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ON A NONLOCAL ELLIPTIC PROBLEM ARISING IN THE MAGNETIC CONFINEMENT OF A PLASMA IN A STELLARATOR

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1. INTRODUCTION

This paper is a survey on some recent results on a two dimensional free boundary problem modeling the magnetic confinement of a plasma in a Stellarator device. The following formulation in form of a *free boundary problem* was introduced in [8]: Let Ω be an open, bounded, regular set of $I\!R^2$, and let $\lambda > 0, F_v > 0, a, b \in L^\infty(\Omega)$, $b > 0$ a.e. in Ω. Given $\gamma < 0$, the problem is to find $u \in H^1(\Omega) \cap L^\infty(\Omega)$ and $F \in C^0(I\!R; [0, \infty))$ such that $F(s) = F_v$ for any $s \leq 0$, $F^2 \in W^{1,\infty}_{loc}(I\!R)$ and (u, F) satisfies the problem

$$(\mathcal{P}_I) \begin{cases} -\Delta u = aF(u) + \left(\frac{F^2}{2}\right)'(u) + \lambda b u_+ \quad \text{in } \Omega, \qquad u = \gamma \quad \text{on } \partial\Omega, \\ \int_{\{u>t\}} [\left(\frac{F^2}{2}\right)'(u) + p'(u)b] dx = j(t_+, \|u_+\|_{L^\infty(\Omega)}), \forall t \in (-\infty, \text{ esssup } u]. \end{cases}$$

Notice $(\mathcal{P}_I)$ can be considered as an *incomplet* and *inverse problem* since its equation is not completely known and, in contrast, we know an extra information on the solution. In the sequel we will refer to the family of integral identities stated in $(\mathcal{P}_I)$ as the *Stellarator condition* (the case $j \equiv 0$ corresponds to the usual zero net current and $j \neq 0$ to the *Current Carrying Stellarators*). Here j is assumed such that $j \in C(I\!R \times I\!R^+)$, $j(\sigma, \sigma) = 0$ for all $\sigma \geq 0$, $j'_t \in C(I\!R^+ \times I\!R^+)$ and $\sup |j'_t(t, \sigma)| < +\infty$. The assumptions on the pressure p are the following: $p \in C^1(I\!R)$, $p(0) = 0$, $0 \leq p'(t) \leq \lambda t_+$, and $|p'(t) - p'(s)| \leq L|t - s|^\alpha$, for some $\lambda > 0$, $L > 0$ and $\alpha \in]0, 1[$. Notice that $j(t, \sigma) \equiv 0$ and $p(t) = \frac{\lambda}{2}(t_+)^2$ satisfy all the requirements.

2. MODELLING

The Stellarators are a class of toroidal plasma confinement devices alternative to the Tokamaks. The currents producing poloidal magnetic fields in Stellarators flow in external conductors allowing a wider range of magnetic configurations than those found in Tokamaks. The geometry of these magnetic configurations is very important since it is directly related to the stability of the plasma. The modelling of the problem starts by considering the ideal magnetohydrodynamics (MHD) system

$$\nabla p = \mathbf{J} \times \mathbf{B}, \quad \nabla \times \mathbf{B} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0$$

where p is the pressure, $\mathbf{B}$ the magnetic field and $\mathbf{J}$ the current density. It follows that $\mathbf{B} \cdot \nabla p = 0$, and $\mathbf{J} \cdot \nabla p = 0$. Then the pressure is constant on each magnetic surface. If such a surface lies in a bounded volume of space and has no edges and if neither $\mathbf{B}$ nor $\mathbf{J}$ vanish anywhere on it then by a well-known theory due to Alexandroff and Hopf it must be a toroid (i.e. a topological torus). Since the magnetic field lines are in toroidal nested surfaces, it is useful to introduce a set of new toroidal coordinates $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$, such that: $\tilde{\rho} = \tilde{\rho}(x, y, z)$ is an arbitrary function which is constant on each nested toroid and $\tilde{\theta} = \tilde{\theta}(x, y, z)$ is the poloidal angle which is constant on any toroidal circuit but changes by 2π over a poloidal circuit. The toroidal angle $\tilde{\phi}$ is defined analogously but interchanging the words poloidal by toroidal.

There are several special choices of $(\tilde{\rho}, \tilde{\theta}, \tilde{\phi})$ which are relevant for different purposes. Here we shall take the Boozer vacuum coordinates system ([6]) which are very useful for Stellarators since magnetic field lines becomes "straights" in the $(\tilde{\theta}, \tilde{\phi})$-plane. In what follows we shall denote this set of coordinates by (ρ, θ, ϕ). For a vacuum configuration (i.e. without any plasma) the magnetic field $\mathbf{B}_v$ may be written in contravariant form as $\mathbf{B}_v = B_0 \rho \nabla \rho \times \nabla(\theta - t_v(\rho)\phi)$ where $t_v(\rho)$ is the so called *vacuum rotational transform* and B_0 is a positive constant. The covariant form of $\mathbf{B}_v$ is $\mathbf{B}_v = F_v \nabla \phi$ where F_v is a constant (which customary is taken as positive).

