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S-Shaped Bifurcation Branch in a Quasilinear Multivalued Model Arising in Climatology

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In this paper we show the existence of a continuous and unbounded connected S-shaped set $\{(Q, u)\}$ where Q is the solar constant and u satisfies a quasilinear eventually multivalued stationary equation on a Riemannian manifold without boundary arising as a stationary energy balance model for the earth surface temperature. © 1998 Academic Press

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1. INTRODUCTION

This work concerns with the study of the sensitivity of a nonlinear stationary model arising in climatology with respect to variations in the so-called solar constant. The model is obtained through an energy balance on the whole surface of the earth leading to a nonlinear partial differential equation on a Riemannian manifold $\mathcal{M}$ for the earth surface temperature. These models were introduced independently by M. I. Budyko and W. D. Sellers in 1969 (see [7, 20] respectively). The diffusion operator considered

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is, some times, nonlinear. The motivation of this fact is the work by Stone [22] where it is suggested that the diffusion coefficient may depend on the temperature gradient. Another characteristic of the model comes from the term representing the feedback effect of the *planetary coalbedo* (the fraction of the incident radiation absorbed). This is modelled by a nonlinearity $\beta(u)$ which can be a discontinuous function of the temperature (see [7]). We shall treat it as a bounded multivalued graph of $\mathbb{R}^2$. A starting simplified evolution model is

$$(P) \quad \begin{cases} ku_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + \mathscr{G}(u) \in QS(x)\,\beta(u) + f & \text{on } (0, T) \times \mathscr{M}, \\ u(0, x) = u_0(x) & \text{on } \mathscr{M} \end{cases}$$

with initial data $u_0 \in L^\infty(\mathscr{M})$. Here, $\mathscr{M}$ is a C^∞ two-dimensional connected compact oriented Riemannian manifold without boundary (see [4]), $\mathscr{G}$ is a strictly increasing function on u, $S(x)$ is a strictly positive and bounded regular function, and k and f are smooth bounded functions. The exponent p is assumed to be $p \geqslant 2$ ($p = 3$ corresponds to the case considered by Stone [22]). The case $1 < p < 2$ could be treated in a similar way. It is useful to introduce the energy space $V = \{v \in L^2(\mathscr{M})\colon \nabla u \in L^p(T\mathscr{M})\}$, where $T\mathscr{M}$ is the tangent space of $\mathscr{M}$. The general theory (existence and uniqueness of weak solutions) for this class of problems was carried out in [8] for the one-dimensional model and then generalized in [10, 11] to the two-dimensional case. The existence of solutions was obtained in the space $C([0, \infty); L^2(\mathscr{M})) \cap L^p_{\mathrm{loc}}(0, \infty; V)$. Later, the stabilization of solutions of the evolution model when time tends to infinity was analyzed in [9] (see also the previous approach made in [13] and [14]).

This paper is a continuation of the previous paper [9], where the multiplicity of the solutions to the problem

$$(P_Q) \quad -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \mathscr{G}(u) + C \in QS(x)\,\beta(u) \qquad \text{on } \mathscr{M}$$

was studied, according the values of the solar constant Q. In [9], the proof of the existence of at least three solutions for a range of the solar parameter Q was found. In the present work, we describe more precisely the bifurcation diagram for Q and in particular, we shall prove that the principal branch (emanating from $(0, \mathscr{G}^{-1}(-C)) \in \mathbb{R}^+ \times L^\infty(\mathscr{M})$) is S-shaped; i.e., it contains at least one turning point to the left and another one to the right. For a turning point to the left (respectively, to the right), we understand a point (Q^*, u^*) in the principal branch such that in a neighbourhood in $\mathbb{R}^+ \times L^\infty(\mathscr{M})$ of it the principal branch is contained in $\{(Q, u) \in \mathbb{R}^+ \times L^\infty(\mathscr{M})/Q \leqslant Q^*\}$ (respectively, $\{(Q, u) \in \mathbb{R}^+ \times L^\infty(\mathscr{M})/Q \geqslant Q^*\}$). A previous result is due to Hetzer [15] for the special case of $p = 2$ and β a C^1 function. He proves that the principal branch of the bifurcation diagram has an even number (including zero) of turning points. Our main result already improves this information showing that indeed this number of turning points is greater than or equal to two. Other references on this case are [16, 21]. See also [18] for the numerical aspects. Semilinear problems with discontinuous forcing terms on an open bounded set and with Dirichlet boundary conditions have been considered in [1–3, 17] and their references.

