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On the Approximate Controllability of Some Semilinear Parabolic Boundary-Value Problems*

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Abstract. We prove the approximate controllability of several nonlinear parabolic boundary-value problems by means of two different methods: the first one can be called a *Cancellation* method and the second one uses the Kakutani fixed-point theorem.

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1. Introduction

In this paper we study the approximate controllability of several semilinear parabolic boundary-value problems for which the nonlinear term appears either in the second-order parabolic equation or in the flux boundary condition. We also distinguish the case in which the control function acts *on the interior* of the set $Q := \Omega \times (0, T)$ from the one in which the control acts *on the boundary* $\Sigma := \partial\Omega \times (0, T)$ (or on a subset $\mathcal{O}$ of Σ). Most of our results concern the control of problems with *final observation*, i.e., our goal is to prove that the set $\{y(T, \cdot; v)\}$ generated by the values of solutions at time T is dense in $L^2(\Omega)$ as v runs through the set of controls. However, we also consider a control problem with a *boundary observation*. In this case we prove that if $\Sigma_1 \subset \Sigma$, then the set $\{y(\cdot, \cdot; v\}|_{\Sigma_1}\}$ generated by the traces of solutions on Σ_1 is a dense subset of $L^2(\Sigma_1)$ as v runs through the set of controls.

This paper grew out of the unpublished thesis of one of the authors [16] at the University of Paris VI. His results were pioneering in the study of the approximate controllability for semilinear parabolic problems. For instance, to the best of our knowledge, the technique of applying the Kakutani fixed-point theorem after a linearization argument, since widely used in works on controllability of nonlinear problems (see, e.g., [23], [13], [14], [6], [11], and [12]), first appeared in [16].

This paper is organized according to the method of proof. Section 2 is devoted to the illustration of the so-called *Cancellation Method*: the nonlinear control problem is solved as a perturbation of a linear control problem canceling the nonlinear term. This method is applied to the study of the approximate controllability of two semilinear parabolic boundary-value problems (see problems $(\mathcal{P}_D)$ and $(\mathcal{P}_N)$ bellow). In the first problem we have a sign constraint on the control.

Section 3 contains the treatment of two different control problems (see $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ below) via application of the Kakutani fixed-point theorem. As indicated above, this general idea has already been used in [16]. However, in order to obtain sharper conditions on the nonlinear terms, we also use some more recent arguments introduced (for other boundary-value problems) in [18] and [13]. This section contains some improvements of the results by Henry [16]. In particular we state and prove all the results of this section under a unique condition on the behavior at infinity of the nonlinear terms $f(y)$: we assume that $f: \mathbb{R} \to \mathbb{R}$ is *sublinear at infinity*, i.e., there exists some nonnegative constants a, b, and M such that

$$|f(s)| \leq a + b|s| \qquad \text{for any} \quad s \in \mathbb{R}, \quad |s| > M. \tag{1.1}$$

For the case $f(s) = |s|^{p-1}s$, the optimality of condition (1.1) (i.e., noncontrollability for $p > 1$) has already been proved in [16] (counterexample due to A. Bamberger) for the case of one-dimensional flux control superlinear problems by using an energy method. Later, the optimality (again for the case $f(s) = |s|^{p-1}s$) was obtained by showing the existence of some *obstruction functions* and, in fact, the approximate controllability for a suitable subclass of desired states may be demonstrated (see [9]).

2. On the Cancellation Method

The main goal of this section is to present some results related to the L^p-approximate controllabiliy of the Dirichlet semilinear problem

$$\begin{cases} y_t - \Delta y + f(y) = v \quad \text{in } Q = \Omega \times (0, T), \end{cases}$$

and the nonlinear Neumann-type problem

$$(\mathcal{P}_N) \quad \begin{cases} y_t - \Delta y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = v & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega, \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^n$ such that $\partial\Omega$ is an $(n-1)$-dimensional infinitely differentiable manifold and Ω is locally on only one side of $\partial\Omega$, $T > 0$, $Q = \Omega \times (0, T)$, f is a continuous real-valued function, $y_0 \in L^2(\Omega)$, ν is the outer unit normal vector to $\partial\Omega$, and in both cases v represents the control.

For problem $(\mathcal{P}_D)$ we show a stronger property than the usual approximate controllability: for suitable desired states we can control the problem by using merely nonnegative controls. In both cases we prove L^p-approximate controllability for any p such that $1 < p < \infty$.

Our treatement of problems $(\mathcal{P}_D)$ and $(\mathcal{P}_N)$ relies on the same general programme: we first establish a conclusion for a linear associated problem and as a second step, we prove the result for the nonlinear case by means of a *cancellation technique* already introduced in [16]. This technique consists in modifying the control associated to the linear case by means of a perturbation which cancels the nonlinearity appearing in the equation or in the boundary condition.

2.1. *Internal Nonnegative Controls*

In spite of the extensive literature on the approximate controllability of nonlinear parabolic problems (see, e.g., the list of references of the survey [7]), the study of the approximate controllability property under a nonnegativity constraint on the controls seems to have been unexplored before the work by Díaz [6] dealing with the parabolic obstacle problem.

We point out that, in constrast to the case of unconstrained control problems (see, e.g., [16] and [10]), constraint on the controls introduces some important difficulties, even if the control v acts on the whole domain Q.

We start by considering the linear case, which we use in the proof of the result for the nonlinear case. In the rest of this paper we always assume $1 < p < \infty$ (the limit cases $p = 1$ and $p = \infty$ can also be treated with some technical modifications). Given a measurable set $\mathcal{M}$ of $\mathbb{R}^d$ $(d \geq 1)$ we define the set $L^p_+(\mathcal{M}) = \{g \in L^p(\mathcal{M}) : g \geq 0\}$.

Theorem 1. *Let $h \in L^p(Q)$, $Y_0 \in L^p(\Omega)$, and $a \in L^\infty(Q)$. We denote by $Y(\cdot; v)$ the solution of*

$$(\mathcal{LP}_D) \quad \begin{cases} Y_t - \Delta Y + aY = h + v & \text{in } Q, \\ Y = 0 & \text{on } \Sigma, \\ Y(0) = Y_0 & \text{on } \Omega. \end{cases}$$

Then, if $\mathcal{U}$ is a dense subset of $L^p_+(Q)$, the set $F := \{Y(T; v);\ v \in \mathcal{U}\}$ is dense in $Y(T; 0) + L^p_+(\Omega)$.

Hahn–Banach theorem (in its geometrical form), we can separate y_d from $\bar{F}$, i.e., there exists $\alpha \in \mathbb{R}$ and $g \in L^{p'}(\Omega)$ (with $1/p + 1/p' = 1$) such that

$$\int_\Omega Y(T; v) g \, dx < \alpha < \int_\Omega y_d g \, dx \qquad \text{for all} \quad v \in \mathcal{U}.$$

Further, if $v \in L^p_+(Q)$ and $\lambda \in \mathbb{R}_+$, then, by linearity, $Y(T, \lambda v) = \lambda Y(T, v) \in \bar{F}$ and so

$$\int_\Omega Y(T; v) g \, dx \leq 0 < \alpha < \int_\Omega y_d g \, dx \qquad \text{for all} \quad v \in \mathcal{U}. \tag{2.1}$$

Now, let $q \in C([0, T] : L^{p'}(\Omega))$ be the solution of the auxiliary backward problem

$$\begin{cases} -q_t - \Delta q + aq = 0 & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = g & \text{on } \Omega. \end{cases} \tag{2.2}$$

Multiplying (2.2) by $Y(v)$, with $v \in \mathcal{U}$ arbitrary, we obtain

$$0 \geq \int_\Omega g(x) Y(T, x; v) \, dx = \int_Q q v \, dx \, dt, \qquad \forall v \in \mathcal{U}.$$

Then $q \leq 0$ in Q. In particular $g \leq 0$, which is a contradiction to (2.1). □

Now we are ready to consider the nonlinear problem $(\mathcal{P}_D)$. For simplicity we assume that

$$f \text{ is a nondecreasing continuous real function} \tag{2.3}$$

and that

$$y_0 \in L^\infty(\Omega). \tag{2.4}$$

Theorem 2. *Assume* (2.3) *and* (2.4). *If $\mathcal{U}$ is a dense subset of $L^p_+(Q)$, then the set $F = \{y(T; v)$ solution of $(\mathcal{P}_D)$; $v \in \mathcal{U}\}$ is dense in $y(T; 0) + L^p_+(\Omega)$.*

Proof. As $y_0 \in L^\infty(\Omega)$, by the maximum principle $y(\cdot; 0) \in L^\infty(Q)$ and $h(\cdot) := -f(y(\cdot; 0)) \in L^\infty(Q)$. Then Theorem 1, with $h = -f(y(\cdot; 0))$, implies that there exists $w_\varepsilon \in L^\infty_+(Q)$ such that

$$\|Y(T; w_\varepsilon) - y_d\|_{L^p(\Omega)} < \varepsilon,$$

with $y_d \in y(T; 0) + L^p_+(\Omega)$. Further, again by means of the maximum principle, $f(Y(w_\varepsilon)) \in L^p(Q)$. Now, given $\delta > 0$, let $\tilde{y}$ be the unique solution of the auxiliary problem

$$\begin{cases} \tilde{y}_t - \Delta \tilde{y} + f(\tilde{y} + Y(w_\varepsilon)) = f(Y(w_\varepsilon)) + \delta & \text{in } Q, \end{cases}$$

