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On the Multiplicity of Equilibrium Solutions to a Nonlinear Diffusion Equation on a Manifold Arising in Climatology

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We analyze the sensitivity of a climatological model with respect to small changes in one of the distinguished parameters: the solar constant. We start by proving the stabilization of solutions of the evolution model when time tends to infinity. Later, we study the stationary problem and obtain uniqueness or a multiplicity of solutions for different values of the solar constant. © 1997 Academic Press

1. INTRODUCTION

In this paper we study a nonlinear parabolic equation and its stationary version, obtained through a global energy balance for the atmosphere surface temperature over relatively long time scales. The so-called climate energy balance models were introduced independently by M. Budyko [9] and W. Sellers [21]. This energy balance is expressed in the following

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simple terms:

$$\text{heat variation} = R_a - R_e + D,$$

where R_a represents the solar energy absorbed by the Earth, R_e is the energy emitted by the Earth into space, and D is heat diffusion. If we denote by u the temperature of the Earth surface, then it is natural to assume that $R_a = QS(x)\beta(u)$, where Q is the solar constant, $S(x)$ is the insolation function, and $\beta(u)$ is the coalbedo function (a nondecreasing function of u of the type $\beta(u) = 0.7$ if $u > -10 + \epsilon$, $\beta(u) = 0.4$ if $u < -10 - \epsilon$, where $\epsilon \geq 0$). The term R_e is usually also assumed to be an increasing function on u and can be expressed according to the Newton cooling law as $Bu + C$, where B and C are positive constants [9], or by the Stefan–Boltzmann law, $R_e(t,x,u) = \sigma u^4$ (where u is in Kelvins) [21]. Assuming (for simplicity) the heat capacity and the diffusion coefficient to be constant (e.g., equal to 1), we obtain the following energy balance model:

$$(P)\begin{cases} u_t - \Delta_p u + R_e(t,x,u) \in QS(x)\beta(u) & \text{in } (0,\infty)\times\mathcal{M}, \\ u(0,x) = u_0(x) & \text{on } \mathcal{M}, \end{cases}$$

where $(\mathcal{M}, \mathbf{g})$ is a C^∞ compact connected oriented two-dimensional Riemannian manifold without boundary (as, e.g., $\mathcal{M} = \mathbf{S}^2$ the unit sphere of $\mathbb{R}^3$) and $\Delta_p u = \operatorname{div}_{\mathcal{M}}(|\operatorname{grad}_{\mathcal{M}} u|^{p-2}\operatorname{grad}_{\mathcal{M}} u)$, $p \geq 2$, where $\operatorname{grad}_{\mathcal{M}}$ is understood in the sense of the Riemannian metric $\mathbf{g}$. Budyko and Sellers considered the linear case $p = 2$, but later, Stone [23] proposed the case $p = 3$, to consider the negative feedback in the eddy fluxes. We also assume that $S: \mathcal{M} \to \mathbb{R}$, $S \in L^\infty(\mathcal{M})$ and that there exists $S_0 \leq S_1$ such that $0 < S_0 \leq S(x) \leq S_1$. The coalbedo function will be treated in the general class of multivalued graphs. More precisely, we shall assume that β is a maximal monotone graph of $\mathbb{R}^2$ such that β is bounded (i.e., $m \leq b \leq M$ for any $b \in \beta(s)$ for any $s \in \mathbb{R}$). We recall that β was assumed to be *multivalued* at $u = -10$ by Budyko [9] and β locally Lipschitz by Sellers [21].

The mathematical analysis for this class of problems was carried out by different authors under special formulations and by Díaz [10] for the one-dimensional associated model and then generalized by Díaz and Tello [11, 12] to the bidimensional case.

One of the main interests of this kind of model is its simplicity for the study of the effect of variations on the data (mainly on the solar constant Q). This study was first carried out for $p = 2$ (see, e.g., [18] and its references) and more rigorously for the Sellers model on a Riemannian manifold by Hetzer [15] (some related work can be found in [19, 20]). Our main goal is to extend those results to the quasilinear case ($p \geq 2$) and to the more general framework of the coalbedo term β. We start by proving (in Section 2) the stabilization of solutions of the evolution problem, when $t \to \infty$, to the solutions of the associated stationary problem. This is proved by using *ad hoc* arguments inspired by Díaz and de Thélin [13]. Section 3 is devoted to the study of the number of stationary solutions according to the parameter Q, when β is not necessarily Lipschitz continuous and $p \geq 2$. We estimate an interval of values for Q where there exist at least three stationary solutions and other complementary intervals for Q where the stationary solution is unique. The proofs are based on super-sub solution techniques combined with the topological index theory. A more precise study of the solution set will be made by Arcoya et al. [3].

2. STABILIZATION OF SOLUTIONS WHEN $t \to +\infty$

In this section we shall assume that $R_e = \mathcal{G}(u) - f(t,x)$. So, problem (P) now becomes

$$(P_f)\begin{cases} u_t - \Delta_p u + \mathcal{G}(u) \in QS(x)\beta(u) + f(t,x) & \text{in } (0,\infty)\times\mathcal{M}, \\ u(0,x) = u_0(x) & \text{on } \mathcal{M}. \end{cases}$$

Here we shall assume that

($H_{\mathcal{G}}$) $\mathcal{G}: \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\mathcal{G}(0) = 0$.

Notice that $\mathcal{G}(s) = Bs$ corresponds to the assumption by Budyko [9] and $\mathcal{G}(s) = B|s|^3 s$ to Sellers [21].

(H_β) β is a bounded maximal monotone graph of $\mathbb{R}^2$.

(H_f) $f \in L^\infty((0,\infty)\times\mathcal{M})$.

In [11, 12] the existence of bounded weak solutions was obtained in the space $C([0,\infty); L^2(\mathcal{M})) \cap L^p_{loc}(0,\infty; V) \cap L^\infty((0,\infty)\times\mathcal{M})$ for initial data $u_0 \in L^\infty(\mathcal{M})$ and under the additional condition $|\mathcal{G}(s)| \geq B|s|^r$ for some $r \geq 1$ and $B > 0$ (as usual, $T\mathcal{M}$ represents the tangent bundle, and the functional spaces $L^p(\mathcal{M})$ and $L^p(T\mathcal{M})$ are defined in a standard way; see, e.g., Aubin [4]). We also assume that

(H_∞) there exists $f_\infty \in V'$ such that

$$\int_{t-1}^{t+1} \|f(\tau,\cdot) - f_\infty(\cdot)\|_{V'}\, d\tau \to 0 \qquad \text{as } t \to \infty,$$

where V' is the dual space of the function space $V := \{u \in L^2(\mathscr{M})$: $\text{grad}_{\mathscr{M}} u \in L^p(T\mathscr{M})\}$. To analyze the stabilization of the evolution solutions, we shall need the following result concerning the global regularity of the solutions on $(0, \infty)$:

LEMMA 1. *Assume that*

$$u_0 \in V \cap L^\infty(\mathscr{M}), \tag{1}$$

$$f \in L^\infty((0,\infty) \times \mathscr{M}) \cap W^{1,1}_{\text{loc}}((0,\infty); L^1(\mathscr{M})) \tag{2}$$

and

$$\int_t^{t+1} \left\| \frac{\partial f}{\partial t}(s, \cdot) \right\|_{L^1(\mathscr{M})} ds \le C_0, \qquad \forall t > 0, \tag{3}$$

where C_0 is a time-independent constant. Then there exists a weak solution of (P_f) verifying

$$u \in L^\infty(0,\infty; V) \text{ and } u_t \in L^2(0,\infty; L^2(\mathscr{M})). \tag{4}$$

