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Symmetrization Techniques on Unbounded Domains: Application to a Chemotaxis System on $\mathbb{R}^N$

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1. INTRODUCTION

This paper is concerned with the initial value problem to the parabolic–elliptic system on $\mathbb{R}^N$:

$$(\mathrm{P})\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } Q_T = (0, T) \times \mathbb{R}^N, \\ 0 = \Delta v - \gamma v + \alpha u & \text{in } \quad Q_T, \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R}^N, \end{cases}$$

where $N \geqslant 2$, and if χ, γ and α are positive numbers. It is always assumed that

$$u_0 \geqslant 0 \text{ on } \mathbb{R}^N \qquad \text{and} \qquad u_0 \in L^1(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N)(p > N). \tag{1.1}$$

For a solution of (P) on Q_T we mean a function (u, v) on Q_T satisfying the following: (i) $u \in C([0, T];\ W^{1,p}(\mathbb{R}^N)) \cap C^1((0, T];\ L^p(\mathbb{R}^N))$, $u(t, \cdot) \in W^{2,p}(\mathbb{R}^N)$ for $0 < t \leqslant T$; (ii) $v \in C((0, T];\ W^{2,p}(\mathbb{R}^N))$, (iii) (u, v) satisfies (P). In Appendix A it is mentioned that there exists uniquely a non-negative solution (u, v) of (P) on Q_T for some $T > 0$, which satisfies:

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(i) $\int_{\mathbb{R}^N} u(t,x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx$ and $\gamma \int_{\mathbb{R}^N} v(t,x)\,dx = \alpha \int_{\mathbb{R}^N} u_0(x)\,dx$; (ii) (u, v) becomes a classical solution on $(0, T) \times \mathbb{R}^N$; (iii) if $u_0 \not\equiv 0$, then $u(t,x) > 0$ and $v(t,x) > 0$ on $(0, T) \times \mathbb{R}^N$. Let $T_{\max}$ be a maximal existence time of the solution (u, v). We see that if $T_{\max} < +\infty$ then $\limsup_{t \to T_{\max}} \|u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} = +\infty$. If u_0 is radially symmetric on $\mathbb{R}^N$, then by the uniqueness of the solution and the symmetry of the problem the corresponding solution (u, v) of (P) is radially symmetric in the space variable x.

The system (P) is a simplified version of the mathematical model of chemotaxis (aggregation of organisms sensitive to a gradient of a chemical substance) proposed by Keller–Segel [20]. Blow-up phenomenon is conjectured to the chemotaxis system by Nanjundiah [25], Childress–Percus [5] and Childress [6]. For the system (P) on a bounded domain in $\mathbb{R}^N$ under homogeneous Neumann boundary conditions, the global existence and blow-up of solutions of (P) on a ball in $\mathbb{R}^N$ have been studied by Nagai [23] in radially symmetric situations. Under homogeneous Dirichlet boundary conditions on u, the global existence of solutions of (P) has been shown by Diaz–Nagai [11] without radially symmetric assumptions by using rearrangement techniques.

In [19], Jäger and Luckhaus have discussed the global existence and blow-up of solutions for another simplified version of the chemotaxis system on a bounded domain in $\mathbb{R}^2$ under homogeneous Neumann boundary conditions. Further study on blow-up problem to the system has been done by Herrero–Velázquez [16] in which they constructed a radial blow-up solution having a δ-function singularity in finite time. For recent study for the complete Keller–Segel system, we refer to Biler [4], Gajewski–Zacharias [13], Herrero–Velázquez [17, 18] and Nagai–Senba–Yoshida [24].

We give an application of symmetrized techniques to the problem (P) on $\mathbb{R}^N$ in Section 3. As an application of such techniques, we get a partial differential inequality on $\int_0^s u_*(t,\sigma)\,d\sigma$ for the solution (u, v) of (P), where $u_*(t,\cdot)$ denotes the decreasing rearrangement of $u(t,\cdot)$ with respect to the spatial variable (see Section 2 for the definition). In consequence of such a result, L^p-bounds of the solution (u, v) of (P) on $\mathbb{R}^2$ are given explicitly in terms of $\|u_0\|_{L^1(\mathbb{R}^2)}$ and $\|u_0\|_{L^\infty(\mathbb{R}^2)}$ under the condition $\alpha\chi \int_{\mathbb{R}^2} u_0\,dx < 8\pi$, and then we obtain the global existence of the solution on $\mathbb{R}^2$.

In applying symmetrization techniques on an unbounded domain, in contrast to Diaz–Nagai [11], the unboundedness of the spatial domain is an important additional difficulty. In particular we extend here a technical but key result concerning the regularity of $\partial u_*/\partial t(t,\cdot)$. The case when $u(t,\cdot)$ is defined on a bounded domain was studied in Mossino–Rakotoson [22] (see also Bandle [2] and Bandle–Stakgold [3]). The proof of [22] was based on the directional derivative of the map $u \mapsto u_*$ and it was needed to rearrange the function $u(t,\cdot) + h\,\partial u/\partial t(t,\cdot)$ which we cannot do anymore. However, we will keep the same trick but instead of using the directional derivative of $u \mapsto u_*$, we will use the directional derivative of the map $u \mapsto u_{+*}$, where $u_+ = \max\{u, 0\}$. In Section 2 we will show that $u_* \in W^{1,q}(0, T; L^p(0, +\infty))$ whenever $u \in W^{1,q}(0, T; L^p(\Omega))$, assumed that Ω is an unbounded open set of $\mathbb{R}^N$, $u \geqslant 0$ and $1 \leqslant q \leqslant +\infty$, $1 \leqslant p < +\infty$. Furthermore, if we set $\mu(t,\theta) = |u(t,\cdot) > \theta| = \text{meas}\{x \in \Omega, u(t,x) > \theta\}$ for $(t,\theta) \in [0, T] \times [0, +\infty)$, then for all $\theta > 0$

$$\int_0^{\mu(t,\theta)} \frac{\partial u_*}{\partial t}(t,\sigma)\,d\sigma = \int_{\{u(t,\cdot)>\theta\}} \frac{\partial u}{\partial t}(t,x)\,dx \qquad \text{for a.a. } t \in (0,T). \tag{1.2}$$

As mentioned above the solution (u, v) of (P) on $\mathbb{R}^2$ exists globally in time under the condition $\alpha\chi \int_{\mathbb{R}^2} u_0\,dx < 8\pi$. In the contrary under the condition $\alpha\chi \int_{\mathbb{R}^2} u_0\,dx > 8\pi$ it is possible for the solution (u, v) to blow up in finite time. In Section 4 under radially symmetric assumptions the following is shown: Let $N \geqslant 2$. Under the condition $\alpha\chi \int_{\mathbb{R}^2} u_0\,dx > 8\pi$ when $N = 2$, if $\int_{\mathbb{R}^N} u_0\,|x|^N\,dx$ is sufficiently small then the solution (u, v) blows up in finite time.

2. REGULARITY OF $\partial u_*/\partial t$ ON UNBOUNDED DOMAINS

Let Ω be a unbounded measurable subset of $\mathbb{R}^N$ and $u: \Omega \to [0, +\infty)$ a measurable function. The distribution function of u is defined by

$$\mu(\theta) = |u > \theta| = \text{meas}\{x \in \Omega, u(x) > \theta\} \qquad (\theta \geqslant 0),$$

and the decreasing rearrangement of u is the generalized inverse of μ that is the function $u_*: [0, +\infty) \to [0, +\infty]$ such that

$$u_*(s) = \inf\{\theta \geqslant 0, \mu(\theta) \leqslant s\}.$$

Let $T > 0$. For $u: [0, T] \times \Omega \to [0, +\infty)$ a measurable function let us set $u(t): \Omega \to [0, +\infty)$ for $t \in [0, T]$ such that $u(t)(x) = u(t,x)$. We denote by u_* the Steiner symmetrization of u with respect to the space variable x of Ω that is $u(t)_*: [0, +\infty) \to [0, +\infty]$ is the decreasing rearrangement of $u(t)$ and $u_*(t,s) = u(t)_*(s)$ for $t \in [0, T]$ and $s \in [0, +\infty)$.

As pointed out in the Introduction, our comparison symmetrization principle needs to use some regularity on the time derivative of the decreasing rearrangement of u. Our main goal in this section is to prove formula (1.2). The proof of (1.2) will rely on the following formula:

$$\frac{\partial u_*}{\partial t} = \frac{\partial w}{\partial s} \qquad \text{in } \mathscr{D}'((0, T) \times (0, +\infty)), \tag{2.1}$$

where, for $t \in (0, T)$ and $s \in (0, +\infty)$,

$$w(t, s) = \int_{\{u(t) > u(t)_*(s)\}} \frac{\partial u}{\partial t}(t, x)\, dx + \int_0^{s - |u(t) > u(t)_*(s)|} \left(\frac{\partial u}{\partial t}(t)_{|\{u(t) = u(t)_*(s)\}} \right)_* (\sigma)\, d\sigma.$$

But, we can only show (2.1) for $u \in W^{1,q}(0, T; L^p(\Omega))$ with $1 \leqslant p \leqslant q \leqslant +\infty$. Then, (1.2) is obtained by density argument for all $(p, q) \in [1, +\infty) \times [1, +\infty]$ knowing that we have

$$\left\| \frac{\partial u_*}{\partial t}(t) \right\|_{L^p(0, +\infty)} \leqslant \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)}.$$

The study of the case $p = +\infty$ is the main object of [31].

