Mathematical treatment of the magnetic confinement in a current carrying stellarator

J.I. Díaz[a,1,2,*], J.F. Padial[2,b], J.M. Rakotoson[c]

Dpt. de Matemática Aplicada, Fac. de Matemáticas, Univ. Complutense de Madrid, Avd. Complutense, s/n, 28040 Madrid, Spain

[b]*Dpt. de Matemática Aplicada, E.T.S. Arquitectura, Univ. Politécnica de Madrid, Avd. Juan de Herrera, 4, 28040 Madrid, Spain*

[c]*Dpt. de Mathematiques, Fac. de Mathematiques, Univ. de Poitiers, 40, Avn. du Recteur Pineau, 86022 Poitiers, France*

. Introduction

The magnetic confinement in a Stellarator can be modeled with help of averaging methods and Boozer vacuum coordinates (see [1, 2]). This leads to a two-dimensional *Grad–Shafranov* type problem for the averaged poloidal flux function. So, we assume that $(\rho, \rho\theta, \phi)$ are the Boozer vacuum coordinates system: i.e. $\rho = \rho(x, y, z)$ is a function which is constant on each nested toroidal and positive except for the magnetic axis where $\rho = 0$; $\theta = \theta(x, y, z)$ is the poloidal angle (i.e. θ is constant on any toroidal loop) and $\phi = \phi(x, y, z)$ is the toroidal angle (i.e. constant on any poloidal circuit). By averaging in ϕ and adding a free boundary formulation to the *Grad–Shafranov* type equation, the problem can be stated in the following terms (see [3]): Let $\Omega = \{(\rho, \theta) : 0 < \rho < R, \theta \in]0, 2\pi[\}$ and define $\partial\Omega = \Gamma_R \cup \Gamma_p \cup \Gamma_0$ by means of $\Gamma_R = \{(\rho, \theta) : \theta \in]0, 2\pi[\}, \Gamma_p = \{(\rho, 0) \text{ or } (\rho, 2\pi) : \rho \in]0, R[\}$ and $\Gamma_0 = \{(0, \theta) : \theta \in]0, 2\pi[\}$. Given $\lambda > 0, F_v > 0, a, b \in L^\infty(\Omega)$ with $\inf_\Omega |a| > 0, b \geq 0$ and $\gamma \in \mathbb{R}$, find

$$u : \Omega \to \mathbb{R} \quad \text{and} \quad F : \mathbb{R} \to \mathbb{R}_+$$

such that $u \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega)$, $p \geq 1$ and $F \in W^{1,\infty}(\inf u, \sup u), F(s) = F_v$ for al $s \leq 0$ and (u, F) satisfy

$$-\mathscr{L}u = a(\rho, \theta)F(u) + F(u)F'(u) + \lambda b(\rho, \theta)p'(u) \quad \text{in } \Omega \tag{1.1}$$

$$u|_{\Gamma_R} = \gamma; \quad u(\rho, 0) = u(\rho, 2\pi) \quad \text{for } \rho \in \,]0, R[. \tag{1.2}$$

Here $-\mathscr{L}$ is a suitable second order elliptic operator (see [4]). Coefficients a anc b are explicitly known in terms of the components of the metric associated to the Boozer coordinate system. The unknown $u = u(\rho, \theta)$ represents the averaged poloida flux, $F(u)$ is the covariant toroidal coordinate of the magnetic field and the constan F_v is the covariant toroidal coordinate of the magnetic field in the vacuum region Function $p(u(x))$ represents the pressure and it cannot be obtained explicitly from the magnetohydrodynamic system. Thus, p is given as a constitutive law. Usually $p(s) = \frac{\lambda}{2}(s_+)^2$ where $s_+ = \max\{s, 0\}$, $s \in \mathbb{R}$ and $\lambda > 0$. The plasma region is defined as $\{(\rho, \theta) \in \Omega : u(\rho, \theta) > 0\} \equiv \Omega^p$ and the vacuum region as $\{(\rho, \theta) \in \Omega : u(\rho, \theta) < 0\} \equiv \Omega^v$ Thus, a free boundary arises as the boundary of the set $\{(\rho, \theta) \in \Omega : u(\rho, \theta) = 0\}$ (see [3, 4]). Notice that if $(\rho, \theta) \in \Omega^v$ then $F(u) \in F_v$.

It is necessary to add another condition typical of any ideal Stellarator to con ditions (1.1) and (1.2). It expresses the zero net current within each flux magnetic surface and can be written as

$$\int_{\{u>t\}} [F(u)F'(u) + \lambda b(\rho, \theta)p'(u)]\rho \, d\rho \, d\theta = 0 \quad \forall t \in \left[\inf_\Omega u, \sup_\Omega u\right]. \tag{1.3}$$

Problem (1.1), (1.2), (1.3) has been treated in [3]. In practice, this condition doe: not hold and some known current arises at the interior of each magnetic surface Some studies on the stability of the equilibrium configuration for the fixed boundary formulation are already in the literature (Cooper et al. [5]). In those works the curren is assumed to have the form

$$J(s) = J(1)(4s^2 - 3s^4) \tag{1.4}$$

within the flux magnetic surface corresponding to the parameter s (i.e. $\{(\rho, \theta) \in \Omega : \iota$ $(\rho, \theta) = s\}$) and where $2\pi J(1)$ is the total current (toroidal). In [5] it is also assumec that the constitutive law for the real pressure function P is

$$P(s) = P(0)(1 - 3s^2 + 2s^3). \tag{1.5}$$

Notice that in such a formulation it is assumed that the magnetic axis corresponds to $s = 0$ and that $s = 1$ corresponds to the free boundary.

We point out that the presence of a *positive total current* is a typical phenomenon of Tokamak devices (see e.g. Temam [6, 7], Berestycki and Brezis [8], Blum [9] Friedman [10], Mossino and Temam [11] and their references). In such a case, this global condition is formulated as

$$\int_\Omega [F(u)F'(u) + \lambda b(\rho, \theta)p'(u)]\rho \, d\rho \, d\theta = I$$

here $I > 0$ is the total current. For an exposition of the important differences between ie mathematical formulations of the magnetic confinement in Tokamak and Stellarator evices see the paper [3] (see also Remark 1).

The main purpose of this paper is to extend the results of [3] to the study of the iove type of Stellarators (the so-called *current carrying Stellarators*). To explain how place (1.3) by a new condition, we need to translate assumptions (1.4) and (1.5) in rms of the formulation used in [3], where the boundary of the plasma region were as-med corresponding to the level $u = 0$ and the magnetic axis to $\max_\Omega u = \|u_+\|_{L^\infty(\Omega)}$.), we define the following change of variable involving $\|u_+\|_{L^\infty(\Omega)}$ with u verify-g (1.1) and (1.2): Let

$$s := \left(1 - \frac{t_+}{\|u_+\|_{L^\infty(\Omega)}}\right) \quad \forall t \in \left[\inf_\Omega u, \sup_\Omega u\right]. \tag{1.6}$$

hus,

$$\{s = 0\} \equiv \{u_+ = \|u_+\|_{L^\infty(\Omega)}\} \quad (\equiv \text{the magnetic axis}),$$
$$\{s = 1\} \equiv \partial\{u_+ = 0\} \quad (\equiv \text{the free boundary}).$$

irthermore, if we fix a current $J(s)$ (as, for instance (1.4)), by the change of variable ven in (1.6) we obtain a new expression

$$J(s) = j(t_+, \|u_+\|_{L^\infty(\Omega)})$$

terms of the new variable $t \in [\inf_\Omega u, \sup_\Omega u]$. Analogously, we will have $P(s) = p(t,$ $t_+\|_{L^\infty(\Omega)})$. Here, and in what follows, we shall assume that

$$\left.\begin{array}{l} j \in \mathscr{C}(\mathbb{R} \times \mathbb{R}^+), \quad j(\sigma, \sigma) = 0 \quad \text{for all } \sigma \geq 0, \\ j'_t \in \mathscr{C}(\mathbb{R}^+ \times \mathbb{R}^+) \quad \text{and} \quad \eta := \sup\{|j'_t(t,\sigma)| : (t,\sigma) \in \mathbb{R}^+ \times \mathbb{R}^+\} < +\infty. \end{array}\right\} \tag{1.7}$$

e shall always use j'_t to denote the derivative of j with respect to the first component e. $j'_t := \frac{\partial j}{\partial t}$). The assumptions on p are the following: $p(t, \sigma) \equiv p(t)$ with $p \in \mathscr{C}^1(\mathbb{R})$ ch that

$$p(0) = 0, \quad 0 \leq p'(t) \leq \lambda t_+ \quad \text{and} \quad |p'(t) - p'(s)| \leq L|t - s|^\alpha \tag{1.8}$$

r some $\lambda > 0, L > 0$ and $\alpha \in]0, 1[$. Notice that $p(t) = \frac{\lambda}{2}(t_+)^2$ satisfies all the require-ents. Finally, the new condition for current carrying Stellarator can be stated as

$$\int_{\{u > t\}} [F(u)F'(u) + p'(u)b(\rho, \theta)]\rho \, d\rho \, d\theta = j(t_+, \|u_+\|_{L^\infty(\Omega)}), \quad t \in \left[\inf_\Omega u, \sup_\Omega u\right]. \tag{1.9}$$

ir the sake of the exposition we shall assume that $\mathscr{L} = \Delta$ (the Laplace operator) d we replace (ρ, θ) by the associated cartesian coordinate $x \in \Omega \subset \mathbb{R}^2$. Say the main rpose of this paper is to prove the existence of a couple (u, F), solution of the llowing problem $(\mathscr{P})$

$$-\Delta u = aF(u) + F(u)F'(u) + bp'(u) \quad \text{in } \Omega, \tag{1.10}$$

$$u - \gamma \in H_0^1(\Omega), \tag{1.11}$$

$$\int_{\{u>t\}} [F(u)F'(u) + bp'(u)]dx = j(t_+ \|u_+\|_{L^\infty(\Omega)}) \quad t \in \left[\inf_\Omega u, \sup_\Omega u \right]. \tag{1.12}$$

Before stating our main result, we introduce the following useful convex cone:

$$V(\Omega) = \{v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \le 0\}.$$

Theorem 1. *Suppose that* $\inf_\Omega |a| > 0$ *and* $\gamma \le 0$. *Then there exists* $\Lambda > 0$ *such that if*

$$\lambda \|b\|_{L^\infty(\Omega)} + \eta < \Lambda$$

there is a couple $(u, F), u \in V(\Omega)$ *and* $F \in W^{1,\infty}(]\inf u, \sup u[)$ *solution of* (1.10) (1.11) *and* (1.12) *satisfying also that* $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$ *and that* F *is entirely determined by* u.

Inspired in [12] and [3] we will reformulate problem ($\mathscr{P}$) in terms of a new problem ($\mathscr{P}_*$), of nonlocal nature, where we eliminate the unknown F by a term involving the function u, its decreasing rearrangement and the relative rearrangement of b with respect of u. However, the lack of regularity of the derivative of the decreasing rearrangement makes it difficult to solve directly problem ($\mathscr{P}_*$). A family of problems ($\mathscr{P}_{*\varepsilon}$) are then introduced. Using a Galerkin method (of interest also for the numerical approach) we will find a solution of ($\mathscr{P}_{*\varepsilon}$). Finally, thanks to a result on the regularity of the derivative of the decreasing rearrangement (Lemma 4) we shall obtain a solution of ($\mathscr{P}_*$) by making $\varepsilon \to 0$. The equivalence of the problems ($\mathscr{P}_*$) and ($\mathscr{P}$) (under a suitable condition) proves that this solution is also a solution of ($\mathscr{P}$). Finally, in Section 6 we give some qualitative properties on the solution.

2. Preliminary results

In this section we recall the notion of relative rearrangement and some useful properties of it. Let Ω be a bounded measurable set of $\mathbb{R}^N, N \ge 1$. For any measurable subset E of Ω, we denote by $|E|$ its Lebesgue measure. Given a measurable function $u : \Omega \to \mathbb{R}$ and any value $t \in \mathbb{R}$, we denote by $\{u = t\}, \{u > t\}$ and $\{u \ge t\}$ the sets $\{x \in \Omega : u(x) = t\}$ $\{x \in \Omega : u(x) > t\}$ and $\{x \in \Omega : u(x) \ge t\}$, respectively. Their measure will be indicated by $|u = t|, |u > t|$ and $|u \ge t|$, respectively. We will say that u *has a flat region at the level* t if $|u = t|$ is strictly positive. It may exist, at most, a countable family of flat regions $P_u(t_i) := \{u = t_i\}$. We denote by $P(u) = \bigcup_{i \in D} P_u(t_i)$ the union of all the flat regions of u.

