),\end{aligned} \tag{47}$$

where

$$A(t, x_2, x_3) = \int_\gamma \left(\frac{\partial q}{\partial x_3}dx_2 - \frac{\partial q}{\partial x_2}dx_3\right) \tag{48}$$

$$B(t, x_1) = \left(\frac{\partial V}{\partial t} + \Delta V\right)(t, x_1, 0, 0). \tag{49}$$

Notice that by (31) function A does not depend on x_1. Moreover

$$\frac{\partial^2 A}{\partial x_2^2} = \frac{\partial^2 q}{\partial x_3 \partial x_2}, \qquad \frac{\partial^2 A}{\partial x_3^2} = -\frac{\partial^2 q}{\partial x_2 \partial x_3}$$

and therefore A is an harmonic function with respect to x_2, x_3. Thus, (38) follows from (47) - (49). Estimate (41) is obtained from (37) and (44). By virtue of the interior and boundary regularity theorems for solutions of the Stokes system the functions a_i, a_j and q are smooth functions and thus the same property holds for functions A and B. Estimate (42) follows from L^2-estimates for solutions of the Stokes system. □

Since the x_1-projection of Ω is a bounded interval we can reduce problem (38), (39), (40) to a family of two-dimensional problems of the form

$$\frac{\partial v}{\partial \tau}(\tau, x_2, x_3) - \Delta v(\tau, x_2, x_3) = g(\tau, x_2, x_3), \qquad (x_2, x_3) \in \mathcal{O}, \quad \tau \in (0, +\infty), \tag{50}$$

$$v = \frac{\partial v}{\partial x_2} = 0, \quad \text{on } (0, +\infty) \times \partial\mathcal{O}, \tag{51}$$

$$v(0, x_2, x_3) = w(x_2, x_3), \quad \text{on } \mathcal{O}, \tag{52}$$

where $\mathcal{O} = \{|x_2| < 1\} \times \mathbb{R}$, $g(\tau, x_2, x_3)$ is an harmonic function with respect to x_2, x_3.

LEMMA 3. – *Let G be rectangle* (20). *Then problem* (38), (39), (40) *can be reduced to problem* (50), (51), (52) *where v,w and g are defined by relations* (58), (59) *below. Besides we have the estimate*

$$\int_{-\infty}^{T} \|v(t, \cdot)\|^2_{H^2(\mathcal{O})} dt + \int_{-\infty}^{T} \|g(t, \cdot)\|^2_{L^2(\mathcal{O})} dt < \infty. \tag{53}$$

Proof. – We decompose functions V, $A + B$ and V_T in their Fourier series:

$$V(t, x_1, x_2, x_3) = V_0(t, x_2, x_3) + \sum_{n=1}^{\infty} [V_{2n+1}(t, x_2, x_3) \cos((2n+1)x_1) + V_{2n}(t, x_2, x_3) \sin(2nx_1)], \tag{54}$$

$$A(t, x_2, x_3) + B(t, x_1) = A_0(t, x_2, x_3) + \sum_{n=1}^{\infty} [A_{2n+1}(t, x_2, x_3) \cos((2n+1)x_1) + A_{2n}(t, x_2, x_3) \sin(2nx_1)], \tag{55}$$

$$\begin{aligned} V_T(x_1, x_2, x_3) = {} & u_0(x_2, x_3) \\ & + \sum_{n=1}^{\infty} [u_{2n+1}(x_2, x_3) \cos((2n+1)x_1) \\ & + u_{2n}(x_2, x_3) \sin(2nx_1)], \end{aligned} \tag{56}$$

where, for instance,

$$V_{2n+1}(t, x_2, x_3) = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} V(t, x_1, x_2, x_3) \cos((2n+1)x_1) dx_1.$$

The rest Fourier coefficients are determinated by similar well-known formulae. Substituting this series into (38) we obtain the equation:

$$\frac{\partial V_n}{\partial t}(t, x_1, x_2) + \Delta V_n(t, x_1, x_2) - n^2 V_n(t, x_1, x_2) = A_n(t, x_1, x_2). \tag{57}$$

Making the change of dependent and independent variables:

$$\tau = T - t, \quad V_n(t, x_1, x_2) = e^{n^2(T-\tau)} v(\tau, x_1, x_2), \tag{58}$$

$$w = e^{-n^2 T} u_n(x_2, x_3), \qquad A_n(t, x_2, x_3) = -e^{n^2(T-\tau)} g(\tau, x_2, x_3), \tag{59}$$

we get equation (50). As A is harmonic the same holds for g. Relations (39) and (40) leads to (51) and (52), with w determined in (59) with help of the Fourier coefficient u_n of the final datum V_T defined in (40). Estimate (53) is easily deduced from (41) and (42). □

The next section will be devoted to prove that any solution v of (50), (51) and (52) must be identically zero leadind to the approximate controllability through the Hahn-Banach theorem. This kind of vanishing result is usually known as the *unique continuation property* (see *e.g.* Mizohata [10] and Saut and Scheurer [11]). A related result for $g \equiv 0$ can be found in [11, Remark 2.3]. Nevertheless we point out that in our case g is not zero and so the technique used in [11] does not seem to work. We shall use as key information that g is an harmonic function.