The first step that we have to do is to prove that w and $\partial w/\partial s$ make sense. This will be a consequence of the existence of directional derivative of $u \mapsto u_{+*}$.

2.1. *Rearrangement and the Function* $F(u) = \min_{k \geqslant 0} \int_\Omega (u-k)_+\, dx$

Let Ω be an unbounded measurable subset of $\mathbb{R}^N$ and $u: \Omega \to [0, +\infty)$ a measurable function. The function u_* enjoys some important properties as, for instance: $\forall\, \theta > 0$, $|u_* > \theta| = |u > \theta|$. Here and in the rest of this paper, $|E|$ denotes the measure of E whenever E is measurable and thus u_* preserves the norm of u. We will always use the following.

PROPOSITION 2.1 (Contraction Property). (i) *Let u be a measurable function on Ω, $v \in L^p(\Omega)$ $1 \leqslant p \leqslant +\infty$, then*

$$\|(u+v)_{+*} - u_{+*}\|_{L^p(0, +\infty)} \leqslant \|v\|_{L^p(\Omega)}.$$

(ii) *If $\Omega_1 \subset \Omega$, with $|\Omega_1|$ being finite, then*

$$\|((u+v)_{|\Omega_1})_* - (u_{|\Omega_1})_*\|_{L^p(0, |\Omega_1|)} \leqslant \|v\|_{L^p(\Omega_1)}.$$

Here, $(u+v)_{|\Omega_1}$ (resp. $u_{|\Omega_1}$) denotes the restriction of $u+v$ (resp. u) to the measurable set Ω_1.

For fixed $s \in [0, +\infty)$, we define, as in [30] and [29], the functions: for $(k, u) \in [0, +\infty) \times L^p(\Omega)$, $1 \leqslant p < +\infty$,

$$F(k, u) \doteq F_s(k, u) \doteq \int_\Omega (u-k)_+\, dx + k\, s,$$

$$F(u) \doteq F_s(u) \doteq \min_{k \geqslant 0} F_s(k, u).$$

The following easy properties are true.

PROPOSITION 2.2. (i) *The maps $(k, u) \in (0, +\infty) \times L^p(\Omega) \mapsto F(k, u)$ and $u \in L^p(\Omega) \mapsto F(u)$ are convex.*

(ii) $\forall k \geqslant 0$, $\forall u \in L^p(\Omega)$: $F_s(k, u) = F_s(k, u_+)$, $F_s(u) = F_s(u_+)$.

(iii) $F_s(u) = F_s(u_{+*}(s), u_+) = \int_0^s u_{+*}(\sigma)\, d\sigma$.

Remark. Statement (ii) can be obtained by a straightforward computation while (i) and (iii) are detailed in [30].

The above proposition shows us that for computing the directional derivative of the map $u \mapsto \int_0^s u_{+*}(\sigma)\, d\sigma$, it suffices to compute the directional derivative of $u \mapsto F_s(u)$ which we state in the following fundamental theorem.

THEOREM 2.1. *For $s \in [0, +\infty)$, we set*

$$F'(u, v) \doteq F'_s(u, v) \doteq \lim_{\lambda \searrow 0} \frac{F(u + \lambda v) - F(u)}{\lambda}$$

with $(u, v) \in L^p(\Omega) \times L^p(\Omega)$, $u \geqslant 0$. Then,

$$F'(u, v) = \begin{cases} \displaystyle\int_{\{u > u_*(s)\}} v(x)\, dx + \int_0^{s - |u > u_*(s)|} (v_{|\{u = u_*(s)\}})_*(\sigma)\, d\sigma \\ \quad \text{if} \quad u_*(s) \neq 0, \\ \displaystyle\int_{\{u > 0\}} v(x)\, dx + \max \left\{ \int_\Omega z(x)\, v(x)\, dx : 0 \leqslant z \leqslant 1, \right. \\ \quad \left. \operatorname{supp} z \subset \{u = 0\}, \quad \text{and} \int_{\{u=0\}} z(x)\, dx \leqslant s - |u > 0| \right\} \\ \quad \text{if} \quad u_*(s) = 0. \end{cases}$$

Here, $v_{\{u = u_*(s)\}}$ denotes the restriction of v to $\{u = u_*(s)\}$, supp z = the support of z and $1 \leqslant p < +\infty$.

A direct and important corollary of Theorem 2.1 is the following.

COROLLARY 2.1. (i) *Let* $u \in L^p(\Omega)$, $u \geqslant 0$. *If* $v \in L^p(\Omega)$ *is such that* $v \geqslant 0$ *on the set* $\{u=0\}$, *then for* $\forall s \in [0, +\infty)$,

$$F_s'(u, v) = \int_{\{u > u_*(s)\}} v(x)\, dx + \int_0^{s - |u > u_*(s)|} (v_{|\{u = u_*(s)\}})_*(\sigma)\, d\sigma = w(s).$$

(ii) *If* $v \in L^r(\Omega)$, $1 \leqslant r < +\infty$, *with* $v \geqslant 0$ *on the set* $\{u=0\}$, *then the conclusion of* (i) *remains true and*

$$\lim_{\lambda \searrow 0} \int_0^s \frac{(u+\lambda v)_{+*} - u_*}{\lambda}(\sigma)\, d\sigma = F_s'(u, v) \qquad \forall s \geqslant 0.$$

Proof. Since $v \geqslant 0$ on the set $\{u=0\}$, we can apply the Hardy–Littlewood equality (see [7] or [30] for instance) to get

$$\int_0^{s-|u>0|} (v_{|\{u=0\}})_*(\sigma)\, d\sigma$$

$$= \max\left\{\int_{\{u=0\}} z(x)\, v(x)\, dx : 0 \leqslant z \leqslant 1 \text{ and } \int_{\{u=0\}} z(x)\, dx \leqslant s - |u>0|\right\}$$

for s such that $u_*(s)=0$. Thus from Theorem 2.1, we derive statement (i) for the expression $F_s'(u, v)$. We recall that it is also true that

$$\int_0^{s-|u>0|} (v_{|\{u=0\}})_*(\sigma)\, d\sigma$$

$$= \max\left\{\int_E v_{|\{u=0\}}(x)\, dx,\ E \subset \{u=0\},\ |E| = s - |u>0|\right\}.$$

To prove that this expression of (i) remains true for $v \in L^r(\Omega)$, $1 \leqslant r < +\infty$, we argue by density. We consider $v_n \in L^\infty(\Omega)$ with support of v_n being compact, v_n converging to v in $L^r(\Omega)$ and $v_n \geqslant 0$ on the set $\{u=0\}$. By the contraction property of the monotone rearrangement, we deduce

$$(v_{n\,|\{u=u_*(s)\}})_* \text{ converges to } v_{|\{u=u_*(s)\}})_* \text{ in } L^1(0, s - |u > u_*(s)|) \tag{2.2}$$

and

$$\lim_{n \to +\infty} \int_{\{u > u_*(s)\}} v_n(x)\, dx = \int_{\{u > u_*(s)\}} v(x)\, dx \tag{2.3}$$

(notice that if there exist $s \in [0, +\infty)$ such that $u_*(s)=0$, then $|u>0|$ is finite and if $u_*(s)>0$, then $|u>u_*(s)|$ is finite). Thus, (2.2) and (2.3) infer that for all $s \geqslant 0$,

$$F_s'(u, v_n) \text{ converges to } F_s'(u, v) \text{ as } n \text{ tends to infinity.} \tag{2.4}$$

If $v \in L^p(\Omega)$, $v \geqslant 0$ on the set $\{u=0\}$, we deduce from Proposition 2.2 and statement (i) that

$$F_s(u+\lambda v) = \int_0^s (u+\lambda v)_{+*}(\sigma)\, d\sigma, \qquad F_s(u) = \int_0^s u_*(\sigma)\, d\sigma.$$

Thus,

$$\lim_{\lambda \searrow 0} w_\lambda^+(s) = \lim_{\lambda \searrow 0} \int_0^s \frac{(u+\lambda v)_{+*} - u_*}{\lambda}(\sigma)\, d\sigma = F_s'(u, v).$$

If $v \in L^r(\Omega)$, $1 \leqslant r < +\infty$, $v \geqslant 0$ on the set $\{u=0\}$, then we consider a sequence $v_n \in L^\infty(\Omega)$ with compact support, converging to v in $L^r(\Omega)$, $v_n \geqslant 0$ on the set $\{u=0\}$. We know already that for v_n

$$F_s'(u, v_n) = \lim_{\lambda \searrow 0} \int_0^s \frac{(u+\lambda v_n)_{+*} - u_*}{\lambda}(\sigma)\, d\sigma. \tag{2.5}$$

But from Hölder's inequality and the contraction property, we have

$$\left|\int_0^s \frac{(u+\lambda v_n)_{+*} - (u+\lambda v)_{+*}}{\lambda}(\sigma)\, d\sigma\right| \leqslant s^{1-(1/r)}\, \|v_n - v\|_{L^r(\Omega)}. \tag{2.6}$$

Thus, from relations (2.2) to (2.6), we easily have

$$\lim_{\lambda \searrow 0} \int_0^s \frac{(u+\lambda v)_{+*} - u_*}{\lambda}(\sigma)\, d\sigma = F_s'(u, v). \qquad \blacksquare$$

The proof of Theorem 2.1 follows an argument made in an unpublished paper of one of the authors [27] and extended in a general framework by Simon [30] (see also [29]).