In contrast to Tokamaks the Stellarators-type configurations are very complicated due to the fully three-dimensional nature of the device. To simplify the model to a two-dimensional problem different averaging methods were used: see [17] and [18]. Following the last reference we may decompose the magnetic field in terms of its toroidally averaged and rapidly varying parts. For a general function f this decomposition takes the form $f = < f > + \tilde{f}$ where $< f > := \frac{1}{2\pi} \int_0^{2\pi} f d\phi$. In our case, motivated by the set of coordinates $(\rho, \rho\theta, \phi)$, the natural way of doing that is

$$\frac{B^i}{D} = < \frac{B^i}{D} > + (\frac{\tilde{B}^i}{D})$$

where B^i are the contravariant components of the vacuum magnetic field, $i = \rho, \theta, \phi$, and D is the Jacobian. Using the Stellarator expansion hypothesis, Hender and Carreras [18] show that

$$\frac{\partial}{\partial \rho}(\rho < \frac{B^\rho}{D} >) + \frac{\partial}{\partial \theta}(< \frac{B^\theta}{D} >) = 0,$$

and thus the *averaged poloidal flux function* $\psi = \psi(\rho, \theta)$ can be defined by

$$< \frac{B^\rho}{D} > = \frac{1}{\rho}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad < \frac{B^\theta}{D} > = -\frac{\partial \psi}{\partial \rho}.$$

They also show that $< B_\phi >$ is a function ψ alone and the same for $< p >$. By introducing the usual notation

$$F(\psi) := < B_\phi > \quad \text{and} \quad p(\psi) := < p >$$

Hender and Carreras obtain a Grad-Shafranov type equation for ψ

$$-\mathcal{L}\psi = a(\rho, \theta)F(\psi) + F(\psi)F'(\psi) + b(\rho, \theta)p'(\psi) \tag{1}$$

where $\mathcal{L}\psi$ is a second order elliptic operator. The coefficients of $\mathcal{L}\psi$, $a(\rho, \theta)$ and $b(\rho, \theta)$ are given in terms of the Riemannian metric associated to the vacuum coordinates system (see [18]).

It is clear that the free boundary (separating the plasma and vacuum regions) is a (toroidal) magnetic surface and, as $p = p(\psi)$, by normalizing, we can identify the free boundary as the level line $\{\psi = 0\}$, the plasma region as $\{\psi > 0\}$ (and thus $\{p > 0\}$) and the vacuum region by $\{\psi < 0\}$ (and $\{p = 0\}$). It is also well-known that the pressure cannot be obtained from the (MHD) system and some constitutive law must be assumed. Usually, for simplicity, it is assumed a quadratic law (see e.g. [34]) $p = \frac{\lambda}{2}[\psi_+]^2$, $\psi_+ = \max\{\psi, 0\}$, which is compatible with the above normalization. We extend the *unknown* $F(\psi)$ to negative values of ψ by $F(\psi) = F_v$ for any $\psi \leq 0$. If the boundary of

Ω^3 (the 3-d domain) is assumed to be a *perfectly conducting wall* thus $\mathbf{B}\cdot\mathbf{n^3}=0$ over $\partial\Omega^3$, where $\mathbf{n^3}$ denotes the outer normal vector to $\partial\Omega^3$. Then we obtain that $\psi=\gamma$ on $\partial\Omega$ (where Ω is the 2-d domain obtained from Ω^3) for some (negative) constant γ. In contrast with Tokamak devices, now is not restrictive to assume γ a priori given.

To complete the formulation of the problem under consideration we must add *the Stellarator condition* imposing a zero net current within each flux magnetic surface. According the averaging method of [18] this condition can be expressed ([8]) as

$$\int_{\{\psi\geq t\}}[F(\psi)F'(\psi)+\lambda b\psi_+]\rho d\rho d\theta=0 \text{ for any } t\in[\,\inf\psi,\ \sup\psi]. \tag{2}$$

For the sake of simplicity in the exposition we have replaced, in $(\mathcal{P}_I)$, the elliptic operator $\mathcal{L}$ by the operator Δ (for a general treatment see [13]). As usual, we have replaced the unknown ψ by the more common u. In practice, the *Stellarator condition* does not always hold and some know current arises at the interior of each magnetic surface. In [7] the current is assumed to have the form $J(s)=J(1)(4s^2-3s^4)$ within the flux magnetic surface corresponding to the parameter s (i.e. $\{(\rho,\theta)\in\Omega: u(\rho,\theta)=s\}$) and where $2\pi J(1)$ is the total (toroidal) current. In [7] it is also assumed that the constitutive law for the pressure is $P(s)=P(0)(1-3s^2+2s^3)$. Notice that in such a formulation it is assumed that the magnetic axis corresponds to $s=0$ and that $s=1$ corresponds to the free boundary. In this type of Stellarators (the so called *Current Carrying Stellarators*) we need, firstly, to change the parametrization s to u for which the boundary of the plasma region corresponds to the level $u=0$ and the magnetic axis to $\max u=\|u_+\|_{L^\infty(\Omega)}$. So, we define the following change of variable, with u verifying the differential equation and the boundary condition:

$$s:=\left(1-\frac{t_+}{\|u_+\|_{L^\infty(\Omega)}}\right)\qquad\forall\, t\in[\inf_\Omega u,\sup_\Omega u]\ .$$

Thus we obtain a new expression $J(s)=j(t_+,\|u_+\|_{L^\infty(\Omega)})$ in terms of the new variable $t\in[\inf u,\inf u]$. Analogously, we will have $P(s)=p(t,\|u_+\|_{L^\infty(\Omega)})$. This justifies the structure of the term j used in $(\mathcal{P}_I)$.