Then, if we define $y = \tilde{y} + Y(w_\varepsilon)$, we easily check that y is the solution of $(\mathcal{P}_D)$ with

$$v_\varepsilon = w_\varepsilon + f(Y(w_\varepsilon)) - f(y(\cdot;0)) + \delta \in L^p(Q).$$

Moreover, $v_\varepsilon \geq 0$ since f is nondecreasing and $Y(\cdot; w_\varepsilon) \geq Y(\cdot; 0) = y(\cdot; 0)$. Using the density of $\mathcal{U}$ and the continuous dependence on the data in problem $(\mathcal{P}_D^*)$, we can choose $v \in \mathcal{U}$ such that $\|v - v_\varepsilon\|_{L^p(Q)} \leq \varepsilon$. Finally, applying Hölder's and Young's inequalities, we conclude (for $\delta > 0$ small enough) that

$$\|\tilde{y}(T)\|_{L^p(\Omega)} \leq C_1\varepsilon$$

and so

$$\|y(T; v) - y_d\|_{L^p(\Omega)} \leq C_2\varepsilon.$$

□

Remark 3. In the above theorem we can replace f by a general maximal monotone graph β of $\mathbb{R}^2$. The proof of existence of a solution in this case can be found, for instance, in [4] and Theorem 2 remains true if we assume $\beta_+(r) < +\infty$ for all $r \in D(\beta)$, where

$$\beta_+(r) := \sup\{b \in \mathbb{R} : \; b \in \beta(r)\}.$$

This assumption holds in many cases: (i) the case of $D(\beta) = \mathbb{R}$ (as, for instance, when β is a continuous nondecreasing function or the Heaviside graph); (ii) the condition is also satisfied in some cases for which $D(\beta) \neq \mathbb{R}$ such as, for instance,

$$\beta(r) = \begin{cases} \emptyset & \text{if} \quad r < 0, \\ (-\infty, 0] & \text{if} \quad r = 0, \\ 0 & \text{if} \quad r > 0. \end{cases}$$

Remark 4. It is easy to see that Theorem 1 with the decomposition $Y = Y_+ - Y_-$ implies the L^p-approximate controllability for the unconstrained linear problem. For the unconstrained nonlinear case the L^p-approximate controllability follows from obvious modifications of Theorem 2. The same property is also proved in [16] without assumption (2.3), but with some additional condition on the behavior of f at infinity.

2.2. *Neumann-Type Boundary Controls*

In this section we study problem $(\mathcal{P}_N)$. The cancellation technique can be applied in order to prove L^p-approximate controllability (under unconstrained controls).

Theorem 5. *Assume* (2.3), (2.4). *For* $v \in L^p(\Sigma)$ *we denote by* $y(v)$ *the unique solution of*

$$(\mathcal{P}_N) \quad \begin{cases} y_t - \Delta y = 0 & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = v & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega. \end{cases}$$

Proof. For $y_d \in L^p(\Omega)$ and $\varepsilon > 0$ fixed, we use the decomposition $y = \tilde{y}_\varepsilon + Y$ with Y the solution of the associated linear problem

$$(\mathcal{LP}_N) \quad \begin{cases} Y_t - \Delta Y = 0 & \text{in } Q, \\ \dfrac{\partial Y}{\partial \nu} = -f(y(\cdot;0)) + v_\varepsilon & \text{on } \Sigma, \\ Y(0) = y_0 & \text{on } \Omega, \end{cases}$$

for a suitable v_ε such that $\|y(T; v_\varepsilon) - y_d\|_{L^p(\Omega)} < \varepsilon$ (we can prove the existence of v_ε again by means of the Hahn–Banach theorem; see [17]). For $\delta > 0$ let $\tilde{y}$ be the solution of the nonlinear problem

$$(\mathcal{P}_N^*) \quad \begin{cases} \tilde{y}_t - \Delta \tilde{y} = 0 & \text{in } Q, \\ \dfrac{\partial \tilde{y}}{\partial \nu} + f(\tilde{y} + Y(v_\varepsilon)) = f(Y(v_\varepsilon)) + \delta & \text{on } \Sigma, \\ \tilde{y}(0) = 0 & \text{on } \Omega. \end{cases}$$

Then, by L^p *a priori* estimates, it is easy to see that if $\delta > 0$ is small enough, there exists $C > 0$ such that

$$\|\tilde{y}(T)\|_{L^p(\Omega)} \leq C\varepsilon.$$

Then using the triangle inequality we obtain the desired result. □

3. Approximate Controllability via the Kakutani Fixed-Point Theorem: Case of Flux Boundary Controls

This section is devoted to proving some controllability results for nonlinear parabolic problems by means of a different method. The key idea is the application of a fixed-point argument for a (possibly) multivalued operator (the Kakutani fixed-point theorem). We consider two different control problems:

$$(\mathcal{P}_1) \quad \begin{cases} y_t - \Delta y + f(y) + a(x,t)\beta(y) \ni h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} = v\chi_{\mathcal{O}} & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{on } \Omega \end{cases}$$

and

$$(\mathcal{P}_2) \quad \begin{cases} y_t - \Delta y + a(x,t)\beta(y) \ni h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Sigma_2, \\ y(x,0) = y_0(x) & \text{on } \Omega. \end{cases}$$

under different criteria: observation in time T for $(\mathcal{P}_1)$ (Section 3.1) and observation on Σ_1 for $(\mathcal{P}_2)$ (Section 3.2). Our method combines some ideas introduced in [16], [18], and [13].

We point out that the controllability results are independent of the uniqueness of the solution for a fixed control: so, for instance, if $a(x,t) < 0$ and β is multivalued there is lack of uniqueness of solutions (see [8]).

3.1. *Observation in Time T*

Let $\mathcal{O}$ be a nonempty open subset of $\Sigma = \partial\Omega \times (0,T)$. Let $a \in L^\infty(Q)$, and consider $h \in L^2(Q)$, $y_0 \in L^2(\Omega)$. We define

$$X^1(Q) = \left\{\varphi\colon \varphi \in H^{1,2}(Q),\ \frac{\partial\varphi}{\partial\nu} = 0,\ \varphi(\cdot,T) \equiv 0\right\}.$$

Here and in what follows we use the notation

$$H^{r,s}(Q) = L^2(0,T;H^s(\Omega)) \cap H^r(0,T;L^2(\Omega))$$

for $r, s \in \mathbb{R}$.

Following Lions and Magenes [19] we define the notion of weak solution in the following manner:

Definition 6. A function $y \in L^2(Q)$ is a solution of problem $(\mathcal{P}_1)$ if there exists $b \in L^2(Q)$ with $b \in \beta(y)$ such that

$$\begin{aligned}&(y, -\varphi_t - \Delta\varphi)_{L^2(Q)}\\ &\quad = (h - f(y) - a(x,t)b, \varphi)_{L^2(Q)} + (v\chi_{\mathcal{O}}, \varphi)_{L^2(\Sigma)} + (y_0, \varphi(\cdot,0))_{L^2(\Omega)},\\ &\quad\ \forall\varphi \in X^1(Q).\end{aligned}$$

Theorem 7. *Let f be a real-valued function satisfying the following two conditions:*

$$f(\cdot) \text{ is continuous and there exists } f'(s_0) \text{ for some } s_0 \in \mathbb{R}, \tag{3.1}$$

$$\text{there exists } M > 0,\ c_1 > 0, \text{ and } c_2 > 0 \text{ such that } |f(s)| \le c_1 + c_2|s|, \text{ for } |s| > M. \tag{3.2}$$

Assume also that β is a bounded maximal monotone graph of $\mathbb{R}^2$ such that $D(\beta) = \mathbb{R}$. Then the set $F := \{y(T;v) : y(T;v) \text{ is a solution of } (\mathcal{P}_1) \text{ with } v \in L^\infty(\mathcal{O})\}$ is dense in $X = L^2(\Omega)$.

Before beginning the proof of Theorem 7 we recall some results.

Proposition 8. *Let $a = a(t,x) \in L^\infty(Q)$. There exists a constant $C > 0$ such that, for each $k \in L^2(\Sigma)$, $h \in L^2(Q)$, and $\omega^0 \in L^2(\Omega)$, the solution ω of*

$$\begin{cases}\omega_t - \Delta\omega + a(t,x)\omega = h & \text{in } Q,\\ \dfrac{\partial\omega}{\partial\nu} = k & \text{on } \Sigma,\end{cases} \tag{3.3}$$

satisfies

$$\|\omega\|_{H^{1/2,1}(Q)} \leq C_a(\|\omega^0\|_{L^2(\Omega)} + \|h\|_{L^2(Q)} + \|k\|_{L^2(\Sigma)}). \tag{3.4}$$

Further, if $\{a_n\} \subset L^\infty(Q)$ *with* $\sup_{n\in\mathbb{N}}\{\|a_n\|_{L^\infty(Q)}\} < \infty$, *then we can choose* $C_{a_n} = C$ *independent of* n.