4. Approximate controllability: case of an infinite rectangular cylinder and orthogonal control directions

The main result of this section is as follows:

THEOREM 2. – *Assumed* $\mathbf{e} = \mathbf{e}_1$ *the Stokes boundary value problem* $(\mathcal{P}_D)$ *on the domain* $\Omega = G \times \mathbb{R}$, *where G is rectangle* (20), *is* $\mathcal{H}^0$*-approximate controllable.*

Suppose, by contradiction, that there exists $\mathbf{f} \in \mathcal{H}^0$, $\mathbf{f} \neq \mathbf{0}$ such that (28) holds. Let (v, g, w) be the functions satisfying (50), (51), (52) associated to the adjoint problem (29). In our case, the set $\mathcal{O}$ has the form

$$\mathcal{O} \doteq \{|x_2| < 1, x_3 \in \mathbb{R}\} = (-1, 1) \times \mathbb{R}. \tag{60}$$

Let $\tilde{v}$ be the Fourier transform of v along x_3:

$$\tilde{v}(t, x_2, \lambda) \doteq \int_{-\infty}^{\infty} e^{-i\lambda x_3} v(t, x_2, x_3)\, dx_3.$$

Analogously, let $\tilde{g}(t, x_2, \lambda)$ and $\tilde{w}(x_2, \lambda)$ be the Fourier transforms of g, w. For λ fixed we define:

$$\varphi(t, x_2) \doteq e^{\lambda^2 t} \tilde{v}(t, x_2, \lambda), \qquad h(t, x_2) \doteq e^{\lambda^2 t} \tilde{g}(t, x_2, \lambda), \qquad \psi(x_2) \doteq \tilde{w}(x_2, \lambda).$$

In the rest of the section we shall denote x_2 by x. Obviously (50), (51), (52) imply the relations

$$\frac{\partial \varphi}{\partial t}(t, x) - \frac{\partial^2 \varphi}{\partial x^2}(t, x) = h(t, x), \quad \text{in } (0, \infty) \times (-1, 1), \tag{61}$$

(62) $$\varphi(t,\pm 1) = \frac{\partial \varphi}{\partial x}(t,\pm 1) = 0, \qquad t \in (0,\infty),$$

(63) $$\varphi(0,x) = \psi(x), \qquad x \in (-1,1).$$

Notice that the functions $(\varphi(t,-x), h(t,-x), \psi(-x))$ and

$$\left(\frac{\varphi(t,x) \pm \varphi(t,-x)}{2}, \frac{h(t,x) \pm h(t,-x)}{2}, \frac{\varphi(x) \pm \varphi(-x)}{2}\right)$$

also satisfy (61), (62), (63). Thus, it is sufficient to study (61), (62), (63) when

(64) $\varphi(t,x), h(t,x), \psi(x)$ are even functions with respect to x

and when

(65) $\varphi(t,x), h(t,x), \psi(x)$ are odd functions with respect to x.

Since g is harmonic then $h(t,x)$ satisfies the equation

$$\left(-\frac{\partial^2}{\partial x^2} + \lambda^2\right) h(t,x) = 0, \qquad x \in (-1,1).$$

Hence, $h(t,x)$ can be rewritten as follows:

(66) $$h(t,x) = \alpha(t)\cosh \lambda x, \quad \text{(case of (64))},$$

(67) $$h(t,x) = \alpha(t)\sinh \lambda x, \quad \text{(case of (65))}.$$

Let us start by considering the case of even solutions of (61), (62), (63).

THEOREM 3. – *Assumed* (64) *and* (66) *the functions* (φ, h, ψ) *are necessarily identically zero.*

Proof. – We extend by zero the functions $\varphi(t,x)$ and $h(t,x)$ outside the set $[0,\infty] \times [-1,1]$ on the whole plane $\mathbb{R}^2$ and the function $w(x)$ outside the interval $[-1,1]$. We denote again those extensions by φ, h and w respectively. Thanks to (62) φ is solution of the Cauchy problem associated to (61) and (63), *i.e.* with $t \in (0,\infty)$ and $x \in \mathbb{R}$. Consider the Fourier transform of those functions with respect to t and x:

(68) $$\hat{\varphi}(p,\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(pt+x\xi)}\varphi(t,x)\,dt\,dx = \int_0^{\infty}\int_{-1}^{1} e^{-i(pt+x\xi)}\varphi(t,x)\,dt\,dx.$$

Taking into account (66), then

(69) $$\hat{\alpha}(p) = \int_{-\infty}^{\infty} e^{-ipt}\alpha(t)\,dt = \int_0^{\infty} e^{-ipt}\alpha(t)\,dt,$$

$$(\chi\widehat{\cosh\lambda x})(\xi) = \int_{-1}^{1} \mathrm{e}^{-ix\xi}\frac{\mathrm{e}^{\lambda x}+\mathrm{e}^{-\lambda x}}{2}dx = \frac{\sin(\xi+i\lambda)}{\xi+i\lambda} + \frac{\sin(\xi-i\lambda)}{\xi-i\lambda}, \tag{70}$$

where $\chi(x) = 1$ for $x \in (-1,1)$, $\chi(x) = 0$, for $x \notin (-1,1)$.