Other differences among $(\mathcal{P}_1)$ and the model for Tokamak devices (see e.g. [34], [35], [3], [5], [15], [23], [21], [26] and their references) concerns the Stellarator condition and the Tokamak condition of positive total current

$$\int_\Omega[F(u)F'(u)+\lambda bu_+]dx=I$$

for a prescribed $I>0$. Due to this fact, in the Tokamak case it seems not possible to determine the function F unless if we have some extra information as, for instance, the value of the normal derivative of u at $\partial\Omega$ (see [2] and its references). Then the mathematical model for Tokamaks assume a state law for F similar to the pressure state law. It is usually assumed that $F(u)F'(u)+\lambda b(x)u_+$ can be written as $\mu c(x)u_+$ for some $\mu\in\mathbb{R}_+$ and $c\in L^\infty(\Omega)$ with $c>0$ in Ω. The coefficient a in $(\mathcal{P}_I)$ is intrinsic to Stellarator configurations and so it does not appear in the Tokamak model.

3. AN EQUIVALENT NONLOCAL FORMULATION

To study problem $(\mathcal{P}_I)$ we shall reformulate it in terms of a new problem $(\mathcal{P}_{NL})$, of nonlocal nature, where we replace the unknown F by a term involving the function u, its decreasing rearrangement and the relative rearrangement of b with respect of u. Let us recall those notions: Let Ω be a bounded measurable set of $\mathbb{R}^N$, $N\geq 1$. For any measurable subset E of Ω, we denote by $|E|$ its Lebesgue measure. Given a measurable function $u:\Omega\to\mathbb{R}$ and any value $t\in\mathbb{R}$, we denote by $\{u=t\}$, $\{u>t\}$ and $\{u\geq t\}$ the sets $\{x\in\Omega: u(x)=t\}$, $\{x\in\Omega: u(x)>t\}$ and $\{x\in\Omega: u(x)\geq t\}$ respectively. Their measure will be indicated by $|u=t|$, $|u>t|$ and $|u\geq t|$ respectively. We denote by Ω_* the interval $]0,|\Omega|[$.

DEFINITION 1 *Let* $u : \Omega \to \mathbb{R}$ *be a Lebesgue measurable function. The* distribution function of u *is defined by* $m_u(t) := |u > t|$ *for any* $t \in \mathbb{R}$. *The generalized inverse of* m_u *is called the* decreasing rearrangement *of* u *and is denoted by* u_*. *That is the function* $u_* : \Omega_* \to \overline{\mathbb{R}}$ *such that* $u_*(s) = \inf\{t \in \mathbb{R} : |u > t| < s\}$ *with* $u_*(0) = \operatorname{ess\,sup}_\Omega u$ *and* $u_*(|\Omega|) = \operatorname{ess\,inf}_\Omega u$.

Some properties of the decreasing rearrangement can be found, for instance, in [33] and [20]. Given $v \in L^1(\Omega)$, we define a function w in $\bar{\Omega}_*$ by:

$$w(s) = \begin{cases} \displaystyle\int_{\{u>u_*(s)\}} v(x)dx & \text{if } |u = u_*(s)| = 0 \\ \displaystyle\int_{\{u>u_*(s)\}} v(x)dx + \int_0^{s-|u>u_*(s)|} \left(v_{|P_u(u_*(s))}\right)_*(\sigma)d\sigma & \text{if } |u = u_*(s)| = 0. \end{cases}$$

Here $v_{|P_u(u_*(s))}$ denotes the restriction of v to the set $P_u(u_*(s)) := \{u = u_*(s)\}$ and $\left(v_{|P_u(u_*(s))}\right)_*$ its decreasing rearrangement. The following lemma was proved in [23, 22].

LEMMA 1 *Let* $u \in L^1(\Omega)$ *and* $v \in L^p(\Omega)$ *for some* $1 \le p \le +\infty$. *Then* $w \in W^{1,p}(\Omega_*)$ *and* $\left\| \dfrac{dw}{ds} \right\|_{L^p(\Omega_*)} \le \|v\|_{L^p(\Omega)}$.

DEFINITION 2 *The function* $\dfrac{dw}{ds}$ *is called the* relative rearrangement of v with respect to u *and it is denoted by* $v_{*u} = \frac{dw}{ds}$.

This function has many properties (see, for instance, [23, 22, 26, 27, 28, 29, 24]). Comming back to the main goal of this section, we mention that in [8]the case $j \equiv 0$ and $p(t) = \frac{\lambda}{2}(t_+)^2$ was reformulated by using the formula

$$\left(F_u^2\right)'(t) = -2\lambda t_+ \frac{\displaystyle\int_{\omega(t)} \frac{b}{|\nabla u|} d\Gamma}{\displaystyle\int_{\omega(t)} \frac{1}{|\nabla u|} d\Gamma} = -2\lambda t_+ b_{*u}(|u > t|),$$

where $\omega(t) := \{x \in \Omega : b(x) = b_{*u}(|u > t|)\}$. More in general, we introduce the nonlocal problem

$$(\mathcal{P}_{NL}) \begin{cases} -\Delta u(x) &= a(x)\mathcal{F}_u(x) + p'(u(x))[b(x) - b_{*u}(|u > u(x)|)] \\ &+ \; j_t'(u_+(x), u_{+*}(0))u_{+*}'(|u > u(x)|) \qquad \text{in } \Omega, \\ u - \gamma. &\in H_0^1(\Omega) \end{cases}$$

where now u is the unique unknown and function $\mathcal{F}_u$ is defined by

$$\mathcal{F}_u(x) := \left[F_v^2 - 2\int_{|u>0|}^{|u>u_+(x)|} [p(u_*)]'(s) b_{*u}(s)ds + 2\int_{|u>0|}^{|u>u_+(x)|} j_t'(u_{+*}(s), u_{+*}(0))(\, u_{+*}'(s))^2 ds\right]_+^{\frac{1}{2}}.$$