The proof is given in two steps.

Lemma 9. *The conclusion of Proposition* 8 *is true if* $\omega^0 \equiv 0$ *and* $k \equiv 0$.

Proof of Lemma 9. By density we can choose a sequence $a^n \in C^\infty(\bar{Q})$ such that $a^n \to a$ in $L^2(Q)$ and $\{a^n\}$ is uniformly bounded in the topology of $L^\infty(Q)$. Then, if we denote by ω^n the solution of

$$\begin{cases} \omega^n_t - \Delta\omega^n + a^n(x,t)\omega^n = h & \text{in } Q, \\ \dfrac{\partial\omega^n}{\partial\nu} = 0 & \text{on } \Sigma, \\ \omega^n(0) = 0 & \text{on } \Omega, \end{cases}$$

by well-known results (see, for instance, Section 6.1 of Chapter 4 of [19]), $\omega^n \in H^{1,2}(Q)$ and

$$\|\omega^n\|_{H^{1,2}(Q)} \leq C(\|h\|_{L^2(Q)} + \|\omega^n\|_{L^2(Q)})$$

with C independent of n. Further, by "multiplying" in the above problem by ω^n and by using Young's inequality it is easy to deduce that

$$\|\omega^n\|_{L^2(Q)} \leq C'\|h\|_{L^2(Q)}$$

with C' independent of n. So, if ω is the limit of ω^n in the weak topology of $H^{1,2}(Q)$, by means of Definition 6 we can pass to the limit in the problem and deduce that ω is the unique solution of the problem

$$\begin{cases} \omega_t - \Delta\omega + a(x,t)\omega = h & \text{in } Q, \\ \dfrac{\partial\omega}{\partial\nu} = 0 & \text{on } \Sigma, \\ \omega(0) = 0 & \text{on } \Omega \end{cases}$$

and that it satisfies

$$\|\omega\|_{H^{1,2}(Q)} \leq C''\|h\|_{L^2(Q)}$$

Proof of Proposition 8. We write $\omega = u + z$, where u satisfies

$$\begin{cases} u_t - \Delta u = h & \text{in } Q, \\ \dfrac{\partial u}{\partial \nu} = k & \text{on } \Sigma, \\ u(0) = \omega_0 & \text{on } \Omega \end{cases}$$

and z is the solution of

$$\begin{cases} z_t - \Delta z + az = -au & \text{in } Q, \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma, \\ z(0) = 0 & \text{on } \Omega. \end{cases} \tag{3.5}$$

Then we have the estimate

$$\|u\|_{H^{1/2,1}(Q)} \leq c_1(\|k\|_{L^2(\Sigma)} + \|\omega_0\|_{L^2(\Omega)} + \|h\|_{L^2(Q)})$$

(see, for instance, Section 15.1 of Chapter 4 of [19]). Finally, by applying Lemma 9, we obtain that

$$\|z\|_{H^{1/2,1}(Q)} \leq \|z\|_{H^{1,2}(Q)} \leq C\|u\|_{L^2(Q)}.$$ □

Proposition 10. *If* ω *is the solution of* (3.3), *then* $\omega \in C([0,T]; L^2(\Omega))$. *Further, if* $k \in H^{1/4,1/2}(\Sigma)$, *then* $\omega \in H^{1,2}((\delta, T) \times \Omega)$ *for all* $0 < \delta < T$.

Proof. We know that $\omega \in L^2(0,T; H^1(\Omega))$ and according to Lions and Magenes [19] (see Proposition 12.1 of Chapter 1 of [19]),

$$\Delta\omega \in L^2(0,T; H^{-1}(\Omega)).$$

Therefore

$$\omega_t = h - a(x,t)\omega + \Delta\omega \in L^2(0,T; H^{-1}(\Omega)).$$

By using Theorem 3.1 of Chapter 3 of [19], we obtain that

$$\omega \in C([0,T]; [H^1(\Omega), H^{-1}(\Omega)]_{1/2}),$$

where $[X, Y]_\theta$ denotes the θ-intermediate space between the Banach spaces X and Y, if X is a dense subset of Y and $X \subset Y$ is a continuous injection (for more details see, for instance, Section 2 of Chapter 1 of [19]). Now, by using Theorem 12.4 of Chapter 1 of [19] we obtain that $[H^1(\Omega), H^{-1}(\Omega)]_{1/2} = L^2(\Omega)$. On the other hand, for all δ, Following Lions [18] and Fabre *et al.* [13], for $\varphi^0 \in L^2(\Omega)$ we introduce the functional

$$J(\varphi^0) = \tfrac{1}{2}\left(\int_{\mathcal{O}} |\varphi(x,t)|\,d\Sigma\right)^2 + \varepsilon|\varphi^0|_{L^2(\Omega)} - \int_\Omega y_d\varphi^0\,dx,$$

with $\varphi(x,t)$ the solution of the backward problem

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \text{in } Q,\\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \text{on } \Sigma,\\ \varphi(T) = \varphi^0 & \text{on } \Omega. \end{cases} \tag{3.6}$$

Remark 11. We point out that, if we reformulate the problem in forward form, by Definition 6, a function φ is said to be a solution of (3.6) if

$$(\varphi, \psi_t - \Delta\psi + a(x,t)\psi)_{L^2(Q)} = (\psi(T), \varphi^0)_{L^2(\Omega)}, \qquad \forall\psi \in X^1(Q).$$

Proposition 12. *If $\mathcal{O}$ is a nonempty open subset of Σ, $a \in L^\infty(Q)$, and φ satisfies*

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \textit{in } Q,\\ \dfrac{\partial\varphi}{\partial\nu} = 0 & \textit{on } \Sigma, \end{cases}$$

with $\varphi(T) \in L^2(\Omega)$ and

$$\varphi = 0 \quad \textit{on } \mathcal{O},$$

then $\varphi \equiv 0$ in Q.

Proof. Let $t^* = \sup\{t \le T : \exists x \in \partial\Omega \text{ such that } (x,t) \in \mathcal{O}\}$. Then, by a unique continuation theorem (see [20] and [21]) and the uniqueness of solutions of this type of problem, $\varphi \equiv 0$ in $Q^* = \Omega \times (0, t^*)$. Finally, by backward uniqueness results (see p. 175 of [15]), $\varphi \equiv 0$ in the whole domain Q. □

Remark 13. We point out that in the proof of Proposition 12, we can apply the unique continuation argument since $\varphi \in L^2(\delta, T; H^2(\Omega))$ for all $0 < \delta < T$ (see Proposition 10).

Proposition 14. *For all $\varepsilon > 0$, $y_d \in L^2(\Omega)$, and $a \in L^\infty(Q)$, the functional $J(\cdot; a, y_d)$: $L^2(\Omega) \to \mathbb{R}$ is strictly convex and satisfies*

$$\liminf_{|\varphi^0|_2\to\infty} \frac{J(\varphi^0; a, y_d)}{|\varphi^0|_2} \ge \varepsilon. \tag{3.7}$$

Further, $J(\cdot; a, y_d)$ attains its minimum at a unique point $\hat\varphi^0$ in $L^2(\Omega)$ and

Proof. If J does not satisfy (3.7), then there exists a sequence $\{\varphi_n^0\} \subset L^2(\Omega)$ such that

$$|\varphi_n^0|_2 \to +\infty \quad \text{and} \quad \liminf_{n\to+\infty} \frac{J(\varphi_n^0; a, y_d)}{|\varphi_n^0|_2} < \varepsilon.$$

Thus, if φ_n is the solution of (3.6) with initial data φ_n^0, we obtain that

$$\liminf_{n\to+\infty} \int_{\mathcal{O}} \frac{|\varphi_n(x,t)|}{|\varphi_n^0|_2} d\Sigma = 0, \tag{3.9}$$

since in the other case

$$\begin{aligned}\liminf_{n\to+\infty} \frac{J(\varphi_n^0; a, y_d)}{|\varphi_n^0|_2} &= \liminf_{n\to+\infty}\left(\tfrac{1}{2}|\varphi_n^0|_2 \left(\int_{\mathcal{O}} \frac{|\varphi_n(t,x)|}{|\varphi_n^0|_2} d\Sigma\right)^2 + \varepsilon - \int_\Omega y_d \frac{\varphi_n^0}{|\varphi_n^0|_2} dx\right)\\ &\geq \liminf_{n\to+\infty}\left(\tfrac{1}{2}|\varphi_n^0|_2 \left(\int_{\mathcal{O}} \frac{|\varphi_n(t,x)|}{|\varphi_n^0|_2} d\Sigma\right)^2 + \varepsilon - |y_d|_2\right) = +\infty.\end{aligned}$$

At the same time, $\varphi_n^0/|\varphi_n^0|_2$ has a unit norm and so it converges weakly in $L^2(\Omega)$ to an element $\psi^0 \in L^2(\Omega)$. Now, from Proposition 8, $\{\varphi_n/|\varphi_n^0|_2\}_{n\in\mathbb{N}}$ converges weakly in $L^2(\Omega)$ to ψ (solution of (3.6) with $\psi(T) = \psi^0$). Then, by (3.9) and the unique continuation property of Proposition 12, $\psi^0 \equiv 0$. Further, since

$$J(\varphi_n^0; a, y_d) \geq |\varphi_n^0|_2 \left(\varepsilon - \int_\Omega y_d \frac{\varphi_n^0}{|\varphi_n^0|_2} dx\right),$$

we deduce that

$$\liminf_{n\to+\infty} \frac{J(\varphi_n^0; a, y_d)}{|\varphi_n^0|_2} \geq \varepsilon,$$

which is a contradiction to the assumption.