It is well-known that

$$F_s'(u, v) = \max\left\{\int_\Omega qv\, dx,\ q \in \partial F_s(u)\right\},$$

where $\partial F_s(u)$ denotes the subdifferential of F_s at a point u (see [21] for instance). But an easy computation shows that $\partial F_s(u) \subset \partial F_s(u_*(s), u)$

(remember that $u \geqslant 0$). We can easily describe the set $\partial F_s(u_*(s), u)$ by the following.

PROPOSITION 2.3. *For all $s \in [0, +\infty)$, we have*

$$\partial F_s(u_*(s), u) = \{q \in L^{p'}(\Omega),\ 0 \leqslant q \leqslant 1,\ q = \mathscr{X}_{\{u > u_*(s)\}} + z,\ \operatorname{supp} z \subset \{u = u_*(s)\}\}.$$

Here $1/p + 1/p' = 1$, $\mathscr{X}_A$ denotes the characteristic function of a set A.

Proof. Let $q \in \partial F_s(u_*(s), u)$ that is: $\forall v \in L^{p'}(\Omega)$, we have

$$\int_\Omega q(v-u)\,dx \leqslant \int_\Omega (v - u_*(s))_+\,dx - \int_\Omega (u - u_*(s))_+\,dx. \tag{2.7}$$

We choose $\varphi \in L^\infty(\Omega)$ with compact support and we set for $\lambda > 0$, $v = u + \lambda\varphi\mathscr{X}_{\{u \neq u_*(s)\}}$. Plugging v in (2.7), dividing by λ and letting λ go to zero, we obtain for all φ with compact support:

$$\int_{\{u \neq u_*(s)\}} q\varphi\,dx = \int_{\{u > u_*(s)\}} \varphi\,dx.$$

Thus, $q(x) = \mathscr{X}_{\{u > u_*(s)\}}(x)$ if $u(x) \neq u_*(s)$. Now, we choose $v = u + \varphi\mathscr{X}_{\{u = u_*(s)\}}$ in (2.7), φ as before. Then, an easy argument shows that

$$\int_{\{u = u_*(s)\}} q\varphi\,dx \leqslant \int_{\{u = u_*(s)\}} \varphi_+\,dx. \tag{2.8}$$

Thus, if $\varphi \geqslant 0$, we deduce $q(x) \leqslant 1$ from (2.8), and if we choose $\varphi \leqslant 0$, then we have $q(x) \geqslant 0$ for x such that $u(x) = u_*(s)$. We easily conclude that for $x \in \Omega$

$$q(x) = \mathscr{X}_{\{u > u_*(s)\}}(x) + z(x), \qquad 0 \leqslant z \leqslant 1,$$

with support of $z \subset \{u = u_*(s)\}$.

Conversely, it is easy to check that any function $q \in L^{p'}(\Omega)$ having the above decomposition belongs to $\partial F_s(u_*(s), u)$. ∎

To get a complete description of $\partial F_s(u)$, we use the polar function F_s^* of F_s. Since it is well-known (see for instance [12]) that

$$\partial F_s(u) = \left\{q \in L^{p'}(\Omega),\ F_s^*(q) + F_s(u) = \int_\Omega qu\,dx\right\}.$$

We will compute $F_s^*(q)$ only for $q \in \partial F_s(u_*(s), u)$. We have the following.

PROPOSITION 2.4. *For $q \in \partial F_s(u_*(s), u)$, we have*

$$F_s^*(q) = \begin{cases} 0 & \text{if } \int_\Omega q(x)\,dx \leqslant s, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. From the definition of F_s^*, we have

$$\begin{aligned} F_s^*(q) &= \sup_{v \in L^p(\Omega)} \left\{\int_\Omega qv\,dx - \min_{k \geqslant 0}\left\{ks + \int_\Omega (v-k)_+\,dx\right\}\right\} \\ &= \sup_{k \geqslant 0}\left\{-ks + \sup_{v \in L^p(\Omega)}\left\{\int_\Omega qv\,dx - \int_\Omega (v-k)_+\,dx\right\}\right\}. \end{aligned}$$

Case 1. $q \in L^1(\Omega)$.

We can write

$$\int_\Omega qv\,dx = \int_\Omega q(v-k)_+\,dx - \int_\Omega q(v-k)_-\,dx + k\int_\Omega q\,dx,$$

and then, the expression of $F_s^*(q)$ becomes

$$\begin{aligned} F_s^*(q) = \sup_{k \geqslant 0}\Bigg\{k\left(\int_\Omega q(x)\,dx - s\right) \\ + \sup_{v \in L^p(\Omega)}\left\{\int_\Omega (q-1)(v-k)_+\,dx - \int_\Omega (v-k)_-\,dx\right\}\Bigg\}. \end{aligned}$$

Since $q \in \partial F_s(u_*(s), u)$, thus

$$\sup_v \left\{\int_\Omega (q-1)(v-k)_+\,dx - \int_\Omega (v-k)_-\,dx\right\} \leqslant 0.$$

A suitable choice of v implies that

$$\sup_v \left\{\int_\Omega (q-1)(v-k)_+\,dx - \int_\Omega (v-k)_-\,dx\right\} = 0,$$

and thus,

$$F_s^*(q) = \sup_{k \geqslant 0}\left\{k\left(\int_\Omega q(x)\,dx - s\right)\right\} = \begin{cases} 0 & \text{if } \int_\Omega q(x)\,dx \leqslant s, \\ +\infty & \text{otherwise.} \end{cases}$$

Case 2. $q \notin L^1(\Omega)$.

Necessarily, we have $u_*(s)=0$, then $|u>0| \leqslant s < +\infty$ and $\int_{\{u=0\}} q(x)\,dx = +\infty$. Since $q \in \partial F_s(u_*(s), u)$, then

$$\int_\Omega qv\,dx = \int_{\{u>0\}} v\,dx + \int_{\{u=0\}} qv\,dx$$

and

$$\int_{\{u>0\}} v\,dx = \int_{\{u>0\}} (v-k)_+\,dx - \int_{\{u>0\}} (v-k)_-\,dx + k\,|u>0|.$$

Replacing the value of the integral $\int_\Omega qv\,dx$ in the expression of $F_s^*(q)$, we get the following expression:

$$F_s^*(q) = \sup_{k \geqslant 0} \left\{ k(|u>0|-s) + \sup_{v \in L^p(\Omega)} \left\{ \int_{\{u=0\}} qv\,dx - \int_{\{u>0\}} (v-k)_-\,dx - \int_{\{u=0\}} (v-k)_+\,dx \right\} \right\}.$$

We want to show that $F_s^*(q) = +\infty$. For this, we consider $v_n = k(\mathscr{X}_{\{u>0\}} + \mathscr{X}_{E_n})$ where $E_n \subset E_{n+1} \subset \bigcup_{j \geqslant 0} E_j = \{u=0\}$, $|E_n| < \infty$, then

$$\int_{\{u=0\}} qv_n\,dx - \int_{\{u>0\}} (v_n-k)_-\,dx - \int_{\{u=0\}} (v_n-k)_+\,dx$$

$$= k \int_{E_n} q\,dx \xrightarrow[n \to +\infty]{} +\infty \qquad \text{if} \quad k \neq 0.$$

So, we get $F_s^*(q) = +\infty$. ∎

Proof of Theorem 2.1.

$$q \in \partial F_s(u) \Leftrightarrow \begin{cases} q(x) = \mathscr{X}_{\{u>u_*(s)\}}(x) + z(x), \qquad 0 \leqslant z \leqslant 1, \\ \quad \operatorname{supp} z \subset \{u = u_*(s)\}, \\ \displaystyle\int_{\{u=u_*(s)\}} z\,dx \leqslant s - |u>u_*(s)|, \\ \displaystyle u_*(s) \int_{\{u=u_*(s)\}} z\,dx = u_*(s)(s - |u>u_*(s)|). \end{cases}$$

So, if $u_*(s) > 0$, then

$$F_s'(u,v) = \int_{\{u>u_*(s)\}} v\,dx + \max \left\{ \int_{\{u=u_*(s)\}} zv\,dx : 0 \leqslant z \leqslant 1, \int_{\{u=u_*(s)\}} z\,dx = s - |u>u_*(s)| \right\}$$

Remember

$$F_s'(u,v) = \max \left\{ \int_\Omega qv\,dx,\ q \in \partial F_s(u) \right\}.$$

But $\{u = u_*(s)\}$ is of finite measure, so the Hardy–Littlewood equality implies

$$\int_0^{s-|u>u_*(s)|} (v_{|\{u=u_*(s)\}})_*(\sigma)\,d\sigma$$

$$= \max \left\{ \int_{\{u=u_*(s)\}} zv\,dx : 0 \leqslant z \leqslant 1, \int_{\{u=u_*(s)\}} z\,dx = s - |u>u_*(s)| \right\}.$$