In order to state the equivalence between problems $(\mathcal{P}_I)$ and $(\mathcal{P}_{NL})$, given $u \in W^{1,\infty}(\Omega)$, we define the function $\mathcal{F} : \mathbb{R} \to \mathbb{R}_+$ by

$$\mathcal{F}(t) = \left[F_v^2 - 2\int_0^{t_+} p'(s) b_{*u}(|u > s|)ds + 2\int_0^{t_+} j_t'(s, u_{+*}(0))\; u_{+*}'(|u > s|)ds\right]_+^{\frac{1}{2}}.$$

We introduce also the convex cone $V(\Omega) = \{v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \le 0\}$. The following theorem was proved in [11] and extends some previous results [8] and [13]

THEOREM 1 *Let $u \in V(\Omega)$ such that meas$\{x \in \Omega : \nabla u(x) = 0\} = 0$. Assume $\hat{m} = \inf u \leq 0$ and $\mathcal{F}_u(x) > 0$ a.e. in Ω. Let $M = \sup u$. Then, if (u, F) is a solution of $(\mathcal{P}_I)$ such that $F : [\hat{m}, M] \to \mathbb{R}^+$, $F \in W^{1,\infty}(]\hat{m}, M[)$ and $F(t) = F_v$ for all $t \leq 0$, then function u is also a solution of $(\mathcal{P}_{NL})$ and necessarily $F = \mathcal{F}$. Conversely, if u is a solution of $(\mathcal{P}_{NL})$ then, the couple $(u, \mathcal{F})$ is a solution of $(\mathcal{P}_I)$ and $\mathcal{F} \in W^{1,\infty}(]\hat{m}, M[)$.*

4. EXISTENCE OF SOLUTIONS

The lack of regularity of the derivative of the decreasing rearrangement makes difficult to solve directly problem $(\mathcal{P}_{NL})$. A family of problems $(\mathcal{P}_{NL\epsilon})$ are then introduced. Using a Galerkin method (of interest also for the numerical approach) we will find a solution of $(\mathcal{P}_{NL\epsilon})$. Finally we shall obtain a solution of $(\mathcal{P}_{NL})$ by making $\epsilon \to 0$, thanks to a result on the regularity of the derivative of the decreasing rearrangement. The equivalence of problems $(\mathcal{P}_{NL})$ and $(\mathcal{P}_I)$ (under a suitable condition) proves that this solution is also a solution of $(\mathcal{P}_I)$.

4.1. The approximate problem $(\mathcal{P}_{NL\epsilon})$. For any fixed $\epsilon > 0$, let us consider the following approximate problem ; find u^ϵ such that

$$(\mathcal{P}_{NL\epsilon}) \left\{ \begin{array}{lcl} -\Delta u^\epsilon & = & aF_\epsilon(x, u^\epsilon, b_{*u^\epsilon}) + H(u^\epsilon, b_{*u^\epsilon}) + J_\epsilon(u^\epsilon) \text{ in } \Omega \\ u^\epsilon - \gamma & \in & H_0^1(\Omega) \cap W^{2,p}(\Omega); \quad \forall\, p \geq 1 \end{array} \right.$$

where

$$\begin{array}{lcl} I(v(x), \sigma) & := & \chi_{[|v > v_+(x)|, |v>0|]}(\sigma), \text{(the characteristic function of } [|v > v_+(x)|, |v > 0|]), \\ F_1(x, v, b_{*v}) & := & \int_{\Omega_*} I(v(x), s)[p(v_*)]'(s) b_{*v}(s) ds, \\ F_{\epsilon,2}(x, v) & := & \int_{\Omega_*} I(v(x), s) h_\epsilon(v'_{+*}(s)) j'_t(v_{+*}(s), v_{+*}(0)) ds, \\ F_\epsilon(x, v, b_{*v}) & := & \left[F_v^2 - 2F_1(x, v, b_{*v}) + 2F_{\epsilon,2}(x, v)\right]_+^{\frac{1}{2}}, \\ H(v(x), b_{*v}) & := & p'(v(x))[b(x) - b_{*v}(|v > v(x)|)], \\ J_\epsilon(v(x)) & := & \xi_\epsilon(v'_{+*}(|v > v_+(x)|)) j'_t(v_+(x), v_{+*}(0)), \end{array}$$

for $\sigma \in \Omega_*$, and a.e. $x \in \Omega$, for any function $v \in H_0^1(\Omega)$. Here, we used the truncation functions $h_\epsilon(t) := \frac{t^2}{1+\epsilon t^2}$, $\xi_\epsilon(t) := \frac{t}{1+\epsilon|t|}$. To simplify the boundary condition we define $w^\epsilon := u^\epsilon - \gamma$. In order to prove the existence of w^ϵ, we find a solution w_m^ϵ of the auxiliary problems $(\mathcal{P}_{NL\epsilon,m})$. We shall search $w_m^\epsilon \in V_m$, where V_m is a finite dimensional space such that $V_m \subset V_{m+1} \subset H_0^1(\Omega)$. Later, using appropriate estimates on the solutions w_m^ϵ of $(\mathcal{P}_{NL\epsilon,m})$, we shall pass to the limit when m goes to infinity.