Definition 1. Let $u : \Omega \to \mathbb{R}$ be the Lebesgue measurable function. The distribution function of u is defined by

$$m_u(t) := |u > t| \quad \text{for any } t \in \mathbb{R}.$$

he generalized inverse of m_u is called the *decreasing rearrangement* of u and is de-
oted by u_*. That is the function $u_* :]0, |\Omega|[\to \overline{\mathbb{R}}$ such that $u_*(s) = \inf \{t \in \mathbb{R} : |u > t| < s\}$
ith $u_*(0) = \text{ess sup}_\Omega u$ and $u_*(|\Omega|) = \text{ess inf}_\Omega u$.

emma 1. *Let* $(v_n)_{n \geq 1}$, $v \in L^p(\Omega)$ *such that* $\text{meas}\{x \in \Omega : v(x) = t\} = 0$ *for all* $t \in \mathbb{R}$
nd $v_n \to v$ *strongly in* $L^p(\Omega)$ *and a.e. in* Ω. *Then*

$$|v_n > v_{n+}(x)| \to |v > v_+(x)| \quad \text{a.e. } x \in \Omega.$$

roof. One has the following chain of inequalities

$$|v > v_+(x)| \leq \liminf_{n \to \infty} |v_n > v_{n+}(x)|$$
$$\leq \limsup_{n \to \infty} |v_n > v_{n+}(x)|$$
$$\leq |v \geq v_+(x)| \quad \text{a.e. } x \in \Omega.$$

ow, since by assumption v has no flat region, we have that $|v = v_+(x)| = 0$ and by
e above inequality we get the conclusion. □

We denote by Ω_* the interval $]0, |\Omega|[$. Later on we shall need to use some properties
˙ the decreasing rearrangement. In particular, we shall use the following classical
sults.

emma 2. *Let* u *and* v *be measurable functions in* Ω. *Then*
For all $s \in \bar{\Omega}_*$, *we have* $m_u(u_*(s)) \leq s$.
If u *has no flat regions, then* m_u *is continuous and* $m_u(u_*(s)) = s$ $\forall s \in \Omega_*$.
u and u_* are equimeasurable, i.e. $|u_* > t| = |u > t|$ $\forall t \in \mathbb{R}$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ *be a Borel function such that* $\varphi(u) \in L^1(\Omega)$, *then*

$$\int_\Omega \varphi(u(x)) \, dx = \int_{\Omega_*} \varphi(u_*(s)) \, ds;$$

If $u \leq v$ *almost everywhere in* Ω, *then* $u_*(s) \leq v_*(s)$ $\forall s \in \Omega_*$;
Let $\varphi : \mathbb{R} \to \mathbb{R}$ *be a nondecreasing function. Then* $\varphi(u_*(s)) = \varphi(u)_*(s)$ *almost ev-
erywhere* $s \in \Omega_*$;
The mapping $u \to u_*$ *applies* $L^p(\Omega)$ *into* $L^p(\Omega_*)$
    $(1 \leq p \leq +\infty)$ *and it is a contraction, i.e.* $\|u_* - v_*\|_{L^p(\Omega_*)} \leq \|u - v\|_{L^p(\Omega)}$
    $(1 \leq p \leq +\infty)$. *Moreover,* $\|u\|_{L^p(\Omega)} = \|u_*\|_{L^p(\Omega_*)}$. □

The proof of this lemma can be found, for instance, in [13–15].
We shall also need some regularity results on the first derivative of u_*. More pre-
:ely we have:

emma 3 ([16, 17]). *Let* Ω *be an open, bounded and connected regular set of* $\mathbb{R}^N$.
$u \in W^{1,p}(\Omega)$ *for some* $1 \leq p \leq +\infty$, *then:*
i) $u_* \in W^{1,p}_{loc}(\Omega_*)$,

(ii) *There exists a constant $Q = Q(\Omega)$ only dependent on Ω such that if we de fine $k(s) = \frac{1}{Q}\min\{s^{1-\frac{1}{N}}, (|\Omega| - s)^{1-\frac{1}{N}}\} \forall s \in \Omega_*$, then one has $\|k\frac{du_*}{ds}\|_{L^p(\Omega_*)} \leq \|\nabla u\|_{L^p(\Omega)}$ $1 \leq p \leq +\infty$.*

(iii) *Finally, if $N < p \leq +\infty$, then $u_* \in W^{1,q}(\Omega_*)$ with $1 \leq q < \frac{Np}{(N-1)p+N}$ and*

$$\left\|\frac{du_*}{ds}\right\|_{L^q(\Omega)_*} \leq C\|\nabla u\|_{L^p(\Omega)}$$

for some $C = C(N, p, q, \Omega)$. $\square$

Making $p \to +\infty$ in (iii) of Lemma 3, we obtain

Corollary 1. *Let Ω as above and $u \in W^{1,\infty}(\Omega)$. Then $u_* \in W^{1,q}(\Omega_*)$ for all $1 \leq q < \frac{N}{N-1}$ and $u_* \in \mathscr{C}(\bar{\Omega}_*)$.* $\square$

The following lemma shows some additional regularity on $\frac{du_{+*}}{ds}$ under suitable as sumptions on u when $N = 2$.

Lemma 4. *Let $\Omega \subset \mathbb{R}^2$. Then, for all $w \in V(\Omega)$ one has*

(i) $\left\|\frac{dw_{+*}}{ds}\right\|_{L^\infty(\Omega_*)} \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi}$,

(ii) $\|w_+\|_{L^\infty(\Omega)_*} \leq \frac{|\Omega|}{4\pi}\|\Delta w\|_{L^\infty(\Omega)}$,

(iii) $\left|\frac{d^+}{ds}w_{+*}(|w_+ > w_+(x)|)\right| \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi}$ a.e. $x \in \Omega$.

Proof. Let $w \in V(\Omega)$. Then

$$\int_\Omega \Delta w(w_+ - t)_+ dx = \int_{\{w_+ > t\}} \Delta w(w_+ - t) dx \quad \text{for all } t > 0 \tag{2.1}$$

since $(w_+ - t)_+ \in H_0^1(\Omega)$, integrating by parts, we have

$$\int_\Omega \Delta w(w_+ - t)_+ dx = -\int_{\{w_+ > t\}} |\nabla w_+|^2 dx. \tag{2.2}$$

By classical arguments (see, for instance [13]), we have

$$\frac{d}{dt}\int_\Omega \Delta w(w_+ - t)_+ dx = -\int_{\{w_+ > t\}} \Delta w dx. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), one has

$$-\frac{d}{dt}\int_{\{w_+ > t\}} |\nabla w_+|^2 dx = -\int_{\{w_+ > t\}} \Delta w dx \leq \|\Delta w\|_{L^\infty(\Omega)}|w_+ > t|. \tag{2.4}$$

Arguing as in Talenti [14] and using the De Giorgi isoperimetric inequality, we get from (2.4)

$$
4\pi|w_+ > t| \leq \left(-\frac{\mathrm{d}}{\mathrm{d}t}|w_+ > t|\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{w_+ > t\}}|\nabla w_+|^2\mathrm{d}x\right)
$$

$$
\leq \left(-\frac{\mathrm{d}}{\mathrm{d}t}|w_+ > t|\right)|w_+ > t|\,\|\Delta w\|_{L^\infty(\Omega)} \tag{2.5}
$$

or a.e. $t \in]\inf_\Omega w_+, \sup_\Omega w_+[$. Thus $4\pi \leq (-\frac{\mathrm{d}}{\mathrm{d}t}|w_+ > t|)\|\Delta w\|_{L^\infty(\Omega)}$. Now, by standard arguments (see [13, 14]) we obtain (i), i.e.

$$
-\frac{\mathrm{d}}{\mathrm{d}s}w_{+*}(s) \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \quad \text{a.e. in } \Omega_*. \tag{2.6}
$$

Since we know already that $w_{+*} \in H^1_{\mathrm{loc}}(\Omega_*)$, (2.6) infers that $w_{+*} \in W^{1,\infty}(\Omega_*)$ and an integration leads to $\|w_+\|_{L^\infty(\Omega)} = w_{+*}(0) \leq \frac{|\Omega|}{4\pi}\|\Delta w\|_{L^\infty(\Omega)}$ (since $w_{+*}(|\Omega|)=0$), i.e. (ii). Let us show (iii). From the equimeasurability, we obtain that

$$
\left[\int_{\Omega\setminus P(w_+)}\left|\frac{\mathrm{d}w_{+*}}{\mathrm{d}s}(|w_+ > w_+(x)|)\right|^p\mathrm{d}x\right]^{\frac{1}{p}} = \left[\int_{\Omega_*\setminus P(w_{+*})}\left|\frac{\mathrm{d}}{\mathrm{d}s}w_{+*}(s)\right|^p\mathrm{d}s\right]^{\frac{1}{p}} \tag{2.7}
$$

or all $p \in [1,+\infty[$. Thus, letting $p \to +\infty$, from (2.6), we deduce that

$$
-\frac{\mathrm{d}^+ w_{+*}}{\mathrm{d}s}(|w_+ > w_+(x)|) \leq \frac{\|\Delta w\|_{L^\infty(\Omega)}}{4\pi} \quad \text{a.e. in } \Omega\setminus P(w_+).
$$

Besides, as w_{+*} is right-continuous,

$$
-\frac{\mathrm{d}^+ w_{+*}}{\mathrm{d}s}(|w_+ > w_+(x)|) = 0 \quad \text{for } x \in P(w_+)
$$

and (iii) holds. $\square$

To pass to the limit in the iterative method that we shall use later we will need the strong convergence of the first derivatives $\frac{\mathrm{d}}{\mathrm{d}s}u_{n*}$. To do that, we will use the notion of coarea regular function (see [18]):

Definition 2. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. For $t \in \mathbb{R}$ we set $m_{u,0}(t) := |\{x \in \Omega : u(x) > t \text{ and } \nabla u(x) = 0\}|$ and $m_{u,1}(t) := m_u(t) - m_{u,0}(t)$. We will say that u is a coarea regular function if the Radon measure $(m_{u,0})'$ is singular with respect to the Lebesgue measure on $\mathbb{R}$.

Now, let us recall two conditions to get a coarea regular function obtained in [19] (see also [3])

Lemma 5. Let Ω be an open set of $\mathbb{R}^2$ and $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $p > 1$. Then u is coarea regular function. $\square$

More in general, if Ω is an arbitrary open set of $\mathbb{R}^N$, we have a simpler statement.

Lemma 6. *Let $u \in W^{1,1}_{loc}(\Omega)$. If meas$\{x \in \Omega : \nabla u(x) = 0\} = 0$ then u is a coarea regula function.*   □

The relation between the notion of coarea regular function and the convergence o the derivative sequence u_{n*} is obtained from an important result due to Almgrem an Lieb [18]:

Lemma 7. *Let u be a coarea regular function of $W^{1,p}(\Omega)$, $1 \le p \le +\infty$ and let Ω b a bounded regular open of $\mathbb{R}^N$. If u_n is a bounded sequence of $W^{1,p}(\Omega)$ converging t u in $W^{1,1}(\Omega)$, then $\frac{d}{ds} u_{n*}$ converges to $\frac{d}{ds} u_*$ a.e. in Ω_*. Furthermore, if $p > N$ $\frac{d}{ds} u_n$ converges to $\frac{d}{ds} u_*$ strongly in $L^q(\Omega_*)$ for any $1 \le q < \bar{q} := \frac{1}{1 + \frac{1}{p} - \frac{1}{N}}$.*   □

The proof of Lemma 7 has been presented in [19].

Now, we recall the notion of relative rearrangement of v with respect to u. Le $v \in L^1(\Omega)$, we define a function w in $\bar{\Omega}_*$ by:

$$
w(s) = \begin{cases} \int_{\{u > u_*(s)\}} v(x)\,dx & \text{if } |u = u_*(s)| = 0, \\ \int_{\{u > u_*(s)\}} v(x)\,dx + \int_0^{s - |u > u_*(s)|} (v_{|P_u(u_*(s))})_*(\sigma)\,d\sigma & \text{if } |u = u_*(s)| \neq 0. \end{cases}
$$

Here $v_{|P_u(u_*(s))}$ denotes the restriction of v to the set $P_u(u_*(s))$ and $(v_{|P_u(u_*(s))})_*$ it decreasing rearrangement. The following lemma was proved in [11, 20].