Proof. Let u be any weak solution of (P_f). Then there exists $z \in \beta(u)$ with $z \in L^2(0,T; L^2(\mathscr{M}))$ for any fixed $T > 0$, such that u coincides with the unique solution of the Cauchy problem:

$$(P_h)\begin{cases} \dfrac{du}{dt}(t) + Au(t) \ni h(t) & t \in (0,T), \text{in } X = L^2(\mathscr{M}) \\ u(0) = u_0, & u_0 \in L^2(\mathscr{M}). \end{cases}$$

associated with $h(t) := QS(\cdot)z(t,\cdot) + f(t,\cdot) \in L^2(\mathscr{M})$, and where A is the operator defined as

$$\begin{aligned} A\colon D(A) \subset L^2(\mathscr{M}) &\to L^2(\mathscr{M}) \\ u &\to -\Delta_p u + \mathscr{G}(u), \end{aligned} \tag{5}$$

and $D(A) = \{u \in L^2(\mathscr{M})$: $-\Delta_p u + \mathscr{G}(u) \in L^2(\mathscr{M})\}$. It is easy to see that A coincides with the subdifferential of the convex lower semicontinuous function ϕ: $D(\phi) \subset L^2(\mathscr{M}) \to \mathbb{R}$, given by

$$\phi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathscr{M}} |\nabla u|^p \, dA + \int_{\mathscr{M}} G(u)\, dA & u \in D(\phi) \\ +\infty & u \notin D(\phi), \end{cases} \tag{6}$$

where $G(u) = \int_0^u \mathscr{G}(\sigma)\, d\sigma$ and $D(\phi) := \{u \in L^2(\mathscr{M})$, $\nabla u \in L^p(T\mathscr{M})$, and $\int_{\mathscr{M}} G(u)\, dA < +\infty\}$. Hence, by the abstract results of Brezis [6], since $u_0 \in V \cap L^\infty(\mathscr{M}) \subset D(\phi)$, we have that the solution u verifies the additional regularity $u \in L^p((0,T); V)$ (and that $u_t \in L^2(0,T; L^2(\mathscr{M}))$, $u \in W^{1,2}((0,T); L^2(\mathscr{M}))$). In what follows we shall improve this regularity, arriving at $L^\infty(0,\infty; V)$. The proof consists of three steps.

Step 1. Let us prove that $\|\nabla u\|_{L^p((t,t+1); L^p(T\mathscr{M}))} \le C_0$ with C_0 independent of t.

Multiplying the equation by u and integrating in $(t, t+1) \times \mathscr{M}$, we obtain

$$\begin{aligned} \int_t^{t+1} \langle u_t, u\rangle_{V' \times V} + \int_t^{t+1}\int_{\mathscr{M}} |\nabla u|^p + \int_t^{t+1}\int_{\mathscr{M}} \mathscr{G}(u) u \\ = \int_t^{t+1}\int_{\mathscr{M}} (QS(x) z u + fu), \end{aligned}$$

where $z \in \beta(u)$. By a well-known result (see [1]),

$$\begin{aligned} \int_t^{t+1} \langle u_t, u\rangle_{V' \times V} &= \frac{1}{2}\int_t^{t+1}\int_{\mathscr{M}} \frac{d}{dt}|u|^2 \\ &= \int_{\mathscr{M}} |u(t+1,x)|^2 - \int_{\mathscr{M}} |u(t,x)|^2. \end{aligned}$$

This expression is bounded independently of t, since by applying the comparison principle, we can prove that $u \in L^\infty((0,\infty) \times \mathscr{M})$. Moreover,

$$\int_t^{t+1}\int_{\mathscr{M}} \mathscr{G}(u) u \ge 0,$$

$$\int_t^{t+1}\int_{\mathscr{M}} QS(x) z u \le Q\|S\|_{L^\infty(\mathscr{M})} M |\mathscr{M}|\, \|u\|_{L^\infty((0,\infty)\times\mathscr{M})},$$

$$\begin{aligned} \int_t^{t+1}\int_{\mathscr{M}} f(s,x) u(s,x)\, dA\, ds &\le \|f\|_{L^\infty(\mathscr{M})} \int_t^{t+1}\int_{\mathscr{M}} |u|\, dA\, ds \\ &\le \|f\|_{L^\infty(\mathscr{M})} |\mathscr{M}|\, \|u\|_{L^\infty((0,\infty)\times\mathscr{M})}. \end{aligned}$$

Combining the previous estimates, we have

$$\int_t^{t+1}\int_{\mathscr{M}} |\nabla u(s,x)|^p \, dA\, ds \le C_0, \tag{7}$$

where C_0 is a positive constant that depends only on $Q, S, \|u\|_{L^\infty((0,\infty)\times\mathscr{M})}$ and $\|f\|_{L^\infty((0,\infty)\times\mathscr{M})}$.

Step 2. Now we replace the data u_0, $S(x)$, and $f(t,x)$ and the graph β to have regular enough solutions by smooth approximations. Then, multiplying the equation by u_t and integrating on $(\sigma,\tau)\times\mathscr{M}$ with $0<\tau-\sigma\le 1$, we obtain

$$\int_\sigma^\tau\int_{\mathscr{M}}|u_t|^2+\frac{1}{p}\int_\sigma^\tau\int_{\mathscr{M}}\frac{d}{dt}|\nabla u|^p+\int_\sigma^\tau\int_{\mathscr{M}}\frac{d}{dt}G(u)$$
$$=\int_\sigma^\tau\int_{\mathscr{M}}QS(x)\frac{d}{dt}j(u)+\int_\sigma^\tau\int_{\mathscr{M}}fu_t,$$

where j is the convex function such that $\partial j=\beta$. Therefore,

$$\int_\sigma^\tau\int_{\mathscr{M}}|u_t|^2+\frac{1}{p}\int_{\mathscr{M}}|\nabla u(\tau)|^p+\int_{\mathscr{M}}G(u(\tau))-\int_{\mathscr{M}}G(u(\sigma))$$
$$=\frac{1}{p}\int_{\mathscr{M}}|\nabla u(\sigma)|^p+\int_{\mathscr{M}}QS(x)j(u(\tau))$$
$$-\int_{\mathscr{M}}QS(x)j(u(\sigma))+\int_\sigma^\tau\int_{\mathscr{M}}fu_t. \qquad (8)$$

From $u\in L^\infty((0,\infty)\times\mathscr{M})$, we deduce that there exists $C_1>0$, which is not dependent of τ, such that

$$\int_{\mathscr{M}}G(u(\tau))\le C_1,\qquad \int_{\mathscr{M}}QS(x)j(u(\tau))\le C_1\qquad\forall\tau. \qquad (9)$$

On the other hand, integrating by parts in the last term of (8) and by using (3), we conclude that

$$\int_\sigma^\tau\int_{\mathscr{M}}fu_t\,dA\,dt=\int_{\mathscr{M}}(f(\tau)u(\tau)-f(\sigma)u(\sigma))\,dA$$
$$-\int_\sigma^\tau\int_{\mathscr{M}}f_tu\,dA\,dt\le C_0\|u\|_{L^\infty((0,\infty)\times\mathscr{M})}. \qquad (10)$$

Taking into account estimates (9), (10) in (8), we get

$$\int_\sigma^\tau\int_{\mathscr{M}}|u_t|^2\,dA\,dt\le C_2+\frac{1}{p}\int_{\mathscr{M}}|\nabla u(\sigma)|^p\,dA\,dt-\frac{1}{p}\int_{\mathscr{M}}|\nabla u(\tau)|^p\,dA\,dt, \quad (11)$$

where C_2 is given by

$$C_2=4C_1+(2|\mathscr{M}|\,\|f\|_{L^\infty((0,\infty)\times(\mathscr{M})}+C_0)\|u\|_{L^\infty((0,\infty)\times\mathscr{M})}.$$

Step 3. To conclude the proof of (4), we shall use the following technical result.