4.2. The Galerkin method. Consider $(\lambda_k, \varphi_k)_{k\geq 1}$ be the eigenvalues and eigenfunctions associated to $-\Delta$ on Ω with Dirichlet boundary conditions, i.e.,

$$-\Delta\varphi_k = \lambda_k \varphi_k\,, \qquad \varphi_k \in H_0^1(\Omega)\,.$$

Let $V_m = \text{span}\{\varphi_1, \ldots, \varphi_m\}$. On V_m, we define the scalar product by $[v, w] := \sum_{k=1}^m v^k w^k$ where $v = \sum_{k=1}^m v^k \varphi_k$ and $w = \sum_{k=1}^m w^k \varphi_k$. Let $\|v\|_{V_m} := [v, v]^{\frac{1}{2}}$ the associated norm. Now, for $\gamma \leq 0$ fixed, we consider the operator $T_m^\epsilon : V_m \to V_m$ defined as

$$\begin{aligned} [T_m^\epsilon v, \varphi] \;=\; & \int_\Omega \nabla v \cdot \nabla\varphi dx - \int_\Omega aF_\epsilon(x, v+\gamma, b_{*(v+\gamma)})\varphi dx \\ & - \int_\Omega H(v+\gamma, b_{*(v+\gamma)})\varphi dx - \int_\Omega J_\epsilon(v+\gamma)\varphi dx \;\; \forall v, \varphi \in V_m\,. \end{aligned} \tag{3}$$

We shall prove that this operator attains zero for some $w_m^\epsilon \in V_m \setminus \{0\}$. It is clear that if w_m^ϵ satisfies $T_m^\epsilon w_m^\epsilon = 0$ in V_m, then w_m^ϵ satisfies the finite dimensional problem $(\mathcal{P}_{NL\epsilon,m})$ given by

$$\begin{aligned} -\Delta(w_m^\epsilon + \gamma) &= P_m[aF_\epsilon(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)}) \\ +H(w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)}) + J_\epsilon(w_m^\epsilon + \gamma)] \text{ in } \Omega, w_m^\epsilon &\in V_m \end{aligned}$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m. It can be shown (see [11]) that T_m^ϵ is coercive assumed

$$\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b > 0. \tag{4}$$

Thus, if we are able to prove the continuity of T_m^ϵ we could apply the Brouwer Fixed Point Theorem (see e.g. [19, Lemma 4.3, p. 55]) obtaining the existence of a solution. The continuity of T_m^ϵ is a more delicate point since we need to prove the continuity of the differential of the rearrangement. The key point here is a variant of an important result due to Almgrem and Lieb [1] obtained by Rakotoson (see [28] and also [13], [29]):

LEMMA 2 *Let $u \in W^{1,p}(\Omega)$, $1 \le p \le +\infty$ such that $meas\{x \in \Omega : \nabla u(x) = 0\} = 0$. If u_n is a bounded sequence of $W^{1,p}(\Omega)$ converging to u in $W^{1,1}(\Omega)$, then $\frac{d}{ds}u_{n*}$ converges to $\frac{d}{ds}u_*$ a.e. in Ω_*. Furthermore, if $p > N$, $\frac{d}{ds}u_{n*}$ converges to $\frac{d}{ds}u_*$ strongly in $L^q(\Omega_*)$ for any $1 \le q < \bar{q} := \frac{1}{1+\frac{1}{p}-\frac{1}{N}}$.*

As consequence we get:

THEOREM 2 *[11] Assume (4). Then there exists at least $w_m^\epsilon \in V_m$ solution of problem $(\mathcal{P}_{NL\epsilon,m})$.*

4.3.A priori estimates on solutions of $(\mathcal{P}_{NL\epsilon,m})$. Taking $\varphi = w_m^\epsilon$,as test function in the weak formulation of $(\mathcal{P}_{NL\epsilon,m})$, we have

$$0 = [T_m^\epsilon w_m^\epsilon, w_m^\epsilon] \ge (\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b - \delta) \int_\Omega |w_m^\epsilon|^2 dx - C_{\epsilon\delta}$$

Thus, choosing δ small enough, one has

$$\|w_m^\epsilon\|_{L^2(\Omega)} \le C_\epsilon \text{ and so } \int_\Omega |\nabla w_m^\epsilon|^2 dx \le C_\epsilon$$

for some positive constant C_ϵ only depending of ϵ. Using this estimate we get

$$\begin{aligned} \|\Delta w_m^\epsilon\|_{L^2(\Omega)} &\le \|aF_\epsilon(x, w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^2(\Omega)} + \|H(w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^2(\Omega)} \\ &\quad + \|J_\epsilon(w_m^\epsilon + \gamma)\|_{L^2(\Omega)} \\ &\le \|a\|_{L^\infty(\Omega)} \left[F_v^2 + \frac{2\eta}{\epsilon}|\Omega|\right]^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \lambda \operatorname*{osc}_{\Omega} b \|w_m^\epsilon\|_{L^2(\Omega)} + \frac{|\Omega|^{\frac{1}{2}}\eta}{\epsilon} \le C_\epsilon. \end{aligned}$$