2. MAIN RESULTS

In this section we study the bifurcation diagram of solutions of (P_Q) for different positive values of Q. In the sequel we denote by Σ the set of pairs $(Q, u) \in \mathbb{R}^+ \times V$, where u verifies the equation (P_Q), that is,

$$\Sigma = \{(Q, u)\colon Q \geqslant 0 \text{ and } u \text{ is solution of } (P_Q)\}.$$

Our goal is to describe qualitatively the solution set Σ in the space $\mathbb{R}^+ \times L^\infty(\mathscr{M})$. We assume that $p \geqslant 2$,

($\mathrm{H}_{\mathscr{M}}$) $\mathscr{M}$ is a C^∞ two-dimensional connected compact oriented Riemannian manifold without boundary,

(H_S) $S\colon \mathscr{M} \to \mathbb{R}$, $S \in L^\infty(\mathscr{M})$, $S_1 \geqslant S(x) \geqslant S_2 > 0$ for some $S_1 > S_2$,

($\mathrm{H}_{\mathscr{G}}$) $\mathscr{G}$ is a continuous increasing function such that $\mathscr{G}(0) = 0$ and $\lim_{|s| \to \infty} |\mathscr{G}(s)| = +\infty$,

(H_β) β is a *bounded* maximal monotone graph of $\mathbb{R}^2$ such that there exist two real numbers $0 < m < M$ and $\varepsilon > 0$ such that $\beta(r) = \{m\}$ for any $r \in (-\infty, -10 - \varepsilon)$ and $\beta(r) = \{M\}$ for any $r \in (-10 + \varepsilon, +\infty)$.

(H_C) $\mathscr{G}(-10 - \varepsilon) + C > 0$ and $\mathscr{G}(-10 + \varepsilon) + C/\mathscr{G}(-10 - \varepsilon) + C \leqslant S_2 M/S_1 m$.

We remark that the above assumptions are fulfiled in the case of the physical models. A function $u \in V \cap L^\infty(\mathscr{M})$ is called a *bounded weak solution* of (P_Q) if there exists $z \in L^\infty(\mathscr{M})$, $z(x) \in \beta(u(x))$ a.e. $x \in \mathscr{M}$ such that

$$\int_{\mathscr{M}} (|\nabla u|^{p-2}\nabla u) \cdot \nabla v \, dA + \int_{\mathscr{M}} \mathscr{G}(u) v \, dA + \int_{\mathscr{M}} Cv \, dA = \int_{\mathscr{M}} QS(x) z v \, dA,$$

for any $v \in V$.

We recall the multiplicity result of [9] specialized to problem (P_Q)

THEOREM 1. *Let (H_S), $(H_{\mathscr{G}})$, and (H_β) be satisfied. Then for any $Q > 0$ there is a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ of problem (P_Q), i.e.,*

$\underline{u}$ and $\bar{u}$ are solutions of (P_Q) and any other solution u of (P_Q) satisfies $\underline{u} \leqslant u \leqslant \bar{u}$ a.e. on $\mathcal{M}$. Moreover, if (H_C) holds then

(i) *if $0 < Q < Q_1$ then (P_Q) has a unique solution,*

(ii) *if $Q_2 < Q < Q_3$, then (P_Q) has at least three solutions,*

(iii) *if $Q_4 < Q$, then (P_Q) has a unique solution,*

where

$$Q_1 = \frac{\mathcal{G}(-10-\varepsilon)+C}{S_1 M} \qquad Q_2 = \frac{\mathcal{G}(-10+\varepsilon)+C}{S_2 M} \tag{1}$$

$$Q_3 = \frac{\mathcal{G}(-10-\varepsilon)+C}{S_1 m} \qquad Q_4 = \frac{\mathcal{G}(-10+\varepsilon)+C}{S_2 m}. \tag{2}$$

Remark 1. Notice that if $Q \leqslant 0$ the associated operator would be monotone in $L^2(\mathcal{M})$ and so the uniqueness of solution holds.