Equation (61) becomes the identity:

$$(ip+\xi^2)\hat{\varphi}(p,\xi) = \hat{\alpha}(p)\left(\frac{\sin(\xi+i\lambda)}{\xi+i\lambda} + \frac{\sin(\xi-i\lambda)}{\xi-i\lambda}\right) + \hat{w}(\xi), \tag{71}$$

where $\hat{w}(\xi)$ is the Fourier transform from x to ξ of the function $\psi(x)$. We point out that since the supports of φ and h are bounded along x-direction then their Fourier transforms are entire analytical functions with respect to ξ. Besides, φ, α are equal to zero for $t < 0$ and therefore their Fourier transforms are analytical functions with respect to p, in the halfplane $\{\mathrm{Im}p < 0\}$. Hence, (71) implies that if $p = i\xi^2$ then the right hand side of (71) vanishes. Consequently we get the equality

$$\hat{\alpha}(i\xi^2)\left(\frac{\sin(\xi+i\lambda)}{\xi+i\lambda} + \frac{\sin(\xi-i\lambda)}{\xi-i\lambda}\right) = -\hat{w}(\xi), \tag{72}$$

for $\xi^2 < 0$. By formula (72) we can construct an analytical extension of the function $\hat{\alpha}(i\xi^2)$ from the halfplane $\{\xi^2 < 0\}$ up to $\{\xi^2 > 0\}$. Hence, this function is meromorphic one with the poles at the roots of the equation:

$$\Phi(\xi,\lambda) = \frac{\sin(\xi+i\lambda)}{\xi+i\lambda} + \frac{\sin(\xi-i\lambda)}{\xi-i\lambda} = 0. \tag{73}$$

These roots are real and simple. Indeed, it is evident if $\lambda = 0$. If $\lambda \neq 0$ then multiplying (73) by $(\xi+i\lambda)(\xi-i\lambda)$ and doing simple transformations we get equation $\xi\tan\xi = -\lambda\tanh\lambda$. Solving this equation by graph method we establish that all its roots, except $\xi = \pm i\lambda$, are real and simple. But straightforward verification shows that $\xi = \pm i\lambda$ are not the roots of (73) if $\lambda \neq 0$. Thus, thanks to (72), we have:

$$\hat{\alpha}(i\xi^2) = -\left(\frac{\sin(\xi+i\lambda)}{(\xi+i\lambda)} + \frac{\sin(\xi-i\lambda)}{(\xi-i\lambda)}\right)^{-1}\hat{w}(\xi). \tag{74}$$

Since $\Phi(\xi,\lambda)$ is even with respect to ξ, we can write for any positive integer N:

$$\begin{aligned}\frac{1}{\Phi(\xi,\lambda)} &= \sum_{k=1}^{N}\left(\frac{1}{\Phi'(\xi_k,\lambda)(\xi+\xi_k)} + \frac{1}{\Phi'(-\xi_k,\lambda)(\xi-\xi_k)}\right) + S_N(\xi)\\ &= \sum_{k=1}^{N}\frac{2\xi_k}{\Phi'(\xi_k,\lambda)(\xi^2-\xi_k^2)} + S_N(\xi),\end{aligned}$$

where $S_N(\xi)$ is an analytic function for $|\mathrm{Re}\xi| < \xi_{N+1}$ and $\{\xi_k, k = 1,2,\ldots\}$ is the set of all positive roots of equation (73). Evidently, $S_N(\xi)$ is even and therefore:

$$\frac{1}{\Phi(\xi,\lambda)} = \sum_{k=1}^{N}\frac{2\xi_k}{\Phi'(\xi_k,\lambda)(\xi^2-\xi_k^2)} + S_{1,N}(\xi^2).$$

Hence, for any positive integer N we can rewrite (74) in the form

$$\hat{\alpha}(i\xi^2) = \sum_{k=1}^{N} \frac{i\alpha_k}{i\xi^2 - i\xi_k^2} + \hat{R}_N(i\xi^2) \tag{75}$$

where $\hat{R}(p)$ is an analytical function defined for $\operatorname{Im} p < \xi_{N+1}^2$ and

$$\alpha_k = -2\xi_k w(\xi_k)/(\partial\Phi(\xi_k,\lambda)/\partial\xi). \tag{76}$$