If $u_*(s) = 0$, we only have

$$F_s'(u,v) = \int_{\{u>0\}} v\,dx + \max \left\{ \int_{\{u=0\}} zv\,dx : 0 \leqslant z \leqslant 1, \int_{\{u=0\}} z\,dx = s - |u>0| \right\}. \ ∎$$

COROLLARY 2.2. *Let* $u \in L^p(\Omega)$, $1 \leqslant p < +\infty$, $u \geqslant 0$ *and* $v \in L^r(\Omega)$, $1 \leqslant r < +\infty$ *such that* $v \geqslant 0$ *on the set* $\{u=0\}$. *We set for* $s \in [0, +\infty)$

$$w_\lambda{}^+(s) = \int_0^s \frac{(u+\lambda v)_{+*} - u_*}{\lambda}(\sigma)\,d\sigma, \qquad \lambda > 0,$$

$$w(s) = \int_{\{u>u_*(s)\}} v(x)\,dx + \int_0^{s-|u>u_*(s)|} (v_{|\{u=u_*(s)\}})_*(\sigma)\,d\sigma.$$

Then

(i) $w \in W^{1,r}(0, M)$ *for all* $M > 0$,

(ii) $dw/ds \in L^r(0, +\infty)$ *and* $\|dw/ds\|_{L^r(0,+\infty)} \leqslant \|v\|_{L^r(\Omega)}$,

(iii) $\lim_{\lambda \searrow 0} dw_\lambda^+/ds = dw/ds$ in $L^r(0, +\infty)$-*weak if* $1 < r < +\infty$ *and* $L^1(0, M)$-*weak for* $r = 1$ *and all* $M > 0$.

Proof. It follows the bounded case, we sketch it: by the contraction property, we have

$$\left\| \frac{dw_\lambda^+}{ds} \right\|_{L^r(0,+\infty)} \leqslant \|v\|_{L^r(\Omega)}, \qquad \|w_\lambda^+(s)\| \leqslant s^{1-(1/r)} \|v\|_{L^r(\Omega)}. \tag{2.9}$$

From Corollary 2.1, we know that $\lim_{\lambda \searrow 0} w_\lambda^+(s) = w(s)$ $\forall s \geqslant 0$. Thus, we deduce easily from (2.9) that statements (i), (ii) and (iii) are true for $1 < r < +\infty$. While for $r = 1$, we use the Dunford–Pettis criteria as for the bounded case (see [29] for instance). ∎

2.2. *Application to the Regularity of a Time Dependent Function*

Let $T > 0$ and Ω an unbounded domain of $\mathbb{R}^N$. For $u: [0, T] \times \Omega \to [0, +\infty)$ a measurable function, we shall study the regularity of the map $t \mapsto u(t)_*$ under some assumptions on u.

THEOREM 2.2. *Let $u \in W^{1,q}(0, T; L^p(\Omega))$, $1 \leqslant q \leqslant +\infty$, $1 \leqslant p < +\infty$ and $u \geqslant 0$. Then*

(i) *$u_* \in W^{1,q}(0, T; L^p(0, +\infty))$ and*

$$\left\| \frac{\partial u_*}{\partial t}(t) \right\|_{L^p(0,+\infty)} \leqslant \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \qquad \text{for a.a. } t,$$

(ii) *If we set $\mu(t, \theta) = |u(t) > \theta|$, then for all $\theta > 0$,*

$$\int_0^{\mu(t,\theta)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = \int_{\{u(t) > \theta\}} \frac{\partial u}{\partial t}(t, x)\, dx \qquad \text{for a.a. } t,$$

(iii) If $1 \leqslant p \leqslant q \leqslant +\infty$, *then*

$$\frac{\partial u_*}{\partial t} = \frac{\partial w}{\partial s} \qquad \text{in } \mathscr{D}'((0, T) \times (0, +\infty)),$$

where $w: (0, T) \times (0, +\infty) \to \mathbb{R}$ is defined as

$$w(t, s) = \int_{\{u(t) > u(t)_*(s)\}} \frac{\partial u}{\partial t}(t, x)\, dx + \int_0^{s - |u(t) > u(t)_*(s)|} \left(\frac{\partial u}{\partial t}(t)_{|\{u(t) = u(t)_*(s)\}} \right)_* (\sigma)\, d\sigma.$$

Proof. We first prove statement (iii) and by density argument we will derive statements (i) and (ii).

First step. Consider $\varphi \in \mathscr{D}((0, T) \times (0, +\infty))$, $u \geqslant 0$ being in $W^{1,q}(0, T; L^p(\Omega))$, $1 \leqslant p \leqslant q \leqslant +\infty$. We shall show that

$$-\int_0^T \int_0^{+\infty} u_*(t, \sigma) \frac{\partial \varphi}{\partial t}(t, \sigma)\, dt\, d\sigma = \int_0^T \int_0^{+\infty} \frac{\partial w}{\partial s}(t, \sigma)\, \varphi(t, \sigma)\, dt\, d\sigma. \tag{2.10}$$

For $h > 0$ consider the following integral

$$I(h) = \int_0^T \int_0^{+\infty} \frac{u(t+h)_*(\sigma) - u(t)_*(\sigma)}{h} \varphi(t, \sigma)\, dt\, d\sigma,$$

and let us introduce the following quantities:

$$\varepsilon(h)\ (0, T) \times \Omega \to R \qquad \text{such that } u(t+h) = u(t) + h \frac{\partial u}{\partial t}(t) + h\varepsilon(h)(t),$$

$$I_1(\varepsilon(h)) = \int_0^T \int_0^{+\infty} \frac{(u(t) + h\,(\partial u/\partial t)(t) + h\varepsilon(h)(t))_{+*} - (u(t) + h(\partial u/\partial t)(t))_{+*}}{h} (\sigma)\, \varphi(t, \sigma)\, dt\, d\sigma,$$

$$I_2(h, t) = \int_0^{+\infty} \frac{(u(t) + h(\partial u/\partial t)(t))_{+*}(\sigma) - u(t)_*(\sigma)}{h} \varphi(t,\sigma)\, d\sigma,$$

$$I_2(h) = \int_0^T I_2(h, t)\, dt.$$

Since $u(t+h) \geqslant 0$ (because $u \geqslant 0$), we can easily check that

$$I(h) = I_1(\varepsilon(h)) + I_2(h). \tag{2.11}$$

Study of $I_1(\varepsilon(h))$. Let $\delta > 0$ and $M > 0$ be such that $\operatorname{supp} \varphi \subset (\delta, T - \delta) \times (0, M)$. From Hölder's inequality and the contraction property, we have

$$|I_1(\varepsilon(h))| \leqslant \|\varphi\|_{L^\infty((\delta, T-\delta) \times (0, M))} M^{1-(1/p)} \|\varepsilon(h)\|_{L^1(\delta, T-\delta; L^p(\Omega))}, \tag{2.12}$$

since $\partial u/\partial t(t) \in L^p(\Omega)$.

The following lemma can be proved using a similar argument as in [22], and we will give the proof in Appendix B.

LEMMA 2.1. *If $\partial u/\partial t \in L^q(0, T; L^p(\Omega))$, $1 \leqslant p \leqslant q \leqslant +\infty$, then $\varepsilon(h)(t) = (u(t+h) - u(t))/h - \partial u/\partial t(t)$ converges to zero in $L^p_{loc}([0, T); L^p(\Omega))$.*

Lemma 2.1 and relation (2.12) imply that

$$\lim_{h \searrow 0} I_1(\varepsilon(h)) = 0. \tag{2.13}$$

Study of $I_2(h)$. A direct computation (or using Stampacchia's result) shows that $\partial u/\partial t(t) = 0$ a.e on $\{x : u(t)(x) = 0\} \doteq \{u(t) = 0\}$ for almost all t. Using statement (iii) of Corollary 2.2, we then have

$$\lim_{h \searrow 0} I_2(h, t) = \int_0^{+\infty} \frac{\partial w}{\partial s}(t, s)\, \varphi(t, s)\, ds \qquad \text{for a.a. } t.$$

Furthermore, applying Hölder's inequality and the contraction property, we have

$$\|I_2(h, t)\| \leqslant \|\varphi(t)\|_{L^{p'}(0, +\infty)} \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \doteq g(t) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

Since $\partial u/\partial t \in L^1(0, T; L^p(\Omega))$ and $\varphi \in \mathscr{D}((0, T) \times (0, +\infty))$, thus $g \in L^1(0, T)$. We now apply the Lebesgue dominate convergence theorem:

$$\lim_{h \searrow 0} I_2(h) = \int_0^T \int_0^{+\infty} \frac{\partial w}{\partial s}(t, s)\, \varphi(t, s)\, dt\, ds. \tag{2.14}$$

From (2.11), (2.13), and (2.14), we then have

$$\lim_{h \searrow 0} I(h) = \int_0^T \int_0^{+\infty} \frac{\partial w}{\partial s}(t, s)\, \varphi(t, s)\, dt\, ds. \tag{2.15}$$