LEMMA (Nakao [17]). *Let $\varphi(t)\ge 0$ be a function locally bounded such that*

$$\varphi(t+1)\le C[\varphi(t)-\varphi(t+1)]+\rho(t)\qquad t>0, \qquad (12)$$

where C is a positive constant and $\rho(t)>0$ when t is large enough. Assume that $\rho(t)=O(1)$ when $t\to\infty$. Then $\varphi(t)=O(1)$.

To apply this lemma, we take

$$\varphi(t)=\frac{1}{p}\int_{\mathscr{M}}|\nabla u(t,x)|^p\,dA$$

and $\rho(t)=C_2$. Let us show that φ verifies the hypothesis (12). From (11) it is easy to see that

$$\varphi(t+1)-\varphi(\sigma)\le C_2+\int_t^{t+1}\int_{\mathscr{M}}|u_t|^2\,dA\,ds, \qquad (13)$$

for any $\sigma\in(t,t+1)$. Moreover, from (7) we obtain that if t is large enough, there exists $t^*\in(t,t+1)$ such that $\varphi(t^*)\le C_0$. Taking $\sigma=t^*$ in (13) and using the above arguments, the inequality (12) holds.

So, by the lemma we have $|\varphi(t)|\le C$, which is equivalent to $u\in L^\infty(0,\infty;V)$; by (11) this implies that $u_t\in L^2(0,\infty;L^2(\mathscr{M}))$. Finally, the above bound depends only on the data, so the conclusion holds for the weak solution u of (P), a limit of such regular approximate solutions. (The proof of the convergence arguments is standard.) ∎

The following theorem proves the stabilization of the solutions u satisfying (4). As usual, let u be a bounded weak solution of (P_f); we define the ω-limit set of u by

$$\omega(u)=\{u_\infty\in V\cap L^\infty(\mathscr{M}):$$
$$\exists t_n\to+\infty\qquad\text{such that } u(t_n,\cdot)\to u_\infty \text{ in } L^2(\mathscr{M})\}.$$

THEOREM 1. *Let $u_0\in L^\infty(\mathscr{M})\cap V$ and let u be any bounded weak solution satisfying* (4). *Then*

(i) $\omega(u)\ne\varnothing$,

(ii) *If $u_\infty\in\omega(u)$, then $\exists t_n\to+\infty$, such that $u(t_n+s,\cdot)\to u_\infty$ in $L^2(-1,1;L^2(\mathscr{M}))$, and $u_\infty\in V$ is a weak solution of the stationary problem*

$$(P_Q)\qquad -\Delta_p u_\infty+\mathscr{G}(u_\infty)\in QS\beta(u_\infty)+f_\infty\qquad\text{in } \mathscr{M},$$

(iii) *Actually, if $u_\infty\in\omega(u)$, then $\exists\{\hat t_n\}\to+\infty$, such that $u(\hat t_n,\cdot)\to u_\infty$ strongly in V.*

Proof.

(i) From (4), $\{u(t_n, \cdot)\}$ is a bounded sequence of V. Since V is a reflexive Banach space, we can extract a subsequence $\{u(t_n, \cdot)\}$ that converges weakly to $v \in V$. As V is compactly embedded into $L^2(\mathscr{M})$, there exists a subsequence $\{u(t_{n_i}, \cdot)\}$ that converges strongly to a function v in $L^2(\mathscr{M})$. Therefore, $v \in \omega(u)$, and so $\omega(u) \neq \varnothing$.

(ii) The first part comes from the integrability of u_t. Let u_∞ be an element of $\omega(u)$. So, there exists a sequence $t_n \to \infty$ such that $u(t_n, \cdot) \to u_\infty$ in $L^2(\mathscr{M})$. Then, for a.e. $x \in \mathscr{M}$,

$$|u(t_n + s, x) - u(t_n, x)| = \left| \int_{t_n}^{t_n+s} u_t(\sigma, x)\, d\sigma \right| \le \sqrt{2} \left(\int_{t_n-1}^{t_n+1} |u_t|^2\, d\sigma \right)^{1/2}$$

and so

$$\|u(t_n + s) - u(t_n)\|^2_{L^2(\mathscr{M})} \le 2\|u_t\|^2_{L^2((t_n-1, t_n+1); L^2(\mathscr{M}))}.$$

Since $u_t \in L^2(0, \infty; L^2(\mathscr{M}))$,

$$\|u_t\|^2_{L^2((t_n-1, t_n+1); L^2(\mathscr{M}))} \to 0 \qquad \text{when } t_n \to \infty. \tag{14}$$

Finally, since $u(t_n + s, \cdot) \to u_\infty$ in $L^2(\mathscr{M})$ a.e. $s \in (-1, 1)$ and $\|u(t_n + s)\|_{L^2(\mathscr{M})} \le \|u\|_{L^\infty(0,\infty; L^2(\mathscr{M}))}$, by the Lebesgue convergence theorem we conclude that

$$u(t_n + \cdot, \cdot) \to u_\infty \text{ in } L^2((-1, 1); L^2(\mathscr{M})).$$

To prove that u_∞ is a solution of (P_Q), we consider the test functions $v(t, x) = \xi(x)\varphi(t - t_n)$, where $\xi \in V \cap L^\infty(\mathscr{M})$ and $\varphi \in \mathscr{D}(-1, 1)$, $\varphi \ge 0$, $\int_{-1}^{1} \varphi = 1$. Then

$$\begin{aligned} &\int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} u_t \xi\varphi(t - t_n) + \int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} |\nabla u|^{p-2} \nabla u \cdot \nabla\xi\varphi(t - t_n) \\ &\quad + \int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} \mathscr{G}(u)\xi\varphi(t - t_n) \\ &= \int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} Qz\xi\varphi(t - t_n) - \int_{t_n-1}^{t_n+1} \int_{\mathscr{M}} f(t, x)\xi\varphi(t - t_n) \\ &\qquad\qquad z \in \beta(u(t, x)), \end{aligned}$$

(here we have denoted by $a \cdot b$ the scalar product on $T\mathscr{M}$). Changing variables, namely $s = t - t_n$ and defining $U_n(s, x) = u(t_n + s, x)$, we obtain the a priori estimates

$$\|U_n\|_{L^\infty(-1,1; V)} \le C_1, \qquad \|\nabla U_n\|_{L^\infty(-1,1; L^p(T\mathscr{M}))} \le C_2,$$
$$\|z_n\|_{L^\infty(-1,1; L^\infty(\mathscr{M}))} \le C_3,$$

and thus the following convergences hold:

$$\begin{aligned} &U_n \rightharpoonup u_\infty && \text{weakly in } L^\sigma((-1, 1); V) && \forall\sigma > 1 \\ &|\nabla U_n|^{p-2}\nabla U_n \rightharpoonup Y && \text{weakly in } L^s((-1, 1); L^{p}(T\mathscr{M})) && \forall\sigma > 1. \end{aligned}$$

Using (14), and applying Aubin's compactness result (see, e.g., Simon [22]), a well-known property of the maximal monotone graphs (see Brezis [7]) and Lebesgue's theorem, we get

$$z_n \rightharpoonup z_\infty \in \beta(u_\infty) \qquad \text{weakly in } L^\sigma((-1, 1) \times \mathscr{M}) \qquad \forall\sigma > 1$$

and

$$\mathscr{G}(U_n) \to \mathscr{G}(u_\infty) \qquad \text{in } L^1((-1, 1) \times \mathscr{M}).$$

Letting $n \to \infty$, we arrive at

$$\int_{-1}^{1} \int_{\mathscr{M}} Y \cdot \nabla\xi\varphi + \int_{\mathscr{M}} \mathscr{G}(u_\infty)\xi = \int_{\mathscr{M}} QSz_\infty\xi + \int_{\mathscr{M}} f_\infty\xi \qquad \forall\xi \in V \cap L^\infty(\mathscr{M}).$$

Now the main difficulty is to prove that $\int_{-1}^{1} Y(s, \cdot)\varphi(s) = |\nabla u_\infty|^{p-2}\nabla u_\infty$. Because of the coercivity of the p-Laplacian operator, we obtain the following inequality:

$$\lim_{n\to\infty} \int_{-1}^{1} \int_{\mathscr{M}} \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla\chi|^{p-2}\nabla\chi \right) \cdot (\nabla u_\infty - \nabla\chi)\varphi(s)\, dA\, ds \ge 0$$
$$\chi \in V. \tag{15}$$

We arrive at the desired convergence by taking $\chi = u_\infty + \lambda\xi$ and applying a Minty-type argument to (15), as in [13].