We substitute $p = i\xi^2$ into (75) and apply to the obtained equality the inverse Fourier transformation $F_{p\to t}^{-1}$. As a result we obtain the equality:

$$\alpha(t) = \begin{cases} 0, & \text{when } t < 0, \\ -\sum\limits_{k=1}^{N} \alpha_k e^{-\xi_k^2 t} + R_N(t), & \text{when } t > 0, \end{cases} \tag{77}$$

where $R_N(t) = F_{p\to t}^{-1}(\hat{R}(p))$ is the function satisfying the relation

$$|R_N(t)e^{\gamma t}| \to 0, \qquad \text{as } t \to \infty, \quad \forall \gamma \in (0, \xi_{N+1}^2). \tag{78}$$

The following assertion is true:

LEMMA 4. – *Let α_k be the numbers given in* (76). *Then for almost all $\lambda \in R$*

$$\alpha_k = 0 \ \textit{for any natural } k. \tag{79}$$

We will prove this Lemma after finishing the proof of Theorem 3.

Continuation of the Theorem 3 Proof. – In virtue of Lemma 4 and (75) the function (74) is an entire analytical function. Let us prove that this function is a constant. We point out that if Φ is defined by (73) then

$$\Phi(\xi,\lambda) = \frac{e^{i\zeta}e^{-(\eta+\lambda)} - e^{-i\zeta}e^{(\eta+\lambda)}}{2i(\zeta + i(\eta+\lambda))} + \frac{e^{i\zeta}e^{-(\eta-\lambda)} - e^{-i\zeta}e^{(\eta-\lambda)}}{2i(\zeta + i(\eta-\lambda))}, \tag{80}$$

where $\xi = \zeta + i\eta$. Let us estimate this function from below when $|Im(\xi)| = |\eta| \geq 2|\lambda|$. Suppose, for instance, that $\eta \geq 2\lambda > 6$ (other situations are treated in a similar way). Then

$$|\Phi(\xi,\lambda)| \geq e^{\eta}\left(\frac{(e^{\lambda} - e^{-3\lambda})}{2\sqrt{\zeta^2 + (\eta+\lambda)^2}} - \frac{(e^{-\lambda} + e^{-3\lambda})}{2\sqrt{\zeta^2 + (\eta-\lambda)^2}}\right) \geq \frac{ce^{\eta}}{\sqrt{\zeta^2+\eta^2}}, \tag{81}$$

for some $c = c(\lambda) > 0$. It is easy to see that (81) (without the middle term) also holds for $\lambda \in [0, 3)$ when $\eta \geq 6$. To estimate $\hat{w}(\xi)$ from above we recall that $\varphi(t,\cdot) \in H^2(0,1)$ and

φ satisfies (62). We can assume also that $w \in H^2(0,1)$ and that w satisfies (62). Indeed, if this is not the case, we consider the problem (61), (62) on the interval (ϵ, ∞), replacing (63) by $v(\epsilon, x_2, x_3) = \varphi(\epsilon, x_2, x_3)$. Using the properties of w we get:

$$
\begin{aligned}
|\hat{w}(\xi)| &= \left| \int_{-1}^{1} \mathrm{e}^{-i\xi x} w(x)\,dx \right| \\
&= \left| \frac{1}{\xi^2} \int_{-1}^{1} \mathrm{e}^{-i(\zeta+i\eta)x} \left(\frac{\partial^2}{\partial x^2} w(x) \right) dx \right| \\
&\le \frac{c\|\frac{\partial^2}{\partial x^2} w(\cdot)\|_{L^2(-1,1)} \mathrm{e}^{|\eta|}}{\zeta^2 + \eta^2},
\end{aligned} \tag{82}
$$

if $\xi = \zeta + i\eta$. From (74), (80), (81), and (82) we conclude that the right hand side term of (74) is an entire function of the variable $\xi = \zeta + i\eta$ which is bounded for $|\eta| \ge \max\{6, 2|\lambda|\}$.

Now we prove its boundedness for $|\eta| < \max\{6, 2|\lambda|\}$. To make it we firstly prove the boundedness of $\hat{\alpha}$ on $\bigcup_{k=1}^{\infty} \{\xi \in \mathbb{C} :\ |\xi - \xi_k| < \varepsilon\}$. Note that in virtue of Lemma 4 and (76) $\hat{w}(\xi_k) = 0$. Since ξ_k are the roots of equation

$$
\tan \xi + \frac{\lambda \tanh \lambda}{\xi} = 0, \tag{83}
$$

then, $\xi_k = k\pi + \alpha_k$ where α_k is small enough for large k. After substitution of this representation for ξ_k into (83) we get that for large k:

$$
\xi_k = k\pi + \alpha_k = k\pi - \frac{\lambda \tanh \lambda}{\pi k} + O\left(\frac{1}{k^2} \right). \tag{84}
$$