By standard change of variable, we also have

$$\lim_{h \searrow 0} I(h) = -\int_0^T \int_0^{+\infty} u_*(t, \sigma) \frac{\partial \varphi}{\partial t}(t, \sigma)\, dt\, d\sigma. \tag{2.16}$$

From (2.15) and (2.16), we get (2.10). Since $\|\partial w/\partial s(t)\|_{L^p(0, +\infty)} \leqslant \|\partial u/\partial t(t)\|_{L^p(\Omega)}$ (see Corollary 2.2), we then have

$$\left\| \frac{\partial u_*}{\partial t}(t) \right\|_{L^p(0, +\infty)} \leqslant \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \qquad \text{for a.a. } t. \tag{2.17}$$

Second step. If $u \in W^{1,q}(0, T; L^p(\Omega))$, $1 \leqslant q < p < +\infty$, then we consider $u_n \in C^1([0, T]; L^p(\Omega))$, $u_n \geqslant 0$ such that u_n converges to u in

$W^{1,q}(0, T; L^p(\Omega))$. Then u_{n*} converges to u_* in $L^q(0, T; L^p(0, +\infty))$. From the preceding result, $u_{n*} \in W^{1,q}(0, T; L^p(0, +\infty))$ and

$$\left\| \frac{\partial u_{n*}}{\partial t}(t) \right\|_{L^p(0, +\infty)} \leqslant \left\| \frac{\partial u_n}{\partial t}(t) \right\|_{L^p(\Omega)} \qquad \text{for a.a. } t.$$

Hence we conclude that $u_* \in W^{1,q}(0, T; L^p(0, +\infty))$ and (2.17).

Third step. We want to show that for all $\theta > 0$

$$\int_0^{\mu(t, \theta)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = \int_{\{u(t) > \theta\}} \frac{\partial u}{\partial t}(t, x)\, dx \qquad \text{for a.a. } t. \tag{2.18}$$

If $1 \leqslant p \leqslant q \leqslant +\infty$, since $\partial u_*/\partial t = \partial w/\partial s$ in $\mathscr{D}'((0, T) \times (0, +\infty))$, we deduce that for almost all t and all $s \geqslant 0$

$$\int_0^s \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = w(t, s). \tag{2.19}$$

For such t, we set $\mu(t, \theta) = |u(t) > \theta| = \text{meas}\{x : u(t, x) > \theta\}$. For almost all $\theta > 0$, $|u(t) = \theta| = 0$. Thus from (2.19) and such $\theta > 0$,

$$\int_0^{\mu(t, \theta)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma = w(t, \mu(t, \theta)) = \int_{\{u(t) > \theta\}} \frac{\partial u}{\partial t}(t, x)\, dx.$$

But the maps

$$\theta \mapsto \int_0^{\mu(t, \theta)} \frac{\partial u_*}{\partial t}(t, \sigma)\, d\sigma \qquad \text{and} \qquad \theta \mapsto \int_{\{u(t) > \theta\}} \frac{\partial u}{\partial t}(t, x)\, dx$$

are continuous from the right, thus we get (2.18) for all $\theta > 0$.

If $1 \leqslant q < p \leqslant +\infty$, we consider $\bar{\Omega}_j \subset \Omega_{j+1} \subset \cdots \subset \bigcup_{j \geqslant 0} \Omega_j = \Omega$, $|\Omega_j| < +\infty$ and $\theta_j(x) = \mathscr{X}_{\Omega_j}(x)$, $x \in \Omega$. Then, we have $\theta_j u \in W^{1,q}(0, T; L^p(\Omega) \cap L^q(\Omega))$. We then apply the preceding result to get

$$\frac{\partial}{\partial t}(\theta_j u)_* = \frac{\partial w_j}{\partial s} \qquad \text{in } \mathscr{D}'((0, T) \times (0, +\infty)), \tag{2.20}$$

where w_j is defined as w but $\partial u/\partial t(t)$ is replaced by $\theta_j\, \partial u/\partial t(t)$ and $u(t)$ by $\theta_j u(t)$. Furthermore, since $\theta_j\, \partial u/\partial t(t) \in L^p(\Omega)$, Corollary 2.2 shows that

$$\left\| \frac{\partial w_j}{\partial s}(t) \right\|_{L^p(0, +\infty)} \leqslant \left\| \theta_j \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \leqslant \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)}.$$

Thus, from (2.20), we deduce $(\theta_j u)_* \in W^{1,q}(0, T; L^p(0, +\infty) \cap L^q(0, +\infty))$ and

$$\left\| \frac{\partial}{\partial t} (\theta_j u)_* (t) \right\|_{L^p(0, +\infty)} \leqslant \left\| \frac{\partial u}{\partial t}(t) \right\|_{L^p(\Omega)} \quad \text{for a.a. } t. \tag{2.21}$$

But, we know that $\lim_{j \to \infty} (\theta_j u)_* = u_*$ in $L^q(0, T; L^p(0, +\infty))$ by the contraction property, thus with relation (2.21), we conclude that

$$\frac{\partial}{\partial t} (\theta_j u)_* \text{ converges weakly to } \frac{\partial u_*}{\partial t} \text{ in } L^q(0, T; L^p(0, +\infty)). \tag{2.22}$$

Is is easy to check that for all $\theta > 0$,

$$\mu_j(t, \theta) = \text{meas}\{x : \theta_j(x)\, u(t, x) > \theta\} \text{ converges to } \mu(t, \theta). \tag{2.23}$$

Relation (2.18) is true for $\theta_j u \in W^{1,q}(0, T; L^q(\Omega) \cap L^p(\Omega))$:

$$\int_0^{\mu_j(t, \theta)} \frac{\partial}{\partial t} (\theta_j u)_* (t, \sigma)\, d\sigma = \int_{\{\theta_j u(t) > \theta\}} \theta_j(x) \frac{\partial u}{\partial t}(t, x)\, dx. \tag{2.24}$$

Letting j go to infinity, we derive from (2.22) to (2.24) that relation (2.18) is true for $u \in W^{1,q}(0, T; L^p(\Omega))$ with $1 \leqslant q < p < +\infty$. ∎

Remark. There are many possible different applications of formula (1.2). So, for instance, now it is possible to extend to unbounded domains some results obtained for bounded domains on variational inequalities (Diaz–Mossino [10]), the Stefan problem (Gustafsson–Mossino [14]) and parabolic quasilinear equations (Diaz [9]), among others.

3. APPLICATION OF SYMMETRIZATION TECHNIQUES TO THE PROBLEM (P)

Let u_0 satisfy (1.1) and let (u, v) be the corresponding solution of (P). Define the function $k(t, s)$ on $[0, T_{\max}) \times [0, \infty)$ by

$$k(t, s) = \int_0^s u_*(t, \sigma)\, d\sigma,$$

where $T_{\max}$ is the maximal existence time of (u, v) and $u_*(t, s)$ is the decreasing rearrangement of $u(t, x)$ with respect to x. Then we have the following.

PROPOSITION 3.1. *The function $k(t, s)$ satisfies*

$$k \in L^\infty([0, T_{\max}) \times (0, +\infty)) \cap H^1(0, T_{\max}: W^{1,p}_{loc}(0, +\infty))$$
$$\cap\, L^2(0, T: W^{2,p}_{loc}(0, +\infty)).$$

and

$$\begin{cases} \dfrac{\partial k}{\partial t} - d(s) \dfrac{\partial^2 k}{\partial s^2} - \alpha \chi k \dfrac{\partial k}{\partial s} \leqslant 0 & \textit{a.e. in} \quad (0, T_{\max}) \times (0, +\infty), \\ k(t, 0) = 0, \qquad k(t, +\infty) = \displaystyle\int_{\mathbb{R}^N} u_0\, dx & \textit{for} \quad t \in [0, T_{\max}), \\ k(0, s) = \displaystyle\int_0^s u_{0*}(\sigma)\, d\sigma & \textit{for} \quad s \geqslant 0, \end{cases} \tag{3.1}$$

where $d(s) = N^2 \kappa_N^{2/N} s^{2(N-1)/N}$ and κ_N is the volume of the unit ball in $\mathbb{R}^N$.

The proof of the differential inequality in (3.1) as well as of the rest of properties follows as in Diaz–Nagai [Lemma 4 in [11]] once the formula (1.2) is established on unbounded domains.

For the proof of the boundedness of solutions (u, v) to (P) on $\mathbb{R}^2$, we need a comparison principle for functions satisfying the differential inequalities in Proposition 3.1. In the following proposition, $C(t)$ denotes a generic positive function in $L^2(0, T)$.

PROPOSITION 3.2. *Let f and g be functions on $Q_T^* = [0, T] \times (0, +\infty)$ such that*

(i) $f, g \in L^\infty(Q_T^*) \cap L^2(0, T; W^{2,2}_{loc}(0, +\infty))$, $\partial f/\partial t$, $\partial g/\partial t \in {}^2(0, T; L^2_{loc}(0, +\infty))$,

(ii) $|\partial f/\partial s(t, s)| \leqslant C(t)$ *and* $|\partial g/\partial s(t, s)| \leqslant C(t) \max\{s^{-\ell}, 1\}$, *where ℓ is a constant satisfying* $0 \leqslant \ell < 1$.