Remark 2. The assumption (H_C) makes it possible to construct some sub and super solutions in order to prove (ii). In [9] it can be found that if $Q_2 < Q < Q_3$ then

$$\bar{u}_1 := \mathcal{G}^{-1}(QS_1 M - C) \qquad \text{and} \qquad \bar{u}_2 := \mathcal{G}^{-1}(QS_1 m - C)$$

are supersolutions of (P_Q) and that

$$\underline{u}_1 := \mathcal{G}^{-1}(QS_2 M - C) \qquad \text{and} \qquad \underline{u}_2 := \mathcal{G}^{-1}(QS_2 m - C)$$

are subsolutions of (P_Q).

In order to get the bifurcation diagram we shall start by considering the problem with β a Lipschitz function (as in the Sellers model). Later, we shall extend our conclusions to the model where β can be multivalued (as in the Budyko model).

THEOREM 2. *Let (H_S), $(H_{\mathcal{G}})$, and (H_C) be satisfied. Let β be a Lipschitz continuous function verifying (H_β). Then Σ contains an unbounded connected component which is S-shaped containing $(0, \mathcal{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1, Q_2) \times L^\infty(\mathcal{M})$ and another one to the left in $(Q_3, Q_4) \times L^\infty(\mathcal{M})$.* ∎

Proof. The proof consists in three steps. In the first step, we shall prove that Σ has an unbounded component containing $(0, \mathcal{G}^{-1}(-C))$, the *principal component.* In the second step we analyze the bifurcation diagrams for two related zero - dimensional models, (P_1) and (P_2). Finally, by using the comparison principle as done, for instance, in [12], for the problem

$$\Delta_p u + \mathcal{G}(u) = f \in L^2(\mathcal{M}) \qquad \text{on } \mathcal{M},$$

we show that a subset of Σ is limited by the bifurcation diagrams of (P_1) and (P_2) for some intervals of Q. The proof will end proving that necessarily the principal component of Σ must be S-shaped.

Step 1. Σ has an unbounded component containing the point $(0, \mathcal{G}^{-1}(-C))$.

We claim that the following result can be applied to our case:

THEOREM 3 (Rabinowitz, [19]). *Let E be a Banach space. If F: $\mathbb{R} \times E \to E$ is compact and $F(0, u) \equiv 0$, then Σ contains a pair of unbounded components C^+ and C^- in $\mathbb{R}^+ \times E$, $\mathbb{R}^- \times E$ respectively and $C^+ \cap C^- = \{(0, 0)\}$.*

In order to apply Theorem 3, we consider the translation of u given by $v := u - \mathcal{G}^{-1}(-C)$. Obviously, v is a solution of

$$-\Delta_p v + \hat{\mathcal{G}}(v) = QS(x)\, \hat{\beta}(v) \qquad \text{on } \mathcal{M}, \tag{3}$$

where $\hat{\mathcal{G}}(\sigma) = \mathcal{G}(\sigma + \mathcal{G}^{-1}(-C)) + C$ and $\hat{\beta}(\sigma) = \beta(\sigma + \mathcal{G}^{-1}(-C))$. We define $\hat{\Sigma}$ in a way analogous to Σ. Let us show that the hypotheses of Theorem 3 hold.

(i) Let $E = L^\infty(\mathcal{M})$ and define

$$F(Q, v) = (-\Delta_p + \hat{\mathcal{G}})^{-1}\, (QS(x)\, \hat{\beta}(v)).$$

Then F is the composition of a continuous operator and a compact operator (recall that $p \geqslant 2$) so F is also compact on E.

(ii) If $Q = 0$, by Theorem 1 the problem (3) has a unique solution $v = 0$. Then $F(0, 0) = 0$.