In order to prove (3.8), we use that $J(\cdot; a, y_d)$ is strictly convex and continuous in $L^2(\Omega)$ and that

$$\lim_{|\varphi_n^0|_2\to+\infty} J(\varphi_n^0; a, y_d) = +\infty.$$

Then $J(.; a, y_d)$ attains its minimum at a unique point $\hat{\varphi}^0 \in L^2(\Omega)$ (see, for instance, [5]).

Further, if $|y_d|_2 \leq \varepsilon$, then

$$\begin{aligned}J(\varphi_n^0; a, y_d) &\geq \varepsilon|\varphi^0|_2 - |y_d|_2|\varphi^0|_2\\ &\geq |\varphi^0|_2(\varepsilon - |y_d|_2)\\ &\geq 0, \qquad \forall \varphi^0 \in L^2(\Omega),\end{aligned}$$

Conversely, if we suppose that $\hat{\varphi}^0 = 0$ and $\varepsilon < |y_d|_2$, we take

$$\gamma = \frac{|y_d|_2 - \varepsilon}{2}$$

and then, as

$$|y_d|_2 = \sup_{|\varphi^0|_2=1} \int_\Omega y_d \varphi^0 \, dx,$$

if $\tilde{\varphi}^0 \in L^2(\Omega)$ with $|\tilde{\varphi}^0|_2 = 1$ and $|y_d|_2 - \int_\Omega y_d \tilde{\varphi}^0 \, dx < \gamma/2$, we obtain for every $\mu > 0$ that

$$\begin{aligned} J(\mu\tilde{\varphi}^0) &= \frac{\mu^2}{2}\left(\int_{\mathcal{O}} |\tilde{\varphi}(t,x)| \, d\Sigma\right)^2 + \mu\left(\varepsilon - \int_\Omega y_d \tilde{\varphi}^0 \, dx\right) \\ &< \frac{\mu^2}{2}\left(\int_{\mathcal{O}} |\tilde{\varphi}(t,x)| \, d\Sigma\right)^2 + \mu\left(\varepsilon - |y_d|_2 + \frac{\gamma}{2}\right) \\ &= \frac{\mu^2}{2}\left(\int_{\mathcal{O}} |\tilde{\varphi}(t,x)| \, d\Sigma\right)^2 + \mu\left(-2\gamma + \frac{\gamma}{2}\right) \\ &< 0, \qquad \text{if } \mu \text{ is small enough.} \end{aligned}$$

However, $\hat{\varphi}^0 = 0$ implies that $J(\mu\tilde{\varphi}^0) \geq J(\hat{\varphi}^0) = 0$, which is a contradiction and so $\varepsilon \geq |y_d|_2$. □

Proposition 15. *Let M be the mapping*

$$\begin{aligned} M: L^\infty(Q) &\to L^2(\Omega), \\ a(x,t) &\to \hat{\varphi}^0. \end{aligned}$$

If B is a bounded subset of $L^\infty(Q)$, then $M(B)$ is a bounded subset of $L^2(\Omega)$.

Proof. Assume, for the sake of a contradiction, that there exists a sequence $(a_n)_n \subset B \subset L^\infty(Q)$ such that

$$|\hat{\varphi}_n^0|_2 = |M(a_n)|_2 \to \infty. \tag{3.10}$$

Now, since B is bounded, there exists $a \in L^\infty(Q)$ such that a subsequence

$$a_n \xrightarrow{n\to+\infty} a \text{ in the weak-* topology of } L^\infty(Q).$$

We see that

$$\liminf_{|\varphi_n^0|_2\to\infty} \frac{J(\varphi_n^0; a_n, y_d)}{|\varphi_n^0|_2} \geq \varepsilon. \tag{3.11}$$

If this is not true, there exists a sequence $(\varphi_n^0)_n$ of $L^2(\Omega)$ such that $|\varphi_n^0|_2 \to \infty$ and

Next, in a way similar to that followed in the proof of Proposition 14 we put $\tilde{\varphi}_n^0 = \varphi_n^0/|\varphi_n^0|_2$ and denote by $\tilde{\varphi}_n$ the solution of (3.6) with respect to a_n with $\tilde{\varphi}_n(T) = \tilde{\varphi}_n^0$. Since $|\tilde{\varphi}_n^0|_2 = 1$, we can suppose that $\tilde{\varphi}_n^0$ converges weakly in $L^2(\Omega)$ to $\tilde{\varphi}^0 \in L^2(\Omega)$.

As in the proof of Proposition 14 it is easy to prove that $\tilde{\varphi}_n$ converges in the weak topology of $L^1(\mathcal{O})$ to $\tilde{\varphi}$ (solution of (3.6)) with respect to a and with $\tilde{\varphi}(T) = \tilde{\varphi}^0$). Then

$$\int_{\mathcal{O}} |\tilde{\varphi}_n| \, dx \, dt \xrightarrow{n \to \infty} 0,$$

and so

$$\tilde{\varphi}^0 = 0.$$

Now if $J_n = J(\varphi_n^0; a_n, y_d)/|\varphi_n^0|_2$, it holds that

$$J_n \geq \left(\varepsilon - \int_{\Omega} y_d^n \tilde{\varphi}_n^0 \, dx \right),$$

and since $\tilde{\varphi}_n^0$ converges in the weak topology of $L^2(\Omega)$ to 0, we obtain

$$\liminf_{n \to +\infty} J_n \geq \varepsilon,$$

which contradicts (3.12) and thus proves (3.11).

Finally, we point out that $J(0; a_n, y_d) = 0$, and so $J(\hat{\varphi}_n^0; a_n, y_d) \leq 0$, which is a contradiction to (3.10) and (3.11). Therefore

$$\sup_{n \in \mathbb{N}} \{|\hat{\varphi}_n^0|_2 : n \in \mathbb{N}\} < +\infty. \qquad \square$$

Definition 16. Given $V \colon X \to \mathbb{R} \cup \{+\infty\}$ is a convex and proper function on the Banach space X, it is said that an element p_0 of V' belongs to the set $\partial V(x_0)$ (subdifferential of V at $x_0 \in X$) if

$$V(x_0) - V(x) \leq (p_0, x_0 - x), \qquad \forall x \in X.$$

Remark 17. Under the conditions of Definition 16, x_0 minimizes V over X (or over a convex subset of X) if and only if

$$0 \in \partial V(x_0).$$

Proposition 18. *Under the above conditions, if V is a lower semicontinuous function, then $p_0 \in \partial V(x_0)$ if and only if*

$$(p_0, x) \leq \lim_{h \to 0^+} \frac{V(x_0 + hx) - V(x_0)}{h} (< +\infty), \qquad \forall x \in X.$$

Remark 19. If V is differentiable its differential coincides with its subdifferential.

Lemma 20. *Let $\varphi^0 \in L^2(\Omega)$, $\varphi^0 \neq 0$. Let φ be the solution of (3.6) verifying $\varphi(T) = \varphi^0$. Then we have that*

$$\partial J(\varphi^0; a, y_d) = \Bigg\{\xi \in L^2(\Omega),\ \exists v \in \operatorname{sgn}(\varphi)\chi_{\mathcal{O}} \text{ satisfying } \int_\Omega \xi(x)\theta^0(x)\,dx = \left(\int_{\mathcal{O}} |\varphi(t,x)|\,d\Sigma\right)\left(\int_{\mathcal{O}} v(t,x)\theta(t,x)\,d\Sigma\right) + \varepsilon\int_\Omega \frac{\varphi^0(x)}{|\varphi^0|_2}\theta^0(x)\,dx - \int_\Omega y_d(x)\theta^0(x)\,dx,\ \forall\theta^0 \in L^2(\Omega)\Bigg\},$$

where θ is the solution of (3.6) verifying $\theta(T) = \theta^0$.

