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On a degenerate parabolic/hyperbolic system in glaciology giving rise to a free boundary

J.I. Diaz[a,*], E. Schiavi[b]

[a] *Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Spain*
[b] *Escuela Superior de Ciencius Experimentales y Tecnologia, Universidad Rey Juan Corlos de Madrid, 28933 Madrid, Spain*

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1. Introduction

According to modern theories on ice sliding over soft and deformable beds such as the till maden of the detrital carbonates of the Hudson Strait, Labrador Sea, Canada, a proper modeling of the Laurentide ice sheet dynamics at the LGM (last glacial maximum, ca. 21 kyr b.p.) must take into account the hydrological effect of the basal drainage system. In fact, while much of the Laurentide ice sheet complex rested on the hard Canadian shield, portions of the Hudson Bay rested on paleozoic carbonates and cretaceous mudstones. Saturated with water, these sediments can deform and behave like a mud: the bed is lubricated and sliding is enhanced. Hence, to model the mechanism whereby (parts of) the Laurentide ice sheet complex may have been surging during the late Pleistocenic ice-age cycle can lead to the analysis of a highly nonlinear coupled system of partial differential equations of mixed type. In fact, in terms of Continuum Mechanics this corresponds to coupling a dimensionless form of the continuity equation with an equilibrium drainage theory together with a prescribed sliding law to arrive at a degenerate evolutive system for the unknown variables of thickness, ice velocity and basal water flux. The mechanism whereby the ice sheet can switch between the slow flow and fast flow regimes can be illustrated following [17]: if the ice is thin, then it is cold based, sluggish and so thickens, due to accumulation. This causes the base to melt and the flow is then hydraulically activated at the bed. If the resultant water production in the basal hydrological system is sufficient enough to allow fast sliding to occur then a surge may result, causing the ice to become thin again. A key

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* Corresponding author. E-mail: jidiaz@sunma4.mat.ucm.es, eschiavi@escet.ursc.es

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role in the mechanism is then the interplay between drainage and sliding. A physically based drainage law which describes the water flux production at the ice/till interface has to be complemented with the prescription of a suitable sliding law relating the basal ice velocity to the basal shear stress and to the effective pressure in a distributed system of canals which drain the bed. This feedback depends critically on the amount of water flux produced at the bed by frictional heating and must be described by a drainage law based on the physics of drainage over wet deformable sediments such as the Fowler and Walder drainage law [14]. The temperature at the bed is then controlled by means of a parameterized heat balance including the competing effects of geothermal and frictional heating as opposed to advective and conductive cooling.

2. Model equations

The first physically-based model of ice sheet motion controlled by basal sliding which itself depends on basal water production, is due to Fowler and Johnson [11] They were able to suggest the possibility of surging in their model by means of a crude 'lumped' version, but this essentially reduced the model to a zero-dimensional one. Nevertheless, their analysis showed that the basal water produced by frictional heat at the base could lead to multiple ice flux/ice thickness relationships, and hence oscillatory behaviour by what they called *hydraulic runaway*, by analogy with the thermal runaway that had been previously suggested as an instability mechanism for ice sheets [7]. The ice sheet model equations we consider here are those proposed by Fowler and Johnson [12], where the authors offered some comments on the possible form of spatially extended surges. As an application Fowler and Schiavi [13] solved numerically those equations to consider in a fixed scaled domain $\Omega=(0,1)$, the transition between an ice sheet and an ice shelf, where $x=0$ locates the divide (the Hudson dome) and $x=1$ is a fixed control section at the downward margin (the Hudson strait). The problem essentially describes the one-dimensional hydraulic activated flow of a 2D ice sheet along a flat, deformable bed. We briefly describe the model (details can be found in [11–13]). Let $x\in[0,1]$ the spatial coordinate being $h(x,t)$ the local thickness of the ice. The ice sheet geometry is then described, at each fixed instant $t<T$, by the points of the x,z plane $\{(x,t)\in\mathbb{R}^2 : x\in[0,1],\ z\in(0,h(x,t))\}$. Let $0<r,\ s<1$, and γ, μ and λ positive dimensionless parameters which have typical values of order one. The values $r=s=1/2$ correspond to the view of Boulton and Hindmarsh [4] that coarse (sandy) tills have values of r,s of $O(1)$. The idea of Kamb [15] is that for clay-rich marine sediments, the sliding law is more nearly plastic, so that $s\approx 1$, $r\ll 1$. Fixed $\delta\in\mathbb{R}^+$, $0<\delta\ll 1$ and given a positive mass balance $a(x,t)$ (accumulation rate), a transition thickness $h_D(1,t)>0$ at the margin and functions $Q_D(t)\geq 0$ representing the water flux at the divide $x=0$, the system we shall solve is as follows:

$$h_t+(hu)_x=a,$$

$$\tau=u^rN^s,$$

$$\tau=-hh_x,$$

$$Q_x = \tau u + \gamma - \mu\frac{u}{\xi^{1/2}} - \frac{\lambda}{h} \quad \text{if } Q > 0,$$

$$N = (\delta + Q)^{-1/3},$$

$$\xi = \int_0^x u \, dx, \tag{1}$$

or the dimensionless variables h, u, τ, N, Q and ξ.