So by applying Theorem 3, we conclude that $\hat{\Sigma}$ contains two unbounded components $\hat{C}^+$ and $\hat{C}^-$ on $\mathbb{R}^+ \times L^\infty(\mathcal{M})$ and $\mathbb{R}^- \times L^\infty(\mathcal{M})$ respectively and $\hat{C}^+ \cap \hat{C}^- = \{(0, 0)\}$. Since Σ is a translation of $\hat{\Sigma}$ then Σ contains two unbounded components C^+ and C^- on $\mathbb{R}^+ \times L^\infty(\mathcal{M})$ and $\mathbb{R}^- \times L^\infty(\mathcal{M})$ respectively and that $C^+ \cap C^- = \{(0, \mathcal{G}^{-1}(-C))\}$. Since $Q \geqslant 0$ in the studied model, we are interested in C^+. In order to establish the behaviour of C^+, we also recall that for every $q > 0$ there exists a constant $L = L(q)$ such that if $0 \leqslant Q \leqslant q$ then every solution u_Q of (P_Q) verifies $\|u_Q\|_{L^\infty(\mathcal{M})} \leqslant L(q)$. Since the principal component is unbounded its projection over Q-axis is $[0, \infty)$. On the other hand, if Q is big enough (P_Q) has a unique solution u_Q and this solution is greater than $\mathcal{G}^{-1}(QS_0 M - C)$. Since $\lim_{|s| \to \infty} |\mathcal{G}(s)| = +\infty$ then C^+ should go to (∞, ∞).

Step 2. Bifurcation diagram for two auxiliary problems.

We consider the auxiliary zero - dimensional models

$$(P_1) \quad \mathscr{G}(u)+C=QS_1\beta(u) \qquad u\in\mathbb{R},$$

$$(P_2) \quad \mathscr{G}(u)+C=QS_2\beta(u) \qquad u\in\mathbb{R}.$$

The number of solutions at these problems depends clearly on the values of Q. In fact, it is easy to obtain explicitly some of the solutions of (P_1) and (P_2). Let us call Σ_1 and Σ_2 the bifurcation diagrams of (P_1) and (P_2), respectively. By assumptions $(H_{\mathscr{G}})$, (H_β), and (H_C) the principal components of Σ_1 and Σ_2 are S-shaped. We also remark that the sets

$$K_1^i := \left\{(Q, u_Q)\in\mathbb{R}^2 : 0\leqslant Q\leqslant \frac{\mathscr{G}(-10-\varepsilon)+C}{S_i m},\ u_Q=\mathscr{G}^{-1}(QS_i m - C)\right\},$$

$$K_2^i := \left\{(Q, u_Q)\in\mathbb{R}^2 : Q\geqslant \frac{\mathscr{G}(-10+\varepsilon)+C}{S_i M},\ u_Q=\mathscr{G}^{-1}(QS_i M - C)\right\}$$

are contained in Σ_i, $i=1, 2$ (notice that there exist values of Q which are in both sets K_1 and K_2 simultaneously).

Step 3. A comparison argument.

From Theorem 1, if $Q < Q_3$, there exists u_Q solution of (P_Q) such that $u_Q < -10-\varepsilon$. Thus u_Q satisfies

$$-\Delta_p u_Q+\mathscr{G}(u_Q)+C=QS(x)m \qquad \text{on } \mathscr{M},$$

and so

$$QS_2 m\leqslant -\Delta_p u_Q+\mathscr{G}(u_Q)+C\leqslant QS_1 m \qquad \text{on } \mathscr{M}.$$

Let u_Q^1 and u_Q^2 be the solutions of the problems

$$\mathscr{G}(u)+C=QS_1 m \qquad \text{on } \mathscr{M}$$

$$\mathscr{G}(u)+C=QS_2 m \qquad \text{on } \mathscr{M},$$

respectively. That is, (Q, u_Q^1) and (Q, u_Q^2) live in Σ_1 and Σ_2, respectively. Now, if $Q<Q_3$,

$$-\Delta_p u_Q^2+\mathscr{G}(u_Q^2)\leqslant -\Delta_p u_Q+\mathscr{G}(u_Q)\leqslant -\Delta_p u_Q^1+\mathscr{G}(u_Q^1),$$

and so by the comparison principle for the monotone problem $-\Delta_p u+\mathscr{G}(u) = f\in L^2(\mathscr{M})$ on $\mathscr{M}$, we have that

$$u_Q^2\leqslant u_Q\leqslant u_Q^1.$$

Therefore, the component of Σ starting in $(0, \mathscr{G}^{-1}(-C))$ lives between Σ_1 and Σ_2 to arrive at (Q_3, u_{Q_3}), where u_{Q_3} is the minimal solution of (P_{Q_3}). Analogously, if we denote by u_{Q_2} to the maximal solution of (P_{Q_2}). We can prove that the component of Σ which connects (Q_2, u_Q^2) with (∞, ∞) lives between Σ_1 and Σ_2. From $Q_2<Q_3$, the branch containing $(0, \mathscr{G}^{-1}(-C))$ is unbounded and by the uniqueness of solution for (P_Q) when $Q>Q_4$ we get that this branch is necessarily S-shaped. ∎

Our next result avoids the Lipschitz assumption made in Theorem 2.