If f and g satisfy the following

$$\begin{cases} \dfrac{\partial f}{\partial t} - d(s) \dfrac{\partial^2 f}{\partial s^2} - \alpha \chi f \dfrac{\partial f}{\partial s} \leqslant \dfrac{\partial g}{\partial t} - d(s) \dfrac{\partial^2 g}{\partial s^2} - \alpha \chi g \dfrac{\partial g}{\partial s} & \text{a.e. } \textit{in } Q_T^*, \\ 0 = f(t, 0) \leqslant g(t, 0) \quad \text{and} \quad f(t, +\infty) \leqslant g(t, +\infty) & \textit{for any} \quad t \in [0, T], \\ f(0, s) \leqslant g(0, s) \quad \textit{on } (0, +\infty) \quad \textit{and} \quad g(t, s) \geqslant 0 & \textit{on } Q_T^*, \end{cases}$$

then $f \leqslant g$ on Q_T^*.

Proof. Put $w = f - g$, which satisfies

$$\frac{\partial w}{\partial t} - d(s)\frac{\partial^2 w}{\partial s^2} - \alpha\chi\left(w\frac{\partial f}{\partial s} + g\frac{\partial w}{\partial s}\right) \leqslant 0 \qquad \text{a.e. in } Q_T^*. \tag{3.2}$$

By multiplying (3.2) by $s^{2(1-N)/N}w_+$ and integrating over (δ, L) $(0 < \delta < 1 < L)$, the integration by parts and $|\partial f/\partial s| \leqslant C(t)$ yield that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\delta^L & s^{2(1-N)/N}(w_+)^2\,ds + N^2\kappa_N^{2/N}\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2 ds \\
&\leqslant \alpha\chi\int_\delta^L s^{2(1-N)/N}\left\{(w_+)^2\frac{\partial f}{\partial s} + w_+\frac{\partial w}{\partial s}g\right\}ds + G(t,\delta,L) \\
&\leqslant C(t)\int_\delta^L s^{2(1-N)/N}(w_+)^2\,ds + \alpha\chi\int_\delta^L s^{2(1-N)/N}w_+\frac{\partial w}{\partial s}g\,ds + G(t,\delta,L)
\end{aligned}$$

a.e. in Q_T^*, where

$$G(t,\delta,L) = Const.\left\{\left|\frac{\partial w}{\partial s}(t,\delta)\right| w_+(t,\delta) + \left|\frac{\partial w}{\partial s}(t,L)\right| w_+(t,L)\right\}$$

which satisfies

$$G(t,\delta,L) \leqslant C(t)^2, \qquad G(t,\delta,L) \to 0 \qquad \text{as} \quad \delta \to 0 \quad \text{and} \quad L \to \infty.$$

By $f(t, 0) = 0$ and (ii), we see that $f^+(t,s) \leqslant C(t)s$ on $(0, \infty)$, from which together with $f \in L^\infty(Q_T^*)$ it follows that $s^{(1-N)/N}f^+(t,s) \leqslant C(t)$ *on* $(0, T) \times (0, +\infty)$. We then obtain

$$\begin{aligned}
\alpha\chi & \int_\delta^L s^{2(1-N)/N}w_+\frac{\partial w}{\partial s}g\,ds \\
&\leqslant \alpha\chi\int_\delta^L s^{2(1-N)/N}w_+\left|\frac{\partial w_+}{\partial s}\right| f^+\,ds \\
&\leqslant \frac{1}{2}N^2\kappa_N^{2/N}\int_\delta^L\left(\frac{\partial w_+}{\partial s}\right)^2 ds + C(t)^2\int_\delta^L s^{2(1-N)/N}(w_+)^2\,ds.
\end{aligned}$$

Hence, for $t \in (0, T)$,

$$\frac{d}{dt}\int_\delta^L s^{2(1-N)/N}w_+^2\,ds \leqslant C(t)^2\int_\delta^L s^{2(1-N)/N}w_+^2\,ds + G(t,\delta,L),$$

from which together with $w_+(0, s) = 0$ on $(0, +\infty)$ it follows that

$$\int_\delta^L s^{2(1-N)/N}w_+^2\,ds \leqslant e^{\int_0^t C(\tau)^2\,d\tau}\int_0^t e^{\int_0^\tau C(\sigma)^2\,d\sigma}G(t,\delta,L)\,d\tau \tag{3.3}$$

for $t \in (0, T)$. Letting $\delta \to 0$ and $L \to \infty$ in (3.3) yields that

$$\int_0^\infty s^{2(1-N)/N}w_+^2\,ds = 0 \qquad \text{for} \quad t \in (0, T),$$

which implies $w_+ = 0$ in Q_T^*. Hence, $f \leqslant g$. ∎

As an application of Proposition 3.1, we give the boundedness of solutions (u, v) to (P) in $\mathbb{R}^2$.

THEOREM 3.1. *Let u_0 be a function on $\mathbb{R}^2$ satisfying* (1.1) *and (u, v) the corresponding solution of* (P). *If $\alpha\chi\int_{\mathbb{R}^2}u_0\,dx < 8\pi$, then $T_{\max} = \infty$ and*

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \leqslant L(u_0,\alpha,\chi,p), \qquad \|v(t)\|_{L^p(\mathbb{R}^2)} \leqslant \frac{\alpha}{\gamma}L(u_0,\alpha,\chi,p) \tag{3.4}$$

for any $t \geqslant 0$ and any $p \in [1, +\infty]$, where

$$L(u_0,\alpha,\chi,p) = \begin{cases}
\dfrac{8\pi}{\alpha\chi}(2p-1)^{-1/p}\|u_0\|_{L^\infty(\mathbb{R}^2)}^{1-1/p}\left(\dfrac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)}\right)^{1/p-1} \\
\qquad \text{if} \quad p \geqslant 1, \\
\dfrac{8\pi}{\alpha\chi}\|u_0\|_{L^\infty(\mathbb{R}^2)}\left(\dfrac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)}\right)^{-1} \\
\qquad \text{if} \quad p = +\infty.
\end{cases}$$

Proof. By Proposition 3.1 and $d(s) = 4\pi s$, the function $k(t,s) = \int_0^s u_*(t,\sigma)\,d\sigma$ satisfies

$$\frac{\partial k}{\partial t} - 4\pi s\frac{\partial^2 k}{\partial s^2} - \alpha\chi k\frac{\partial k}{\partial s} \leqslant 0 \qquad \text{in} \quad (0, T_{\max}) \times (0, +\infty).$$

Let us define the function $w(s)$ by

$$w(s) = \frac{8\pi q s}{\alpha\chi(1 + qs)} \qquad \text{for} \quad s \geqslant 0,$$

where q is a positive constant determined below. The function w satisfies

$$4\pi s w'' + \alpha\chi w w' = 0 \qquad \text{on} \quad (0, +\infty).$$

By noting that $k(0, +\infty) = \int_{\mathbb{R}^2} u_0\, dx < 8\pi/(\alpha\chi) = w(+\infty)$, it is shown that $k(0, s) \leqslant w(s)$ on $[0, +\infty)$ whenever

$$q \geqslant \|u_0\|_{L^\infty(\mathbb{R}^2)} \left(\frac{8\pi}{\alpha\chi} - \|u_0\|_{L^1(\mathbb{R}^2)}\right)^{-1}. \tag{3.5}$$

For any $t > 0$ we also have $k(t, +\infty) = \int_{\mathbb{R}^2} u_0\, dx < w(+\infty)$. Hence, applying Proposition 3.2 gives.

$$k(t, s) \leqslant w(s) \quad \text{on} \quad (0, T_{\max}) \times (0, +\infty). \tag{3.6}$$

By a simple application of a lemma in [2, p. 74] or Lemma 1.33 in [8], (3.6) implies

$$\|u_*(t)\|_{L^p(0,\infty)} \leqslant \|w'\|_{L^p(0,\infty)} \qquad (0 \leqslant t < T_{\max}) \tag{3.7}$$

for any $p \in [1, +\infty]$. We take q as the equal sign in (3.5) so that $\|w'\|_{L^p(0,\infty)} = L(u_0, \alpha, \chi, p)$. Then it follows from (3.7) and $\|u_*(t)\|_{L^p(0,\infty)} = \|u(t)\|_{L^p(\mathbb{R}^2)}$ that the desired inequality on u in (3.4) is derived. Finally, the inequality on v in (3.4) is derived from the inequality on u in (3.4) and the following inequality

$$\|v(t)\|_{L^p(\mathbb{R}^2)} \leqslant \frac{\alpha}{\gamma} \|u(t)\|_{L^p(\mathbb{R}^2)} \qquad (0 \leqslant t < T_{\max})$$

for any $p \in [1, +\infty]$. Therefore, $T_{\max} = +\infty$ and the proof is complete. ∎

4. BLOW-UP OF RADIALLY SYMMETRIC SOLUTIONS

Throughout this section, we always assume that u_0 is radially symmetric in x. Hence, the solution (u, v) of (P) with the initial function u_0 is radially symmetric in x. In order to show the blow-up of radially symmetric solution (u, v) under the condition such that $\int_{\mathbb{R}^N} u_0(x)\, |x|^N\, dx$ is sufficiently small, we use a method in [23] where the blow-up problem to (P) on bounded domains is considered. Following the method in [23] we make an inequality on $\int_{\mathbb{R}^N} u(t, x)\, |x|^N\, dx$.