Proof. We introduce the following notation:

$$J(\varphi^0; a, y_d) = \tfrac{1}{2}\left(\int_{\mathcal{O}} |\varphi(t,x)|\,d\Sigma\right)^2 + \varepsilon|\varphi^0|_2 - \int_\Omega y_d\varphi^0\,dx = J_1(\varphi^0) + J_2(\varphi^0) + J_3(\varphi^0).$$

Let $P := \{(t,x) \in \mathcal{O}$ such that $\varphi(t,x) = 0\}$ and $\xi \in \partial J_1(\varphi^0)$. Since J_1 satisfies conditions of Proposition 18, for every $\theta^0 \in L^2(\Omega)$ we have

$$\begin{aligned}
(\xi, \theta^0) &\le \lim_{h\to 0^+} \frac{J_1(\varphi^0 + h\theta^0) - J_1(\varphi^0)}{h}\\
&= \lim_{h\to 0^+} \frac{1}{2h}\left[\left(\int_{\mathcal{O}-P} |\varphi + h\theta|\,d\Sigma\right)^2 - \left(\int_{\mathcal{O}-P} |\varphi|\,d\Sigma\right)^2\right]\\
&\quad + \lim_{h\to 0^+} \frac{1}{2h}\left[\left(\int_P |\varphi + h\theta|\,d\Sigma\right)^2 - \left(\int_P |\varphi|\,d\Sigma\right)^2\right]\\
&\quad + \lim_{h\to 0^+} \frac{1}{h}\left[\left(\int_{\mathcal{O}-P} |\varphi + h\theta|\,d\Sigma\right)\cdot\left(\int_P |\varphi + h\theta|\,d\Sigma\right)\right]\\
&= \lim_{h\to 0^+} \frac{1}{2h}\left[\left(\int_{\mathcal{O}-P} (|\varphi| + \operatorname{sgn}(\varphi)h\theta)\,d\Sigma\right)^2 - \left(\int_{\mathcal{O}-P} |\varphi|\,d\Sigma\right)^2\right]\\
&\quad + \lim_{h\to 0^+} \frac{1}{h}\left[\left(\int_{\mathcal{O}-P} (|\varphi| + \operatorname{sgn}(\varphi)h\theta)\,d\Sigma\right)\cdot\left(\int_P h|\theta|\,d\Sigma\right)\right]\\
&= \lim_{h\to 0^+} \frac{1}{2h}\Bigg[h^2\left(\int_{\mathcal{O}-P} \operatorname{sgn}(\varphi)\theta\,d\Sigma\right)^2\\
&\qquad + 2h\int_{\mathcal{O}-P} |\varphi|\,d\Sigma\int_{\mathcal{O}-P} \operatorname{sgn}(\varphi)\theta\,d\Sigma\Bigg]
\end{aligned}$$

$$= \int_{\mathcal{O}-P} |\varphi|\, d\Sigma \int_{\mathcal{O}-P} \operatorname{sgn}(\varphi)\theta\, d\Sigma + \int_{\mathcal{O}} |\varphi|\, d\Sigma \int_P |\theta|\, d\Sigma$$
$$= \int_{\mathcal{O}} |\varphi|\, d\Sigma \int_{\mathcal{O}-P} \operatorname{sgn}(\varphi)\theta\, d\Sigma + \int_{\mathcal{O}} |\varphi|\, d\Sigma \int_P |\theta|\, d\Sigma.$$

Then

$$\xi \in \partial J_1(\varphi^0) \quad \Leftrightarrow \quad \forall \theta^0 \in L^2(\Omega),$$

$$(\xi, \theta^0) \leq |\varphi|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}-P} \operatorname{sgn}(\varphi(t,x))\theta(t,x)\, d\Sigma + \int_P |\theta(t,x)|\, d\Sigma \right). \tag{3.13}$$

Now we put

$$G = \{\theta \in L^1(\mathcal{O}) : \theta \text{ is solution of (3.6) with } \theta^0 \in L^2(\Omega)\}.$$

Then the mapping $\theta \to \theta^0 \to (\xi, \theta^0)$ is a linear mapping on G and so applying the Hahn–Banach theorem there exists a linear mapping V on $L^1(\mathcal{O})$, such that,

$$\forall \theta^0 \in L^2(\Omega), \qquad (\xi, \theta^0) = V(\theta)$$

and, for each $\Theta \in L^1(\mathcal{O})$,

$$V(\Theta) \leq |\varphi|_{L^1(\mathcal{O})} \left(\int_{\mathcal{O}-P} \operatorname{sgn}(\varphi(t,x))\Theta(t,x)\, d\Sigma + \int_P |\Theta(t,x)|\, d\Sigma \right). \tag{3.14}$$

From (3.14), V is continuous on $L^1(\mathcal{O})$ and then $V \in L^\infty(\mathcal{O})$ and

$$\left| \int_{\mathcal{O}} V(t,x)\Theta(t,x)\, d\mathcal{O} - |\varphi|_{L^1(\mathcal{O})} \int_{\mathcal{O}-P} \operatorname{sgn}(\varphi(t,x))\Theta(t,x)\, d\Sigma \right|$$
$$\leq |\varphi|_{L^1(\mathcal{O})} \int_P |\Theta(t,x)|\, d\Sigma, \qquad \forall \Theta \in L^1(\mathcal{O}). \tag{3.15}$$

If we choose $\Theta \in L^1(\mathcal{O})$ with support contained in $\mathcal{O} - P$, we obtain

$$V = |\varphi|_{L^1(\mathcal{O})} \frac{\varphi}{|\varphi|} \qquad \text{a.e. in} \quad \mathcal{O} - P.$$

Next, if we take $\Theta \in L^1(P)$ we obtain

$$\left| \int_P V(t,x)\Theta(t,x)\, d\Sigma \right| \leq |\varphi|_{L^1(\mathcal{O})} \int_P |\Theta(t,x)|\, d\Sigma,$$

and then

$$|V(t,x)| \leq \|V\|_{L^\infty(P)} \leq |\varphi|_{L^1(\mathcal{O})} \qquad \text{almost every} \quad (x,t) \in P.$$

This proves that there exists $v \in \operatorname{sgn}(\varphi)\chi_{\mathcal{O}}$ such that

Reciprocally, if $V \in |\varphi|_{L^1(\mathcal{O})} \operatorname{sgn}(\varphi)\chi_{\mathcal{O}}$, then

$$\theta^0 \to \int_{\mathcal{O}} V(t,x)\theta(t,x)\, d\Sigma$$

is a continuous linear function on $L^2(\Omega)$ and so there exists a unique $\xi \in L^2(\Omega)$ such that

$$(\xi, \theta^0) = \int_{\mathcal{O}} V(t,x)\theta(t,x)\, d\Sigma, \qquad \forall \theta^0 \in L^2(\Omega).$$

Obviously ξ satisfies (3.13) and then $\xi \in \partial J(\varphi^0)$.

As a second step, consider

$$J_2(\varphi^0) = \varepsilon \left(\int_\Omega |\varphi^0(x)|^2\, dx \right)^{1/2}.$$

By Remark 19,

$$\begin{aligned}(\partial J_2(\varphi^0), \theta^0) &= \frac{\varepsilon}{2} \left(\int_\Omega |\varphi^0(x)|^2\, dx \right)^{-1/2} 2 \int_\Omega \varphi^0(x)\theta^0(x)\, dx \\ &= \varepsilon |\varphi^0|_2^{-1} \int_\Omega \varphi^0(x)\theta^0(x)\, dx.\end{aligned}$$

Finally, by linearity,

$$(\partial J_3(\varphi^0), \theta^0) = -\int_\Omega y_d(x)\theta^0(x)\, dx. \qquad \square$$

Now we are ready to prove a linear version of Theorem 1.

Theorem 21. *If* $|y_d|_2 > \varepsilon$ *and* $\hat{\varphi}$ *is the solution of* (3.6) *satisfying* $\hat{\varphi}(T) = \hat{\varphi}^0$, *then there exists* $v \in \operatorname{sgn}(\hat{\varphi})\chi_{\mathcal{O}}$ *such that, for every* $h \in L^2(Q)$, *the solution of*

$$\begin{cases} y_t - \Delta y + a(t,x)y = h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} = |\hat{\varphi}|_{L^1(\mathcal{O})} v \chi_{\mathcal{O}} & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega \end{cases} \tag{3.16}$$

satisfies

$$y(T) = y_d - \varepsilon \frac{\hat{\varphi}^0}{|\hat{\varphi}^0|_2},$$

and then $|y(T) - y_d|_2 = \varepsilon$.

Proof of Theorem 21. Using linearity and Proposition 15, we can assume $y_0 \equiv 0$ and $h \equiv 0$. Now, because of the subdifferentiability of $J(.; a, y_d)$ at $\hat{\varphi}^0$ ($\neq 0$ by (3.8)), we know (see Remark 17) that

$$0 \in \partial J(\hat{\varphi}^0),$$

which is equivalent, from Lemma 20, to the existence of $v \in \operatorname{sgn}(\hat{\varphi})\chi_{\mathcal{O}}$, such that

$$-|\hat{\varphi}|_{L^1(\mathcal{O})}\left(\int_{\mathcal{O}} v(x,t)\theta(x,t)\,dx\,dt\right) = \frac{\varepsilon}{|\hat{\varphi}^0|_2}\int_\Omega \hat{\varphi}^0(x)\theta^0(x)\,dx - \int_\Omega y_d(x)\theta^0(x)\,dx. \tag{3.17}$$

On the other hand, it follows from Remark 11 that since θ is the solution of (3.6) and y is an admissible "test function" in $X^1(Q)$,

$$(y(T), \theta^0) = |\hat{\varphi}|_{L^1(\mathcal{O})}\left(\int_{\mathcal{O}} v(x,t)\theta(x,t)\,dx\,dt\right). \tag{3.18}$$

Then, from (3.17) and (3.18), we obtain

$$(y(T), \theta^0) = \left(y_d - \varepsilon\frac{\hat{\varphi}^0}{|\hat{\varphi}^0|_2}, \theta^0\right), \qquad \forall \theta^0 \in L^2(\Omega),$$

and we conclude that $y(T) = y_d - \varepsilon(\hat{\varphi}^0/|\hat{\varphi}^0|_2)$. □

For the study of the nonlinear case we need to apply a fixed-point theorem for multivalued operators:

Definition 23. Let X and Y be Banach spaces and let $\Lambda\colon X \to \mathcal{P}(Y)$ be a multivalued function. We say that Λ is *upper hemicontinuous* at $x_0 \in X$, if, for every $p \in Y'$, the function

$$x \to \sigma(\Lambda(x), p) = \sup_{y \in \Lambda(x)} \langle p, y\rangle_{Y' \times Y}$$

is upper semicontinuous at x_0. We say that the multivalued function is upper hemicontinuous on a subset K of X if it satisfies this property for every point of K.