(iii) This part also uses the coercivity of the operator and the fact that

$$\int_{-1}^{1} \int_{\mathscr{M}} \left(|\nabla U_n|^{p-2}\nabla U_n - |\nabla u_\infty|^{p-2}\nabla u_\infty \right) \cdot (\nabla U_n - \nabla u_\infty)\varphi(s)\, dA\, ds \to 0.$$

The inequality $|\zeta - \hat{\zeta}|^p \le (|\zeta|^{p-2}\zeta - |\hat{\zeta}|^{p-2}\hat{\zeta}) \cdot (\zeta - \hat{\zeta})\ \forall\zeta, \hat{\zeta} \in \mathbb{R}^N$ allows us to obtain

$$\lim_{n\to\infty} \int_{-1}^{1} \int_{\mathscr{M}} |\nabla U_n - \nabla u_\infty|^p \varphi(s)\, dA\, ds = 0,$$

$\forall\varphi \in \mathscr{D}(-1, 1)$, $\varphi \ge 0$ and $\int_{-1}^{1} \varphi = 1$. This means that there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ where $s_n \in (-1, 1)$, such that

$$\lim_{n\to\infty} \int_{\mathscr{M}} |\nabla u(t_n + s_n, \cdot) - \nabla u_\infty|^p\, dA = 0,$$

and so we obtain the assertion. ∎

Remark 1. If u_∞ is an isolated point of $\omega(u)$, it is easy to see that in fact the above convergences hold as $t \to +\infty$ (and not merely for a sequence $\{t_n\} \to +\infty$). The proof of this convergence is an open problem in the remaining cases.

3. MULTIPLICITY OF EQUILIBRIUM SOLUTIONS

We consider the problem (P_Q) obtained in the above section. More precisely, we shall deal with the following elliptic quasilinear problem:

$$(P_Q) \qquad -\Delta_p u + \mathscr{G}(u) \in QS(x)\beta(u) + f_\infty(x) \qquad \text{on } \mathscr{M}.$$

We assume in this section that

(H_S) $S: \mathscr{M} \to \mathbb{R}$, $S \in L^\infty(\mathscr{M})$, $S_1 \geq S(x) \geq S_0 > 0$ for some $S_1 > S_0$.

$(\mathrm{H}_{\mathscr{G}}^*)$ $\mathscr{G}$ satisfies $(\mathrm{H}_{\mathscr{G}})$ and $\lim_{|s| \to \infty} |\mathscr{G}(s)| = +\infty$.

(H_{f_∞}) $f_\infty \in L^\infty(\mathscr{M})$ and there exist $C_f > 0$ such that $-\|f_\infty\|_{L^\infty(\mathscr{M})} \leq f(x) \leq -C_f$ a.e. $x \in \mathscr{M}$.

We also assume that

(H_β^*) There exist two real numbers $0 < m < M$ and $\epsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10 - \epsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10 + \epsilon, +\infty)$.

(H_{C_f}) $\mathscr{G}(-10 - \epsilon) + C_f > 0$ and

$$\frac{\mathscr{G}(-10 + \epsilon) + \|f_\infty\|_{L^\infty(\mathscr{M})}}{\mathscr{G}(-10 - \epsilon) + C_f} \leq \frac{S_0 M}{S_1 m}.$$

A function $u \in V \cap L^\infty(\mathscr{M})$ is called a *bounded weak solution* of (P_Q) if there exists $z \in L^\infty(\mathscr{M})$, $z(x) \in \beta(u(x))$ a.e. $x \in \mathscr{M}$ such that

$$\int_{\mathscr{M}} (|\nabla u|^{p-2} \nabla u) \cdot \nabla v \, dA + \int_{\mathscr{M}} \mathscr{G}(u) v \, dA = \int_{\mathscr{M}} QS(x) z v \, dA + \int_{\mathscr{M}} f_\infty v \, dA,$$

for any $v \in V$.

The main result of this section is the following multiplicity theorem.

THEOREM 2. *Let (H_S), $(H_{\mathscr{G}}^*)$, (H_{f_∞}), and (H_β^*) be satisfied. Let u_m, u_M be the (unique) solutions of the problems*

$$(P_m) \qquad -\Delta_p u + \mathscr{G}(u) = QS(x)m + f_\infty(x) \qquad \text{on } \mathscr{M},$$

and

$$(P_M) \qquad -\Delta_p u + \mathscr{G}(u) = QS(x)M + f_\infty(x) \qquad \text{on } \mathscr{M},$$

respectively. Then

(i) *For any $Q > 0$ there is a minimal solution $\underline{u}$ (resp. a maximal solution $\overline{u}$) of problem (P_Q). Moreover, any other solution u must satisfy*

$$u_m \leq \underline{u} \leq u \leq \overline{u} \leq u_M \tag{16}$$

and

$$\mathscr{G}^{-1}(QS_0 m - \|f_\infty\|_{L^\infty(\mathscr{M})}) \leq u_m \leq \mathscr{G}^{-1}(QS_1 m - C_f), \tag{17}$$

$$\mathscr{G}^{-1}(QS_0 M - \|f_\infty\|_{L^\infty(\mathscr{M})}) \leq u_M \leq \mathscr{G}^{-1}(QS_1 M - C_f). \tag{18}$$

(ii) *For any Q there is, at least, a solution u of (P_Q) that is a global minimum of the functional*

$$J(w) = \frac{1}{p} \int_{\mathscr{M}} |\nabla w|^p \, dA + \int_{\mathscr{M}} G(w) \, dA - \int_{\mathscr{M}} f_\infty w \, dA - \int_{\mathscr{M}} QS(x) j(w) \, dA,$$

on the set $K = \{w \in V, G(w) \in L^1(\mathscr{M})\}$, where $\beta = \partial j$.

Moreover, if (H_{C_f}) holds, and if we define

$$Q_1 = \frac{\mathscr{G}(-10 - \epsilon) + C_f}{S_1 M} \qquad Q_2 = \frac{\mathscr{G}(-10 + \epsilon) + \|f_\infty\|_{L^\infty(\mathscr{M})}}{S_0 M} \tag{19}$$

$$Q_3 = \frac{\mathscr{G}(-10 - \epsilon) + C_f}{S_1 m} \qquad Q_4 = \frac{\mathscr{G}(-10 + \epsilon) + \|f_\infty\|_{L^\infty(\mathscr{M})}}{S_0 m}, \tag{20}$$

then

(iii) *If $0 < Q < Q_1$, then (P_Q) has a unique solution $u = u_m$, $u < -10$, u is the minimum of J on K, and*

$$\mathscr{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathscr{M})}) \leq \liminf_{Q \searrow 0} \|u\|_{L^\infty(\mathscr{M})} \leq \limsup_{Q \searrow 0} \|u\|_{L^\infty(\mathscr{M})} \leq \mathscr{G}^{-1}(-C_f).$$

(iv) *If $Q_2 < Q < Q_3$, then (P_Q) has at least three solutions, u_i, $i = 1, 2, 3$, where $u_1 = u_M$, $u_1 > -10$, $u_2 = u_m$, $u_2 < -10$, and $u_1 \geq u_3 \geq u_2$ on $\mathscr{M}$. Moreover, u_1 and u_2 are local minima of J on $K \cap L^\infty(\mathscr{M})$ and, if $p > 2$, on K.*

(v) *If $Q_4 < Q$, then (P_Q) has a unique solution $u = u_M$, $u > -10$, u is the minimum of J on K, and $\|u\|_{L^\infty(\mathscr{M})} \to +\infty$ when $Q \to +\infty$.*

Proof. (i) Let u be any bounded weak solution of (P_Q); then

$$QS(x)m + f_\infty(x) \le -\Delta_p u + \mathscr{G}(u) \le QS(x)M + f_\infty(x) \qquad \text{on } \mathscr{M}.$$

Since $\mathscr{G}$ is strictly increasing, the comparison principle holds for problems (P_m) and (P_M), and so, necessarily, $u_m \le u \le u_M$ on $\mathscr{M}$. On the other hand, since u_m (resp. u_M) is a subsolution (resp. supersolution) of (P_Q), then the conclusion follows in a standard way from the method of sub- and supersolutions (see, e.g., [2]).