The first equation is the mass conservation equation where $u(x,t)$ is a depth averaged orizontal velocity which is essentially a sliding velocity (at least when the base is at ielting point). The second equation is a sliding law of the type proposed in glaciolog- :al models (see, e.g. [3,6,10]): it describes the sliding velocity in terms of the shear τ nd the effective pressure N. The third equation comes from the momentum equation ut it is essentially a balance force. It is the classical shear stress formula and relates ie shear stress τ to the thickness h and the slope h_x of the ice sheet free surface. he fourth equation represents an equilibrium theory for the basal drainage system. The 'ater flux is supposed to increase along a system of canals which drain the bed. The ght-hand side is a parameterized heat balance at the bed between the heating terms of ictional heating τu and geothermal heat flux γ and the cooling effects of advection and onduction. It has been deduced by Fowler and Johnson [12] by means of a boundary yer analysis of the thermal problem and it provide the switching mechanism between old and temperate based dynamics. The extra variable in the sliding law is the effec- ve pressure in the canals. It has been introduced in glaciological models by Lliboutry 6] and it must be described by a Walder and Fowler type law where the important oint is that N increase when Q decrease. The small parameter δ which appears in e drainage law (1_5) has been introduced in order to get a bounded effective pressure hen Q tends towards zero. Finally, ξ represents the accumulated ice velocity.

The implicit discretized system

Let $T > 0$, $\Omega = (0,1)$, $R = 1/r$, $S = s/3r$ and define $p = R + 1$, $m = (2R+1)/R$. otice that $p > 2$ and $m > 1$. In fact, our results will be valid for p and m arbitrary this range of parameters. Elimination of N and τ in (1) allows us to consider the llowing initial and boundary value problem: given h_0, h_D, Q_D and an *accumulation te* function $a(t,x)$, find three functions, h, Q and ξ satisfying

$$(\mathrm{S})\begin{cases} \partial_t h - [(\delta+Q)^S |(h^m)_x|^{p-2}(h^m)_x]_x = a & \text{in } (0,T)\times\Omega, \\ \partial_x Q + \beta(Q) \ni (\delta+Q)^S h^p |h_x|^p - \mu\xi_x\xi^{-1/2} + \gamma - \lambda h^{-1} & \text{in } (0,T)\times\Omega, \\ \partial_x \xi = (\delta+Q)^S h^{p-1}|h_x|^{p-1} & \text{in } (0,T)\times\Omega, \\ h(t,1) = h_D(t), & t\in(0,T), \\ h_x(t,0) = 0, \quad Q(t,0) = Q_D(t), \quad \xi(t,0) = 0, & t\in(0,T), \\ h(0,x) = h_0(x) & \text{on } \Omega. \end{cases}$$

Here β denotes the maximal monotone graph defined by

$$\beta(r)=\emptyset \quad \text{if } r<0, \qquad \beta(0)=(-\infty,0], \qquad \beta(r)=0 \quad \text{if } r>0. \tag{2}$$

This graph is introduced to deal with the cold ($Q=0$)/temperate ($Q>0$) transition a the bed. It allows only non-negative (physically meaningful) water flux amounts.

The coefficients γ, μ, λ are $O(1)$ dimensionless parameters. The small, real positiv constant δ, $0<\delta\ll 1$ represents the ice shearing component in the flow when $Q=($ We shall indicate later the notion of solution for solving the system and, in particula the multivalued equation for Q. Since $m>1$ and $p>2$ the equation for h become degenerate and so, as is well known, we cannot expect to have classical solution Notice also the singular term $\xi^{-1/2}$ arising in the equation for Q. In order to intrc duce the notion of weak solution we shall follow the usual procedure of the *abstrac semigroup theory* which relies on the idea of considering the associated Euler implic semidiscretized scheme. More precisely, given a positive integer number N and lettin $k=T/N$ (the time step of the discretization) we denote by $I_{k,n}=(t_{n-1},t_n)=((n-1)k,nk$ ($n=1,\dots,N$, $t_n=nk$) the associated sub-intervals of $(0,T)$. Let $V\doteq V_h\times V_\xi\times V_Q$ be th Banach space defined by $V_h\doteq\{\phi\in W^{1,p}(\Omega)\colon \phi(1)=0\}$, $V_\xi\doteq\{\psi\in W^{1,p'}(\Omega)\colon \psi(0)=0$ $V_Q\doteq\{\eta\in W^{1,1}(\Omega)\colon \eta(0)=0\}$ (as usual $p'\doteq p/(p-1)$). We shall assume [13] th following hypothesis on the data of the problem:

$$Q_D, h_D\in C[0,T], \qquad h_0\in W^{1,p}(0,1), \quad h_D(0)=h_0(1), \tag{3}$$

$$\begin{gathered} M_D\ge h_D\ge m_D>0,\ M_0\ge h_0\ge m_0>0, \quad \text{and } Q_D\ge 0,\\ \text{for some constants } M_D>m_D>0 \quad \text{and } M_0>m_0>0. \end{gathered} \tag{4}$$

$$a\in L^\infty((0,T)\times\Omega), \quad a>0, \tag{5}$$

$$\begin{gathered} a(t,\cdot) \text{ is a nonincreasing function, for a.e. } t\in(0,T),\\ h_0'(x)<0 \quad \text{a.e. } x\in(0,1). \end{gathered} \tag{6}$$