Lemma 8. *Let $u \in L^1(\Omega)$ and $v \in L^p(\Omega)$ for some $1 \le p \le +\infty$. Then $w \in W^{1,p}(\Omega_*$ and $\|\frac{dw}{ds}\|_{L^p(\Omega_*)} \le \|v\|_{L^p(\Omega)}$.*   □

Definition 3. The function $\frac{dw}{ds}$ is called the *relative rearrangement of v with respec to the u* and it is denoted by

$$
v_{*u} = \frac{dw}{ds}.
$$

This function has many properties as we stated below (see, for instanc [11, 15, 20, 21]):

Lemma 9. *Consider u, v_1, v_2 three elements of $L^1(\Omega)$. Then:*
1. *If $v_1 \le v_2$ a.e. in Ω, then $v_{1*u} \le v_{2*u}$ a.e. in Ω_*.*
2. *If $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel function such that $\varphi(u) \in L^1(\Omega)$, then $(v_1 + \varphi(u))_{*u} = v_{1*u}$ + $\varphi(u_*)$. In particular, if we take $\varphi \equiv k$ constant, then $(v_1 + k)_{*u} = v_{1*u} + k$.*
3. *The mapping $v \to v_{*u}$ applies $L^p(\Omega)$ into $L^p(\Omega_*)$ for any $1 \le p \le +\infty$ and it is ϵ contraction. In particular*

$$
\|v_{*u}\|_{L^p(\Omega_*)} \le \|v\|_{L^p(\Omega)} \quad \text{for any } v \in L^p(\Omega) \text{ and } u \in L^1(\Omega). \quad \square
$$

We shall be interested in expressing the function $[p'(u)b]_{*u}$ in terms of u_* and b_{*u} To do that, we need the notion of mean operator introduced in [11].

Definition 4. (*Second class of mean value operators*). Let $u, v \in L^1(\Omega)$, and $g \in L^1(\Omega_*)$. Let us define $\mathcal{M}_{u,v}(g)$ as the function

$$\mathcal{M}_{u,v}(g)(x) = \begin{cases} g(m_u(u(x))) & \text{if } x \in \Omega - P(u) \\ \frac{1}{|v_i = v_i(x)|} \int_{|v_i > v_i(x)|}^{|v_i \geq v_i(x)|} g(\sigma + |u > t_i|) \, d\sigma & \text{if } x \in P_u(t_i) \end{cases}$$

i.e. $x \in \Omega$ where $v_i = v_{|P_u(t_i)}$ the restriction of v to $P_u(t_i)$.

In particular, we will use the following lemmas (see [11, 19, 22]).

Lemma 10. *The operator $\mathcal{M}_{u,v}(g)$ is well defined and one has*
i) *if $g \in L^1(\Omega_*)$, then $\mathcal{M}_{u,v}(g) \in L^1(\Omega)$, besides*

$$\int_{\Omega_*} g(s) \, ds = \int_{\Omega} \mathcal{M}_{u,v}(g)(x) \, dx,$$

i) *$\mathcal{M}_{u,v}$ is a linear continuous map with norm one from $L^p(\Omega_*)$ to $L^p(\Omega)$ for any $1 \leq p \leq +\infty$.* $\square$

We need also the following result which gives a relation between the relative rearrangement and the mean value operators:

Lemma 11. *Let $u \in L^1(\Omega)$ and $v \in L^p(\Omega)$, $1 \leq p \leq +\infty$. For any $g \in L^{p'}(\Omega_*)$, $\frac{1}{p} + \frac{1}{?} = 1$, we have*

$$\int_{\Omega_*} v_{*u}(s)g(s) \, ds = \int_{\Omega} \mathcal{M}_{u,v}(g)(x)v(x) \, dx.$$

If $|P(u)| = 0$ the last equality is reduced to

$$\int_{\Omega_*} v_{*u}(s)g(s) \, ds = \int_{\Omega} g(m_u(u(x)))v(s) \, dx.$$ $\square$

Using the last two lemmas, we have

Lemma 12 ([3]). *Let $u \in L^1(\Omega)$ and such that $u_* \in \mathscr{C}(\Omega_*)$. Let $F_0: \mathbb{R} \to \mathbb{R}$ be a Borel function such that $F_0(u) \in L^1(\Omega)$. Then, if $b \in L^\infty(\Omega)$ we have*

$$[F_0(u)b]_{*u} = F_0(u_*)b_{*u}.$$ $\square$

Since we shall use later an approximate method, we shall need to know the behavior of the relative rearrangement of a fixed function b with respect to a sequence of functions u_n converging in $L^1(\Omega)$. In that sense we have

Lemma 13 ([3]). *Let u_n, u be in $L^1(\Omega)$ and assume that u_n converges to u in $L^1(\Omega)$. Then, for all $v \in L^p(\Omega)$ (for a given p, $1 < p \leq +\infty$) we have*

$$(v\chi_{\Omega \setminus P(u)})_{*u_n} \rightharpoonup (v\chi_{\Omega \setminus P(u)})_{*u}$$

weakly in $L^p(\Omega_*)$ if $p < +\infty$ and weakly in $L^\infty(\Omega_*)$ if $p = +\infty$ (by χ_E we denote the characteristic function of the set E).* $\square$

Under some additional regularity conditions it is possible to improve the above convergence:

Lemma 14 ([23]). *Let $v \in L^p(\Omega)$ with $1 \le p < +\infty$ and u, $u_n \in W^{1,\infty}(\Omega)$ such that $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = \mathrm{meas}\{x \in \Omega : \nabla u_n(x) = 0\} = 0$. If u_n converges to u in $W^{1,q}(\Omega)$ for some $q > N$, then $v_{*u_n} \xrightarrow[n \to \infty]{} v_{*u}$ strongly in $L^p(\Omega_*)$. Furthermore, $v_{*u_n}(|u_n > u_n(\cdot)|)$ converges to $v_{*u}(|u > u(\cdot)|)$ strongly in $L^p(\Omega)$.* $\square$

As a direct consequence of this lemma, we have the following result

Lemma 15 ([23]). *Let $v \in L^p(\Omega)$ with $1 \le p < +\infty$ and let $(\lambda_k, \varphi_k)_{k=1}^{+\infty}$ be the sequence of eigenvalues and eigenfunctions of $-\Delta$ on Ω with Dirichlet conditions; i.e. $-\Delta \varphi_k = \lambda_k \varphi_k$, $\varphi_k \in H_0^1(\Omega)$. Consider the finite dimensional vector space $V_m = \mathrm{span}\{\varphi_1, \dots \varphi_m\}$ (the vector space spanned by $(\varphi_k)_{k=1}^m$). Then, the maps given by*

$$u \in V_m \backslash \{0\} \to v_{*u} \in L^p(\Omega_*) \quad 1 \le p < +\infty$$

and

$$u \in V_m \backslash \{0\} \to v_{*u}(|u > u(\cdot)|) \in L^p(\Omega) \quad 1 \le p < +\infty$$

are strongly continuous over $L^p(\Omega_*)$ and $L^p(\Omega)$, respectively. $\square$

As a complement of Lemma 7, we have the following useful result.

Lemma 16. *Let u_n, $u \in W^{1,p}(\Omega)$, $1 \le p \le +\infty$ and Ω be a regular open subset of $\mathbb{R}^N$ such that*

$$\mathrm{meas}\{x \in \Omega : \nabla u_n(x) = 0\} = 0 = \mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\}. \tag{2.8}$$

Then, if u_n converges to u in $W^{1,p}(\Omega)$ for some $p > N$;

$$u'_{n*}(|u_n > u_n(\cdot)|) \text{ converges strongly to } u'_*(|u > u(\cdot)|) \text{ in } L^q(\Omega)$$

for all $1 \le q < \bar{q} := \frac{1}{1 + \frac{1}{p} - \frac{1}{N}}$. $\square$

The proof of this lemma is included in the proof of Lemma 14.

3. An equivalent formulation as a nonlocal problem

The main goal of this section is to show that if u satisfies the family of conditions (1.12), i.e.

$$\int_{\{u > t\}} \{F(u)F'(u) + p'(u)b\} \, dx = j(t_+, u_{+*}(0)) \quad \forall t \in \left[\inf_\Omega u, \sup_\Omega u\right] \tag{3.1}$$

then it is possible to express F in terms of u decoupling in this way, the equation of such a family of conditions. This fact gives us the key to reformulate the problem (1.10), (1.11) and (1.12) like a nonlocal problem where we can eliminate the unknown function F. In this way, we reduce the original problem to the simpler problem $(\mathscr{P}_*)$

$$-\Delta u(x) = a(x)\mathscr{F}_u(x) + p'(u(x))[b(x) - b_{*u}(|u > u(x)|)]$$
$$+ j'_t(u_+(x), u_{+*}(0))u'_{+*}(|u > u(x)|) \quad \text{in } \Omega, \tag{3.2}$$

$$u - \gamma \in H_0^1(\Omega), \tag{3.3}$$

where now u is the unique unknown and function $\mathscr{F}_u$ is defined as follows:

$$\mathscr{F}_u(x) := \left[F_v^2 - 2 \int_{|u > 0|}^{|u > u_+(x)|} [p(u_*)]'(s) b_{*u}(s) \, ds \right.$$
$$\left. + 2 \int_{|u > 0|}^{|u > u_+(x)|} j'_t(u_{+*}(s), u_{+*}(0))(u'_{+*}(s))^2 \, ds \right]_+^{\frac{1}{2}}. \tag{3.4}$$

In order to prove the equivalence between problems $(\mathscr{P})$ and $(\mathscr{P}_*)$, we will need some few lemmas. Their proofs are easy modifications of the corresponding lemmas in [3]. Here, we shall only show the significant differences. In all this section, we set $\hat{m} = \inf_\Omega u$ and $M = \sup_\Omega u$ which are justified since $u \in L^\infty(\Omega)$.

Given $u \in W^{1,\infty}(\Omega)$, we define the function $\mathscr{F} : \mathbb{R} \to \mathbb{R}_+$ by

$$\mathscr{F}(t) = \left[F_v^2 - 2 \int_0^{t_+} p'(s) b_{*u}(|u > s|) \, ds \right.$$
$$\left. + 2 \int_0^{t_+} j'_t(s, u_{+*}(0)) u'_{+*}(|u > s|) \, ds \right]_+^{\frac{1}{2}}. \tag{3.5}$$

Then, we have

Lemma 17. *Let* $u \in W^{1,\infty}(\Omega)$ *such that* $\text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. *Then* $\mathscr{F}_u(x) = \mathscr{F}(u(x))$ *for all* $x \in \bar{\Omega}$.

Proof. Similar to Proposition 1 of [3]. $\square$

Lemma 18. *Let* $u \in V(\Omega)$ *such that* $\text{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. *Then*

$$-\frac{d^+ u_{+*}}{ds}(|u > t|) \leq \frac{\|\Delta u\|_{L^\infty(\Omega)}}{4\pi} \quad \text{for a.e. } t. \tag{3.6}$$

Proof. It suffices to show (3.6) for $t \in]\inf_\Omega u, \sup_\Omega u[$. Let $p \in [1, +\infty[$. Making the change of variables $t = u_*(s)$ we have

$$\int_{\inf_\Omega u}^{\sup_\Omega u} |u'_{+*}(|u > t|)|^p \, dt = \int_0^{|\Omega|} |u'_{+*}(s)|^{p+1} \, ds.$$

From Lemma 4, we then have for all $p \in [1, +\infty[$

$$\left(\int_{\inf_\Omega u}^{\sup_\Omega u} |u'_{+*}(|u > t|)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \leq \frac{\|\Delta u\|_{L^\infty(\Omega)}^{1+\frac{1}{p}}}{(4\pi)^{1+\frac{1}{p}}}.$$

Letting $p \to +\infty$ we obtain (3.6). $\square$

Lemma 19. *Assume $u \in V(\Omega)$ such that $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$ and $\min\{\mathscr{F}(t) : t \in [\hat{m}, M]\} > 0$, $\hat{m} := \inf_\Omega u$, $M := \sup_\Omega u$. Then (i) $\mathscr{F} \in W^{1,\infty}(]\hat{m}, M[)$ and (ii) for almost every $t \in [\hat{m}, M]$, we have*

$$\mathscr{F}(t)\mathscr{F}'(t) + p'(t)b_{*u}(|u > t|) = j'_t(t_+ u_{+*}(0))u'_{+*}(|u > t|).$$

Proof. Since $b_{*u} \in L^\infty(\Omega_*)$, $p \in \mathscr{C}^1(\mathbb{R})$, then, the map

$$t \to \int_0^{t_+} p'(\sigma)b_{*u}(|u > \sigma|) \, \mathrm{d}\sigma$$

is in $W^{1,\infty}(]\hat{m}, M[)$. From the assumptions on j'_t and Lemma 18, we deduce that the map

$$t \to \int_0^{t_+} j'_t(\sigma_+, u_{+*}(0))u'_{+*}(|u > \sigma|) \, \mathrm{d}\sigma$$

belongs to $W^{1,\infty}(]\hat{m}, M[)$. Finally, proceeding as in [3, Lemma 12], we obtain the conclusion. $\square$