By standard regularity results, $(w_m^\epsilon)_{m\ge1}$ is uniformly bounded in $W^{2,2}(\Omega)$ with respect to m.
4.4.Passing to the limit $m \to \infty$: Existence of solution of $(\mathcal{P}_{NL\epsilon})$. By the above estimates, there exists a subsequence of $\{w_m^\epsilon\}$, which we also denote by $\{w_m^\epsilon\}$, and $w^\epsilon \in W^{2,2}(\Omega)$ such that

$$w_m^\epsilon \rightharpoonup w^\epsilon \qquad \text{weakly in } W^{2,2}(\Omega) \text{ , as } m \to \infty$$

and so,

$$w_m^\epsilon \longrightarrow w^\epsilon \qquad \text{strongly in } W^{1,p}(\Omega),\ \forall p \in [1, +\infty[, \text{ and in } \mathcal{C}(\overline{\Omega}) \text{ .}$$

Our next step is to verify that $T^\epsilon w^\epsilon = 0$. Here $T^\epsilon : H_0^1(\Omega) \to H_0^1(\Omega)$ is defined as T_m but by extending it to the whole $H_0^1(\Omega)$. We have $\|aF_\epsilon(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^\infty(\Omega)} \leq C_\epsilon \ \forall\ m$ and $\|b_{*(w_m^\epsilon+\gamma)}(|w_m^\epsilon+\gamma > (w_m^\epsilon+\gamma)(\cdot)|)\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)}$. Using the strong convergence of $[p((w_m^\epsilon+\gamma)_*)]'$ to $[p((w^\epsilon+\gamma)_*)]'$ in $L^1(\Omega_*)$ (thanks to Lemma 2) and the continuity of p', we finally get that w^ϵ verifies the weak formulation of $(\mathcal{P}_{NL\epsilon})$ assumed that $\text{meas}\{x \in \Omega : \nabla w^\epsilon(x) = 0\} = 0$.

4.5. Condition on the data in order to get $\text{meas}\{x \in \Omega : \nabla w^\epsilon(x) = 0\} = 0$ *and the existence of solution of* $(\mathcal{P}_{NL})$. Setting, again, $u^\epsilon := w^\epsilon + \gamma$, arguing by contradiction and using Stampacchia result we get

THEOREM 3 *[11] If $\lambda\|b\|_{L^\infty(\Omega)}$ and η are small enough so that*

$$\left[\lambda\|b\|_{L^\infty(\Omega)} + \frac{\eta}{|\Omega|}\right]S < \inf_\Omega |a| \left[F_v^2 - 2\lambda\|b\|_{L^\infty(\Omega)}S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}} \tag{5}$$

with

$$\eta := \sup|j_t'(t,\sigma)|, \text{ and } S := \frac{\|a\|_{L^\infty(\Omega)}F_v|\Omega|}{4\pi(1-\nu)}$$

then meas$\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0$. In particular, u^ϵ satisfies problem $(\mathcal{P}_{NL\epsilon})$.

The a priori estimates that we need to pass to the limit are given in the next crucial lemma

LEMMA 3 *[11] Let $\{u^\epsilon\}$ verifying $(\mathcal{P}_{NL\epsilon})$ and such that $u^\epsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. If*

$$\nu := \frac{1}{4\pi}\left[2^{1/2}\eta^{1/2}|\Omega|^{1/2}\|a\|_{L^\infty(\Omega)} + \lambda|\Omega| \operatorname*{osc}_\Omega b + \eta\right] < 1 \tag{6}$$

then

$$\|\Delta u^\epsilon\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)}F_v}{1-\nu} \tag{7}$$

and

$$\|u_+^\epsilon\|_{L^\infty(\Omega)} \leq S \tag{8}$$

uniformly in ϵ.

The proof of this result is based in an auxiliary result relative to dimension two:

LEMMA 4 *[11] Let $\Omega \subset I\!R^2$. Then, for all $w \in V(\Omega)$ one has*
i) $\left\|\frac{dw_{+*}}{ds}\right\|_{L^\infty(\Omega_*)} \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi}$, *ii)* $\|w_+\|_{L^\infty(\Omega_*)} \leq \frac{|\Omega|}{4\pi}\|\Delta w\|_{L^\infty(\Omega)}$ *and*
iii) $\left|\frac{d^+}{ds}w_{+*}(|w_+ > w_+(x)|)\right| \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi}$ *a.e.* $x \in \Omega$.