It is useful to introduce the following notation:

$$\begin{gathered} A=(\delta+Q)^S, \qquad B=h^p|h_x|^p, \qquad C=\mu\,\xi_x\,\xi^{-1/2},\\ D=\lambda h^{-1}, \qquad E=h^{p-1}|h_x|^{p-1}. \end{gathered} \tag{7}$$

Term A comes from the sliding law and reflects the importance of the lubricatin water flux at the ice/till interface; B represents the frictional heating term, C and are (respectively) the advective and conductive cooling terms while E is the amou of shear in the sliding law. Fixed $n<N$ and $k<T$, we define the piecewise consta in time approximations of the data in the usual manner (see, e.g., [1]):

$$h_{D_{k,n}}(t)\doteq\frac{1}{k}\int_{(n-1)k}^{nk}h_D(s)\,\mathrm{d}s, \qquad Q_{D_{k,n}}(t)\doteq\frac{1}{k}\int_{(n-1)k}^{nk}Q_D(s)\,\mathrm{d}s \quad \forall t\in I_{k,n}. \tag{8}$$

Let also

$$a_{k,n}(t,x)\doteq\frac{1}{k}\int_{(n-1)k}^{nk}a(s,x)\,\mathrm{d}s \quad \text{a.e. } t\in I_{k,n}, \quad \text{and a.e. } x\in\Omega. \tag{9}$$

Ve can now consider the stationary system

$$
(\mathrm{S}_{k,n})\begin{cases}
\partial_t^{-k} h_{k,n} - [A_{k,n}|(h_{k,n}^m)_x|^{p-2}(h_{k,n}^m)_x]_x = a_{k,n} & \text{in } (0,T)\times\Omega,\\
\partial_x Q_{k,n} + \beta(Q_{k,n}) \ni (\delta + Q_{k,n})^S B_{k,n} - C_{k,n} + \gamma - D_{k,n} & \text{in } (0,T)\times\Omega,\\
\partial_x \xi_{k,n} = A_{k,n}E_{k,n} & \text{in } (0,T)\times\Omega,\\
h_{k,n}(t,1) = h_{D_{k,n}}(t), & t\in(0,T),\\
(h_{k,n})_x(t,0) = 0, \quad Q_{k,n}(t,0) = Q_{D_{k,n}}(t), \quad \xi_{k,n}(t,0)=0, & t\in(0,T),\\
h_{k,0}(0,x) = h_0(x) & \text{on } \Omega,
\end{cases}
$$

'here

$$
\partial_t^{-k} h_{k,n}(t,.) \doteq \frac{h_{k,n}(.) - h_{k,n-1}(.)}{k} \quad \forall n = 1,\dots,N, \quad \text{if } t\in I_{k,n}
$$

ıd $A_{k,n}$, $B_{k,n}$, $C_{k,n}$, $D_{k,n}$ and $E_{k,n}$ are piecewise constant in time functions defined as ı (7) replacing h, ξ and Q by $h_{k,n}$, $\xi_{k,n}$ and $Q_{k,n}$.

'efinition 3.1. Given a, h_D, Q_D and h_0 satisfying hypothesis (3)–(6) and $a_{k,n}$, $h_{D_{k,n}}$, $'_{D_{k,n}}$, $\xi_{D_{k,n}}$ the associated discretized functions, we say that $(h_{k,n}, \xi_{k,n}, Q_{k,n})$ is a weak ılution of $(\mathrm{S}_{k,n})$ if

$$
(h_{k,n}^m(t,.), \xi_{k,n}(t,.), Q_{k,n}(t,.)) \in [h_D^m + V_h] \times V_\xi \times [Q_D + V_Q] \quad \text{a.e. } t\in(0,T),
$$

ıere exists $b_{k,n}$ with $b_{k,n}(t,x)\in\beta(Q_{k,n}(t,x))$ a.e. $t\in(0,T)$ and $x\in(0,1)$, $b_{k,n}(t,\cdot)$ $L^1(\Omega)$, $h_{k,n}^{-1}(t,\cdot)\in L^\infty(\Omega)$, $(\xi_{k,n}(t,\cdot))^{-1/2}\in L^1(\Omega)$ a.e. $t\in(0,T)$ and the following in- gral conditions hold:

$$
\int_0^1 \partial_t^{-k} h_{k,n}(t)\phi + \int_0^1 (\delta + Q_{k,n})^S |(h_{k,n}^m)_x|^{p-2}(h_{k,n}^m)_x\phi_x = \int_0^1 a\phi,
$$

$$
\int_0^1 \xi_{k,n}\psi_x + \int_0^1 (\delta + Q_{k,n})^S h_{k,n}^{p-1}|(h_{k,n})_x|^{p-1}\psi = \xi_{k,n}(1,t)\psi(1,t),
$$

$$
\begin{aligned}
&\int_0^1 Q_{k,n}\eta_x + \int_0^1 [(\delta + Q_{k,n})^S h_{k,n}^p |(h_{k,n})_x|^p + \gamma]\eta \\
&\quad = Q_{k,n}(1,t)\eta(1,t) + \mu\int_0^1 (\xi_{k,n})_x(\xi_{k,n})^{-1/2}\eta + \lambda\int_0^1 h_{k,n}^{-1}\eta + \int_0^1 b_{k,n}\eta,
\end{aligned}
$$

r any vectorial test function $(\phi,\psi,\eta)\in V_h\times V_\xi\times V_Q$.