Theorem 24 (Kakutani's Fixed-Point Theorem). *Let $K \subset X$ be a convex and compact subset and let $\Lambda\colon K \to K$ be an upper hemicontinuous application with convex, closed, and nonempty values. Then there exists a fixed point x_0 of Λ.*

Proof of Theorem 7. We fix $y_d \in L^2(\Omega)$, $\varepsilon > 0$, and we define

$$g(s) = \begin{cases} \dfrac{f(s) - f(s_0)}{s - s_0}, & s \neq s_0, \\ f'(s_0), & s = s_0. \end{cases}$$

Then, from the assumption made on f, we have that $g \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$.

Now, using Theorem 21, for each $z \in L^2(Q)$, $b \in \beta(z)$, and $\varepsilon > 0$ it is possible to find two functions $\varphi(z, b) \in L^2(Q)$ and $v(z, b) \in \operatorname{sgn}(\varphi(z, b))\chi_{\mathcal{O}}$ such that the solution $y = y_b^z$ of

$$\begin{cases} y_t - \Delta y + g(z)y = -f(s_0) + g(z)s_0 - a(x,t)b + h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} = u\chi_{\mathcal{O}} & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega \end{cases} \tag{3.19}$$

(where $u = \|\varphi(z, b)\|_{L^1(\mathcal{O})} v(z, b)$) satisfies

$$|y(T) - y_d|_{L^2(\Omega)} \leq \varepsilon. \tag{3.20}$$

Now, as $g(\cdot)$ is bounded, from Propositions 15 and 8 we obtain that

$$\{\|\varphi(z, b)\|_{L^1(\mathcal{O})} v(z, b), z \in L^2(Q),\ b \in \beta(z)\} \qquad \text{is bounded in } L^\infty(Q). \tag{3.21}$$

Let

$$M = \sup_{\substack{z \in L^2(Q) \\ b \in \beta(z)}} \|\varphi(z, b)\|_{L^1(\mathcal{O})} < \infty. \tag{3.22}$$

Obviously $u = \|\varphi(z, b)\|_{L^1(\mathcal{O})} v(z, b)$ satisfies

$$\|u\|_{L^\infty(\Sigma)} \leq M. \tag{3.23}$$

Therefore, if we define the operator

$$\Lambda \colon L^2(Q) \to \mathcal{P}(L^2(Q))$$

by

$$\Lambda(z) = \{y \text{ satisfying (3.19), (3.20) for some } b \in \beta(z) \text{ and } u \text{ verifying (3.23)}\},$$

we have seen that, for each $z \in L^2(Q)$, $\Lambda(z) \neq \emptyset$. In order to apply Kakutani's fixed-point theorem, we must verify that the following properties hold:

(i) There exists a compact subset U of $L^2(Q)$, such that, for every $z \in L^2(Q)$, $\Lambda(z) \subset U$.

(ii) For every $z \in L^2(Q)$, $\Lambda(z)$ is a convex, compact, and nonempty subset of

The proof that these properties hold is as follows:

(i) From Proposition 8 we know that there exists a bounded subset U of $H^{1/2,1}(Q)$ such that, for every $z \in L^2(Q)$, $\Lambda(z) \subset U$. Now, to see that we can choose U compact we prove that the set

$$\mathcal{Y} = \{y \text{ satisfying (3.19) for some } z \in L^2(Q),\ b \in \beta(z), \text{ and } u \text{ verifying (3.23)}\}$$

is a relatively compact subset of $L^2(Q)$. However, this is easy to prove using Proposition 8 and the fact that

$$H^{1/2,1}(Q) \subset L^q([0,T]; L^2(\Omega)) \qquad \text{with compact embedding} \quad \forall q < \infty \tag{3.24}$$

(see Lemma 5, p. 78, and Theorem 3, p. 80, of [22]).

(ii) We have already seen that, for every $z \in L^2(Q)$, $\Lambda(z)$ is a nonempty subset of $L^2(Q)$. Further, $\Lambda(z)$ is obviously convex because $B(y_d, \varepsilon)$, $\beta(z)$, and $\{u \in L^\infty(\Sigma) : u$ satisfies (3.23)$\}$ are convex sets. Then we have to see that $\Lambda(z)$ is a compact subset of $L^2(Q)$. In (i) we have proved that $\Lambda(z) \subset U$ with U compact. Let $(y^n)_n$ be a sequence of elements of $\Lambda(z)$ which converges in $L^2(Q)$ to $y \in U$. We have to prove that $y \in \Lambda(z)$. We know that there exist $b^n \in \beta(z)$ and $u^n \in L^\infty(\Sigma)$ satisfying (3.23) such that

$$\begin{cases} y_t^n - \Delta y^n + g(z)y^n = -f(s_0) + g(z)s_0 - ab^n + h & \text{in } Q, \\ \dfrac{\partial y^n}{\partial \nu} = u^n \chi_{\mathcal{O}} & \text{on } \Sigma, \\ y^n(0) = y_0 & \text{on } \Omega, \\ |y^n(T) - y_d|_2 \le \varepsilon. \end{cases} \tag{3.25}$$

Now, using the facts that β is a bounded maximal monotone graph and that the controls u^n are uniformly bounded, we deduce that $u^n \rightharpoonup u$ and $b^n \rightharpoonup b$ in the weak topology of $L^2(\Sigma)$ and of $L^2(Q)$, respectively. Moreover, u satisfies (3.23) and since any maximal monotone graph is strongly–weakly closed (see Proposition 3.5, p. 75, of [3]) over any Banach space with uniformly convex dual (as, for instance, $L^2(Q)$) we obtain that $b \in \beta(z)$. Therefore, if we pass to the limit in (3.25) we obtain

$$\begin{cases} y_t - \Delta y + g(z)y = -f(s_0) + g(z)s_0 - ab + h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} = u \chi_{\mathcal{O}} & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega. \end{cases}$$

Further, we shall see that $y^n(T)$ converges to $y(T)$ in $L^2(\Omega)$. Let $\omega^n = y - y^n$ be the solution of

$$\begin{cases} \omega_t^n - \Delta \omega^n + g(z)\omega^n = -a(b - b^n) & \text{in } Q, \\ \dfrac{\partial \omega^n}{\partial \nu} \ldots \end{cases}$$

Then, if we choose $\gamma^n \in H^{1/4,1/2}(\Sigma)$ such that $\|\gamma^n - (u - u^n)\chi_{\mathcal{O}}\|_{L^2(\Sigma)} \le 1/n$, then $\gamma^n \to 0$ in the weak topology of $L^2(\Sigma)$ and the solution $\bar{\omega}^n$ of

$$\begin{cases} \bar{\omega}^n_t - \Delta\bar{\omega}^n + g(z)\bar{\omega}^n = A(b - b^n) & \text{in } Q, \\ \dfrac{\partial \bar{\omega}^n}{\partial \nu} = \gamma^n & \text{on } \Sigma, \\ \bar{\omega}^n(0) = 0 & \text{on } \Omega \end{cases}$$

satisfies $\bar{\omega}^n \in H^{1,2}(Q)$ (see [19]). Therefore $\bar{\omega}^n$ is a strong solution and if we "multiply" by $\bar{\omega}^n$ and integrate, we obtain that

$$\|\bar{\omega}^n(T)\|^2_{L^2(\Omega)} \le k_1 \int_Q |(b - b^n)\bar{\omega}^n|\, dx\, dt + k_2 \int_\Sigma |\gamma^n \bar{\omega}^n|\, dx\, dt \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus, $\bar{\omega}^n(T)$ converges to 0 in $L^2(\Omega)$ and by again using regularity results (see [19]) we see that

$$\|\bar{\omega}^n - \omega^n\|_{H^{3/4,3/2}(Q)} \le k\|\gamma^n - (u - u^n)\chi_{\mathcal{O}}\|_{L^2(\Sigma)}.$$

Finally, as $H^{3/4,3/2}(Q) \subset \mathcal{C}([0,T]; L^2(\Omega))$ is a continuous injection (even a compact embedding; see Theorem 3 of [22]) we have

$$\|\bar{\omega}^n(T) - \omega^n(T)\|_{L^2(\Omega)} \to 0 \qquad \text{as} \quad n \to \infty.$$

Thus $\omega^n(T) \to 0$ in $L^2(\Omega)$, which implies that $|y(T) - y_d|_2 \le \varepsilon$. This proves that $y \in \Lambda(z)$ and concludes the proof of (ii).