As a final conclusion we have

THEOREM 4 *[11] Assume $\inf_\Omega |a| > 0$, $\gamma \in I\!R^-$ and $\lambda\|b\|_{L^\infty(\Omega)} + \eta < \Lambda$ for a suitable $\Lambda > 0$. Then there is u solution of $(\mathcal{P}_{NL})$. Moreover $u \in V(\Omega)$.*

Proof. Our aim is to let $\epsilon \to 0$. By the uniform estimate on $\|\Delta u^\epsilon\|_{L^\infty(\Omega)}$ given in Lemma 3, there exists some subsequence of (u^ϵ) (which we will again denote by u^ϵ) and a function $\alpha \in L^\infty(\Omega)$ such that

$$\Delta u^\epsilon \rightharpoonup \alpha \qquad \text{weakly* in } L^\infty(\Omega)\ .$$

By standard regularity, u^ϵ belongs to a bounded set of $W^{2,p}(\Omega)$, for all $p \in [1,+\infty[$. Then, we have (for some subsequence) that

$$\begin{aligned} u^\epsilon &\rightharpoonup u \qquad \text{weakly in } W^{2,p}(\Omega),\\ u^\epsilon &\longrightarrow u \qquad \text{strongly in } \mathcal{C}^1(\bar\Omega) \end{aligned}$$

In particular, $\alpha = \Delta u$, $\Delta u \in L^\infty(\Omega)$, $u \in V(\Omega)$ and the estimates of Lemma 3 remain true replacing u^ϵ by u. On the other hand, we have

$$\|\hat b^\epsilon\|_{L^\infty(\Omega_*)} \le \|b\|_{L^\infty(\Omega)} \quad \text{and} \quad \|\tilde b^\epsilon\|_{L^\infty(\Omega)} \le \|b\|_{L^\infty(\Omega)}$$

and so, $\hat b^\epsilon \rightharpoonup \hat b$ weakly* in $L^\infty(\Omega_*)$ and $\tilde b^\epsilon \rightharpoonup \tilde b$ weakly* in $L^\infty(\Omega)$. Furthermore, one has

$$u^{\epsilon\,\prime}_{+*} \longrightarrow u'_{+*} \qquad \text{in } L^p(\Omega_*)\ \forall p \in [1,+\infty[\tag{9}$$

(use Lemma 4 and Lemma 2). From this convergence, we can identify $\tilde b(x) = b_{*u}(|u > u(x)|)$ in Ω, $\hat b(s) = b_{*u}(s)$ in Ω_* and, in conclusion, u is a solution of $(\mathcal{P}_{NL})$.

5. SOME QUALITATIVE PROPERTIES.

5.1. A partial result on the uniqueness of solutions. In [10] we considered the question of the uniqueness of solutions of $(\mathcal{P}_I)$, with $j \equiv 0$ and $p(t) = \frac{\lambda}{2}(t_+)^2$, under the following special conditions: given (u_1, F), (u_2, F) solutions of $(\mathcal{P}_I)$ (i.e. having a common second component) with $u_i \in \mathcal{U}$, $i = 1,2$, find assumptions on F implying that $u_1 = u_2$. The special structure of the equation of $(\mathcal{P}_I)$ makes reasonable to assume the function $(F^2)'$ to be locally Lipschitz continuous (i.e. $F^2 \in W^{2,\infty}_{loc}(I\!R)$). Notice that, obviously, this assumption is stronger than condition $F^2 \in W^{1,\infty}_{loc}(I\!R)$ included in the definition of solution of $(\mathcal{P}_I)$ and that no one of both imply the local Lipschitz continuity of F since it is not a priori known either the property

$$F^2(u_i(x)) > 0 \qquad \text{a.e. } x \in \Omega\,, \qquad i = 1,2, \tag{10}$$

holds or not. Notice also that necessarily $F_{u_1} = F_{u_2}\ \ (= F)$ on $(-\infty, m]$ (where $m := \min\{\sup u_i\}$) and that

$$\left(F^2_{u_i}\right)'(t) = -2\lambda t_+ b_{*u_i}(|u_i > t|) \qquad \text{a.e. } t \in \left(-\infty, \|u_{i+}\|_{L^\infty(\Omega)}\right]\,. \tag{11}$$

Moreover, if for instance $m = \sup u_1$ then F_{u_1} can be prolonged to $(-\infty, M]$, with $M := \max\{\sup u_i\}$ ($= \sup u_2$ in this case) by means of F_{u_2} and this prolongation still verifies the requirements ($F_{u_2} \in \mathcal{C}^0((-\infty, M] : [0,\infty))$ and $F^2_{u_2} \in W^{1,\infty}_{loc}(-\infty, M)$).

The λ-dependence in (11) is the motivation of formulating the mentioned condition $F^2 \in W^{2,\infty}_{loc}(I\!R)$ in the following quantitative terms:

$$\left|\left(F^2(t)\right)' - \left(F^2(\hat t)\right)'\right| \le \lambda K \left|t - \hat t\right| \qquad \forall t, \hat t \in (-\infty, M] \tag{12}$$

for some positive constant K independent of λ. The uniqueness result can be stated in the following terms

THEOREM 5 *[10] Let (u_1, F), (u_2, F) be solutions of $(\mathcal{P}_I)$, with $j \equiv 0$, and $p(t) = \frac{\lambda}{2}(t_+)^2$, $F \in \mathcal{C}^0(I\!R; [0,\infty))$ and $F^2 \in W^{2,\infty}_{loc}(I\!R)$ satisfying (12). Then, there exists a positive constant δ such that if $\lambda < \delta$ necessarily $u_1 \equiv u_2$.*

We point out that if b is a positive constant then for any (u, F) solution of $(\mathcal{P}_I)$, with $j \equiv 0$, $p(t) = \frac{\lambda}{2}(t_+)^2$ and $u \in \mathcal{U}$, we have that

$$\left(F_u^2\right)'(t) = -2\lambda t_+ b \qquad \text{a.e. } t \in (-\infty, \|u_+\|_{L^\infty(\Omega)}]$$

and so assumption (12) trivially holds with $K = 2b$. The Lipschitz continuity of $(F_u^2)'$ is related to the Lipschitz continuity of the functions $s \to b_*(s)$ and $t \to |u > t|$. In [11, Lemma 4] it is shown that if $b \in H^1(\Omega), \Delta b \in L^\infty(\Omega), b = B$ on $\partial\Omega$ and $b(x) \geq B$ a.e. $x \in \Omega$, for some constant B, then the function $s \to b_*(s)$ is Lipschitz continuous (recall that $\Omega \subset \mathbb{R}^2$). The Lipschitz regularity of $t \to |u > t|$ is a more delicate which remains open as far as we know.