The main goal of this work is to present a mathematical analysis of the above 'stem by showing the existence of a weak solution. In which follows, given a function $:(0,T)\times\Omega \to \mathbb{R}$ we shall write $f\in L^1(\Omega)$ to express that $f(t,\cdot)\in L^1(\Omega)$ for a.e. $\in(0,T)$.

4. Existence of weak solutions via an iterative scheme

In order to prove the existence of a weak solution of $(\mathrm{S}_{k,n})$ we shall use an iterative process which allows the system to be uncoupled into three separate problems: $\mathrm{P}(h^j_{k,n})$, $\mathrm{P}(\xi^j_{k,n})$ and $\mathrm{P}(Q^j_{k,n})$ (see the definition below). Later we shall obtain some a priori estimates which allow us to prove the convergence of such an iterative scheme. The decoupled problem is the following: For each j we shall find three functions $(h^j_{k,n})$, $(\xi^j_{k,n})$ and $(Q^j_{k,n})$ satisfying

$$(\mathrm{S}^j_{k,n})\begin{cases} \partial_t^{-k} h^j_{k,n} - [A^{j-1}_{k,n}|\partial_x[(h^j_{k,n})^m]|^{p-2}\partial_x[(h^j_{k,n})^m]]_x = a_{k,n} & \text{in } (0,T)\times\Omega,\\ \partial_x \xi^j_{k,n} = A^{j-1}_{k,n} E^j_{k,n} & \text{in } (0,T)\times\Omega,\\ \partial_x Q^j_{k,n} + \beta(Q^j_{k,n}) \ni (\delta + Q^j_{k,n})^S B^j_{k,n} - C^j_{k,n} + \gamma - D^j_{k,n} & \text{in } (0,T)\times\Omega,\\ h^j_{k,n}(t,1) = h_{D_{k,n}}(t,1) & t\in(0,T),\\ (h^j_{k,n})_x(t,0) = 0, Q^j_{k,n}(t,0) = Q_{D_{k,n}}(t), \xi^j_{k,n}(t,0) = 0 & t\in(0,T),\\ h^j_{k,n}(0,x) = h_0(x) & \text{on } \Omega \end{cases}$$

where the coefficient $A^{j-1}_{k,n} \doteq (\delta + Q^{j-1}_{k,n})^S$, $A^0_{k,n} \doteq \delta^S$, is assumed to be known, positive and uniformly bounded at each j-step of the iterative process: $A^{j-1}_{k,n} \geq \delta^S > 0$, $\|A^{j-1}_{k,n}\|_{L^\infty} \leq C$. By using suitable supersolutions of the multivalued water flux equation it is proved in Section 4 that these conditions hold uniformly in j, k and n. The main result is as follows:

Theorem 4.1. *Assume the data satisfying* (3)–(6). *Then for any* $j\in\mathbb{N}$ *there exist* $(h^j_{k,n}, \xi^j_{k,n}, Q^j_{k,n})$ *verifying* $(\mathrm{S}^j_{k,n})$. *Moreover the sequence* $(h^j_{k,n}, \xi^j_{k,n}, Q^j_{k,n})$ *converges to* $(h_{k,n}, \xi_{k,n}, Q_{k,n})$, *solution of* $(\mathrm{S}_{k,n})$, *when* $j\to\infty$.

In order to prove this result we study, for fixed j, three decoupled problems:

First step: *Problem* $\mathrm{P}(h^j_{k,n})$. We introduce the change of variable $w \doteq (h^j_{k,n})^m$ and define $V_w := V_h$. Let $w_D \doteq (h_{D_{k,n}})^m$, $\hat{A} \doteq kA^{j-1}_{k,n}$ and $f \doteq ka_{k,n} + h^j_{k,n-1}$. For each j-step of the iterative process we have to find a function w satisfying

$$\mathrm{P}(w)\begin{cases} -\partial_x(\hat{A}|w_x|^{p-2}w_x) + |w|^{(1/m)-1}w = f & \text{on } \Omega,\\ w_x(0) = 0, \quad w(1) = w_D. \end{cases} \tag{10}$$

The usual notion of weak solution of $P(w)$ is the following:

Definition 4.1. Given $\hat{A}\in L^\infty(\Omega)$, $\hat{A} \geq k\delta > 0$ a.e. $x\in\Omega$ and $f\in L^\infty(\Omega)$, we say that w is a bounded weak solution of $P(w)$ if $w\in w_D + V_w$ and it satisfies

$$-\int_\Omega \hat{A}|w_x|^{p-2}w_x\phi_x + \int_\Omega |w|^{1/m}w\phi = \int_\Omega f\phi, \quad \forall\phi\in V_w.$$

Lemma 4.1. *Let k,n,j,δ be fixed real positive constants. Given $\hat{A}$ and f as before, there exists a unique weak bounded solution of problem* P(w). *Moreover, if f, $\hat{f}\in L^\infty(\Omega)$ are such that $f\le\hat{f}$ in Ω and $w,\hat{w}$ are the bounded weak solutions of the associated problems* (10) *then we have*

$$w\le\hat{w}.$$

In particular, if $f\ge 0$ then $w\ge 0$.

Proof. A being strictly positive and bounded, the existence of a unique bounded weak solution $w=(h^j_{k,n})^m\in w_D+V_w$ and the comparison principle are well-known results in the literature (see, e.g., the exposition made in [8]). □

As usual we introduce the notion of super and subsolutions.

Definition 4.2. Let $A^{j-1}_{k,n}$, $a_{k,n}$, $h_{D_{k,n}}$ and $h_{k,0}$ given functions verifying hypothesis (3)–(6). We say that a function $w\in W^{1,p}(\Omega)$, is a *subsolution* (*resp. supersolution*) of problem $P(w)$ if it verifies

$$-\frac{\partial}{\partial x}(\hat{A}|w_x|^{p-2}w_x)+|w|^{(1/m)-1}w\le(\text{resp.}\ge)f\quad\text{in }\mathscr{D}'(\Omega),$$

$$w\le(\text{resp.}\ge)w_D=h^m_{D_{k,n}}\quad\text{at }x=1\quad\text{and }w_x(0)=0\quad\text{at }x=0.$$

We can now obtain some a priori estimates on w (i.e. on $(h^j_{k,n})^m$). Firstly, by using the comparison principle and suitable super and subsolutions, which will be built up thanks to the assumptions on data (4), we obtain

Lemma 4.2. *Let w be a weak bounded solution of problem* P(w). *Then there exist two real positive numbers m^*, M^* (depending only on the data of the problem) such that*

$$0<(m^*)^m\le w\le(M^*+T\|a\|_{L^\infty((0,T)\times(\Omega))})^m<+\infty\qquad a.e.\ x\in\Omega. \tag{11}$$

Moreover $w\in W^{2,2}_{\mathrm{loc}}(\Omega)$ and if f is not constant on any positive measured set then $|w_x|>0$ a.e. $x\in\Omega$. Finally, $D^j_{k,n}>0$, a.e. $x\in\Omega$, $D^j_{k,n}\in L^\infty(\Omega)$ and $D^j_{k,n}$ is uniformly bounded in $L^\infty(\Omega)$ with respect to j.

Remark 4.1. *As $f=ka_{k,n}+h^j_{k,n-1}$, the condition f not constant, for any k, n and j is verified, for instance, under hypothesis* (5) *and* (6). *Such conditions usually hold in most of the glaciological models.*

Proof. Let $\underline{w}\doteq(m^*)^m$, $\bar{w}\doteq(M^*+T\|a\|_{L^\infty((0,T)\times(\Omega))})^m$ being M^* and m^* defined as $0<m^*\doteq\text{Min}\,\{m_D,m_0\}<M^*\doteq\text{Max}\,\{M_D,M_0\}<\infty$ where constants M_D,M_0 and m_D, m_0 verify hypothesis (3) and (4). It is straightforward to show that $\underline{w}$ and $\bar{w}$ are (respectively) sub and supersolutions. In fact, we have

$$-\partial_x[\hat{A}|\bar{w}_x|^{p-2}\bar{w}_x] + \bar{w}^{1/m} = M^* + T\|a\|_{L^\infty((0,T)\times\Omega)} \geq f(x) \quad \text{in } \mathscr{D}'(0,1),$$

$$\bar{w}(1) \geq (M^*)^m \geq (h_{D_{k,1}})^m(t,1), \quad \bar{w}_x(0) = 0,$$

$$-\partial_x[\hat{A}\,|\underline{w}_x|^{p-2}\underline{w}_x] + \underline{w}^{1/m} = m^* \leq f(x) \quad \text{in } \mathscr{D}'(0,1),$$

$$\underline{w}(1) = (m^*)^m \leq w_D \doteq (h_{D_{k,n}})^m \quad \text{at } x = 1, \quad \text{and } \underline{w}_x(0) = 0.$$