(iii) We must prove that, for each $z_0 \in L^2(Q)$,

$$\limsup_{z_n \to z_0} \sigma(\Lambda(z_n), k) \le \sigma(\Lambda(z_0), k), \qquad \forall k \in L^2(Q).$$

We have seen in (ii) that $\Lambda(z)$ is a compact set, which implies that for every $n \in \mathbb{N}$ there exists $y^n \in \Lambda(z_n)$ such that

$$\sigma(\Lambda(z_n), k) = \int_Q k(t,x) y^n(t,x)\, dx\, dt.$$

Now by (i) $(y^n)_n \subset U$, a compact set. Thus, there exists $y \in L^2(Q)$ such that (after extracting a subsequence) $y^n \to y$ on $L^2(Q)$. We prove that $y \in \Lambda(z_0)$. We know that there exist $b^n \in \beta(z_n)$ and $u^n \in L^\infty(\Sigma)$ satisfying (3.23) such that

$$\begin{cases} y^n_t - \Delta y^n + g(z_n)y^n = -f(s_0) + g(z_n) - ab^n + h & \text{in } Q, \\ \dfrac{\partial y^n}{\partial \nu} = u^n \chi_{\mathcal{O}} & \text{on } \Sigma, \end{cases} \tag{3.26}$$

Hence there exists $u \in L^\infty(\Sigma)$ satisfying (3.23) such that $u^n \to u$ in the weak-$*$ topology of $L^\infty(\Sigma)$. Moreover, by again using that β is a bounded strongly–weakly closed graph and that the heat equation has a smoothing effect (as in the proof of (ii)), we deduce that y satisfies (3.19) and (3.20) with $z = z_0$ for some $u \in L^\infty(\Sigma)$ satisfying (3.23) and some $b \in \beta(z_0)$, which implies that $y \in \Lambda(z_0)$. Then, for every $k \in L^2(Q)$,

$$\begin{aligned}\sigma(\Lambda(z_n), k) &= \int_Q k(t,x) y^n(t,x)\,dx\,dt \to \int_Q k(t,x) y(t,x)\,dx\,dt \\ &\le \sup_{\bar{y}\in\Lambda(z_0)} \int_Q k(t,x)\bar{y}(t,x)\,dx\,dt = \sigma(\Lambda(z_0), k),\end{aligned}$$

which proves that Λ is upper hemicontinuous and concludes the proof of (iii). Finally, the restriction of Λ to $K = \mathrm{conv}(U)$ (the convex envelope of U), which is a compact set in $L^2(Q)$, satisfies the assumptions of Kakutani's fixed-point theorem. Thus, Λ has a fixed point $y \in K$. Furthermore, by construction, there exists a control $u \in L^\infty(\Sigma)$ satisfying (3.23) such that

$$\begin{cases} y_t - \Delta y + f(y) + a(x,t)\beta(y) \ni h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} = u\chi_O & \text{on } \Sigma, \\ y(0) = y_0 & \text{on } \Omega, \\ |y(T) - y_d|_2 \le \varepsilon. \end{cases} \tag{3.27}$$

Therefore, y is the solution that we were looking for. □

3.2. *Boundary Observation*

Let Ω be a bounded smooth subset of $\mathbb{R}^n$, let $Q = \Omega \times (0,T)$, let Σ_1 be a subset of $\Sigma = \partial\Omega \times (0,T)$ such that $\Sigma_2 = \Sigma \backslash \Sigma_1$ has nonempty interior set, let $a(\cdot,\cdot) \in L^\infty(Q)$, let f be a real function, let $h \in L^2(Q)$, let $y_0 \in L^2(\Omega)$, and let β be a bounded maximal monotone graph of $\mathbb{R}^2$ such that $D(\beta) = \mathbb{R}$. We study "the approximate controllability" with observation on Σ_1 of the problem

$$(\mathcal{P}) \quad \begin{cases} y_t - \Delta y + a(x,t)\beta(y) \ni h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Sigma_2, \\ y(x,0) = y_0(x) & \text{on } \Omega. \end{cases}$$

Here we define the weak solutions of $(\mathcal{P})$ in a way similar to that in which we defined weak solutions in Section 3.1.

Theorem 25. *If f is a nondecreasing continuous function satisfying*

$$|f(r)| \le C(1 + |r|) \tag{3.28}$$

Proof. Given $y_d \in L^2(\Sigma_1)$ and $\varepsilon > 0$, for $0 < \alpha < \frac{1}{2}$ we take

$$\mathcal{U} = \{z \in H^{1/2-\alpha/2,1-\alpha}(Q) : \|z_{|\Sigma_1} - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon\}.$$

We point out that, since the trace operator from $H^{1/2-\alpha/2,1-\alpha}(Q)$ to $L^2(\Sigma)$ is continuous (see, for instance, Section 2.2 of Chapter 4 of [19]), it is easy to prove that $\mathcal{U}$ is a closed set with the topology of $H^{1/2-\alpha/2,1-\alpha}(Q)$ and, so, it is a Banach space. We define the multivalued mapping

$$\mathcal{F}: \mathcal{U} \to \mathcal{P}(\mathcal{U})$$

by

$$\mathcal{F}(z) = \{y_z^b(v) \text{ for some } b \in \beta(z),\ \|v\|_{L^2(\Sigma_2)} \leq R,\ \|y_z^b(v)_{|\Sigma_1} - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon\},$$

where $y_z^b(v)$ is the solution of the associated problem

$$\begin{cases} y_t - \Delta y + a(x,t)b = h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(z_{|\Sigma_1}) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Sigma_2, \\ y(x,0) = y_0(x) & \text{on } \Omega. \end{cases} \tag{3.29}$$

By regularity results (see [19]), as $f(z|_{\Sigma_1}) \in L^2(\Sigma)$ (by (3.28)), we obtain that $y_z^b(v) \in H^{1/2,1}(Q)$ for every $z \in H^{1/2-\alpha/2,1-\alpha}(Q)$, $b \in \beta(z)$, and $v \in L^2(\Sigma_2)$. Moreover,

$$\|y_z^b(v)\|_{H^{1/2,1}(Q)} \leq C(1 + \|h\|_{L^2(Q)} + \|v\|_{L^2(\Sigma_2)} + \|y_0\|_{L^2(\Omega)}) \tag{3.30}$$

(see Proposition 8). To prove Theorem 25 we use Kakutani's fixed-point theorem. Next, we prove that the hypotheses of that theorem are satisfied.

From (3.30) we know that there exists a bounded subset $\mathcal{V}$ of $\mathcal{U} \cap H^{1/2,1}(Q)$ such that, for each $z \in \mathcal{U}$, $\mathcal{F}(z) \subset \mathcal{V}$. Then, using the facts that $H^{1/2,1}(Q) \subset H^{1/2-\alpha/2,1-\alpha}(Q)$ is a compact embedding and $\mathcal{U}$ is a closed set, we can choose $\mathcal{V}$ to be a compact set of $\mathcal{U}$. Also, it is easy to prove that $\mathcal{F}(z)$ is a convex set for all $z \in \mathcal{U}$. To see that $\mathcal{F}(z)$ is a compact set for every $z \in \mathcal{U}$ we suppose that $(y^n)_n$ is a sequence of elements of $\mathcal{F}(z)$ which converges in $H^{1/2-\alpha/2,1-\alpha}(Q)$ to $y \in \mathcal{V}$. We have to prove that $y \in \mathcal{F}(z)$. We know that there exist $b^n \in \beta(z)$ and $u^n \in L^2(\Sigma_2)$ satisfying $\|u^n\|_{L^2(\Sigma_2)} \leq R$, such that

$$\begin{cases} y_t^n - \Delta y^n + a(x,t)b^n = h & \text{in } Q, \\ \dfrac{\partial y^n}{\partial \nu} + f(z_{|\Sigma_1}) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y^n}{\partial \nu} = u^n & \text{on } \Sigma_2, \\ y^n(0) = y_0 & \text{on } \Omega, \\ \|y^n - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon. \end{cases} \tag{3.31}$$

of $L^2(\Sigma_2)$ and of $L^2(Q)$, respectively. Also u satisfies $\|u\|_{L^2(\Sigma_2)} \leq R$ and since any maximal monotone graph is strongly–weakly closed (see, e.g., Proposition 3.5, p. 75, of [3]) over any Banach space with uniformly convex dual (as, for instance, $L^2(Q)$) we obtain that $b \in \beta(z)$. Therefore, if we pass to the limit in (3.31) (taking into account that $f(y^n) \to f(y)$ in $L^2(\Sigma)$), we obtain

$$\begin{cases} y_t - \Delta y + a(x,t)b = h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(z_{|\Sigma_1}) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial \nu} = u & \text{on } \Sigma_2, \\ y(0) = y_0 & \text{on } \Omega. \end{cases}$$