One of the main steps of the proof of Theorem 5 is to show that if $u_i \in \mathcal{U}$ and λ is small enough then (10) holds and the Lipschitz constant of F on $(-\infty, M]$ is also small. As a consequence, we can apply a general uniqueness criterion for semilinear problems implying that necessarily $u_1 = u_2$. In order to prove (10) we obtain previously some L^∞-estimates on u_i in terms of the parameter λ.

Another uniqueness result, under a sharper bound on λ, can be obtained by a different technique assuming a sign condition on the coefficient a.

THEOREM 6 *[10] Assume $a > 0$ on Ω and let (u_1, F), (u_2, F) as in Theorem 6 with F satisfying (12). Then there exists $\hat{\delta} > 0$ (with $\hat{\delta}$ depending on the second eigenvalue of a certain weighted problem) such that if $\lambda < \hat{\delta}$ necessarily $u_1 \equiv u_2$.*

The proof follows a technique similar to the one of [25]. We remark that, in both results, the pressure term can be generalized to any C^1 function $p(u)$ such that $0 \leq p'(u) \leq \lambda u_+$ on $\mathbb{R}_+$ and that (as in [31]) if λ is large enough we do not expect to have any uniqueness result.

5.2. An estimate on the plasma region. In [11] we estimate the measure of the plasma region $|u > 0| = \int_{\{u>0\}} dx$. We already know that $\max u_+ \leq S$ (see Lemma 3). So, we have $|u > 0| \geq \frac{1}{S}\int_\Omega u_+ dx$. In order to estimate the L^1–norm of u_+ , as in [32, 8, 11], we use the identity

$$\lambda_1 \int_\Omega u\varphi_1 dx - \gamma = \int_\Omega a\mathcal{F}_u\varphi_1 dx + \int_\Omega H(u, b_{*u})\varphi_1 dx + \int_\Omega J(u)\varphi_1 dx$$

where we rewrite the equation of (P_{NL}) as $-\Delta u = a\mathcal{F}_u + H(u, b_{*u}) + J(u)$ in a similar way to $(P_{NL\varepsilon})$ and φ_1 is the normalized eigenfunction associated to the first eigenvalue λ_1 of the operator $-\Delta$ on Ω with Dirichlet boundary condition, i.e., $\varphi_1 \in H_0^1(\Omega)$ and $-\Delta\varphi_1 = \lambda_1\varphi_1$ on Ω. We obtain

THEOREM 7 *([11]) Assume the hypotheses of Theorem 4 and $\gamma > \gamma_0 := F_v \int_\Omega a\varphi_1 dx$. Let (u, F) be the solution of $(\mathcal{P}_I)$ obtained in Theorem 4. Then there exits a constant $L(\lambda)$*

$$|u > 0| \geq \frac{\gamma - \gamma_0 - O(\eta^{1/2})}{SL(\lambda)} > 0$$

if η is small enough.

5.3.On the numerical approach. The numerical approach of problem $(\mathcal{P}_I)$ with $j \equiv 0$, has been considered recently in [4] by using a direct algorithm approximating simultaneously u and F. More concretely, given u^0 it is calculated F^0 and for $k = 0, 1, 2, ..$the following linear problems are solved

$$(\mathcal{P}_I^{(k)}) \quad -\Delta u^{(k+1)} = aF^{(k)}(u^{(k)}) + \left(\frac{F^{(k)2}}{2}\right)'(u^{(k)}) + \lambda b p'(u^{(k)}) \quad \text{in } \Omega, \quad u^{(k+1)} = \gamma \qquad \text{on } \partial\Omega,$$

and calculating $F^{(k+1)}$ by using the integral condition

$$\int_{\{u^{(k+1)}>t\}} [\left(\frac{F^{(k+1)2}}{2}\right)'(u^{(k+1)}) + bp'(u^{(k+1)})]dx \quad = \quad 0, \quad \forall t \in (-\infty, \ \text{esssup} u^{(k+1)}].$$

Problems $(\mathcal{P}_I^{(k)})$ are discretized by finite elements and the family of integral identities is replaced by using a suitable interpolation of $\left(\frac{F^2}{2}\right)'$ which, roughly speaking, is obtained from the condition

$$\int_{\{u^{(k+1)}>t_i\}}[\left(\frac{F^{(k+1)2}}{2}\right)'(u^{(k+1)})+bp'(u^{(k+1)})]dx \quad = \quad 0,$$

for $\{t_i\}$ originating a partition of the interval $(0, \operatorname{esssup} u^{(k+1)}]$. Several numerical experiences can be found in the mentioned work.

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