Lemma 20. *Let $u \in V(\Omega)$ such that $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$ and let $\mathscr{F}$ given by (3.5). Assume that $\min\{\mathscr{F}(t), t \in [\hat{m}, M]\} > 0$. Then, for all $t \in [\hat{m}, M]$*

$$\int_{\{u > t\}} \{\mathscr{F}(u)\mathscr{F}'(u) + p'(u)b\} \, \mathrm{d}x = j(t_+, u_{+*}(0)).$$

Proof. By Lemma 19, if we set $N = \{t \in [\hat{m}, M] : \mathscr{F}(t)\mathscr{F}'(t) + p'(t)b_{*u}(|u > t|) \neq j'_t(t_+, u_{+*}(0))u'_{+*}(|u > t|)\}$, it has zero measure. Then $\{x \in \bar{\Omega} : u(x) \in N\}$ and $\{s \in \bar{\Omega}_* : u_*(s) \in N\}$ have zero measure (use equimeasurability, that $\{s \in \bar{\Omega}_* : u_*(s) \in N\} \subset m_u(N)$ and that m_u is absolutely continuous). Therefore,

$$\mathscr{F}(u(x))\mathscr{F}'(u(x)) + p'(u(x))b_{*u}(|u > u(x)|) = j'_t(u_+(x), u_{+*}(0))u'_{+*}(|u > u(x)|)$$

a.e. $x \in \bar{\Omega}$. Integrating on $\{u > t\}$, one has

$$\int_{\{u > t\}} \{\mathscr{F}(u(x))\mathscr{F}'(u(x)) + p'(u(x))b_{*u}(|u > u(x)|)\} \, \mathrm{d}x$$

$$= \int_{\{u > t\}} j'_t(u_+(x), u_{+*}(0))u'_{+*}(|u > u(x)|) \, \mathrm{d}x. \tag{3.7}$$

By equimeasurability

$$\int_{\{u>t\}} j_t'(u_+(x), u_{+*}(0)) u_{+*}'(|u > u(x)|) \, dx = \int_{\{u_*>t\}} j_t'(u_{+*}(s), u_{+*}(0)) u_{+*}'(s) \, ds.$$

If $t \geq 0$, one has $u_{+*}'(s) = u'(s)$ for almost every $s \in \{u_* > t\}$ since meas$\{x \in \Omega : \nabla u(x) = 0\} = 0$ and $u \in V(\Omega)$. By the change of variable $\theta = u_*(s)$

$$\int_0^{|u_*>t|} j_t'(u_{+*}(s), u_{+*}(0)) u_{+*}'(s) \, ds = -\int_t^{\sup u_+} j_t'(\theta, u_{+*}(0)) \, d\theta$$

$$= -j(u_{+*}(0), u_{+*}(0)) + j(t, u_{+*}(0)). \tag{3.8}$$

Moreover $j(u_{+*}(0), u_{+*}(0)) = 0$ by Assumption (1.7). Thus, from (3.7) and (3.8), for all $t \geq 0$

$$\int_{\{u>t\}} [\mathscr{F}(u)\mathscr{F}'(u) + p'(u)b_{*u}(|u > u(x)|)] \, dx = j(t_+, u_{+*}(0)).$$

Let $t < 0$. From relation (3.7)

$$\int_{\{u>t\}} [F(u)F'(u) + p'(u)b_{*u}(|u > u(x)|)] \, dx$$

$$= \int_{\{u\geq 0\}} j_t'(u_+(x), u_{+*}(0)) u_{+*}'(|u > u(x)|) \, dx$$

$$+ \int_{\{t<u<0\}} j_t'(u_+(x), u_{+*}(0)) u_{+*}'(|u > u(x)|) \, dx \tag{3.9}$$

(notice that the last integral is zero). By the same change of variable as before

$$\int_{\{u\geq 0\}} j_t'(u_+(x), u_{+*}(0)) u_{+*}'(|u > u(x)|) \, dx = -\int_0^{\sup_\Omega u_+} j_t'(\theta, u_{+*}(0)) \, d\theta$$

$$= j(0, u_{+*}(0)) - j(u_{+*}(0), u_{+*}(0))$$

$$= j(0, u_{+*}(0)). \tag{3.10}$$

From (3.9) and (3.10), we have

$$\int_{\{u>t\}} [\mathscr{F}(u)\mathscr{F}'(u) + p'(u)b_{*u}(|u > u(x)|)] \, dx = j(0, u_{+*}(0)) = j(t_+, u_{+*}(0))$$

for $t < 0$ and the conclusion holds. □

Lemma 20 shows that, given u, the function $\mathscr{F}$ defined by (3.5) satisfies the original Stellarator condition (1.12), i.e. (3.1). Let us prove a result in the reciprocal way:

Lemma 21. *Let* $u \in V(\Omega)$ *such that* meas$\{x \in \Omega : \nabla u(x) = 0\} = 0$. *If* $F \in W^{1,\infty}(]\hat{m}, M[)$ *is a function satisfying* $F : [\hat{m}, M] \to \mathbb{R}^+$, $F(t) = F_v$ *for all* $t \leq 0$ *and for all* $t \in [\hat{m}, M]$

the relation (3.1) *holds, then, necessarily* $F(t) = \mathcal{F}(t)$ *for all* $t \in [\hat{m}, M]$ *with* $\mathcal{F}$ *given by* (3.5).

Proof. We follow the same type of arguments as in [3]. From relation (3.1) we deduce that $j(u_{+*}(s), u_{+*}(0)) = w(s) := \int_{\{u > u_*(s)\}} \{F(u)F'(u) + p'(u)b\}\, dx$ for all $s \in \bar{\Omega}_*$. Thus, differentiating with respect to s this relation and using Lemma 12, for all $s \in \Omega_*$

$$j'_t(u_{+*}(s), u_{+*}(0))u'_{+*}(s) = F(u_*(s))F'(u_*(s)) + p'(u_*(s))b_{*u}(s)$$

since $s = |u > u_*(s)|$ (because $|P(u)| = 0$). The rest of the proof follows as in [3, Lemma 15]. □

From the above lemmas, we can obtain the main result of this section:

Theorem 2. *Let* $u \in V(\Omega)$ *such that* $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. *Assume* $\hat{m} = \inf_\Omega u \leq 0$ *and* $\mathcal{F}_u(x) > 0$ *a.e. in* Ω. *Then, if* (u, F) *is a solution of* $(\mathcal{P})$ *such that* $F : [\hat{m}, M] \to \mathbb{R}^+$, $F \in W^{1,\infty}(]\hat{m}, M[)$ *and* $F(t) = F_v$ *for all* $t \leq 0$, *then function* u *is also a solution of* $(\mathcal{P}_*)$ *and necessarily* $F = \mathcal{F}$ *with* $\mathcal{F}$ *given by* (3.5). *Conversely, if* u *is a solution of* $(\mathcal{P}_*)$ *and* $\mathcal{F}$ *is given by* (3.5) *then, the couple* $(u, \mathcal{F})$ *is a solution of* $(\mathcal{P})$ *and* $\mathcal{F} \in W^{1,\infty}(]\hat{m}, M[)$.

Proof. If (u, F) is a solution of $(\mathcal{P})$, then thanks to the assumptions, F is defined necessarily by relation (3.5). Now, by Lemma 17, one has that $\mathcal{F}_u(x) = F(u(x))$ for all $x \in \Omega$. The assumption $\mathcal{F}_u(x) > 0$ and the regularity $u \in \mathscr{C}(\bar{\Omega})$ imply that $\min\{F(t) : t \in [\hat{m}, M]\} > 0$. Then, the conditions of Lemma 19 are fulfilled and so

$$F(t)F'(t) + p'(t)b_{*u}(|u > t|) = j'_t(t_+, u_{+*}(0))u'_{+*}(|u > t|)$$

for almost every $t \in [\hat{m}, M]$. In particular, we deduce that u satisfies (3.2) and so u is a solution of $(\mathcal{P}_*)$. Conversely, if u satisfies $(\mathcal{P}_*)$ with $\mathcal{F}$ given by (3.5), then $\mathcal{F}(u(x)) = \mathcal{F}_u(x) > 0$, $x \in \bar{\Omega}$ by Lemma 17. Then Lemma 20 can be applied to get the relation (3.1) of $(\mathcal{P})$. Equation (1.10) is verified by u thanks to Lemma 19. □

4. The approximate problem $(\mathcal{P}_{*\varepsilon})$

Here and in what follows, we use the notation:

$$I(v(x), \sigma) := \chi_{[|v > v_+(x)|, |v > 0|]}(\sigma) \qquad \sigma \in \Omega_*$$

(4.1)

(the characteristics function of $[|v > v_+(x)|, |v > 0|]$),

$$F_1(x, v, b_{*v}) := -\int_{\Omega_*} I(v(x), s)[p(v_*)]'(s)b_{*v}(s)\, ds, \qquad (4.2)$$

$$F_{\varepsilon,2}(x, v) := -\int_{\Omega_*} I(v(x), s)h_\varepsilon(v'_{+*}(s))j'_t(v_{+*}(s), v_{+*}(0))\, ds, \qquad (4.3)$$

$$F_\varepsilon(x, v, b_{*v}) := [F_v^2 - 2F_1(x, v, b_{*v}) + 2F_{\varepsilon,2}(x, v)]_+^{\frac{1}{2}}, \tag{4.4}$$

$$H(v(x), b_{*v}) := p'(v(x))[b(x) - b_{*v}(|v > v(x)|)], \tag{4.5}$$

$$J_\varepsilon(v(x)) := \xi_\varepsilon(v'_{+*}(|v > v_+(x)|))j'_t(v_+(x), v_{+*}(0)), \tag{4.6}$$

lways for a.e. $x \in \Omega$ and for any function $v \in H_0^1(\Omega)$. Here, we used the truncation unctions

$$h_\varepsilon(t) := \frac{t^2}{1 + \varepsilon t^2}, \qquad \xi_\varepsilon(t) := \frac{t}{1 + \varepsilon|t|}. \tag{4.7}$$

Ve shall adopt the notation $F_2 := F_{0,2}$ and $J := J_0$ (for $\varepsilon = 0$) concerning problem (3.2) nd (3.3): that is

$$F_2(x, v) := F_{0,2}(x, v) = \int_{|v > 0|}^{|v > v_+(x)|} (v'_{+*})^2(s)j'_t(v_{+*}(s), v_{+*}(0))\,\mathrm{d}s \tag{4.8}$$

nd

$$J(v) := J_0(v) = v'_{+*}(|v > v_+(x)|)j'_t(v_+(x),\ v_+ * (0)). \tag{4.9}$$

Now, let us consider the following approximate problem $(\mathscr{P}_{*\varepsilon})$ for any fixed $\varepsilon > 0$; nd u^ε such that

$$-\Delta u^\varepsilon = aF_\varepsilon(x, u^\varepsilon, b_{*u^\varepsilon}) + H(u^\varepsilon, b_{*u^\varepsilon}) + J_\varepsilon(u^\varepsilon) \quad \text{in } \Omega, \tag{4.10}$$

$$u^\varepsilon - \gamma \in H_0^1(\Omega) \cap W^{2,p}(\Omega); \quad \forall p \geq 1. \tag{4.11}$$

'o simplify the boundary condition we define $w^\varepsilon := u^\varepsilon - \gamma$. In order to prove the xistence of w^ε we shall use a Galerkin method as in [24]. First, we shall find a olution w_m^ε of some auxiliary problems $(\mathscr{P}_{*\varepsilon,m})$. We shall search $w_m^\varepsilon \in V_m$, where V_m ; a finite dimensional space such that $V_m \subset V_{m+1} \subset H_0^1(\Omega)$. Later, using appropriate stimates on the solutions w_m^ε of $(\mathscr{P}_{*\varepsilon,m})$, we shall pass to the limit when m goes to ıfinity and so, we shall find a function w^ε such that $w^\varepsilon + \gamma$ is a solution of $(\mathscr{P}_{*\varepsilon})$.

.1. The Galerkin method. Existence of solution for a family of finite dimensional roblems $(\mathscr{P}_{*\varepsilon,m})$

Consider $(\lambda_k, \varphi_k)_{k \geq 1}$ be the eigenvalues and eigenfunctions associated to $-\Delta$ on Ω /ith Dirichlet boundary conditions, i.e.

$$-\Delta \varphi_k = \lambda_k \varphi_k, \quad \varphi_k \in H_0^1(\Omega).$$

et $V_m = \mathrm{span}\{\varphi_1, \ldots, \varphi_m\}$. On V_m, we define the scalar product by $[v, w] := \sum_{k=1}^m v^k w^k$ /here $v = \sum_{k=1}^m v^k \varphi_k$ and $\omega = \sum_{k=1}^m w^k \varphi_k$. Let $\|v\|_{V_m} := [v, v]^{\frac{1}{2}}$ the associated norm.