Finally, it is easy to see that

$$\bar{u}_1 := \mathscr{G}^{-1}(QS_1M - C_f) \text{ is a supersolution of } (P_M),$$

$$\underline{u}_1 := \mathscr{G}^{-1}(QS_0M - \|f_\infty\|_{L^\infty(\mathscr{M})}) \text{ is a subsolution of } (P_M),$$

$$\bar{u}_2 := \mathscr{G}^{-1}(QS_1m - C_f) \text{ is a supersolution of } (P_m),$$

$$\underline{u}_2 := \mathscr{G}^{-1}(QS_0m - \|f_\infty\|_{L^\infty(\mathscr{M})}) \text{ is a subsolution of } (P_m).$$

Since $\mathscr{G}$ is strictly increasing, then $\underline{u}_1 \le u_M \le \bar{u}_1$ and $\underline{u}_2 \le u_m \le \bar{u}_2$.

(ii) Since β is a bounded maximal monotone graph of $\mathbb{R}^2$, its primitive j is a real continuous function. Thus J is lower semicontinuous for the weak topology in V and $J(w) \to +\infty$ if $\|w\|_V \to +\infty$. The conclusion then follows from the Weierstrass argument by using the Lebesgue convergence theorem.

(iii) From assumption (H_{C_f}) and since $0 < Q < Q_1$, we obtain

$$\bar{u}_1 < -10 - \epsilon \text{ and } \bar{u}_2 < -10 - \epsilon.$$

Finally, from the comparison $u_m \le u \le u_M$ we conclude that, in this case, every bounded weak solution of (P_Q) is smaller than -10. Thus, u satisfies

$$-\Delta_p u + \mathscr{G}(u) = QS(x)m + f_\infty(x) \qquad \text{on } \mathscr{M},$$

and so u must coincide with the unique solution $\underline{u}$ of this problem, and by (ii) u is the (unique) global minimum of J on K. Finally, from (16) and (17),

$$\mathscr{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathscr{M})}) \le \liminf_{Q\searrow 0} \|u\|_{L^\infty(\mathscr{M})} \le \limsup_{Q\searrow 0} \|u\|_{L^\infty(\mathscr{M})} \le \mathscr{G}^{-1}(-C_f),$$

(iv) This proof will be divided into several steps. First we shall construct some super- and subsolutions of (P_Q) proving the existence of, at least, two solutions of (P_Q). Later we shall prove the existence of a solution for an

approximate model (P_Q^λ) by a topological fixed point argument. The proof will end by showing the convergence of the mentioned solution of (P_Q^λ) to a third solution of (P_Q).

First step. *Construction of super- and subsolutions of* (P_Q).

(a) We fix our attention on *supersolutions* U of (P_Q) that are constants, i.e., verifying

$$(P_1) \qquad \mathscr{G}(U) \in QS_1\beta(U) - C_f.$$

It is clear that any solution of (P_1) is a supersolution of (P_Q), since $S(x) \le S_1$ and $f_\infty(x) \le -C_f$. From $Q > Q_2$ and $S_0 < S_1$, we deduce that

$$U_1 := \mathscr{G}^{-1}(QS_1M - C_f) > -10 + \epsilon.$$

So U_1 is a solution of (P_1) (and then a supersolution of (P_Q)). From $Q < Q_3$,

$$U_2 := \mathscr{G}^{-1}(QS_1m - C_f) < -10 - \epsilon.$$

So U_2 is a solution of (P_1) (and then a supersolution of (P_Q)).

(b) Analogously, we can find constant *subsolutions* V of (P_Q) verifying

$$(P_2) \qquad \mathscr{G}(V) \in QS_0\beta(V) - \|f_\infty\|_{L^\infty(\mathscr{M})}.$$

We know that every solution of (P_2) is a subsolution of (P_Q). From $Q_2 < Q < Q_3$ and $S_0 < S_1$, we obtain that

$$V_1 := \mathscr{G}^{-1}(QS_0M - \|f_\infty\|_{L^\infty(\mathscr{M})}) > -10 + \epsilon \qquad \text{and}$$

$$V_2 := \mathscr{G}^{-1}(QS_0m - \|f_\infty\|_{L^\infty(\mathscr{M})}) < -10 - \epsilon$$

are solutions of (P_2), and so they are subsolutions of (P_Q).

So we have constructed two constant subsolutions and two constant supersolutions. It is clear that $U_1 > U_2$ and $V_1 > V_2$. Moreover, by (H_S),

$$U_1 - V_1 > 0 \qquad \text{and} \qquad U_2 - V_2 > 0.$$

In conclusion,

$$V_2 < U_2 < -10 - \epsilon < -10 + \epsilon < V_1 < U_1. \tag{21}$$

Second step. *Existence of solutions of* (P_Q) *that do not cross the level* $u = -10$.

From the super- and subsolution method we know the existence of two solutions u_1, u_2 of the problem (P_Q), verifying

$$V_1 \le u_1(x) \le U_1 \qquad \text{on } \mathscr{M},$$

$$V_2 \le u_2(x) \le U_2 \qquad \text{on } \mathscr{M}.$$

Since $u_1(x) > -10+\epsilon$ on $\mathscr{M}$, u_1 satisfies (P_M), which has a unique solution. In fact, every weak solution of (P_Q) greater than $-10+\epsilon$ is also a solution of (P_M). Then u_1 is the unique solution of (P_Q) greater than $-10+\epsilon$. Analogously, u_2 is the unique solution of (P_Q) less than $-10-\epsilon$.

Third step. *The approximate model.* The behavior of β on $(-10-\epsilon, -10+\epsilon)$ is not prescribed, and so β may be multivalued on this interval (this is the case of the Budyko model). Let us consider the family of problems (P_Q^λ),

$$(P_Q^\lambda) \qquad -\Delta_p u + \mathscr{G}(u) = QS(x)\beta_\lambda(u) + f_\infty(x) \qquad \text{on } \mathscr{M},$$

where β_λ is the Lipschitz function $\beta_\lambda = \frac{1}{\lambda}(I-(I-\lambda\beta)^{-1})$, $\lambda > 0$ (the Yosida approximation of β). Since β verifies $(\mathrm{H}_\beta^\#)$, we get that

β_λ is a bounded and nondecreasing function $\forall\lambda > 0$,

$\beta_\lambda(s) = \beta(s)$ for any $s \notin [-10-\epsilon, -10+\epsilon+\lambda M]$, $\forall\lambda > 0$,

$\beta_\lambda(s) \to \beta(s)$ in the sense of maximal monotone graphs when $\lambda \to 0$ (see [7]).