According to Lemma 4.1 we get $\underline{w} \leq w \leq \bar{w}$, where w is the bounded solution of problem P(w) and we deduce inequality (11) by hypothesis (4) and the definition of M^* and m^*. From the equation and the boundedness of f and w we deduce that $(\hat{A}|w_x|^{p-2}w_x)_x \in L^\infty(\Omega)$. So, by well-known results (see e.g. [8, p. 261]) we have that $w \in W^{2,2}_{\text{loc}}(\Omega)$. Then by Stampacchia's theorem we have that if f is not constant then $|w_x| > 0$ a.e. $x \in \Omega$. Indeed, if by the contrary there exists a positively measured subset B of Ω such that $|w_x| = 0$ on B then, as $w \in W^{2,2}_{\text{loc}}(\Omega)$, by using the Stampacchia's theorem we deduce that $(\hat{A}|w_x|^{p-2}w_x)_x = 0$ a.e. $x \in B$ and so $f = w^{1/m}$ a.e. on B. This implies that f is constant on B which is a contradiction. The properties stated on $D^j_{k,n}$ are consequence of the inequality (11). □

A second a priori estimate can be obtained by using an energy method:

Lemma 4.3. *The function $w - w_D \doteq (h^j_{k,n})^m - (h_{D_{k,n}})^m$ is uniformly bounded in the energy space V_w. In particular, $B^j_{k,n}$ and $E^j_{k,n}$ are uniformly bounded in their respective spaces.*

Proof. The function $\phi = w - w_D \in V_w$ is an admissible test function. Substituting ϕ in Definition 4.1 we get

$$\int_\Omega \hat{A}|w_x|^{p-2}w_x(w_x - w_{D_x}) + \int_\Omega w^{1/m}(w - w_D) = \int_\Omega a\,(w - w_D). \tag{12}$$

We define $I_1 = \int_\Omega \hat{A}|w_x|^{p-2}w_x(w_x - w_{D_x})$, $I_2 = \int_\Omega w^{1/m}(w - w_D)$ and $I_3 = \int_\Omega f\,(w - w_D)$ to write Eq. (12) as $I_1 + I_2 = I_3$. We shall use the following inequality (see [18, Lemma 1.1])

$$|\zeta|^{p-2}\zeta(\zeta - \eta) \geq \frac{1}{p'}(|\zeta|^p - |\eta|^p) \quad \forall \zeta, \eta \in \mathbb{R}, \qquad p' = \frac{p}{p-1}, \quad p \in \mathbb{R},\ p > 1.$$

Let $\zeta = w_x$ and $\eta = w_{D_x}$. Then

$$I_1 = \int_\Omega \hat{A}|w_x|^{p-2}w_x(w_x - w_{D_x}) \geq \frac{k\delta}{p'}\int_\Omega |w_x|^p - \frac{k\delta}{p'}\int_\Omega |w_{D_x}|^p. \tag{13}$$

We write I_2 in the form

$$I_2 = \|w\|^{(m+1)/m}_{(m+1)/m} - \int_\Omega w^{1/m}w_D. \tag{14}$$

The integral term in (14) can be bounded by using Young's inequality:

$$\int_\Omega w^{1/m} w_D \le \frac{1}{p}\int_\Omega w^{p/m} + \frac{1}{p'}\int_\Omega w_D^{p'} = \frac{1}{p}\|w\|_{p/m}^{p/m} + \frac{1}{p'}\|w_D\|_{p'}^{p'}. \tag{15}$$

We get

$$\frac{k\delta}{p'}\|w_x\|_p^p - \frac{k\delta}{p'}\|w_{D_x}\|_{p'}^{p'} + \|w\|_{(m+1)/m}^{(m+1)/m} - \frac{1}{p}\|w\|_{p/m}^{p/m} - \frac{1}{p'}\|w_D\|_{p'}^{p'} \le I_1 + I_2. \tag{16}$$

The integral term I_3, can be bounded using the Hölder and Poincaré inequalities to get

$$\begin{aligned} I_3 &\le \|f\|_{p'}\|w - w_D\|_p \le C_p\|f\|_{p'}\|w_x - w_{D_x}\|_p \\ &\le C_p\|f\|_{p'}\{\|w_x\|_p + \|w_{D_x}\|_p\}. \end{aligned} \tag{17}$$

By Young's inequality

$$I_3 \le C(\varepsilon)C_p\|f\|_{p'}^{p'} + \varepsilon C_p\|w_x\|_p^p + C_p\|f\|_{p'}\|w_{D_x}\|_p. \tag{18}$$

Using the relations (13)–(18), we get

$$\begin{aligned} \frac{k\delta}{p'}\|w_x\|_p^p + \|w\|_{(m+1)/m} &\le \frac{1}{p'}\|w_D\|_{p'}^{p'} + \frac{1}{p}\|w\|_{p/m}^{p/m} + \frac{k\delta}{p'}\|w_{D_x}\|_p^p \\ &\quad + C(\varepsilon)C_p\|f\|_{p'}^{p'} + \varepsilon C_p\|w_x\|_p^p + C_p\|f\|_{p'}\|w_{D_x}\|_p. \end{aligned} \tag{19}$$

Having suitably reordered the terms in (19) we get the following inequality:

$$\begin{aligned} \left(\frac{k\delta}{p'} - \varepsilon C_p\right)\|w_x\|_p^p + \|w\|_{(m+1)/m} &\le C_p\|f\|_{p'}\|w_{D_x}\|_p + \frac{1}{p}\|w\|_{p/m}^{p/m} + \frac{k\delta}{p'}\|w_{D_x}\|_p^p \\ &\quad + C(\varepsilon)C_p\|f\|_{p'}^{p'} + C_p\|f\|_{p'}\|w_{D_x}\|_p. \end{aligned}$$