Now, for $\gamma \le 0$ fixed, we consider the operator $T_m^\varepsilon : V_m \to V_m$ defined as

$$[T_m^\varepsilon v, \varphi] = \int_\Omega \nabla v \cdot \nabla \varphi \, dx - \int_\Omega a F_\varepsilon(x, v + \gamma, b_{*(v+\gamma)}) \varphi \, dx$$

$$- \int_\Omega H(v + \gamma, b_{*(v+\gamma)}) \varphi \, dx - \int_\Omega J_\varepsilon(v + \gamma) \varphi \, dx \quad \forall v, \varphi \in V_m. \qquad (4.12)$$

We shall prove that this operator attains zero for some $w_m^\varepsilon \in V_m \setminus \{0\}$. It is clear that i w_m^ε satisfies $T_m^\varepsilon w_m^\varepsilon = 0$ in V_m then w_m^ε satisfies the finite dimensional problem $(\mathscr{P}_{*\varepsilon, m}$ given by

$$-\Delta(w_m^\varepsilon + \gamma) = P_m[a F_\varepsilon(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)}) + H(w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)})$$

$$+ J_\varepsilon(w_m^\varepsilon + \gamma)] \quad \text{in } \Omega,$$

$$w_m^\varepsilon \in V_m,$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m. To prove that T_n' has a zero in $V_m \setminus \{0\}$ we shall use Lemma 4.3 of [25]. We need to check that T_m^ε i: a coercive and continuous map. We notice that $[T_m^\varepsilon 0, \varphi] = F_v \int_\Omega a\varphi \, dx \ne 0$ for some φ provided that $a \not\equiv 0$.

Proposition 1. *If*

$$\lambda_1 - \lambda \operatorname*{osc}_\Omega b > 0 \qquad (4.13)$$

then

$$[T_m^\varepsilon v, v] \xrightarrow[\|v\|_{V_m} \to \infty]{} +\infty.$$

In particular, T_m^ε is a coercive map.

Proof. Let $\gamma \le 0$ fixed. We shall estimate

$$[T_m^\varepsilon v, v] = \int_\Omega |\nabla v|^2 \, dx - \int_\Omega a F_\varepsilon(x, v + \gamma, b_{*(v+\gamma)}) v \, dx - \int_\Omega H(v + \gamma, b_{*(v+\gamma)}) v \, dx$$

$$- \int_\Omega J_\varepsilon(v + \gamma) v \, dx \quad \forall v \in V_m \qquad (4.14)$$

term by term. Let us start with $\int_\Omega H(v + \gamma, b_{*(v+\gamma)}) v \, dx$. Then,

$$\left| \int_\Omega H(v + \gamma, b_{*(w_m^\varepsilon + \gamma)}) v \, dx \right| \le \lambda \operatorname*{osc}_\Omega b \int_\Omega v^2 \, dx \qquad (4.15)$$

where we used the assumptions $0 \le p'(s) \le \lambda s_+$, $\lambda > 0$ (see (1.8)) and that $\gamma \le 0$. It is also clear that

$$\left| \int_\Omega J_\varepsilon(v + \gamma) v \, dx \right| \le \frac{\eta}{\varepsilon} \int_\Omega |v| \, dx \qquad (4.16)$$

hanks to (1.7) and that function ξ_ε is bounded by $1/\varepsilon$. Let us estimate the term $\int_\Omega aF_\varepsilon(x, v + \gamma, b_{*(v+\gamma)})v\,\mathrm{d}x$. Dropping nonpositive terms and using the same argument s above,

$$\int_\Omega aF_\varepsilon(x, v + \gamma, b_{*(v+\gamma)})v\,\mathrm{d}x \leq \|a\|_{L^\infty(\Omega)} \left[F_v^2 + \frac{2\eta}{\varepsilon}|\Omega|\right]^{1/2} \int_\Omega |v|\,\mathrm{d}x, \qquad (4.17)$$

ince $|h_\varepsilon| \leq 1/\varepsilon$. Using the above estimates, we have

$$[T_m^\varepsilon v, v] \geq \int_\Omega |\nabla v|^2\,\mathrm{d}x - C_\varepsilon \int_\Omega |v|\,\mathrm{d}x - \lambda \operatorname*{osc}_\Omega b \int v^2\,\mathrm{d}x \qquad (4.18)$$

rith $C_\varepsilon = \|a\|_{L^\infty(\Omega)}[F_v^2 + \frac{2\eta}{\varepsilon}|\Omega|]^{1/2} + \frac{\eta}{\varepsilon}$ and for all $v \in V_m$. Applying Young's inequality $\int_\Omega |v|\,\mathrm{d}x$ we get

$$\int_\Omega |v|\,\mathrm{d}x \leq \frac{K^2}{2}\int_\Omega v^2\,\mathrm{d}x + \frac{|\Omega|}{2K^2} \quad \forall K > 0.$$

hoosing $K^2 = \frac{2\delta}{C_\varepsilon}$ and $C_{\varepsilon\delta} = \frac{C_\varepsilon^2|\Omega|}{4\delta}$, we obtain that

$$[T_m^\varepsilon v, v] \geq \lambda_1 \int_\Omega v^2\,\mathrm{d}x - \delta \int_\Omega v^2\,\mathrm{d}x - C_{\varepsilon\delta} - \lambda \operatorname*{osc}_\Omega b \int_\Omega v^2\,\mathrm{d}x$$

$$= (\lambda_1 - \delta - \lambda \operatorname*{osc}_\Omega b) \int_\Omega v^2\,\mathrm{d}x - C_{\varepsilon\delta} \quad \forall v \in V_m. \qquad (4.19)$$

rom the Assumption (4.13) we obtain the coercivity of operator T_m^ε assumed we take small enough. □

roposition 2. T_m^ε *is a continuous map.*

roof. Since $(\varphi_k)_{k=1}^m$ is an orthogonal base of $(V_m, [\cdot, \cdot])$, $T_m^\varepsilon v$ can be expressed in the ollowing way

$$T_m^\varepsilon v = \sum_{k=1}^m [T_m^\varepsilon v, \varphi_k]\varphi_k. \qquad (4.20)$$

o, the continuity of T_m^ε on V_m is equivalent to the continuity of the application

$$V_m \ni v \to [T_m^\varepsilon v, \varphi]$$

here $\varphi \in V_m$ is an arbitrary function. We shall prove the continuity of the different unctions appearing in Definition (4.12) of T_m^ε once φ is fixed. In the following, we iall take a sequence of functions $v_n \in V_m \backslash \{0\}$ and $v \in V_m$ such that

$$v_n \to v \quad \text{in } V_m.$$

/e need a previous lemma.

Lemma 22. *Let $(v_n)_{n \geq 1}$ be a sequence of $V_m \backslash \{0\}$ and let v be in $V_m \backslash \{0\}$, such that*

$$v_n \rightarrow v \quad \text{in } V_m. \tag{4.21}$$

Then one has

$$v_n \underset{n \to \infty}{\longrightarrow} v \quad \text{strongly in } \mathscr{C}^k(\bar{\Omega}) \forall k \in \mathbb{N} \cup \{0\}, \tag{4.22}$$

$$v_{n*} \underset{n \to \infty}{\longrightarrow} v_* \quad \text{strongly in } \mathscr{C}(\bar{\Omega}_*), \tag{4.23}$$

$$v'_{n+*} \underset{n \to \infty}{\longrightarrow} v'_{+*} \quad \text{strongly in } L^q(\Omega_*) \forall 1 \leq q < \infty, \tag{4.24}$$

$$v'_{n+*}(|v_n > v_n(\cdot)|) \underset{n \to \infty}{\longrightarrow} v'_{+*}(|v > v(\cdot)|) \quad \text{strongly in } L^q(\Omega) \forall 1 \leq q < \infty. \tag{4.25}$$

Proof. On V_m all the norms are equivalent and since $V_m \subset \mathscr{C}^\infty(\bar{\Omega})$, we have (4.22). As Ω is connected and v_n, $v \in \mathscr{C}^0(\bar{\Omega})$, then v_{n*} and v_* are continuous in $\bar{\Omega}_*$. From the contraction property and (4.22) we get (4.23). On the other hand, since any element of $V_m \backslash \{0\}$ is coarea regular (notice that it is analytic in Ω), one has, from Lemma 7,

$$\frac{\mathrm{d}v_{n*}}{\mathrm{d}s} \rightarrow \frac{\mathrm{d}v_*}{\mathrm{d}s} \quad \text{in } L^q(\Omega_*), \quad \forall q \in [1, 2[. \tag{4.26}$$

So,

$$\frac{\mathrm{d}v_{n+*}}{\mathrm{d}s} \rightarrow \frac{\mathrm{d}v_{+*}}{\mathrm{d}s} \quad \text{in } L^q(\Omega_*), \quad \forall q \in [1, 2[. \tag{4.27}$$

Moreover, as the functions v_n are such that $v_{n|\partial\Omega} = 0$, they belong to $V(\Omega)$. By (4.22) v is also in $V(\Omega)$ and we know that $|\Delta(v_n - v)|_{\mathscr{C}(\bar{\Omega})} \underset{n \to +\infty}{\longrightarrow} 0$. Thus, from Lemma 4, we have

$$\left| \frac{\mathrm{d}v_{n+*}}{\mathrm{d}s} \right|_{L^\infty(\Omega_*)} \leq \frac{\|\Delta v_n\|_{L^\infty(\Omega)}}{4\pi} \leq \text{constant independent of } n. \tag{4.28}$$

We easily conclude from this estimate and (4.27) that

$$\frac{\mathrm{d}v_{n+*}}{\mathrm{d}s} \rightarrow \frac{\mathrm{d}v_{+*}}{\mathrm{d}s} \quad \text{in } L^p(\Omega_*), \quad \forall p \in [1, +\infty[. $$

Finally, the last statement comes from Lemma 16 and (iii) of Lemma 4. $\square$

Proof of Proposition 2. *Continuity of the map*

$$V_m \ni v \rightarrow \int_\Omega H(v + \gamma, b_{*(w_m^\varepsilon + \gamma)}) \varphi \, \mathrm{d}x. \tag{4.29}$$

It is equivalent to prove that the following equality holds

$$\lim_{n \to \infty} \int_\Omega \varphi(x) p'((v_n + \gamma)(x))[b(x) - b_{*(v_n + \gamma)}(|v_n + \gamma > (v_n + \gamma)(x)|)] \, \mathrm{d}x$$

$$= \int_\Omega \varphi(x) p'((v + \gamma)(x))[b(x) - b_{*(v + \gamma)}(|v + \gamma > (v + \gamma)(x)|)] \, \mathrm{d}x \tag{4.30}$$

for all $\varphi \in V_m$. We distinguish the case $v \not\equiv 0$ from the case $v \equiv 0$. Since any element of $V_m \backslash \{0\}$ is an analytical function (recall that φ_k are analytical functions), then meas$\{x \in \Omega : \nabla v(x) = 0\} = 0 = $ meas$\{x \in \Omega : \nabla v_n(x) = 0\}$ and by Lemma 14, we get

$$b_{*(v_n+\gamma)}(|v_n + \gamma > (v_n + \gamma)(\cdot)|) \underset{n\to\infty}{\longrightarrow}$$

$$b_{*(v+\gamma)}(|v + \gamma > (v + \gamma)(\cdot)|) \quad \text{strongly in } L^q(\Omega) \tag{4.31}$$

for all $q \geq 1$ since b is in $L^\infty(\Omega)$. By Assumption (1.8) we obtain, from the convergence Lemma 22, that

$$p'(v_n + \gamma) \underset{n\to\infty}{\longrightarrow} p'(v + \gamma) \quad \text{strongly in } L^r(\Omega) \tag{4.32}$$

for all $r \geq 1$. Now, by the convergences (4.31) and (4.32) we obtain (4.30) when $v \not\equiv 0$. Finally, if $v \equiv 0$, the $L^r(\Omega)$-strong convergence of $p'(v_n + \gamma)$ to 0, the uniform boundedness in $L^\infty(\Omega)$ of $b_{*(v_n+\gamma)}(|v_n + \gamma > (v_n + \gamma)(\cdot)|)$ and the weak convergence in $L^p(\Omega)$ of $b_{*(v_n+\gamma)}(|v_n + \gamma > (v_n + \gamma)(\cdot)|)$ to $b_{*(v+\gamma)}(|v + \gamma > (v + \gamma)(\cdot)|)$ (see Lemma 13) imply (4.30). Thus, the map defined by (4.29) is continuous.