THEOREM 4. *Let (H_S), $(H_{\mathscr{G}})$, (H_β), and (H_C) be satisfied. Then Σ has an unbounded S-shaped component containing $(0, \mathscr{G}^{-1}(-C))$ with at least one turning point to the right contained in the region $(Q_1, Q_2)\times L^\infty(\mathscr{M})$ and another one to the left in $(Q_3, Q_4)\times L^\infty(\mathscr{M})$.*

Our idea to prove Theorem 4 is based on [3] (see also [2]) and consists in approximating the problem (P_Q) when β is not Lipschitz continuous. We only need to show the *convergence* of the principal branches C_n of these approximating problems to a S-shaped unbounded connected set C of solutions of (P_Q). For this reason, let us recall the notions of *lim inf* and *lim sup* of a sequence of subsets C_n of a metric space X:

$$\liminf_{n\to\infty} C_n := \{p\in X\text{: for any neighbourhood } U(p) \text{ of } p \text{ in } X$$
$$\exists n_0\in\mathbb{N}\text{: } U(p)\cap C_n\neq\varnothing\ \forall n\geqslant n_0\},$$
$$\limsup_{n\to\infty} C_n := \{p\in X\text{: for any neighbourhood } U(p) \text{ of } p \text{ in } X$$
$$U(p)\cap C_n\neq\varnothing \text{ for infinitely many } n\},$$

and the following topological lemma:

LEMMA 1 (Whyburn, [23]). *Let $\{C_n\}$ be a sequence of connected sets in a metric space X such that*

(i) $\liminf_{n\to\infty} C_n\neq\varnothing$
(ii) $\bigcup_{n=1}^{\infty} C_n$ *is precompact.*

Then jlim $\sup_{n\to\infty} C_n$ *is a nonempty, precompact, closed and connected set.*

Proof of Theorem 4. The method of super and sub solutions used in [9] for (P_Q) proves that if $Q>Q_2$ then there exists a solution of (P_Q) greater than $-10+\varepsilon$ given by the unique solution

$$(P_Q^M) \quad -\Delta_p u+\mathscr{G}(u)+C=QS(x)M \qquad \text{on } \mathscr{M}.$$

Analogously, we know that if $0 \leqslant Q < Q_3$ then (P_Q) has a solution smaller than $-10-\varepsilon$ given by the unique solution of

$$(P_Q^m) \quad -\Delta_p u + \mathscr{G}(u) + C = QS(x)m \qquad \text{on } \mathscr{M}.$$

By Theorem 1, it is clear that these functions are not the unique solutions of (P_Q) in those intervals and that the uniqueness holds at least in the Q-intervals $[0, Q_1)$ and (Q_4, ∞). Since we can not apply directly Theorem 3 to our problem, we consider the family $\beta_n = n(I-(I-(1/n)\beta)^{-1})$, $n \in \mathbb{N}$ to approximate β in the sense of maximal monotone graphs when $n \to \infty$. Notice that since β verifies (H_β) then β_n is a Lipschitz bounded nondecreasing function (see [6]) and that $\beta_n(s) = \beta(s)$ for any $s \notin [-10-\varepsilon, -10+\varepsilon+(M/n)]$, $\forall n$.