The boundedness of w, w_D and f in $L^\infty(\Omega)$ implies that it is possible to choose ε small enough such that $k\delta > \varepsilon p' C_p$ and in conclusion the energy $\|w_x\|_p^p$ is bounded. The uniform estimates on $B_{k,n}^j$ and $E_{k,n}^j$ are consequence of the information obtained on $w = h_{k,n}^m$. □

As a consequence of Lemma 4.1 we have the following regularity result:

Lemma 4.4. *Let $h_{k,n}^j$ be the weak solution of problem $\mathrm{P}(h_{k,n}^j)$. Then the coefficient*

$$A_{k,n}^{j-1}E_{k,n}^j \doteq (\delta + Q_{k,r}^{j-1})^S (h_{k,r}^j)^{p-1}|(h_{k,r}^j)_x|^{p-1}$$

is uniformly bounded in $L^{p'}(\Omega)$. Moreover, if $f = ka_{k,n} + h_{k,n-1}$ is not constant (see Remark 4.1), $A_{k,n}^{j-1}E_{k,n}^j(x) > 0$ for a.e. $x \in \Omega$.

Proof. It is straightforward by Definition 7, Lemmata 4.1–4.3. □

Second step: Problem $\mathrm{P}(\xi^j_{k,n})$. We consider the problem

$$\mathrm{P}(\xi^j_{k,n}) \quad \begin{cases} (\xi^j_{k,n})_x = A^{j-1}_{k,n}E^j_{k,n} & \text{on } \Omega, \\ \xi^j_{k,n}(0) = 0. \end{cases}$$

Definition 4.3. We say that $\xi^j_{k,n} \in W^{1,p'}(\Omega)$ is a weak solution of problem $\mathrm{P}(\xi^j_{k,n})$ i $\xi^j_{k,n} \in V_\xi$ and

$$\int_0^1 \xi^j_{k,n}\psi_x + \int_0^1 A^{j-1}_{k,n}E^j_{k,n}\psi = \xi_{D_{k,n}}(1)\psi(1), \quad \forall \psi \in L^p(\Omega) \text{ such that } \psi_x \in L^1(\Omega).$$

The existence of a unique function $\xi^j_{k,n}$ weak solution of problem $\mathrm{P}(\xi^j_{k,n})$ is obtaine by a simple integration. Since $(\xi^j_{k,n})_x(x) \geq 0$, *a.e.* $x \in \Omega$, we have the following result

Lemma 4.5. *Let k,n and j be fixed and assume that $f = ka_{k,n} + h_{k,n-1}$ is not con stant on a positive measured set. Then there exists a unique solution $\xi^j_{k,n}$ of prob lem $\mathrm{P}(\xi^j_{k,n})$ in the sense of Definition 4.3. Moreover $\xi^j_{k,n}(x) > 0$, $\forall x \in \Omega$. In particula $C^j_{k,n} \geq 0$, a.e. $x \in \Omega$ and $C^j_{k,n}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. The existence and uniqueness of the weak solution is obtained by direct in tegration. By Lemma 4.4 $A^{j-1}_{k,n}E^j_{k,n}(x) > 0$ for a.e. $x \in \Omega$ hence $\xi_x > 0$ a.e. $x \in \Omega$ an then $\xi > 0$ $\forall x \in \Omega$. We deduce also (set $\psi \equiv 1$ in the weak formulation of problen $\mathrm{P}(\xi^j_{k,n})$) that $\xi^j_{k,n}(1) = \|A^{j-1}_{k,n}E^j_{k,n}\|_{L^1(\Omega)}$. Then by Lemma 4.4 we can apply the Hölde inequality to get

$$\xi^j_{k,n}(1) \leq C_1\|(\xi^j_{k,n})_x\|_{L^{p'}(\Omega)} = C_1\|A^{j-1}_{k,n}E^j_{k,n}\|_{L^{p'}(\Omega)} \leq C_2$$

for some positive constant C_1, C_2. The regularity on $C^j_{k,n}$ is obtained by making th change of variable $s = \xi^j_{k,n}(x,t)$. Then

$$\|C^j_{k,n}\|_{L^1(\Omega)} = \int_0^1 \frac{(\xi^j_{k,n})_x}{(\xi^j_{k,n})^{1/2}}\,\mathrm{d}x = \int_0^{\xi^j_{k,n}(1)} \frac{1}{s^{1/2}}\,\mathrm{d}s = 2\sqrt{\xi^j_{k,n}(1)} \leq C_3 < +\infty. \qquad \square$$