Continuity of the map

$$V_m \ni v \to \int_\Omega J_\varepsilon(v + \gamma)\varphi \, dx. \tag{4.33}$$

Following the same steps as before we verify that

$$\lim_{n\to\infty} \int_\Omega \xi_\varepsilon((v_n + \gamma)'_{+*}(|v_n + \gamma > (v_n + \gamma)(x)|))j'_t((v_n + \gamma)_+(x), (v_n + \gamma)_{+*}(0))\varphi dx$$

$$= \int_\Omega \xi_\varepsilon((v + \gamma)'_{+*}(|v + \gamma > (v + \gamma)(x)|))j'_t((v + \gamma)_+(x), (v + \gamma)_{+*}(0))\varphi dx \tag{4.34}$$

for all $\varphi \in V_m$. Since by Lemma 22

$$v_n + \gamma \underset{n\to\infty}{\longrightarrow} v + \gamma \quad \text{strongly in } \mathscr{C}^k(\bar{\Omega}) \; \forall k \in \mathbb{N} \cup \{0\},$$

we have

$$(v_n + \gamma)_+(x) \underset{n\to\infty}{\longrightarrow} (v + \gamma)_+(x) \quad \forall x \in \Omega$$

and

$$(v_n + \gamma)_{+*}(0) = \|(v_n + \gamma)_+\|_{L^\infty(\Omega)} \underset{n\to\infty}{\longrightarrow} \|(v + \gamma)_+\|_{L^\infty(\Omega)} = (v + \gamma)_{+*}(0).$$

The continuity of j'_t in $\mathbb{R}^+ \times \mathbb{R}^+$ and the above convergences give

$$j'_t((v_n + \gamma)_+(x), (v_n + \gamma)_{+*}(0)) \underset{n\to\infty}{\longrightarrow} j'_t((v + \gamma)_+(x), (v + \gamma)_{+*}(0)) \quad \forall x \in \Omega.$$

Applying again Lemma 22, the Lipschitz continuity and the boundedness of function ξ_ε, one has

$$\xi_\varepsilon((v_n + \gamma)'_{+*}(|v_n + \gamma > (v_n + \gamma)(\cdot)|)) \underset{n\to\infty}{\longrightarrow} \xi_\varepsilon((v + \gamma)'_{+*}(|v + \gamma > (v + \gamma)(\cdot)|))$$

strongly in $L^q(\Omega)$ with $1 \le q < \infty$. The above convergences and assumption (1.7), lead to the conclusion.

Continuity of the map

$$V_m \ni v \to \int_\Omega aF_\varepsilon(x, v + \gamma, b_{*(v+\gamma)}) \varphi \, \mathrm{d}x. \tag{4.35}$$

As before, the first integral which appears in the definition of F_ε (see (4.4)) converges to

$$2 \int_{|v+\gamma>0|}^{|v+\gamma>(v+\gamma)_+(x)|} [p((v + \gamma)_*)]'(s) b_{*(v+\gamma)}(s) \, \mathrm{d}s$$

from the strong convergence of Lemmas 22, 1, 14 and assumption (1.8). The convergence of the second integral to

$$2 \int_{|v+\gamma>0|}^{|v+\gamma>(v+\gamma)_+(x)|} j_t'((v + \gamma)_{+*}(s), (v + \gamma)_{+*}(0)) h_\varepsilon((v + \gamma)_{+*}'(s)) \, \mathrm{d}s$$

is obtained in the same way by using the continuity of j_t'.

The above steps show the continuity of the map T_m^ε. □

Theorem 3. *Assume* (4.13). *Then there exists at least* $\omega_m^\varepsilon \in V_m$ *solution of problem* $(\mathscr{P}_{*\varepsilon,m})$, *i.e. satisfying* $\forall \varphi \in V_m$

$$[T_m^\varepsilon w_m^\varepsilon, \varphi] = \int_\Omega \nabla w_m^\varepsilon \cdot \nabla \varphi \, \mathrm{d}x - \int_\Omega aF_\varepsilon(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon+\gamma)}) \varphi \, \mathrm{d}x$$

$$- \int_\Omega H(w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon+\gamma)}) \varphi \, \mathrm{d}x - \int_\Omega J_\varepsilon(w_m^\varepsilon + \gamma) \varphi \, \mathrm{d}x = 0. \tag{4.36}$$

Proof. From the coercivity of T_m^ε and the continuity of T_m^ε given by Propositions 1 and 2, we can apply the Brouwer Fixed Point Theorem (see e.g. [23, Lemma 4.3, p. 55]). □

4.2. A priori estimates on solutions of $(\mathscr{P}_{*\varepsilon,m})$

In this section, w_m^ε is assumed to be a solution of $(\mathscr{P}_{*\varepsilon,m})$. Taking $\varphi = w_m^\varepsilon$, from (4.18) and (4.36), we have

$$0 = [T_m^\varepsilon w_m^\varepsilon, w_m^\varepsilon] \ge \left(\lambda_1 - \lambda \operatorname*{osc}_\Omega b - \delta \right) \int_\Omega |w_m^\varepsilon|^2 \, \mathrm{d}x - C_{\varepsilon\delta}.$$

Thus, choosing δ small enough, one has that

$$\|w_m^\varepsilon\|_{L^2(\Omega)} \le C_\varepsilon \tag{4.37}$$

and from (4.17), we deduce

$$\int_\Omega |\nabla w_m^\varepsilon|^2 \, \mathrm{d}x \le C_\varepsilon \tag{4.38}$$

for some positive constant C_ε only depending of ε. From estimate (4.37) and using the same arguments used to obtain estimates (4.15), (4.16) and (4.17), we get

$$\|\Delta w_m^\varepsilon\|_{L^2(\Omega)} \le \|aF_\varepsilon(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)})\|_{L^2(\Omega)} + \|H(w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)})\|_{L^2(\Omega)}$$

$$+ \|J_\varepsilon(w_m^\varepsilon + \gamma)\|_{L^2(\Omega)}$$

$$\le \|a\|_{L^\infty(\Omega)} \left[F_v^2 + \frac{2\eta}{\varepsilon}|\Omega|\right]^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \lambda \operatorname*{osc}_\Omega b \|w_m^\varepsilon\|_{L^2(\Omega)} + \frac{|\Omega|^{\frac{1}{2}}\eta}{\varepsilon} \le C_\varepsilon$$

$$(4.39)$$

where C_ε also denotes a positive constant only depending on ε. Thus, by standard regularity results, $(w_m^\varepsilon)_{m \ge 1}$ is uniformly bounded in $W^{2,2}(\Omega)$ with respect to m.

4.3. Passing to the limit $m \to \infty$: Existence of solution of $(\mathscr{P}_{*\varepsilon})$

By the above estimates, there exists a subsequence of $\{w_m^\varepsilon\}$, which we also denote by $\{w_m^\varepsilon\}$, and $w^\varepsilon \in W^{2,2}(\Omega)$ such that

$$w_m^\varepsilon \underset{m\to\infty}{\rightharpoonup} w^\varepsilon \quad \text{weakly in } W^{2,2}(\Omega),$$

and so,

$$w_m^\varepsilon \underset{m\to\infty}{\longrightarrow} w^\varepsilon \quad \text{strongly in } W^{1,p}(\Omega), \ \forall p \in [1, \infty[, \text{ and in } \mathscr{C}(\bar{\Omega}).$$

Our next step is to verify that $T^\varepsilon w^\varepsilon = 0$. Here $T^\varepsilon : H_0^1(\Omega) \to H_0^1(\Omega)$ is the operator defined by

$$[T^\varepsilon v, \varphi] = \int_\Omega \nabla v \cdot \nabla \varphi \, dx - \int_\Omega aF_\varepsilon(x, v + \gamma, b_{*(v+\gamma)}) \varphi$$

$$- \int_\Omega H(v + \gamma, b_{*(v+\gamma)}) \varphi \, dx - \int_\Omega J_\varepsilon(v + \gamma) \varphi \, dx \qquad (4.40)$$

if $v, \varphi \in H_0^1(\Omega)$. As before we have

$$\|aF_\varepsilon(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)})\|_{L^\infty(\Omega)} \le C_\varepsilon \quad \forall m. \qquad (4.41)$$

Let us show that

$$F_1(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)}) \underset{m\to+\infty}{\longrightarrow} F_1(x, w^\varepsilon + \gamma, \hat{b}^\varepsilon) \quad \text{a.e. in } \Omega$$

where

$$b_{*(w_m^\varepsilon + \gamma)} \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \hat{b}^\varepsilon \quad \text{weakly-star in } L^\infty(\Omega_*)$$

due to the uniform boundedness $\|b_{*(w_m^\varepsilon + \gamma)}\|_{L^\infty(\Omega_*)} \le \|b\|_{L^\infty(\Omega)}$. Indeed, as in [3], one has

$$\lim_{m\to+\infty} I(w_m^\varepsilon(x) + \gamma, \sigma)[p((w_m^\varepsilon + \gamma)_*)]'(\sigma) = I(w^\varepsilon(x) + \gamma, \sigma)[p((w^\varepsilon + \gamma)_*)]'(\sigma)$$

a.e. in Ω_*. Thus, by the strong convergence of $[p((w_m^\varepsilon + \gamma)_*)]'$ to $[p((w^\varepsilon + \gamma)_*)]'$ in $L^1(\Omega_*)$ thanks to Lemma 7 and the continuity of p', we deduce that

$$I(w_m^\varepsilon(x) + \gamma, \cdot)[p((w_m^\varepsilon + \gamma)_*)]'(\cdot) \xrightarrow[m\to+\infty]{} I(w^\varepsilon(x) + \gamma, \cdot)[p((w^\varepsilon + \gamma)_*)]'(\cdot)$$

strongly in $L^1(\Omega_*)$ and since $b_{*(w_m^\varepsilon + \gamma)} \xrightarrow[m\to\infty]{*} \hat{b}^\varepsilon$ we then obtain

$$F_1(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon)}) \xrightarrow[m\to+\infty]{} F_1(x, w^\varepsilon + \gamma, \hat{b}^\varepsilon) \quad \text{a.e. in } \Omega.$$

By the same argument applied to $F_{\varepsilon,2}(x, w_m^\varepsilon)$, noting that, first,

$$\lim_{m\to+\infty} I(w_m^\varepsilon(x) + \gamma, \cdot)h_\varepsilon((w_m^\varepsilon + \gamma)_*'(\cdot)) = I(w^\varepsilon(x) + \gamma, \cdot)h_\varepsilon((w^\varepsilon + \gamma)_*'(\cdot))$$

a.e. in Ω_* and later, strongly in $L^1(\Omega_*)$, we obtain that

$$F_{\varepsilon,2}(x, w_m^\varepsilon + \gamma) \xrightarrow[m\to\infty]{} F_{\varepsilon,2}(x, w^\varepsilon + \gamma) \quad \text{a.e. in } \Omega.$$

Thus, we have (for some subsequence) that

$$F_\varepsilon(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)}) \xrightarrow[m\to\infty]{*} F_\varepsilon(x, w^\varepsilon + \gamma, \hat{b}^\varepsilon) \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{4.42}$$

Analogously, one has that $\|b_{*(w_m^\varepsilon + \gamma)}(|w_m^\varepsilon + \gamma > (w_m^\varepsilon + \gamma)(\cdot)|)\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)}$ and thus, for some subsequence, there exists a function $\tilde{b}^\varepsilon \in L^\infty(\Omega)$ such that

$$b_{*(w_m^\varepsilon + \gamma)}(|w_m^\varepsilon + \gamma > (w_m^\varepsilon + \gamma)(\cdot)|) \xrightarrow[m\to\infty]{*} \tilde{b}^\varepsilon \quad \text{weakly-star in } L^\infty(\Omega).$$

As before, $p'((w_m^\varepsilon + \gamma)(\cdot))$ converges to $p'((w^\varepsilon + \gamma)(\cdot))$ strongly in $L^1(\Omega)$ and thus,

$$H(x, w_m^\varepsilon + \gamma, b_{*(w_m^\varepsilon + \gamma)}(|w_m^\varepsilon + \gamma > (w_m^\varepsilon + \gamma)(x)|)) \xrightarrow[m\to\infty]{*} H(x, w^\varepsilon + \gamma, \tilde{b}^\varepsilon(x))$$

weakly* in $L^\infty(\Omega)$.