Let u_n be the solutions of the approximated problem

$$(P_Q^n) \quad -\Delta_p u_n + \mathscr{G}(u_n) + C = QS(x)\,\beta_n(u_n) \qquad \text{on } \mathscr{M}$$

and let Σ_n be the bifurcation diagrams for (P_Q^n). Let us denote by S_n the component of Σ_n containing $(0, \mathscr{G}^{-1}(-C))$. By Theorem 2 every S_n is an unbounded, connected and S-shaped set. First of all, we are going to prove that $\limsup S_n$ is a connected and closed set of solutions to the problem (P_Q). In order to apply Lemma 1 we consider the sets C_n^j $(j > Q_4)$ defined as $S_n \cap ([0, j] \times L^\infty(\mathscr{M}))$, $\forall n \in \mathbb{N}$ containing $(0, \mathscr{G}^{-1}(-C))$. It is easy to see that these sets are connected. Moreover

(i) $\liminf_{n\to\infty} C_n^j \neq \varnothing$. From $(0, \mathscr{G}^{-1}(-C)) \in C_n^j$ for all $n \in \mathbb{N}$ then every neighbourhood U of $(0, \mathscr{G}^{-1}(-C))$ in $X := ([0, j] \times L^\infty(\Omega))$ verifies

$$U \cap C_n^j \neq \varnothing \qquad \forall n \in \mathbb{N}.$$

Therefore $(0, \mathscr{G}^{-1}(-C)) \in \liminf_{n\to\infty} C_n^j$.

(ii) $\bigcup_{n=1}^\infty C_n^j$ is precompact. Since X is a Banach space, it suffices to prove that every sequence $\{(Q_l, u_l)\}_{l\in\mathbb{N}} \subset \bigcup_{n=1}^\infty C_n^j$ contains a subsequence $\{(Q_{l_k}, u_{l_k})\}$ converging in X. From $Q_l \in [0, j]$ then there exists $Q \in [0, j]$ and a subsequence of $\{Q_l\}$ which is still denoted by $\{Q_l\}$ such that $Q_l \to Q$. On the other hand, u_l is a solution of the problem

$$-\Delta_p u_l + \mathscr{G}(u_l) + C = QS(x)\,\beta_l(u_l) \qquad \text{on } \mathscr{M}.$$

Taking u_l as test function in this equation we obtain the estimate

$$\int_{\mathscr{M}} |\nabla u_l|^p \, dA \leqslant (j\,\|S\|_\infty M + C)\,|\mathscr{M}|\,C^*, \tag{4}$$

where $|\mathscr{M}|$ is the Hausdorff measure of $\mathscr{M}$ and C^* is an upper bound of $\|u_l\|_\infty$. Then u_l is a bounded sequence in V. From the compact embedding

$V \subset L^\infty(\mathscr{M})$ when $p > 2$, there exists $u \in L^\infty(\mathscr{M})$ and a subsequence $\{u_{l_k}\}$ of $\{u_l\}$ such that $u_{lk} \to u$ in $L^\infty(\mathscr{M})$. If $p = 2$ then $\{u_l\}$ is a bounded sequence in the Sobolev space $H^2(\mathscr{M})$. From the compact embedding $H^2(\mathscr{M}) \subset C(\mathscr{M})$ we deduce the existence of a subsequence $\{u_{lk}\}$ and $u \in C(\mathscr{M})$ such that $u_{lk} \to u$ in $L^\infty(\mathscr{M})$. Thus $\bigcup_{n=1}^\infty C_n^j$ is precompact.

Then $\{C_{n_k}\}$ satisfies conditions (i) and (ii) of Lemma 1 and therefore

$$C^j \equiv \limsup_{n\to\infty} C_n^j$$

is a connected and compact set in X. Moreover, since every S_n is unbounded and fixed $\bar{Q}$ the solutions u_Q are uniformly bounded in $L^\infty(\mathscr{M})$ for $Q < \bar{Q}$, we have that

$$C_k^j \cap (\{j\} \times L^\infty(\mathscr{M})) \neq \varnothing \qquad \text{for all} \quad j \in \mathbb{N}.$$

Now, we prove that the resulting set C^j is contained in Σ. Let us see that for every $Q \in [Q_1, Q_4]$ we have that every $(Q, u) \in C^j$ verifies that u is a solution of (P_Q) (notice that it is true for every $Q \in (0, Q_1] \cup [Q_4, +\infty)$ from $C_n^j = C^j$ in these intervals).