We differentiate the function $\Phi(\xi, \lambda)$ in (73) and make simple transformation taking into account (84). Then, we have:

$$
\begin{aligned}
\frac{\partial \Phi(\xi_k, \lambda)}{\partial \xi} &= (-1)^k \left(\frac{\cos(i\lambda) - \alpha_k \sin(i\lambda)}{\pi k + \alpha_k + i\lambda} + \frac{\cos(i\lambda) + \alpha_k \sin(i\lambda)}{\pi k + \alpha_k - i\lambda} \right) + O\left(\frac{1}{k^2} \right) \\
&= (-1)^k \frac{2\cosh \lambda (\pi k + \alpha_k) - 2\lambda \alpha_k \sinh \lambda}{(\pi k + \alpha_k)^2 + \lambda^2} + O\left(\frac{1}{k^2} \right) \\
&= (-1)^k \frac{2\cosh \lambda}{\pi k} + O\left(\frac{1}{k^2} \right).
\end{aligned} \tag{85}
$$

Therefore, we obtain:

$$
\left| \frac{\partial \Phi(\xi_k, \lambda)}{\partial \xi} \right| \ge \frac{\cosh \lambda}{\pi k} \text{ for large } k. \tag{86}
$$

Differentiating on $\Phi(\xi,\lambda)$ two times yields:

$$\left|\frac{\partial^2\Phi(\xi,\lambda)}{\partial\xi^2}\right| \leq \frac{c}{|\xi|}, \tag{87}$$

where c does not depend on ξ. As in (82) we obtain:

$$\left|\frac{\partial^l\hat{w}(\xi)}{\partial\xi^l}\right| = \left|\int_{-1}^{1} e^{-i\xi x}x^l w(x)dx\right| \leq \frac{c}{|\xi|^2}. \tag{88}$$

Let $Q_{k,\varepsilon} = \{\xi \in \mathbb{C} : \ |\xi-\xi_k| < \varepsilon\}$ where ε is small enough and $k \geq k_0$, with k_0 sufficiently large. Applying to (74) the Taylor formula with the remainder Lagrange term and taking into account equality $\hat{w}(\xi_k)=0$ and estimates (86)-(88) we get:

$$\begin{aligned} |\hat{\alpha}(i\xi^2)| &= \left|\frac{\partial\Phi(\xi_k,\lambda)}{\partial\xi}(\xi-\xi_k) + \frac{1}{2}\frac{\partial^2\Phi(\hat{\xi}_k,\lambda)}{\partial\xi}(\xi-\xi_k)^2\right|^{-1} \\ &\quad\times\left|w'(\xi_k)(\xi-\xi_k) + \frac{w''}{2}(\tilde{\xi}_k)(\xi-\xi_k)^2\right| \\ &\leq \left(\left|\frac{\partial\Phi(\xi_k,\lambda)}{\partial\xi}\right| - \frac{1}{2}\left|\frac{\partial^2\Phi(\hat{\xi}_k,\lambda)}{\partial\xi^2}(\xi-\xi_k)\right|\right)^{-1} \\ &\quad\times\left(|w'(\xi_k)| + |w''(\tilde{\xi}_k)(\xi-\xi_k)|\right) \\ &\leq \left(\frac{\cosh\lambda}{\pi k} - \frac{c_1\varepsilon}{k}\right)^{-1}\frac{c_2}{k^2} \leq \frac{c_3}{k} \quad \forall\xi\in Q_{k,\varepsilon},\ k>k_0,\ \varepsilon\ll 1. \end{aligned} \tag{89}$$

We prove now boundedness of $\hat{\alpha}$ on the set:

$$\mathcal{D}_\varepsilon = \{\xi=\zeta+i\eta : \ \zeta\in R,\ |\eta| < \max\{6, 2|\lambda|\}\} \setminus \bigcup_{k=1}^{\infty}\{\xi\in\mathbb{C} : \ |\xi-\xi_k|<\varepsilon\} \tag{90}$$

where $\{\xi_k\}$ is the set of all roots of equation (73) and $\varepsilon > 0$ is sufficiently small.

To make it we rewrite function (73) in an other form. After elementary transformations we get:

$$\begin{aligned} \Phi(\xi,\lambda) &= \frac{\xi(\sin(\xi+i\lambda)+\sin(\xi-i\lambda)) + i\lambda(\sin(\xi-i\lambda)-\sin(\xi+i\lambda))}{\xi^2+\lambda^2} \\ &= \frac{2\xi\sin\xi\cos(i\lambda) - 2i\lambda\cos\xi\sin(i\lambda)}{\xi^2+\lambda^2} \\ &= \frac{2\xi\cosh\lambda}{\xi^2+\lambda^2}\sin\xi + \frac{2\lambda\sinh\lambda}{\xi^2+\lambda^2}\cos\xi \\ &= 2\frac{\sqrt{\xi^2\cosh^2\lambda+\lambda^2\sinh^2\lambda}}{\xi^2+\lambda^2}\sin(\xi+\theta(\xi,\lambda)), \end{aligned} \tag{91}$$

where:

$$\theta(\xi,\lambda)=\arccos\frac{\xi\cosh\lambda}{\sqrt{\lambda^2\sinh^2\lambda+\xi^2\cosh^2\lambda}}\,,\qquad \theta\in[0,\pi],\ \xi\in R. \tag{92}$$