Finally, since $w_{m*}^{\varepsilon\prime}$ converges in $L^1(\Omega_*)$ to $w_*^{\varepsilon\prime}$ (see Lemma 22) we have

$$J_\varepsilon(w_m^\varepsilon + \gamma) \to J_\varepsilon(w^\varepsilon + \gamma) \quad \text{strongly in } L^1(\Omega), m \to \infty.$$

Then, w^ε verifies the weak formulation of the following problem

$$-\Delta w^\varepsilon = aF_\varepsilon(x, w^\varepsilon + \gamma, \hat{b}^\varepsilon) + p'(w^\varepsilon + \gamma)[b - \tilde{b}_\varepsilon] + J_\varepsilon(w^\varepsilon + \gamma) \quad \text{in } \Omega, \tag{4.43}$$

$$w^\varepsilon \in H_0^1(\Omega) \cap W^{2,2}(\Omega) \tag{4.44}$$

for any $\varepsilon > 0$. To obtain a solution of $(\mathscr{P}_{*\varepsilon})$, we only need to identify $F_\varepsilon(x, w^\varepsilon + \gamma, \hat{b}^\varepsilon)$ as $F_\varepsilon(x, w^\varepsilon + \gamma, b_{*(w^\varepsilon + \gamma)})$ and $\tilde{b}^\varepsilon$ as $b_{*(w^\varepsilon + \gamma)}(|w^\varepsilon + \gamma > (w^\varepsilon + \gamma)(x)|)$. In this sense we have

Proposition 3. *If* $\text{meas}\{x \in \Omega : \nabla w^\varepsilon(x) = 0\} = 0$ *then* $\hat{b}^\varepsilon = b_{*(w^\varepsilon + \gamma)}$ *in* Ω_* *and* $\tilde{b}^\varepsilon = b_{*(w^\varepsilon + \gamma)}(|w^\varepsilon + \gamma > (w^\varepsilon + \gamma)(x)|)$. *In particular,* w^ε *is a solution of* $(\mathscr{P}_{*\varepsilon})$.

Proof. Use the analyticity of w_m^ε and Lemma 14. $\square$

Next, we want to obtain a sufficient condition on the data in order to have the property meas $\{x \in \Omega : \nabla w^\varepsilon(x) = 0\} = 0$. After that, we will let $\varepsilon \to 0$ getting a solution u of $(\mathscr{P}_*)$.

5. Condition on the data in order to get meas$\{x \in \Omega : \nabla w^\varepsilon(x) = 0\} = 0$ and the existence of solution of $(\mathscr{P}_*)$

From the above section, we already know the existence of $w^\varepsilon \in W^{2,2}(\Omega) \cap H_0^1(\Omega)$ verifying (4.43) and (4.44) for γ fixed. Then, setting $u^\varepsilon := w^\varepsilon + \gamma$, function u^ε verifies that

$$-\Delta u^\varepsilon = a[F_v^2 - 2F_1(x, u^\varepsilon, \hat{b}^\varepsilon) + 2F_{\varepsilon,2}(x, u^\varepsilon)]_+^{\frac{1}{2}} + H(u^\varepsilon, \tilde{b}^\varepsilon) + J_\varepsilon(u^\varepsilon) \quad \text{in } \Omega, \qquad (5.1)$$

$$u^\varepsilon - \gamma \in H_0^1(\Omega) \cap W^{2,2}(\Omega). \qquad (5.2)$$

We recall that F_1, $F_{\varepsilon,2}$, H and J_ε were defined in (4.2), (4.3), (4.5) and (4.6), and the truncation functions h_ε and ξ_ε were defined in (4.7).

Let us start by giving a condition on the data in order to have meas$\{x \in \Omega : \nabla w^\varepsilon(x) = 0\} = 0$. We need the following technical result:

Lemma 23. *Let* $\{u^\varepsilon\}$ *verifying* (5.1) *and such that* $u^\varepsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$. *If*

$$v := \frac{1}{4\pi} \left[2^{1/2} \eta^{1/2} |\Omega|^{1/2} \|a\|_{L^\infty(\Omega)} + \lambda |\Omega| \operatorname*{osc}_\Omega b + \eta \right] < 1 \qquad (5.3)$$

then

$$\|\Delta u^\varepsilon\|_{L^\infty(\Omega)} \le \frac{\|a\|_{L^\infty(\Omega)} F_v}{1 - v}. \qquad (5.4)$$

In particular

$$\|u_+^\varepsilon\|_{L^\infty(\Omega)} \le \frac{\|a\|_{L^\infty(\Omega)} F_v |\Omega|}{4\pi(1 - v)} := S \qquad (5.5)$$

uniformly in ε.

Proof. We need some a priori estimates. Let $\varepsilon > 0$ and let u^ε be any solution of $(\mathscr{P}_{*\varepsilon})$. The function

$$F_\varepsilon(x, u^\varepsilon, \hat{b}^\varepsilon) = [F_v^2 - 2F_1 + 2F_{\varepsilon,2}]_+^{\frac{1}{2}}$$

with F_1 and $F_{\varepsilon,2}$ defined by (4.2) and (4.3) is bounded in Ω because the integral F_1 is positive and $F_{\varepsilon,2}$ is bounded since $|\xi_\varepsilon| \le 1/\varepsilon$ and $|j_t'| \le \eta$ (see (1.7)). In the same way, the term $J_\varepsilon(u^\varepsilon)$, defined by (4.6), is bounded in Ω and finally $H(u^\varepsilon, \tilde{b}^\varepsilon)$, given by (4.5), is majored by $\lambda \|u_+^\varepsilon\|_{L^\infty(\Omega)} \operatorname{osc}_\Omega b$. So, $\Delta u^\varepsilon \in L^\infty(\Omega)$ for any ε. Now, applying Lemma 4, we have that

$$0 \le -u_{+*}^{\varepsilon\prime}(s) \le \frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}}{4\pi}. \qquad (5.6)$$

Thus

$$\|u_+^\varepsilon\|_{L^\infty(\Omega)} \le \frac{|\Omega|}{4\pi}\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}. \tag{5.7}$$

Our following task is to prove that Δu^ε is bounded in $L^\infty(\Omega)$ uniformly with respect to the parameter ε. As before, using (5.7), we have the following estimates:

$$0 \le F_1(x, u^\varepsilon, \hat{b}^\varepsilon) \le \lambda\|b\|_{L^\infty(\Omega)}\|u_+^\varepsilon\|_{L^\infty(\Omega)} \le \frac{\lambda|\Omega|}{4\pi}\|b\|_{L^\infty(\Omega)}\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}, \tag{5.8}$$

$$|F_{\varepsilon,2}(x, u^\varepsilon)| \le \eta|\Omega|\frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}^2}{(4\pi)^2}, \tag{5.9}$$

$$|H(u^\varepsilon, \tilde{b}^\varepsilon)| \le \lambda \operatorname*{osc}_\Omega b\|u_+^\varepsilon\|_{L^\infty(\Omega)} \le \lambda \operatorname*{osc}_\Omega b\frac{|\Omega|}{4\pi}\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}, \tag{5.10}$$

$$|J_\varepsilon(u^\varepsilon)| \le \frac{\eta}{4\pi}\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}. \tag{5.11}$$

Then

$$\|\Delta u^\varepsilon\|_{L^\infty(\Omega)} \le \|a\|_{L^\infty(\Omega)}F_v + \frac{1}{4\pi}\left[2^{\frac{1}{2}}\eta^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}\|a\|_{L^\infty(\Omega)} + \lambda|\Omega|\operatorname*{osc}_\Omega b + \eta\right]\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}.$$

We obtain the conclusion from assumption (5.3) and the above estimate. □

In order to get a solution of problem $(\mathscr{P}_{*\varepsilon})$, we need to verify the assumption $\operatorname{meas}\{x \in \Omega : \nabla u^\varepsilon(x) = 0\} = 0$. Indeed, by Proposition 3 we would have that

$$\hat{b}^\varepsilon = b_{*u^\varepsilon} \quad \text{in } L^q(\Omega_*)$$

and

$$\tilde{b}^\varepsilon(x) = b_{*u^\varepsilon}(|u^\varepsilon > u^\varepsilon > (x)|) \quad \text{in } L^\infty(\Omega)$$

and so u^ε would satisfy $(\mathscr{P}_{*\varepsilon})$ thanks to (5.1). The following theorem gives a sufficient condition for this property.

Theorem 4. *If $\lambda\|b\|_{L^\infty(\Omega)}$ and η are small enough, that is*

$$\left[\lambda\|b\|_{L^\infty(\Omega)} + \frac{\eta}{|\Omega|}\right]S < \inf_\Omega |a|\left[F_v^2 - 2\lambda\|b\|_{L^\infty(\Omega)}S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}} \tag{5.12}$$

then

$$\operatorname{meas}\{x \in \Omega : \nabla u^\varepsilon(x) = 0\} = 0.$$

In particular, u^ε satisfies problem $(\mathscr{P}_{*\varepsilon})$.

Proof. We argue by contradiction. Suppose that

$$\operatorname{meas}\{x \in \Omega : \nabla u^\varepsilon(x) = 0\} \ne 0. \tag{5.13}$$

Then, from the equation $(\mathscr{P}_{*\varepsilon})$

$$0 = a[F_v^2 - 2F_1(x, u^\varepsilon, \hat{b}^\varepsilon) + 2F_{\varepsilon,2}(x, u^\varepsilon)]_+^{\frac{1}{2}} + H(u^\varepsilon, \tilde{b}^\varepsilon) + J_\varepsilon(u^\varepsilon)$$

a.e. on $\{x \in \Omega : \nabla u^\varepsilon(x) = 0\}$. Using the estimates (5.8), (5.9), (5.10) and (5.11), we get that

$$\lambda S \operatorname*{osc}_\Omega b \geq |H(u^\varepsilon, \tilde{b}^\varepsilon)| \geq \inf_\Omega |a| [F_v^2 - 2F_1(x, u^\varepsilon, \hat{b}^\varepsilon) + 2F_{\varepsilon,2}(x, u^\varepsilon)]_+^{\frac{1}{2}} - J_\varepsilon(u^\varepsilon),$$

$$\left[\lambda \operatorname*{osc}_\Omega b + \frac{\eta}{|\Omega|}\right] S \geq \inf_\Omega |a| \left[F_v^2 - 2\lambda\|b\|_{L^\infty(\Omega)}S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}}.$$

This contradicts assumption (5.12) proving the lemma. $\square$

Now, we have

Theorem 5. *Assume* $\inf_\Omega |a| > 0$, $\gamma \in \mathbb{R}^-$ *and that* $\lambda\|b\|_{L^\infty(\Omega)} + \eta < \Lambda$ *for a suitable* $\Lambda > 0$. *Then there is* u *solution of* $(\mathscr{P}_*)$. *Moreover* $u \in V(\Omega)$.

Proof. Our aim is to let $\varepsilon \to 0$. By the uniform estimate on $\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}$ given in Lemma 23, there exists some subsequence of (u^ε) (which we will again denote by u^ε) and a function $\alpha \in L^\infty(\Omega)$ such that

$$\Delta u^\varepsilon \underset{\varepsilon \to 0}{\overset{*}{\rightharpoonup}} \alpha \quad \text{weakly}^* \text{ in } L^\infty(\Omega).$$

By standard regularity, u^ε belongs to a bounded set of $W^{2,p}(\Omega)$, for all $p \in [1, +\infty[$. Then, we have (for some subsequence) that

$$u^\varepsilon \underset{\varepsilon \to 0}{\rightharpoonup} u \quad \text{weakly in } W^{2,p}(\Omega),$$

$$u^\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} u \quad \text{strongly in } \mathscr{C}^1(\bar{\Omega}).$$

In particular, $\alpha = \Delta u$, $\Delta u \in L^\infty(\Omega)$, $u \in V(\Omega)$ and the estimates (5.4) and (5.5) of Lemma 23 remain true replacing u^ε by u. Then

$$\|\Delta u\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)}F_v}{1-v}, \tag{5.14}$$

$$\|u_+\|_{L^\infty(\Omega)} \leq \frac{\|a\|_{L^\infty(\Omega)}F_v|\Omega|}{4\pi(1-v)} := S. \tag{5.15}$$

Now, by Lemma 9, we have

$$\|\hat{b}^\varepsilon\|_{L^\infty(\Omega_*)} \leq \|b\|_{L^\infty(\Omega)} \quad \text{and} \quad \|\tilde{b}^\varepsilon\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\Omega)}$$

and so, $\hat{b}^\varepsilon \underset{\varepsilon \to 0}{\overset{*}{\rightharpoonup}} \hat{b}$ weakly* in $L^\infty(\Omega_*)$ and $\tilde{b}^\varepsilon \underset{\varepsilon \to 0}{\overset{*}{\rightharpoonup}} \tilde{b}$ weakly* in $L^\infty(\Omega)$. Furthermore, one has

$$u_{+*}^{\varepsilon\prime} \underset{\varepsilon \to 0}{\longrightarrow} u_{+*}^\prime \quad \text{in } L^p(\Omega_*) \, \forall p \in [1, +\infty[\tag{5.16}$$

(Lemmas 4 and 7). From this convergence, we easily have:

$$h_\varepsilon(u_{+*}^{\varepsilon\prime}) \xrightarrow[\varepsilon\to 0]{} (u_{+*}')^2 \quad \text{in } L^q(\Omega_*) \,\forall q \in [1, +\infty[.$$