In the case of β is a Lipschitz function, we take $\beta_\lambda = \beta$. From (21) there exists $\lambda_0 = \lambda_0(Q)$ such that

$$V_2 < U_2 < -10-\epsilon < -10+\epsilon+\lambda_0 M < V_1 < U_1.$$

If $\lambda < \lambda_0$, $\beta_\lambda(U_i) = \beta(U_i)$ and $\beta_\lambda(V_i) = \beta(V_i)$ for $i = 1,2$. So U_1, U_2 are supersolutions of (P_Q^λ) and V_1, V_2 are subsolutions of (P_Q^λ).

Arguing as in the second step, we obtain two families of solutions $\{u_1^\lambda\}$ and $\{u_2^\lambda\}$ of (P_Q^λ) for $\lambda < \lambda_0$ such that

$$-10+\epsilon+\lambda_0 M < V_1 \le u_1^\lambda \le U_1$$

$$V_2 \le u_2^\lambda \le U_2 < -10-\epsilon.$$

Moreover, since $u_1^\lambda > -10+\epsilon+\lambda_0 M > -10+\epsilon+\lambda M$ and $\beta_\lambda(u_1^\lambda) = \beta(u_1^\lambda)$, we deduce that $u_1^\lambda = u_1$. Analogously, we conclude that $u_2^\lambda = u_2$ if $\lambda < \lambda_0$. To prove that (P_Q^λ) has a solution u_3^λ different from u_1^λ and u_2^λ, we use the following result.

LEMMA 2 (Amann [2]). *Let X be a retract of some Banach space E and let F: $X \to X$ be a compact map. Suppose that X_1 and X_2 are disjoint retracts of X, and let Y_k, $k = 1,2$ be an open subset of X such that $Y_k \subset X_k$. Moreover, suppose that $F(X_k) \subset X_k$ and that F has no fixed points on $X_k - Y_k$, $k = 1,2$. Then F has at least three distinct fixed points x, x_1, x_2 with $x_k \in X_k$ and $x \in X - (X_1 \cup X_2)$.*

We are going to prove that the assumptions of this lemma are satisfied. Any solution u of the problem (P_Q^λ) is a fixed point of the equation

$$u = (-\Delta_p + \mathscr{G})^{-1}(QS(\cdot)\beta_\lambda(u) + f_\infty(\cdot)).$$

Let $E = L^\infty(\mathscr{M})$, which is an ordered Banach space with respect to the natural ordering, whose positive cone is given by

$$L_+^\infty(\mathscr{M}) = \{v \in L^\infty(\mathscr{M}) : v(x) \ge 0 \text{ a.e. } x \in \mathscr{M}\},$$

having a nonempty interior. Let us define the intervals $X = [V_2 - \delta, U_1 + \delta]$, $X_1 = [V_1 - \delta, U_1 + \delta]$, and $X_2 = [V_2 - \delta, U_2 + \delta]$, where $\delta > \lambda_0 M$ is taken such that $V_1 > -10+\epsilon+\delta$, $U_2 > -10-\epsilon-\delta$. So there exists an open set Y_k of $L^\infty(\mathscr{M})$ containing u_k^λ for $k = 1,2$ such that $Y_k \subset X_k$.

Let $F(v) := (-\Delta_p + \mathscr{G})^{-1}(QS(\cdot)\beta_\lambda(v) + f_\infty(\cdot))$ for $v \in L^\infty(\mathscr{M})$. Let us see that X, X_1, X_2, and F verify the assumptions of the lemma:

1. X, X_1, and X_2 are retracts of $L^\infty(\mathscr{M})$ (resp. X), since they are nonempty closed convex subsets of $L^\infty(\mathscr{M})$ (resp. X).

2. $F(X) \subset X$ and $F(X_k) \subset X_k$. Indeed:

- Let $v \in X = [V_2 - \delta, U_1 + \delta]$ and $F(v) = (-\Delta_p + \mathscr{G})^{-1}(QS(x)\beta(v) + f_\infty(x))$, then

  $$QS_0 m - \|f_\infty\|_{L^\infty(\mathscr{M})} \le -\Delta_p(F(v)) + \mathscr{G}(F(v)) \le QS_1 M - C_f.$$

  By the comparison principle for the operator $-\Delta_p + \mathscr{G}$, we have that $F(v) \in [V_2, U_1] \subset X$.
- If $v \in X_1 = [V_1 - \delta, U_1 + \delta]$, then

  $$v > -10+\epsilon+\lambda_0 M \qquad \text{and} \qquad \beta_\lambda(v) = M,$$

  so $F(v)$ is the solution of $-\Delta_p u + \mathscr{G}(u) = QS(x)M + f_\infty(x)$. Since U_1 and V_1 are solutions of the problems $-\Delta_p u + \mathscr{G}(u) = QS_1 M - C_f$ and $-\Delta_p u + \mathscr{G}(u) = QS_0 M - \|f_\infty\|_{L^\infty(\mathscr{M})}$, respectively, we have that $F(v) \in [V_1, U_1] \subset X_1$.
- Similarly, we can verify that $F(X_2) \subset X_2$.

3. F: $X \to X$ is a compact map. Since $N = \dim \mathscr{M} = 2$, we should consider separately the cases $p = 2$ and $p > 2$. If $p = 2$, we define the operators

$$G: [V_2 - \delta, U_1 + \delta] \subset L^\infty(\mathscr{M}) \to L^2(\mathscr{M})$$

$$v \longrightarrow QS(x)\beta_\lambda(v) + f_\infty(x)$$

and

$$(-\Delta + \mathscr{G})^{-1}: L^2(\mathscr{M}) \to H^2(\mathscr{M})$$

$$g \to (-\Delta + \mathscr{G})^{-1}(g),$$

which are continuous. Moreover, the embedding $H^2(\mathscr{M}) \subset L^\infty(\mathscr{M})$ is compact. Then the composition of the two operators is compact, and clearly, $F = (-\Delta + \mathscr{G})^{-1} \circ G$.

In the case $p > 2$, we define

$$G\colon [V_2 - \delta, U_1 + \delta] \subset L^\infty(\mathscr{M}) \to L^p(\mathscr{M})$$
$$v \longrightarrow QS(x)\beta_\lambda(v) + f_\infty(x)$$

and

$$(-\Delta_p + \mathscr{G})^{-1} \colon L^p(\mathscr{M}) \to W^{1,p}(\mathscr{M})$$
$$g \to (-\Delta_p + \mathscr{G})^{-1}(g),$$

which are continuous. Moreover, the embedding $W^{1,p}(\mathscr{M}) \subset L^\infty(\mathscr{M})$ is compact if $p > N = 2$. Now we define F as the composition of the above operators, $F = (-\Delta_p + \mathscr{G})^{-1} \circ G$, and hence F is compact.

4. Existence of Y_k, $k = 1,2$. (P_Q^λ) has a unique solution in $X_1 = [V_1 - \delta, U_1 + \delta]$; in particular, we know that this solution $u_1^\lambda \in [V_1, U_1]$, then $Y_1 = B_{L^\infty(\mathscr{M})}(u_1^\lambda, \delta/2)$ (open ball in the $L^\infty(\mathscr{M})$ topology whose center is u_1^λ, and with radius $\delta/2$) and it is embedded in X_1. (P_Q^λ) has no solution in $X_1 - Y_1$; that is, F has no fixed points in $X_1 - Y_1$. Analogously, we can construct an open set Y_2 in X_2 such that F has no fixed points in $X_2 - Y_2$.

So, by the lemma we conclude that F has at least three fixed points or, equivalently, (P_Q^λ) has at least three solutions: $u_1^\lambda \in X_1$, $u_2^\lambda \in X_2$, and $u_3^\lambda \in X - (X_1 \cup X_2)$.