Let $(Q, u) \in C^j = \limsup_{n\to\infty} C_n^j$, that is, let there exist a subsequence of $(Q_n, u_n) \in C_n^j$ such that $(Q_{n_k}, u_{n_k}) \to (Q, u)$ in $\mathbb{R} \times L^\infty(\mathscr{M})$. From the estimate (4) and the compact embedding $H^2(\mathscr{M}) \subset L^\infty(\mathscr{M})$ (for $p = 2$) and $V \subset L^\infty(\mathscr{M})$ (for $p > 2$), we deduce the existence of $u \in L^\infty(\mathscr{M})$ and a subsequence of $\{(Q_{n_k}, u_{n_k})\}$ which we call $\{(Q_{n_k}, u_{n_k})\}$ such that

$$(Q_{n_k}, u_{n_k}) \to (Q, u) \qquad \text{in} \quad \mathbb{R} \times L^\infty(\mathscr{M}).$$

Since $\beta_n \to \beta$ in the sense of maximal monotone graphs of $\mathbb{R}^2$, we have that

$$\beta_{n_k}(u_{n_k}) \rightharpoonup z \in \beta(u) \qquad \text{weakly in} \quad L^2(\mathscr{M})$$

(see, e.g., [5]). Due to the coercivity of the p-laplacian operator when $p \geqslant 2$, we obtain the following inequality:

$$\lim_{n_k\to\infty} \int_{\mathscr{M}} (|\nabla u_{n_k}|^{p-2}\,\nabla u_{n_k} - |\nabla\chi|^{p-2}\,\nabla\chi)\cdot(\nabla u - \nabla\chi)\,dA \geqslant 0 \qquad \forall \chi \in V.$$

From this, taking $\chi = u + \lambda\xi$ $\forall \xi \in V$, we obtain that

$$|\nabla u_{n_k}|^{p-2}\,\nabla u_{n_k} \rightharpoonup |\nabla u|^{p-2}\,\nabla u \qquad \text{weakly in } L^{p'}(T\mathscr{M}).$$

So, passing to the limit in the equations that u_{n_k} satisfy, we deduce that u is a solution of the problem (P_Q). Thus $(Q, u) \in \Sigma$ and $C^j \subset \Sigma$.

Since for all n and j

$$C_n^j \cap (\{j\} \times L^\infty(\mathscr{M})) \neq \varnothing,$$

then there exists $\{(j, u_n)\}_{n \in \mathbb{N}}$ such that $(j, u_n) \in C_n^j$; that is,

$$-\Delta_p u_n + \mathscr{G}(u_n) = jS(x)\,\beta_n(u_n) - C \qquad \text{in } \mathscr{M}.$$

Using that the operator $(\Delta_p + \mathscr{G})^{-1}$ is compact in $L^\infty(\mathscr{M})$, there exists a subsequence

$$u_{n_k} \to u \qquad \text{in } L^\infty(\mathscr{M}).$$

Thus $(j, u) \in C^j$ and $C^j \cap (\{j\} \times L^\infty(\mathscr{M})) \neq \varnothing$. Since $j > Q_4$, u_j is the unique solution of (P_Q) and also of $(P_Q^{\mathscr{M}})$, but we know that $\Sigma \cap (j, \infty) \times L^\infty(\mathscr{M}) = \Sigma_M \cap (j, \infty) \times L^\infty(\mathscr{M})$. So, we have obtained a connected unbounded set which starts in $(0, \mathscr{G}^{-1}(-C))$. The proof ends with the argument used in the proof of Theorem 2. The S-shaped remains after passing to the limit since we have again $Q_2 < Q_3$. ∎

Remark. We point out that our results remain true for the more realistic equation

$$-\operatorname{div}(k(x)\,|\nabla u|^{p-2}\,\nabla u) + \mathscr{G}(u) + C \in QS(x)\,\beta(u) \qquad \text{on } \mathscr{M}$$

where $k(x)$ is a given bounded function with $k(x) \geqslant k_0 > 0$ a.e. $x \in \mathscr{M}$ representing the eddy diffusion coefficient. When $\mathscr{M} = \mathscr{S}^1$ it is usually assumed that $S(x) = S(\lambda)$ and $k(x) = k(\lambda, \varphi)$ with λ the latitude and φ the longitude. So, in that case, the corresponding solutions are not φ-invariant.

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