Thus,

$$I(u^\varepsilon(x), \cdot) h_\varepsilon(u_{+*}^\varepsilon(\cdot)) \xrightarrow[\varepsilon\to+\infty]{} I(u(x), \cdot)(u_{+*}')^2.$$

In $L^q(\Omega_*)$ for all $q \in [1, +\infty[$ and a.e. in x. We get for all $x \in \Omega$

$$F_{\varepsilon,2}(x, u^\varepsilon) \xrightarrow[\varepsilon\to 0]{} F_2(x, u) = \int_{|u>0|}^{|u>u_+(x)|} [u_{+*}'(\sigma)]^2 j_t'(u_{+*}(\sigma), u_{+*}(0)) \, d\sigma$$

(because j_t' is continuous on $\mathbb{R}^+ \times \mathbb{R}^+$). Since $|F_{\varepsilon,2}(x, u^\varepsilon)| \leq \eta \|\Delta u\|_{L^\infty(\Omega)}^2 |\Omega|$, from Lebesgue's theorem we have

$$F_{\varepsilon,2}(\cdot, u^\varepsilon) \xrightarrow[\varepsilon\to 0]{} F_2(\cdot, u) \quad \text{in } L^p(\Omega) \text{ for all } p < +\infty.$$

With respect to $J_\varepsilon(u^\varepsilon)$, we first obtain from Lemmas 4 and 23

$$|u_{+*}^{\varepsilon\prime}(|u_+^\varepsilon > u_+^\varepsilon(x)|)| \leq \frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}}{4\pi} \leq \frac{S}{|\Omega|}.$$

Since $|\xi^\varepsilon(u_{+*}^{\varepsilon\prime}(|u^\varepsilon > u_+^\varepsilon(x)|)) j_t'(u^\varepsilon(x), u_{+*}^\varepsilon(0))| \leq \eta \frac{\|\Delta u^\varepsilon\|_{L^\infty(\Omega)}}{4\pi} \leq \eta \frac{S}{|\Omega|}$ by Assumption (1.7) and again by Lemma 23. Then, we may assume that

$$\xi^\varepsilon(u_{+*}^{\varepsilon\prime}(|u^\varepsilon > u_+^\varepsilon(x)|)) j_t'(u_+^\varepsilon(x), u_{+*}^\varepsilon(0)) \xrightarrow[n\to\infty]{*} \ell_u j_t'(u_+(x), u_{+*}(0))$$

weakly* in $L^\infty(\Omega)$ for some $\ell_u \in L^\infty(\Omega)$. Thus u is a solution of

$$-\Delta u = a[F_v^2 - 2F_1(x, u, \hat{b}) + 2F_2(x, u)]_+^{\frac{1}{2}} + p'(u)[b - \tilde{b}] + \ell_u j_t'(u_+(x), u_{+*}(0)) \tag{5.17}$$

with $\|\ell_u\|_{L^\infty(\Omega)} \leq \frac{\eta}{|\Omega|} S$ and $|p'(u)[b - \tilde{b}]| \leq \lambda \operatorname{osc}_\Omega b \|u_+\|_{L^\infty(\Omega)}$. Arguing as in Theorem 4, but now using equation (5.17), then if

$$\left[\lambda \|b\|_{L^\infty(\Omega)} + \frac{\eta}{|\Omega|}\right] S < \inf_\Omega |a| \left[F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}}$$

we obtain that $\operatorname{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. Thus, we deduce $\xi^\varepsilon(u_{+*}^{\varepsilon\prime}(|u^\varepsilon > u_+^\varepsilon(x)|)) \xrightarrow[\varepsilon\to 0]{} u_{+*}'(|u > u_+(x)|)$ a.e. $x \in \Omega$ (see Lemma 1) and then $\ell_u(\cdot) = u_{+*}'(|u > u_+(\cdot)|)$ a.e. in Ω proving in that way that

$$\xi^\varepsilon(u_{+*}^{\varepsilon\prime}(|u^\varepsilon > u_+^\varepsilon(\cdot)|)) j_t'(u_+^\varepsilon(\cdot), u_{+*}^\varepsilon(0)) \xrightarrow[\varepsilon\to 0]{} u_{+*}'(|u > u_+(\cdot)|) j_t'(u_+(\,\cdot\,), u_{+*}(0))$$

strongly in $L^p(\Omega) \,\forall p \in [1, +\infty[$. Applying Proposition 3 we can identify $\tilde{b}(x) = b_{*u}(|u > u(x)|)$ in Ω, $\hat{b}(s) = b_{*u}(s)$ in Ω_* and, in conclusion, u is a solution of $(\mathscr{P}_*)$. □

The main result of the paper, stated in the Introduction, is now an easy consequence f the above result:

'roof of Theorem 1. By Theorem 5 there exists u solution of $(\mathscr{P}_*)$ such that $u \in V(\Omega)$. Moreover, the assumptions of Theorem 2 are fulfilled and so the couple $(u, \mathscr{F})$ is a solution of $(\mathscr{P})$. $\square$

. Qualitative properties

In this section we shall give some qualitative properties on the founded solution u f $(\mathscr{P})$. They are the following: a condition for the existence of the free boundary, an estimate on the measure of the plasma region and an estimate of the L^1-norm of u_+. Let φ_1 be a normalized eigenfunction associated to the first eigenvalue λ_1 of the operator $-\Delta$ on Ω with Dirichlet boundary condition, i.e. $\varphi_1 \in H_0^1(\Omega)$ and $-\Delta\varphi_1 = \varphi_1$ on Ω. We know that $\varphi_1 > 0$ on Ω. Besides, we can renormalizate it such that $\int_\Omega \varphi_1 \, dx = 1$.

heorem 6. *Assume that*

$$-\gamma < F_v \int_\Omega a(x)\varphi_1(x)\,dx := -\gamma_0,$$

en any solution u of $(\mathscr{P}_*)$ satisfies $u_+ \not\equiv 0$.

roof. We argue as in [3, Theorem 8]. The proof relies on the identity

$$\lambda_1 \int_\Omega u\varphi_1 \, dx - \gamma = \int_\Omega a\mathscr{F}_u\varphi_1 \, dx + \int_\Omega H(u, b_{*u})\varphi_1 \, dx + \int_\Omega J(u)\varphi_1 \, dx \qquad (6.1)$$

here $\mathscr{F}_u(x) := [F_v^2 - 2F_1(x, u, b_{*u}) + 2F_2(x, u)]_+^{\frac{1}{2}}$ (see (4.2), (4.8) and (4.9)). Recall at u satisfies the equation (3.2), i.e. $-\Delta u = a\mathscr{F}_u + H(u, b_{*u}) + J(u)$ and then

$$\int_\Omega (u - \gamma)\nabla v \, dx = \int_\Omega a\mathscr{F}_u v \, dx + \int_\Omega H(u, b_{*u})v \, dx + \int_\Omega J(u)v \, dx \quad \forall v \in H_0^1(\Omega).$$

choosing $v = \varphi_1$ and using that $\int_\Omega \nabla(u - \gamma)\nabla\varphi_1 \, dx = \int_\Omega (u - \gamma)(-\Delta\varphi_1)\,dx$ we get (6.1) om the above identity. To end the proof of Theorem 6, we argue by contradiction. assume that $u_+ \equiv 0$. Thus, (6.1) reduces to

$$\lambda_1 \int_\Omega u\varphi_1 \, dx = \gamma + F_v \int_\Omega a\varphi_1 \, dx \qquad (6.2)$$

ice $F_1(x, u, b_{*u}) = H(u, b_{*u}) = F_2(x, u) = J(u) = 0$ if $u_+ \equiv 0$ (see Definitions (4.2), .5), (4.8) and (4.9)). In that case, the first integral of (6.2) is nonpositive, and us relation (6.2) implies that $-\gamma \geq F_v \int_\Omega a\varphi_1 \, dx$. This relation contradicts the choice γ. $\square$

Now, we shall estimate the measure of the plasma region

$$|u > 0| = \int_{\{u>0\}} dx.$$

This quantity can be estimated in terms of $\int_\Omega u_+ \, dx$ by using that

$$\int_\Omega u_+ \, dx \leq |u > 0| \max_\Omega u_+.$$

We already know that $\max_\Omega u_+ \leq S$ (see (5.5) Lemma 23). So, we have

$$|u > 0| \geq \frac{1}{S} \int_\Omega u_+ \, dx. \tag{6.3}$$

In order to estimate the L^1-norm of u_+ from below we use the identity (6.1). If w write $u = u_+ - u_-$ we obtain that

$$\int_\Omega [\mathscr{F}_u + \lambda_1 u_-] \varphi_1 \, dx + \gamma = \int_\Omega \varphi_1 [\lambda_1 - \lambda(b - b_{*u}(|u > u(x)|))] u_+ \, dx$$

$$- \int_\Omega J(u) \varphi_1 \, dx. \tag{6.4}$$

From (6.4), estimate (5.11) and the Lemma 23 we deduce that

$$\gamma - \gamma_0 + \int_\Omega a(x)[\mathscr{F}_u(x) - F_v] \varphi_1(x) \, dx \leq [\lambda_1 + \lambda \operatorname*{osc}_\Omega b] \|\varphi_1\|_{L^\infty(\Omega)}$$

$$\times \int_\Omega u_+(x) \, dx + \eta S \|\varphi_1\|_{L^\infty(\Omega)}. \tag{6.5}$$

Now, we define $\mathscr{F}_u^1$ by

$$\mathscr{F}_u^1(x) := [F_v^2 - 2F_1(x, u, b_{*u})]_+^{1/2}.$$

Then, it is clear that

$$\int_\Omega a[\mathscr{F}_u - F_v] \varphi_1 \, dx = \int_\Omega a[\mathscr{F}_u - \mathscr{F}_u^1] \varphi_1 \, dx + \int_\Omega a[\mathscr{F}_u^1 - F_v] \varphi_1 \, dx.$$

We estimate the last two integrals:

$$\cdot \quad \left| \int_\Omega a[\mathscr{F}_u^1 - F_v] \varphi_1 \, dx \right| \leq \lambda^{1/2} \|b\|_{L^\infty(\Omega)}^{1/2} \|a\|_{L^\infty(\Omega)} \int_\Omega u_+ \, dx$$

and

$$\left| \int_\Omega a[\mathscr{F}_u - \mathscr{F}_u^1] \varphi_1 \, dx \right| \leq 2^{1/2} \eta^{1/2} |\Omega|^{3/2} \|a\|_{L^\infty(\Omega)} \|\varphi_1\|_{L^\infty(\Omega)}.$$

hus, we get to

$$\left| \int_\Omega a[\mathscr{F}_u - F_v]\varphi_1 \, dx \right| \le \lambda^{\frac{1}{2}} \|b\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|a\|_{L^\infty(\Omega)} \int_\Omega u_+ \, dx$$
$$+ 2^{\frac{1}{2}} \eta^{\frac{1}{2}} |\Omega|^{\frac{3}{2}} \|a\|_{L^\infty(\Omega)} \|\varphi_1\|_{L^\infty(\Omega)}.$$

he last inequality and (6.5) becomes

$$\gamma - \gamma_0 \le L(\lambda) \int_\Omega u_+ \, dx + \eta S \|\varphi_1\|_{L^\infty(\Omega)} + 2^{1/2} \eta^{1/2} |\Omega|^{3/2} \|a\|_{L^\infty(\Omega)} \|\varphi_1\|_{L^\infty(\Omega)} \quad (6.6)$$

ith $L(\lambda) := [\lambda_1 + \lambda \operatorname{osc}_\Omega b + \lambda^{1/2} \|b\|_{L^\infty(\Omega)}^{1/2} \|a\|_{L^\infty(\Omega)}] \|\varphi_1\|_{L^\infty(\Omega)}$. From (6.6) we obtain

$$\frac{\gamma - \gamma_0 - O(\eta^{1/2})}{L(\lambda)} \le \int_\Omega u_+(x) \, dx.$$

conclusion (combining this estimate and (6.3)) we have shown

heorem 7. *Assume the hypotheses of Theorems 1 and 6. Let (u, F) be the solution $\ulcorner (\mathscr{P})$ obtained in Theorem 1. Then*

$$|u > 0| \ge \frac{\gamma - \gamma_0 - O(\eta^{1/2})}{SL(\lambda)} > 0$$

η *is small enough.* $\quad\square$

emark 1. *The model that we studied here concerns the case of a Stellarator ma-ine. Other models related on the magnetic confinement in Plasma Physics are rived for Tokamak machines. Most of these models have a different formulation. evertheless, the questions raised by those machines are just the same. For the sake completeness, we provide here a few references on Tokamak machines that can ide the reader: Description of the Tokamak machine and derivation of the model 26–29]); Existence, uniqueness and Control theory for local models ([6–9,30–35,?]).*

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