LEMMA 4.1. *Assume that* $\int_{\mathbb{R}^N} u_0(x)\, |x|^N\, dx < +\infty$. *Then for* $t \in (0, T_{\max})$ $M(t) = \int_{\mathbb{R}^N} u(t, x)\, |x|^N\, dx < +\infty$, *and the following inequality holds*:

$$M(t) \leqslant M(0) + \int_0^t F(M(s))\, ds \qquad \textit{for} \quad t \in [0, T_{\max}),$$

where

$$F(M) = 2N(N-1)\, \|u_0\|_{L^1(\mathbb{R}^N)}^{2/N} M^{(N-2)/N} - \frac{\alpha\chi N}{2\omega_N} \|u_0\|_{L^1(\mathbb{R}^N)}^2 + G(M),$$

$$G(M) = C(\alpha, \gamma, \chi) \times \begin{cases} \|u_0\|_{L^1(\mathbb{R}^N)}^{3/2} M^{1/2} & \textit{if} \quad N = 2, \\ \|u_0\|_{L^1(\mathbb{R}^N)}^{(2N-2)/N} M^{2/N} & \textit{if} \quad N \geqslant 3, \end{cases}$$

and ω_N *is the surface area of the unit sphere in* $\mathbb{R}^N$ *and* $C(\alpha, \gamma, \chi)$ *is a positive constant depending only on* α, γ *and* χ.

Proof. For $m = 1, 2, 3, \ldots$, let us take functions $\psi_m \in C^2([0, +\infty))$ such that

$$0 \leqslant \psi_m(r) \leqslant 1 \quad (r \geqslant 0), \qquad \psi_m(r) = 1 \quad (r \leqslant m),$$
$$\psi_m'(r) \leqslant 0, \qquad |\psi_m'(r)| \leqslant C\psi_m(r), \qquad |\psi_m''(r)| \leqslant C\psi_m(r),$$

where C is a constant independent of m. Multiplying the first equation in (P) by $|x|^N \psi_m(|x|)$ and integrating over $\mathbb{R}^N$ gives

$$\frac{d}{dt} M_m(t) = \int_{\mathbb{R}^N} u\Delta(|x|^N \psi_m)\, dx + \chi \int_{\mathbb{R}^N} u\nabla v \cdot \nabla(|x|^N \psi_m)\, dx, \tag{4.1}$$

where

$$M_m(t) = \int_{\mathbb{R}^N} u(t, x)\, |x|^N \psi_m(|x|)\, dx.$$

We use the properties of ψ_m and Hölder's inequality to estimate the first term on the right-hand side of (4.1) as follows:

$$\begin{aligned} &\int_{\mathbb{R}^N} u\Delta(|x|^N \psi_m)\, dx \\ &\leqslant 2N(N-1) \int_{\mathbb{R}^N} u\, |x|^{N-2} \psi_m\, dx + C \int_{|x| \geqslant m} u\, |x|^N \psi_m\, dx \\ &\leqslant 2N(N-1)\, \|u_0\|_{L^1(\mathbb{R}^N)}^{2/N} \{M_m(t)\}^{(N-2)/N} + C \int_{|x| \geqslant m} u\, |x|^N \psi_m\, dx. \end{aligned} \tag{4.2}$$

By using the second equation in (P) and the radially symmetric property of (u, v) with respect to x, calculations similar to those in the proof of Lemma 3.2 in [23] give us the following:

$$\chi \int_{\mathbb{R}^N} u\, \nabla v \cdot \nabla(|x|^N \psi_m)\, dx$$

$$\leqslant -\frac{\alpha\chi N}{2\omega_N} \|u_0\|^2_{L^1(\mathbb{R}^N)} + \frac{\alpha\chi N}{\omega_N} \int_{\mathbb{R}^N} uB(t, |x|)\, \psi_m(|x|)\, dx$$

$$+ C \int_{|x| \geqslant m} u\, |x|^N \psi_m\, dx + C\left(\int_{|x| \geqslant m} u\, dx\right)^2, \tag{4.3}$$

where $B(t, r) = \int_{|x| \leqslant r} v(t, y)\, dy$. For each t the function $B(t, \cdot)$ satisfies

$$\begin{cases} r^{N-1} \dfrac{\partial}{\partial r}\left(r^{1-N} \dfrac{\partial B}{\partial r}\right) = \alpha A - \gamma B & \text{for} \quad r > 0, \\ B(t, 0) = 0, \qquad B(t, +\infty) = \|v(t)\|_{L^1(\mathbb{R}^N)} = \dfrac{\alpha}{\gamma} \|u_0\|_{L^1(\mathbb{R}^N)}, \end{cases}$$

where $A(t, r) = \int_{|x| \leqslant r} u(t, y)\, dy$. Then $B(t, r)$ is estimated as

$$B(t, r) \leqslant C\, \|u_0\|_{L^1(\mathbb{R}^N)} \times \begin{cases} r & \text{if} \quad N = 2, \\ r^2 & \text{if} \quad N \geqslant 3, \end{cases}$$

where C is a positive constant depending only on α and γ. Using this inequality and Hölder's inequality, we have

$$\frac{\alpha\chi N}{\omega_N} \int_{\mathbb{R}^N} uB\psi_m\, dx$$

$$\leqslant G(M_m) = C(\alpha, \gamma, \chi) \times \begin{cases} \|u_0\|^{3/2}_{L^1(\mathbb{R}^N)} M_m^{1/2} & \text{if} \quad N = 2, \\ \|u_0\|^{(2N-2)/N}_{L^1(\mathbb{R}^n)} M_m^{2/N} & \text{if} \quad N \geqslant 3. \end{cases}$$

Hence, by (4.1) to (4.3),

$$\frac{d}{dt} M_m(t) \leqslant F(M_m(t)) + C \int_{|x| \geqslant m} u\, |x|^N \psi_m\, dx + C\left(\int_{|x| \geqslant m} u\, dx\right)^2. \tag{4.4}$$

From (4.4) it follows that

$$\frac{d}{dt} M_m(t) \leqslant C(M_m(t) + 1),$$

which implies that $M_m(t) \leqslant (M(0) + 1) \exp\{Ct\}$ $(m \geqslant 1)$. Letting $m \to \infty$, we have $M(t) \leqslant (M(0) + 1) \exp\{Ct\}$ for $t \in (0, T_{\max})$. By integrating (4.4) on $(0, t)$ and letting $m \to \infty$, the desired inequality on $M(t)$ is obtained. ∎

THEOREM 4.1. *Let $N \geqslant 2$. If $F(M(0)) < 0$, then $T_{\max} < +\infty$ and*

$$\limsup_{t \to T_{\max}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = \infty.$$

Proof. Assume $T_{\max} = +\infty$. Define the function $H(t)$ on $[0, +\infty)$ by

$$H(t) = M(0) + \int_0^t F(M(s))\, ds.$$

From Lemma 4.1 it follows that $M(t) \leqslant H(t)$ for $t > 0$. Since $F(M)$ is increasing in M, we see that $F(M(t)) \leqslant F(H(t))$ and

$$H'(t) \leqslant F(H(t)). \tag{4.5}$$

By $F(H(0)) = F(M(0)) < 0$, (4.5) implies that there exists $T_0 > 0$ such that $H(t) = 0$ for $t \geqslant T_0$. Hence, we get $M(t) = 0$ for $t \geqslant T_0$, which contradicts $u(t, x) > 0$ for $x \in \mathbb{R}^N$ add $t > 0$. Therefore, we obtain the conclusion of the theorem. ∎

The following corollary is an immediate consequence of Theorem 4.1.

COROLLARY 4.1. *Let $N \geqslant 2$. Assume that $\alpha\chi \int_{\mathbb{R}^2} u_0\, dx > 8\pi$ when $N = 2$. Then there exists a positive constant c depending on $\int_{\mathbb{R}^N} u_0\, dx$ such that if $0 < \int_{\mathbb{R}^N} u_0\, |x|^N\, dx < c$ then $T_{\max} < +\infty$ and*

$$\limsup_{t \to T_{\max}} \|u(t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.$$

APPENDIX A: LOCAL EXISTENCE IN TIME

Let (u, v) be a solution of (P) on Q_T. Since $u \in C([0, T]; W^{1,p}(\mathbb{R}^N))$ and $W^{1,p}(\mathbb{R}^N) \subset L^\infty(\mathbb{R}^N)$ with continuous inclusion (see [15]) because of $p > N$, we have

$$\sup_{0 \leqslant t \leqslant T} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant Const. \sup_{0 \leqslant t \leqslant T} \|u(t)\|_{W^{1,p}(\mathbb{R}^N)} < \infty.$$

Similar arguments in Lemmas 1 and 2 in [11] give the following.

PROPOSITION A1. (i) *The uniqueness holds for the problem (P) on Q_T.*

(ii) *Let (u, v) be a solution of (P) on Q_T. Then $u(t, x) \geqslant 0$ and $v(t, x) \geqslant 0$ on Q_T.*

The following proposition shows that the conservation of total mass on $\mathbb{R}^N$ holds.