Fourth step. *Convergence.* Our aim now is to obtain a priori estimates that allow us to prove the convergence of a subsequence of u_3^λ to a solution of (P_Q). Taking u_λ as a test function in the weak formulation of (P_Q^λ), we obtain

$$\int_{\mathscr{M}} |\nabla u_\lambda|^p \, dA + \int_{\mathscr{M}} \mathscr{G}(u_\lambda) u_\lambda \, dA = \int_{\mathscr{M}} QS(x)\beta_\lambda(u_\lambda) u_\lambda \, dA + \int_{\mathscr{M}} f_\infty u_\lambda \, dA.$$

Let us now estimate the term appearing on the right-hand side of the equation. By using the Hölder and Young inequalities, we obtain

$$\int_{\mathscr{M}} QS(x)\beta_\lambda(u_\lambda) u_\lambda \, dA + \int_{\mathscr{M}} f_\infty u_\lambda \, dA \le C_1$$

for some $C_1 > 0$, since by the maximum principle u_λ are uniformly bounded. Then, by the monotonicity of $\mathscr{G}$,

$$\int_{\mathscr{M}} |\nabla u_\lambda|^p \, dA \le C_1.$$

This leads to the estimate

$$\|u_\lambda\|_V \le C_2.$$

Since V is a reflexive Banach space, we can extract a subsequence of $\{u_\lambda\}$ (which we label again by $\{u_\lambda\}$) such that

$$\exists u \in V \qquad \text{such that } u_\lambda \rightharpoonup u \text{ weakly in } V.$$

From the compact embedding $V \subset L^2(\mathscr{M})$,

$$u_\lambda \to u \qquad \text{strongly in } L^2(\mathscr{M}).$$

Taking limits in the weak formulation of (P_Q^λ), we conclude that u is a solution of (P_Q). Indeed, from the convergence $u_\lambda \to u$ in $L^2(\mathscr{M})$, we have

$$\int_{\mathscr{M}} \mathscr{G}(u_\lambda) v \, dA \to \int_{\mathscr{M}} \mathscr{G}(u) v \, dA.$$

Moreover, since

$$\|\beta_\lambda(u_\lambda)\|^2_{L^2(\mathscr{M})} \le M^2 |\mathscr{M}|,$$

we obtain that $\{\beta_\lambda(u_\lambda)\}$ is bounded in $L^2(\mathscr{M})$, and so there exists $z \in L^2(\mathscr{M})$ such that

$$\beta_\lambda(u_\lambda) \rightharpoonup z \qquad \text{weakly in } L^2(\mathscr{M}).$$

Since $\beta_\lambda \to \beta$ in the sense of maximal monotone graphs, we deduce that, necessarily, $z \in \beta(u)$ (see, e.g., [5]).

To prove that

$$|\nabla u_\lambda|^{p-2} \nabla u_\lambda \rightharpoonup |\nabla u|^{p-2} \nabla u \text{ in } L^{p'}(T\mathscr{M}),$$

we consider the estimate

$$\| |\nabla u_\lambda|^{p-2} \nabla u_\lambda \|_{L^{p'}(T\mathscr{M})} = \left(\int_{\mathscr{M}} \left(|\nabla u_\lambda|^{p-1} \right)^{p/p-1} dA \right)^{(p-1)/p}$$
$$= \left(\int_{\mathscr{M}} |\nabla u_\lambda|^p \, dA \right)^{(p-1)/p} \le C_3.$$

Hence there exists a subsequence of u_λ (which we label again as u_λ) such that

$$|\nabla u_\lambda|^{p-2} \nabla u_\lambda \rightharpoonup Y \text{ weakly in } L^{p'}(T\mathscr{M}).$$

Arguing as in the above section, we conclude that $Y = |\nabla u|^{p-2} \nabla u$.

Fifth step. *Existence of a third solution of (P_Q) if β is not Lipschitz.*

In the last step we have constructed three families $\{u_1^\lambda\}$, $\{u_2^\lambda\}$ and $\{u_3^\lambda\}$ of solutions of (P_Q^λ) converging in $L^2(\mathscr{M})$ to u_1, u_2, and u_3 solutions of (P_Q). We recall that $u_1 > -10 > u_2$. Now our purpose is to prove that, in fact, u_3^λ converges uniformly on $\mathscr{M}$ to u_3. Let z be the same as in the last step, that is, $\beta_\lambda(u_\lambda) \rightharpoonup z \in \beta(u)$; then taking $u_\lambda - u$ as a test function in the weak formulation of (P_Q^λ) and (P_Q), we obtain

$$\int_{\mathscr{M}} \left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda - |\nabla u|^{p-2}\nabla u\right)\cdot(\nabla u_\lambda - \nabla u)\, dA$$

$$+\int_{\mathscr{M}} (\mathscr{G}(u_\lambda) - \mathscr{G}(u))(u_\lambda - u)\, dA$$

$$= \int_{\mathscr{M}} QS(x)(\beta_\lambda(u_\lambda) - z)(u_\lambda - u)\, dA.$$

By the inequality $(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta)\cdot(\xi - \eta) \geq |\xi - \eta|^p$ for any $\xi, \eta \in T\mathscr{M}$ and the Hölder inequality, we see that

$$\int_{\mathscr{M}} |\nabla u_\lambda - \nabla u|^p\, dA + \int_{\mathscr{M}} (\mathscr{G}(u_\lambda) - \mathscr{G}(u))\,(u_\lambda - u)\, dA$$

$$\leq QS_1 \|\beta_\lambda(u_\lambda) - z\|_{L^2(\mathscr{M})} \|u_\lambda - u\|_{L^2(\mathscr{M})}. \tag{22}$$

Taking limits when $\lambda \to 0$ in (22), we arrive at

$$\lim_{\lambda\to 0} \|\nabla u_\lambda - \nabla u\|_{L^p(T\mathscr{M})} = 0.$$

If $p > 2$, then $V \subset C(\mathscr{M})$ with continuous embedding and

$$\lim_{\lambda\to 0} \|u_\lambda - u\|_{L^\infty(\mathscr{M})} = 0.$$

If $p = 2$, then $V \subset L^q(\mathscr{M})$ for any $q < +\infty$ with continuous embedding and

$$\lim_{\lambda\to 0} \|u_\lambda - u\|_{L^q(\mathscr{M})} = 0.$$

Nevertheless, we can still obtain uniform convergence because in fact, by regularity, $u_\lambda \in H^2(\mathscr{M})$ and $u \in H^2(\mathscr{M})$. Then, using the continuous embedding $H^2(\mathscr{M}) \subset C(\mathscr{M})$, we conclude that

$$\lim_{\lambda\to 0} \|u_\lambda - u\|_{L^\infty(\mathscr{M})} = 0 \qquad \forall p \geq 2.$$

Now we have that

$$u_i^\lambda \to u_i \text{ in } L^\infty(\mathscr{M}), \qquad \text{for } i = 1,2,3.$$

Then, for any $\lambda < \lambda_0$, there exists a open set $\Omega_\lambda \subset \mathscr{M}$ such that

$$u^\lambda_{3|_{\Omega_\lambda}} \subset [-10 - \epsilon, -10 + \epsilon + \lambda_0 M]. \tag{23}$$

Let us suppose, on the contrary, that $u_1 = u_3$. From $u_3^\lambda \to u_3$ uniformly and $u_1 > -10 + \epsilon_0$, there exists ϵ_1 such that $\forall \epsilon < \epsilon_1$, $u_3^\epsilon > -10 + \epsilon_0$, which is a contradiction to (23). Analogously, we prove that u_3 is different from u_2. In particular, u_3 necessarily crosses the level -10.