This function can be extended up to a bounded analytical function $\theta(\xi,\lambda)$ for $\xi\in\{|\mathrm{Re}\xi|>\xi_0(\lambda),\ |\mathrm{Im}\xi|<2|\lambda|\}$, where $\xi_0(\lambda)>0$ is a certain number and Im $\theta(\xi,\lambda)\neq 0$ if Im $\xi\neq 0$. Hence, function (74) can be rewritten in the form:

$$\hat{\alpha}(i\xi^2)=2\frac{\xi^2+\lambda^2}{\sqrt{\xi^2\mathrm{ch}^2\lambda+\lambda^2\mathrm{sh}^2\lambda}}\frac{\hat{w}(\xi)}{(\sin(\xi+\theta(\xi,\lambda)))}. \tag{93}$$

The representation (93), (74) and estimate (82) imply that, for an arbitrary small $\varepsilon>0$, the function (85) is bounded on $\mathcal{D}_\varepsilon$.

Estimate (89) and the boundedness of function (93) on the set (90) prove the boundedness of $\alpha(i\xi^2)$ on the set $\{\xi\in\mathbb{C}:\ |\mathrm{Im}\xi|<\max\{6,2|\lambda|\}\}$. Hence the function $\alpha(i\xi^2)$ is bounded on whole complex plane and by the Phragmen-Lindeloff principle (see Hille [3]) this function must be a constant. Then, from (74), (81) and (82) we deduce that

$$|\hat{\alpha}(i\xi)^2|\leq\frac{c}{\sqrt{\eta^2+\zeta^2}}$$

and so $\hat{\alpha}(p)\equiv 0$. Therefore, by (74) $\hat{w}(\xi)\equiv 0$ and from (71) $\hat{\varphi}(p,\xi)\equiv 0$. This implies that $\varphi(t,x)\equiv 0,\forall t\in(\epsilon,T)$ with $\epsilon>0$ arbitrary. Thus, by continuity, $\varphi(t,x)\equiv 0$ for any $t\in(0,T)$ and hence $w(x)\equiv 0$. $\square$

Proof of Lemma 4. – We consider the problem (61)-(63) where h is defined by (66), (77) and $\psi(x)$ is an even function. Obviously, the solution $\varphi(t,x)$ of this problem is even with respect to x. That is why we can develop $\varphi(t,x)$ in its Fourier series with respect to functions $\{\cos\frac{\pi}{2}(2k+1),\ k=0,1,2,\ldots\}$:

$$\begin{cases}\varphi(t,x)=\displaystyle\sum_{k=0}^{\infty}\varphi_k(t)\cos\left(\frac{\pi}{2}(2k+1)x\right),\\ \text{where}\\ \varphi_k(t)=\displaystyle\int_{-1}^{1}\varphi(t,x)\cos\left(\frac{\pi}{2}(2k+1)x\right)dx.\end{cases} \tag{94}$$

Suppose that $\lambda\neq\frac{\pi}{2}(2k+1),\ k\in\mathbb{Z}_+$. Then

$$\cos\lambda x=\sum_{k=0}^{\infty}\mu_k\cos\frac{\pi}{2}(2k+1)x, \tag{95}$$

where:

$$\mu_k = \int_{-1}^{1} \cos(\lambda x) \cos\left(\frac{\pi}{2}(2k+1)x\right) dx \tag{96}$$
$$= (-1)^k \left(\frac{\sin(\lambda + \frac{\pi}{2}(2k+1))}{\lambda + \frac{\pi}{2}(2k+1)} + \frac{\sin(\lambda - \frac{\pi}{2}(2k+1))}{\lambda - \pi 2(2k+1)} \right)$$
$$= \frac{(-1)^k (\cos\lambda) \frac{\pi}{2}(2k+1)}{\left(\frac{\pi}{2}(2k+1)\right)^2 - \lambda^2}.$$

The substitution (94), (95) into (61) yields the equalities for φ_k:

$$\frac{d}{dt}\varphi_k(t) + \left(\frac{\pi}{2}(2k+1)\right)^2 \varphi_k = \alpha(t)\mu_k. \tag{97}$$

Moreover, if ψ_k are the Fourier coefficients of the decomposition

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k \cos\left(\frac{\pi}{2}(2k+1)x\right)$$

then in virtue of (63), we have:

$$\varphi_k(t)|_{t=0} = \psi_k. \tag{98}$$

Equalities (97), (98), (77) imply the identity

$$\varphi_k(t) = e^{-(\frac{\pi}{2}(2k+1))^2 t}\psi_k + \int_0^t e^{-(\frac{\pi}{2}(2k+1))^2(t-\tau)} \left(-\sum_{j=1}^{N} \alpha_j e^{-\xi_j^2 \tau} + R_N(\tau) \right) d\tau \mu_k \tag{99}$$
$$= e^{-(\frac{\pi}{2}(2k+1))^2 t}\psi_k - \sum_{j=1}^{N} \alpha_j \mu_k \frac{e^{-\xi_j^2 t} - e^{-(\frac{\pi}{2}(2k+1))^2 t}}{(\frac{\pi}{2}(2k+1))^2 - \xi_j^2}$$
$$+ \int_0^t e^{-(\frac{\pi}{2}(2k+1))^2(t-\tau)} R(\tau) d\tau \mu_k.$$