Further, $\|y^n - y\|_{L^2(\Sigma_1)} \leq \|y^n - y\|_{H^{1/2-\alpha/2,1-\alpha}(Q)} \to 0$ and so $\|y - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon$. To show that $\mathcal{F}(z)$ is not the empty set we use the following lemma:

Lemma 26. *We consider the linear problem*

$$(\mathcal{LP}) \quad \begin{cases} Y_t - \Delta Y = F & \textit{in } Q, \\ \dfrac{\partial Y}{\partial \nu} = G & \textit{on } \Sigma_1, \\ \dfrac{\partial Y}{\partial \nu} = v & \textit{on } \Sigma_2, \\ Y(x,0) = y_0(x) & \textit{on } \Omega, \\ \|v\|_{L^2(\Sigma_2)} \leq R, \end{cases}$$

where F and G lie in bounded sets B of $L^2(Q)$ and E of $L^2(\Sigma_1)$, respectively. If $v_{F,G} \in L^2(\Sigma_2)$ is the optimal control of this problem relative to the functional

$$J(v) = \int_{\Sigma_1} |Y(v) - y_d|^2 \, d\Sigma$$

($v_{F,G}$ exists by the compactness result pointed out earlier), then for every $\varepsilon > 0$ we can choose R large enough to obtain

$$|Y(F, G, v_{F,G}) - y_d|_{L^2(\Sigma_1)} \leq \varepsilon \qquad \textit{for all} \quad F \in B \quad \textit{and} \quad G \in E.$$

Proof. By linearity we can assume $y_0 \equiv 0$. We take $\gamma > 0$ small enough. Then, if $F \in H^{-(1/4+\gamma),-2(1/4+\gamma)}(Q)$ and $G \in H^{-\gamma,-2\gamma}(\Sigma_1)$, we obtain (see, for instance, [19]) that the solution $Y(F, G)$ of $(\mathcal{LP})$ belongs to $H^{3/4-\gamma,2(3/4-\gamma)}(Q)$.

Let P_R: $H^{-(1/4+\gamma),-2(1/4+\gamma)}(Q) \times H^{-\gamma,-2\gamma}(\Sigma_1) \to L^2(\Sigma_1)$ be the mapping defined by

The optimality condition for $v_{F,G}$ is

$$\int_{\Sigma_1} (Y(F, G, v_{F,G}) - y_d) Y(0, 0, v - v_{F,G})\, d\Sigma \geq 0,$$
$$\forall v \in L^2(\Sigma_2), \quad \|v\|_{L^2(\Sigma_2)} \leq R.$$

Then, by adding the inequalities

$$\int_{\Sigma_1} (Y(F_1, G_1, v_{F_1,G_1}) - y_d) Y(0, 0, v_{F_1,G_1} - v_{F_2,G_2})\, d\Sigma \leq 0$$

and

$$\int_{\Sigma_1} (Y(F_2, G_2, v_{F_2,G_2}) - y_d) Y(0, 0, v_{F_2,G_2} - v_{F_1,G_1})\, d\Sigma \leq 0,$$

we obtain

$$\int_{\Sigma_1} \left[Y(F_1, G_1, v_{F_1,G_1}) - Y(F_2, G_2, v_{F_2,G_2})\right] Y(0, 0, v_{F_1,G_1} - v_{F_2,G_2})\, d\Sigma$$
$$\leq 0. \tag{3.32}$$

Now, as

$$Y(F_1, G_1, v_{F_1,G_1}) - Y(F_2, G_2, v_{F_2,G_2})$$
$$= Y(F_1 - F_2, G_1 - G_2, 0) + Y(0, 0, v_{F_1,G_1} - v_{F_2,G_2}),$$

using (3.32) we see that

$$\|Y(F_1, G_1, v_{F_1,G_1}) - Y(F_2, G_2, v_{F_2,G_2})\|^2_{L^2(\Sigma_1)}$$
$$\leq \int_{\Sigma_1} [Y(F_1, G_1, v_{F_1,G_1}) - Y(F_2, G_2, v_{F_2,G_2})] Y(F_1 - F_2, G_1 - G_2, 0)\, d\Sigma.$$

By the Hölder and Young inequalities we conclude that

$$\|Y(F_1, G_1, v_{F_1,G_1}) - Y(F_2, G_2, v_{F_2,G_2})\|^2_{L^2(\Sigma_1)}$$
$$\leq \|Y(F_1 - F_2, G_1 - G_2, 0)\|^2_{L^2(\Sigma_1)}$$
$$\leq \|Y(\cdot; F_1 - F_2, G_1 - G_2, 0)\|^2_{H^{3/4-\gamma, 2(3/4-\gamma)}(Q)}$$
$$\leq C(\|F_1 - F_2\|^2_{H^{-(1/4+\gamma), -2(1/4+\gamma)}(Q)} + \|G_1 - G_2\|^2_{H^{-\gamma, -2\gamma}(\Sigma_1)}),$$

where the constant C is independent of R. Therefore, P_R is equicontinuous and, by Ascoli's theorem, $(P_R)_{R>0}$ converges uniformly over the compact sets of $H^{-(1/4+\gamma), -2(1/4+\gamma)}(Q) \times H^{-\gamma, -2\gamma}(\Sigma_1)$. Thus, as

$$L^2(Q) \times L^2(\Sigma_1) \subset H^{-(1/4+\gamma), -2(1/4+\gamma)}(Q) \times H^{-\gamma, -2\gamma}(\Sigma_1)$$

with compact embedding, the result is obtained by using the fact that the approximate controllability of problem $(\mathcal{LP})$ implies

End of the Proof of Theorem 25. Applying (3.28) and the previous lemma we can choose R such that $\mathcal{F}(z)$ is a nonempty set, for all $z \in \mathcal{U}$. Finally, in order to apply Kakutani's fixed-point theorem, we have to prove that $\mathcal{F}$ is upper hemicontinuous. Therefore, we prove that, for every $z_0 \in H^{1/2-\alpha/2,1-\alpha}(Q)$,

$$\limsup_{z_n \to z_0} \sigma(\mathcal{F}(z_n), k) \leq \sigma(\mathcal{F}(z_0), k), \qquad \forall k \in (H^{1/2-\alpha/2,1-\alpha}(Q))'.$$

Since we have already seen that $\mathcal{F}(z)$ is a compact set, we know that for each $n \in \mathbb{N}$ there exists $y_n \in \mathcal{F}(z_n)$ such that

$$\sigma(\mathcal{F}(z_n), k) = \langle k, y_n \rangle.$$

Now, by using that $(y_n)_n \subset \mathcal{V}$, we know that there exists $y \in H^{1/2-\alpha/2,1-\alpha}(Q)$ such that (after extracting a subsequence) $y_n \to y$ in the topology of $H^{1/2-\alpha/2,1-\alpha}(Q)$. We show that $y \in \mathcal{F}(z_0)$.

We know that there exists $b^n \in \beta(z_n)$ and $v^n \in L^2(\Sigma_2)$ satisfying

$$\|v\|_{L^2(\Sigma_2)} \leq R, \tag{3.33}$$

such that

$$\begin{cases} y_t^n - \Delta y^n + a(x,t)b^n = h & \text{in } Q, \\ \dfrac{\partial y^n}{\partial \nu} + f(z_n) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y^n}{\partial \nu} = v^n & \text{on } \Sigma_2, \\ y^n(x,0) = y_0(x) & \text{on } \Omega, \\ \|y^n - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon. \end{cases}$$

Thus there exists $v \in L^2(\Sigma_2)$ satisfying (3.33) such that $v^n \to v$ in the weak topology of $L^2(\Sigma_2)$. Moreover, by using that β is a bounded strongly–weakly closed graph and that the heat equation has a smoothing effect, we deduce (as before) that y satisfies (3.29) and

$$\|y^n - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon,$$

with $z = z_0$ for some $v \in L^2(\Sigma_2)$ satisfying (3.33) and some $b \in \beta(z_0)$, which implies that $y \in \mathcal{F}(z_0)$. Then, for every $k \in (H^{1/2-\alpha/2,1-\alpha}(Q))'$, we obtain

$$\sigma(\mathcal{F}(z_n), k) = \langle k, y^n \rangle \to \langle k, y \rangle \leq \sup_{\bar{y} \in \mathcal{F}(z_0)} \langle k, \bar{y} \rangle = \sigma(\mathcal{F}(z_0), k),$$

which proves that $\mathcal{F}$ is upper hemicontinuous. Finally, the restriction of $\mathcal{F}$ to $K =$

Further, by construction, there exists a control $v \in L^2(\Sigma_2)$ satisfying (3.33) such that

$$\begin{cases} y_t - \Delta y + a(x,t)\beta(y) \ni h & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu} + f(y) = 0 & \text{on } \Sigma_1, \\ \dfrac{\partial y}{\partial \nu} = v & \text{on } \Sigma_2, \\ y(x,0) = y_0(x) & \text{on } \Omega, \\ \|y - y_d\|_{L^2(\Sigma_1)} \leq \varepsilon. \end{cases}$$

Therefore, y is the solution that we were looking for. □

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