PROPOSITION A2. *Let (u, v) be a solution of* (P) *on Q_T. Then*

$$\int_{\mathbb{R}^N} u(t, x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx,$$

$$\int_{\mathbb{R}^N} v(t, x)\,dx = \frac{\alpha}{\gamma} \int_{\mathbb{R}^N} u_0(x)\,dx \qquad \textit{for} \quad 0 < t \leqslant T.$$

Proof. Let us take $\varphi_m(x)$ such that

$$\varphi_m(x) = 1 \quad (|x| \leqslant m), \qquad \varphi_m(x) = (|x| - m + 1)^{-N} \quad (|x| > m).$$

By multiplying the first equation in (P) by φ_m and integrating over $\mathbb{R}^N \times (0, t)$, the integration by parts yields that

$$\int_{\mathbb{R}^N} u(t, x)\,\varphi_m(x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,\varphi_m(x)\,dx + \int_0^t H_m(s)\,ds, \tag{A.1}$$

where

$$H_m(t) = \int_{\mathbb{R}^N} \{-\nabla u(t, x) + \chi u(t, x)\,\nabla v(t, x)\} \cdot \nabla \varphi_m(x)\,dx.$$

Since $H_m(t) \to 0$ as $m \to +\infty$, we first get $u(t) \in L^1(\mathbb{R}^N)$ from (A.1), and then by letting $m \to +\infty$ in (A.1) we get

$$\int_{\mathbb{R}^N} u(t, x)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx.$$

Similarly, integrating the second equation of (P) on $\mathbb{R}^N$, we have

$$\gamma \int_{\mathbb{R}^N} v(t, x)\,dx = \alpha \int_{\mathbb{R}^N} u(t, x)\,dx = \alpha \int_{\mathbb{R}^N} u_0(x)\,dx,$$

which completes the proof of the proposition. ∎

As concerns the local existence of solutions to the problem (P) on Q_T and its regularity, using similar arguments in Theorem 1 in [11] we obtain

PROPOSITION A3. (i) *There exists a positive number T such that* (P) *has a unique solution (u, v) on Q_T, which becomes a classical solution.*

(ii) *If $u_0 \not\equiv 0$, then $u(t, x) > 0$ and $v(t, x) > 0$ on Q_T.*

(iii) *Let $T_{\max}$ be a maximal existence time of (u, v). If $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant$ Const. on $(0, T_{\max})$, then $T_{\max} = +\infty$.*

APPENDIX B: PROOF OF LEMMA 2.1

We set

$$g_h(t, x) = \frac{u(t+h) - u(t)}{h}(x), \qquad h > 0.$$

Let us recall the proof of the following result (see [26]).

LEMMA B1. *Let $u \in W^{1,q}(0, T; L^p(\Omega))$, $1 \leqslant q \leqslant +\infty$, $1 \leqslant p \leqslant +\infty$. Then, if $\Omega_1 \subset \Omega$ with measure of Ω_1 being finite, then g_h converges to $\partial u/\partial t$ in $L^\alpha_{loc}([0, T); L^\alpha(\Omega_1))$ with $\alpha = \min(p, q)$.*

Proof. Let $\alpha = \min(p, q)$, then $\partial u/\partial t \in L^\alpha(0, T; L^\alpha(\Omega_1))$. One can check that for $0 < h < \delta < T$,

$$\left(\int_0^{T-\delta} \int_{\Omega_1} |g_h(t, x)|^\alpha\,dx\,dt\right)^{1/\alpha} \leqslant \left(\int_0^{T-\delta+h} \int_{\Omega_1} \left|\frac{\partial u}{\partial t}(t, x)\right|^\alpha dx\,dt\right)^{1/\alpha} \leqslant \text{Const.}$$

If $\alpha > 1$, using standard argument, we have that g_h converges weakly to $\partial u/\partial t$ in $L^\alpha(0, T-\delta; L^\alpha(\Omega_1))$. From the above ienquality, we know that

$$\limsup_{h \to 0} \|g_h\|_{L^\alpha(Q^1_\delta)} \leqslant \left\|\frac{\partial u}{\partial t}\right\|_{L^\alpha(Q^1_\delta)}, \tag{B.1}$$

where we set $Q^1_\delta = (0, -\delta) \times \Omega_1$. Since $L^\alpha(Q^1_\delta)$ is uniformly convex, the weak-convergence and (B.1) infer that g_h tends to $\partial u/\partial t$ in $L^\alpha(Q^1_\delta)$-strong.

Now, if $\alpha = 1$, we consider $u_n \in W^{1,2}(0, T; L^2(\Omega_1))$ such that u_n converges to u in $W^{1,1}(0, T; L^1(\Omega_1))$ as $n \to +\infty$. We set

$$\varepsilon^n(h)(t) = \frac{u_n(t+h) - u_n(t)}{h} - \frac{\partial u_n}{\partial t}(t),$$

then $\varepsilon^n(h)$ converge to 0 in $L^1(Q^1_\delta)$ as $h \to 0$. Let us put

$$\varepsilon(h)(t) = \frac{u(t+h) - u(t)}{h} - \frac{\partial u}{\partial t}(t).$$

One has

$$\|\varepsilon(h)\|_{L^1(Q^1_\delta)} \leqslant \|\varepsilon^n(h)\|_{L^1(Q^1_\delta)} + \|\varepsilon^n(h) - \varepsilon(h)\|_{L^1(Q^1_\delta)}$$

and

$$\|\varepsilon^n(h) - \varepsilon(h)\|_{L^1(Q^1_\delta)} \leqslant 2 \left\| \frac{\partial}{\partial t}(u_n - u) \right\|_{L^1(0,\, T;\, L^1(\Omega_1))}.$$

With these relations, we deduce

$$\limsup_{h \to 0} \|\varepsilon(h)\|_{L^1(Q^1_\delta)} \leqslant 0 : \varepsilon(h) \xrightarrow[h \to 0]{} 0 \qquad \text{in } L^1(Q^1_\delta). \quad ▮$$

Proof of Lemma 2.1. Using Jensen's inequality, we have

$$\int_\Omega |g_h(t, x)|^p \, dx \leqslant \frac{1}{h} \int_t^{t+h} d\sigma \int_\Omega \left| \frac{\partial u}{\partial t}(\sigma, x) \right|^p dx. \tag{B.2}$$

If $1 \leqslant p \leqslant q \leqslant +\infty$, we can apply Jensen's inequality to (B.2) to get

$$\left(\int_\Omega |g_h(t, x)|^p \, dx \right)^{q/p} \leqslant \frac{1}{h} \int_t^{t+h} d\sigma \left(\int_\Omega \left| \frac{\partial u}{\partial t}(\sigma, x) \right|^p dx \right)^{q/p}. \tag{B.3}$$

If we integrate this relation (B.3) from 0 to $T - \delta$, we deduce

$$\int_0^{T-\delta} dt \left(\int_\Omega |g_h(t, x)|^p \, dx \right)^{q/p} \leqslant \int_0^{T-\delta+h} dt \left(\int_\Omega \left| \frac{\partial u}{\partial t}(t, x) \right|^p dx \right)^{q/p}. \tag{B.4}$$

Thus,

$$\limsup_{h \to 0} \|g_h\|_{L^q(0,\, T-\delta;\, L^p(\Omega))} \leqslant \left\| \frac{\partial u}{\partial t} \right\|_{L^q(0,\, T-\delta;\, L^p(\Omega))}. \tag{B.5}$$

If $p > 1$, from Lemma B.1, relations (B.4) and (B.5), we deduce that g_h converges to $\partial u/\partial t$ strongly in $L^q(0, T-\delta; L^p(\Omega)) \subset L^p(0, T-\delta; L^p(\Omega))$ as $h \to 0$.

If $p = 1$, we use as before a density argument. Considering $u_n \in W^{1,2}(0, T; L^2(\Omega))$ such that u_n converges to u in $W^{1,1}(0, T; L^1(\Omega))$ as $n \to +\infty$. Then,

$$\varepsilon^n(h)(t) = \frac{u_n(t+h) - u_n(t)}{h} - \frac{\partial u_n}{\partial t}(t)$$

converge to zero in $L^1(0, T-\delta; L^1(\Omega))$ when h tends to zero. But one has

$$\|\varepsilon(h)\|_{L^1(Q_\delta)} \leqslant \|\varepsilon^n(h)\|_{L^1(Q_\delta)} + \|\varepsilon^n(h) - \varepsilon(h)\|_{L^1(Q_\delta)}$$

and

$$\|\varepsilon^n(h) - \varepsilon(h)\|_{L^1(Q_\delta)} \leqslant 2 \left\| \frac{\partial}{\partial t}(u_n - u) \right\|_{L^1(Q_\delta)}.$$

Here,

$$Q_\delta = (0, T-\delta) \times \Omega, \qquad \varepsilon(h)(t) = \frac{u(t+h) - u(t)}{h} - \frac{\partial u}{\partial t}(t).$$

The above inequalities imply that

$$\limsup_{h \to 0} \|\varepsilon(h)\|_{L^1(Q_\delta)} = 0. \quad ▮$$

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