Taking into account (62) and (94) we get the equality:

$$\left.\frac{\partial}{\partial x}\varphi(t,x)\right|_{|x|=1} = -\sum_{k=0}^{\infty} \varphi_k(t)(-1)^k \frac{\pi}{2}(2k+1) = 0. \tag{100}$$

To justificate equality (100) we have to prove the absolute convergence of series (100). The solution $\varphi(t,x)$ of problem (61)-(63) belongs to $C^\infty(R_+ \times [-1,1]) \cap C^\infty(R_+; H^2(-1,1) \cap H_0^1(-1,1))$. Therefore for any $t > 0$, we obtain:

$$\sum_{k=0}^{\infty} |\varphi_k(t)|^2 k^4 < C < \infty,$$

where C does not depend on t. Hence

$$\sum_{k=0}^{\infty}|\varphi_k(t)|\frac{\pi}{2}(2k+1) \leq \frac{\pi}{2}\left(\sum_{k=0}^{\infty}|\varphi_k(t)|^2(2k+1)^4\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\right) < \infty.$$

The substitution of (96) into (99), and, after that, (99) into (100) yields the desired identity:

$$\begin{aligned}(101)\quad & \sum_{k=0}^{\infty}\Bigg((-1)^k\frac{\pi}{2}(2k+1)\Bigg(e^{-\frac{\pi}{2}(2k+1))^2t}\psi_k - \sum_{j=1}^{N}\alpha_j\frac{(-1)^k(\cos\lambda)\pi(2k+1)}{((\frac{\pi}{2}(2k+1))^2-\lambda^2)}\\ &\times\frac{(e^{-\xi_j^2t}-e^{-(\frac{\pi}{2}(2k+1))^2t})}{((\frac{\pi}{2}(2k+1))^2-\xi_j^2)}\\ &+\frac{(-1)^k\pi(2k+1)\cos\lambda}{(\frac{\pi}{2}(2k+1))^2-\lambda^2}\int_0^t e^{-(\frac{\pi}{2}(2k+1))^2(t-\tau)}R_N(t)dt\Bigg)\Bigg)=0.\end{aligned}$$

Since ξ_j in (101) are the positive roots of the equation $\tan\xi = -\lambda(\tanh\lambda)/\xi$, the following inequalities are true:

$$(102)\qquad \frac{\pi}{2} < \xi_1 < \ldots < \xi_j < \frac{\pi}{2}(2j+1) < \xi_{j+1} < \frac{\pi}{2}(2j+3)k\ldots$$

We can rewrite (101) in the form:

$$\begin{aligned}(103)\quad & \sum_{k=0}^{\infty}e^{-(\frac{\pi}{2}(2k+1))^2t}\Bigg[(-1)^k\frac{\pi}{2}(2k+1)\psi_k + \Bigg(\frac{1}{2}\int_0^t e^{(\frac{\pi}{2}(2k+1))^2\tau}R_N(\tau)d\tau\\ &+\sum_{j=1}^{N}\frac{\alpha_j}{(\frac{\pi}{2}(2k+1))^2-\xi_j^2}\Bigg)\frac{\pi^2(2k+1)^2\cos\lambda}{2((\frac{\pi}{2}(2k+1))^2-\lambda^2)}\Bigg]\\ &-\frac{1}{2}\sum_{m=1}^{\infty}\frac{(\pi(2m+1))^2\cos\lambda}{(\frac{\pi}{2}(2m+1))^2-\lambda^2}\sum_{j=1}^{N}\frac{\alpha_je^{-\xi_j^2t}}{(\frac{\pi}{2}(2m+1))^2-\xi_j^2}=0.\end{aligned}$$

We will use (103) to prove, by induction, that $\alpha_j = 0$ for any j. Suppose that we prove already that α_k as well as the coefficients contained in square brackets in (103) with an arbitrary integer $N > k$ are equal zero for $k = 1, 2, \ldots, l-1$. (The first induction step $k = 1$ is proved as well as the step $k = l$ which is realized below.) Let prove the same assertion for $k = l$. For this we take in (103)$N = l$, transfer the term containing $e^{-\xi_l^2t}$ to the right side of (103) and divide the obtained equality on $e^{-\xi_l^2t}$. Then we get:

$$\begin{aligned}&\frac{1}{2}\sum_{m=1}^{\infty}\frac{\pi(2m+1)^2\cos\lambda}{((\frac{\pi}{2}(2m+1))^2-\lambda^2)}\frac{\alpha_l}{((\frac{\pi}{2}(2m+1))^2-\xi_l^2)}\\ &=\sum_{k=l}^{\infty}e^{-((\frac{\pi}{2}(2k+1))^2-\xi_l^2)t}\Bigg((-1)^k\frac{\pi}{2}(2k+1)\psi_k\\ &+\left(\frac{1}{2}\int_0^t e^{(\frac{\pi}{2}(2k+1))^2\tau)}R_N(\tau)d\tau+\frac{\alpha_l}{\frac{\pi}{2}(2k+1))^2-\xi_l^2}\right)\frac{\pi^2(2k+1)^2\cos\lambda}{2((\frac{\pi}{2}(2k+1))^2-\lambda^2)}\Bigg).\end{aligned}$$

Passing to the limit, as $t \to \infty$, in this equality we obtain with help of relation (78) that:

$$\frac{\alpha_l}{2}\sum_{m=1}^{\infty}\frac{(\pi(2m+1))^2\cos\lambda}{((\frac{\pi}{2}(2m+1))^2-\lambda^2)((\frac{\pi}{2}(2m+1))^2-\xi_l^2)}=0.$$

The function

$$Z(\lambda)=\sum_{m=1}^{\infty}\frac{(\pi(2m+1))^2\cos\lambda}{((\frac{\pi}{2}(2m+1))^2-\lambda^2)((\frac{\pi}{2}(2m+1))^2-\xi_l^2(\lambda))} \tag{104}$$

is analytical in a neighbourhood of real straight line R. Indeed, when the term $(\frac{\pi}{2}(2m+1))^2-\lambda^2$ in the denominator is equal zero, then $\cos\lambda$ is equal to zero also. The factor $(\frac{\pi}{2}(2m+1))^2-\xi_e^2(\lambda)$ in denominator does not equal zero in virtue of (102). Since $Z(\lambda)$ is an analytical function, the equation $Z(\lambda)=0$ has not more than a countable set Z_0 of roots. We suppose that λ satisfies the property

$$\lambda \notin Z_0.$$

Then (104) implies the equality $\alpha_l = 0$.

Let $N > l+1$ in (103) be arbitrary. Transfer now the term from (103) containing $e^{-(\frac{\pi}{2}(2l+1))^2 t}$ to the right side, divide the obtained equality on $e^{-(\frac{\pi}{2}(2l+1))^2 t}$ and pass on to the limit as $t\to\infty$. As a result we get that the coefficient before $e^{-(\frac{\pi}{2}(2l+1)^2 t}$ is equal to zero, that completes the induction step. □

Let us consider now the case of odd solutions of (61), (62) and (63).

THEOREM 4. – *Assume* (65) *and* (67). *Then the equality* $(\varphi, h, \psi)\equiv(0,0,0)$ *holds.*

Proof. – It follows the same line as the Theorem 3 proof. Taking Fourier transforms we obtain:

$$(\widehat{\sinh\lambda x})=\int_{-1}^{1}\frac{\mathrm{e}^{-ix\xi}}{2}(\mathrm{e}^{\lambda x}-\mathrm{e}^{-\lambda x})dx=\frac{\sin(\xi+i\lambda)}{\xi+i\lambda}-\frac{\sin(\xi-i\lambda)}{\xi-i\lambda}.$$

Therefore, instead of (74), we get:

$$\hat{\alpha}(i\xi^2)=-\left(\frac{\sin(\xi+i\lambda)}{\xi+i\lambda}-\frac{\sin(\xi-i\lambda)}{\xi-i\lambda}\right)^{-1}\hat{w}(\xi).$$

It is easy to see that we can repeat the arguments of the proof of Theorem 3 and the conclusion holds. □

The proof of Theorem 2 is now a trivial consequence of Theorems 3 and 4 and the Hahn-Banach theorem.

Acknowledgements

This paper was started during the stay of the second author at the Universidad Complutense de Madrid as Visiting Profesor, in April and May of 1993. The research of the first author was partially supported by the DGICYT (Spain), project PB 93/0443.

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(Manuscript received January 1996.)

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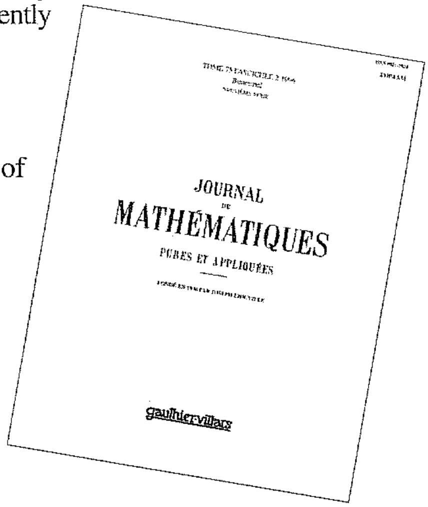

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Export subscription rate: US $ 640 (3 200 FF)

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