To show that the solutions $u_1 > -10$ and $u_2 < -10$ are local minima of the functional J, we assume first that $p > 2$. Since $V \subset L^\infty(\mathscr{M})$ (recall that $\mathscr{M}$ is a bidimensional manifold), there exists $\delta > 0$ such that if $w \in V$ and

$$\|w - u_1\|_{W^{1,p}(\mathscr{M})} < \delta,$$

then $w(x) < -10$ for a.e. $x \in \mathscr{M}$, since

$$\|w - u_1\|_{L^\infty(\mathscr{M})} < C\delta$$

for some C (only dependent on $\mathscr{M}$). In that case, from assumption (H_β^*), u_1 is the unique solution of (P_M), and so u_1 is the (unique) minimum of the convex functional

$$J_1(w) = \frac{1}{p}\int_{\mathscr{M}} |\nabla w|^p\, dA + \int_{\mathscr{M}} G(w)\, dA - \int_{\mathscr{M}} (f_\infty + QS(x)M) w\, dA$$

on V. But

$$J_{1|_{B_\delta(u_1)}} = J_{|_{B_\delta(u_1)}}$$

(here $B_\delta(u_1)$ denotes the open ball in V centered at u_1 and radius δ). So u_1 is the unique minimum of J on $B_\delta(u_1)$. The case of u_2 is similar. When $p = 2$, the result follows by taking a small enough ball centered at u_i, $i = 1, 2$, in $K \cap L^\infty(\mathscr{M})$, with the norm

$$|||w||| = \|\nabla w\|_{L^p(T\mathscr{M})} + \|w\|_{L^\infty(\mathscr{M})}.$$

(v) It is analogous to the proof of (i). From (16) and (18), $\|u\|_{L^\infty(\mathscr{M})} \to +\infty$ when $Q \to \infty$. ∎

COROLLARY 1. *Let $R_e(u) = Bu + C$ and*

$$\beta(u) = \begin{cases} M & \text{if } u > -10 \\ [m, M] & \text{if } u = -10 \\ m & \text{if } u < -10, \end{cases}$$

$-10B + C > 0$ and $S_1/S_0 \leq M/m$. Then we have

(i) *If $0 < Q < (-10B + C)/S_1 M$, then (P_Q) has a unique solution.*

(ii) *If $(-10B + C)/S_0 M < Q < (-10B + C)/S_1 m$, then (P_Q) has at least three solutions.*

(iii) *If $(-10B + C)/S_0 m < Q$, then (P_Q) has a unique solution.*

Remark 2. As noted in [16], the uniqueness of solutions for Q small and Q large still holds if conditions (H_β^*) and $(H_{\mathscr{G}})$ are replaced by

$$\mathscr{G} \in C^1(\mathbb{R}), \qquad \beta \in C^1(\mathbb{R} - \{-10\}),$$

$$m \leq \beta(r) \leq M \ \forall r \in \mathbb{R} - \{-10\},$$

$$\inf\left\{ \frac{\mathscr{G}'(r)}{\beta'(r)}, \ r \in [\underline{U}, -10 - \epsilon] \right\} > 0 \tag{24}$$

where $\underline{U} := \mathscr{G}^{-1}(-\|f_\infty\|_{L^\infty(\mathscr{M})})$.

$$\inf\left\{ \frac{\mathscr{G}'(r)}{\beta'(r)}, \ r \in [-10 + \epsilon, +\infty) \right\} > 0. \tag{25}$$

Indeed, if Q is small enough, we can construct a supersolution showing that any possible solution u satisfies $\underline{U} \leq u \leq -10 - \epsilon$ on $\mathscr{M}$. Then any solution u must satisfy

$$-\Delta_p u + \mathscr{F}(x, u) = f_\infty(x) \tag{26}$$

where

$$\mathscr{F}(x, u) := \mathscr{G}(u) - QS(x)\beta(u).$$

From assumption (24) we conclude that $\mathscr{F}(x, u)$ is a strictly increasing function on $[\underline{U}, -10 - \epsilon]$, for a.e. $x \in \mathscr{M}$, which implies the uniqueness of solutions for (26). Assumption (25) leads to a similar conclusion when Q is large enough.

Remark 3. The variational arguments of Theorem 2 still can be applied to the generalisation mentioned in Remark 2. When $p = 2$, the conclusion that u_i $(i = 1, 2)$ are minima of J on K can be obtained when we replace $\mathscr{M}$ by an open bounded set Ω and we add zero boundary Dirichlet conditions. In that case, an easy modification of the arguments of [8] leads to the conclusion (notice that $u_i \in C^{1,\alpha}(\mathscr{M})$, $i = 1, 2$, although β may be multivalued, since $f_\infty \in L^\infty(\mathscr{M})$ and β is bounded).

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REFERENCES

1. H. W. Alt and S. Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.* **183** (1983), 311–341.
2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.
3. D. Arcoya, J. I. Díaz, and L. Tello, S-shaped bifurcation branch in a model arising in climatology, 1997 (submitted).
4. T. Aubin, "Nonlinear Analysis on Manifolds: Monge-Ampère Equations," Springer Verlag, New York, 1982.
5. Ph. Benilan, M. G. Crandall, and P. Sachs, Some L^1 existence and dependence results for semilinear elliptic equations under nonlinear boundary conditions, *Appl. Math. Optim.* **17** (1988), 203–224.
6. H. Brezis, Proprietés régularisantes de certains semi-groupes nonlinéaires, *Israel J. Math.* **9** (1971), 513–534.
7. H. Brezis, "Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," North Holland, Amsterdam, 1973.
8. H. Brezis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris, Série I* **317** (1993), 465–472.
9. M. I. Budyko, The effects of solar radiation variations on the climate of the Earth, *Tellus* **21** (1969), 611–619.
10. J. I. Díaz, Mathematical analysis of some diffusive energy balance climate models, *in* "Mathematics, Climate and Environment" (J. I. Díaz and J. L. Lions, Eds.), pp. 28–56, Masson, Paris, 1993.
11. J. I. Díaz and L. Tello, Sobre un modelo bidimensional en Climatología, *in* "Actas del, XIII CEDYA/III Congreso de Matemática Aplicada" (A. Casal et al., Eds.), pp. 310–315, 1995.
12. J. I. Díaz and L. Tello, A nonlinear parabolic problem on a Riemannian manifold without boundary arising in Climatology. *Collectanea Mathematica*, 1997 (to appear).
13. J. I. Díaz and F. de Thélin, On a nonlinear parabolic problem arising in some models related to turbulent flows, *SIAM J. Math. Anal.* **25**.(1994), 1085–1111.
14. J. Hernández, Qualitative methods for nonlinear diffusion equations, *in* "Nonlinear Diffusion Equations," Lecture Notes, (A. Fasano and M. Primicerio, Eds.), pp. 47–118, Springer Verlag, New York, 1986.
15. G. Hetzer, The structure of the principal component for semilinear diffusion equations from energy balance climate models, *Houston J. Math.* **16** (1990), 203–216.
16. G. Hetzer, S-shapedness for energy balance climate models of Sellers type, *in* "The Mathematics of Models for Climatology and Environment" (J. I. Díaz, Ed.), pp. 253–288, NATO ASI Series I: Global Environmental Change, No. 48, Springer Verlag, Heidelberg, 1996.
17. M. Nakao, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan* **30** (1978), 747–762.
18. G. R. North, Introduction to simple climate models, *in* "Mathematics, Climate and Environment (J. I. Díaz and J. L. Lions, Eds.), pp. 139–159, Masson, Paris, 1993.
19. T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, *Trans. Amer. Math. Soc.* **331** (1992), 503–527.
20. T. Ouyang, On the positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, part II, *Indiana Univ. Math. J.* **40** (1991), 1083–1141.
21. W. D. Sellers, A global climatic model based on the energy balance of the earth-atmosphere system, *J. Appl. Meteorol.* **8** (1969), 392–400.
22. J. Simon, Compact sets in the space $L^p(0, T; B)$, *Annali Mat. Pura Appl.* CXLVI (1987), 65–96.
23. P. H. Stone, A simplified radiative-dynamical model for the static stability of rotating atmospheres, *J. Atmospheric Sci.* **29** (1972), 405–418.