Third step: Problem $\mathrm{P}(Q^j_{k,n})$. Let $h^j_{k,n}$ and $\xi^j_{k,n}$ be the weak solutions of $\mathrm{P}(h^j_{k,n}$ and $\mathrm{P}(\xi^j_{k,n})$ respectively. We consider $B^j_{k,n}$, $C^j_{k,n}$ and $D^j_{k,n}$ defined by (7). Notice tha $D^j_{k,n} \in C([0,1])$, and that $B^j_{k,n}$, $C^j_{k,n} \in L^1(\Omega)$ are, in general, merely in $L^1(\Omega)$. Given Q_l satisfying (3) and (4) we introduce the problem

$$\mathrm{P}(Q^j_{k,n}) \quad \begin{cases} \partial_x Q^j_{k,n} + \beta(Q^j_{k,n}) \ni (\delta + Q^j_{k,n})^S B^j_{k,n} + \gamma - C^j_{k,n} - D^j_{k,n} & \text{in } \Omega, \\ Q^j_{k,n}(0) = Q_{D_{k,n}}. \end{cases}$$

Definition 4.4. We say that $Q^j_{k,n} \in Q_{D_{k,n}} + V_Q$ is a weak solution if there exists a function $z \in L^1(\Omega)$ such that $z(x) \in \beta(Q^j_{k,n}(x))$, a.e. $x \in \Omega$ and

$$\int_0^1 Q^j_{k,n}\eta_x + \int_0^1 (\delta + Q^j_{k,n})^S B^j_{k,n}\eta + \gamma\int_0^1 \eta$$
$$= \int_0^1 C^j_{k,n}\eta + \int_0^1 D^j_{k,n}\eta + \int_0^1 z\eta + Q^j_{k,n}(1)\eta(1)$$

for each $\eta \in V_Q$.

We have

Theorem 4.2. *Let k, n, j, δ be real fixed positive constants. Given the coefficients $B^j_{k,n}$, $C^j_{k,n}$, $D^j_{k,n}$ as in Lemmata* 4.4 *and* 4.5, *there exist a unique weak solution of $P(Q^j_{k,n})$.*

Proof. We shall drop the sub-index $_{k,n}$ and the super-index ^{j} denoting $Q^j_{k,n} \doteq Q$, $B^j_{k,n} \doteq B$, $C^j_{k,n} \doteq C$, $D^j_{k,n} \doteq D$. We approximate the maximal monotone graph β and the function $b(Q) \doteq (\delta + Q)^S$ by some sequences of Lipschitz functions generating some approximating regularized problems having solutions $Q_m \in W^{1,1}(\Omega)$ (the existence of solutions to such problems is a consequence of a Banach fixed point argument). Moreover, we get the a priori estimates $\|Q_m\|_{L^\infty(\Omega)} \le C$ and $\|\partial_x Q_m\|_{L^1(\Omega)} \le C$. Passing to the limit in the weak formulation of the regularizing problems we obtain that $Q_m \to Q$ *strongly* in $L^2(\Omega)$ when $m \to \infty$. The limit function $Q = Q^j_{k,n}$ is a solution of problem $P(Q^j_{k,n})$. More precisely we introduce

$$\begin{aligned} &Q^+ \doteq Q \quad \text{if } Q > 0, \qquad Q^- \doteq -Q \quad \text{if } Q < 0,\\ &Q^+ \doteq 0 \quad \text{if } Q \le 0. \qquad Q^- \doteq 0 \quad \text{if } Q \ge 0. \end{aligned} \tag{20}$$

Notice that $Q \doteq Q^+ - Q^-$ and $Q^- \ge 0$ by definition. We regularize the graph $\beta(Q)$ by means of a sequence of continuous non-decreasing functions

$$\beta_m(Q_m) \doteq -m\,Q_m^- \le 0, \tag{21}$$

where $\beta_m \equiv \beta$ if $s \in \mathbb{R}^+$ (e.g. $\forall (t,x) \in Q_T$ such that $Q > 0$). Finally, we approximate $(\delta + Q)^S$, $0 < S < 1$ by a sequence of Lipschitz functions $b_m \in W^{1,\infty}(\mathbb{R})$ defined by

$$\begin{aligned} &b_m(Q_m) = (\delta + Q_m)^S \quad \text{if } Q_m \ge -\delta + 1/m,\\ &b_m(Q_m) = m^{1-S}(\delta + Q_m) \quad \text{if } -\delta \le Q_m < -\delta + 1/m,\\ &b_m(Q_m) = 0 \quad \text{if } Q_m < -\delta. \end{aligned} \tag{22}$$

The sequence (b_m) represents a regular Lipschitz continuous approximation of the positive part $[(\delta + Q)^S]^+$ when $m \to \infty$. We generated the approximating regularizing problems $P(Q_m)$:

4.1. Approximating problems

Let k, n fixed. Given functions B, C, D and γ a positive real constant we shall determine, for each fixed $m \in \mathbb{N}$ a function Q_m satisfying

$$\mathrm{P}(Q_m) \quad \begin{cases} \partial_x Q_m + \beta_m(Q_m) = b_m(Q_m)B - C + \gamma - D & \text{on } \Omega, \\ Q_m(0) = Q_D \geq 0. \end{cases} \tag{23}$$

We observe that the problems $\mathrm{P}(Q_m)$ are no longer of multivalued type and that solving $\mathrm{P}(Q_m)$ is equivalent to finding a solution of the ODE $\partial_x Q_m = f_m(x, Q_m)$ with

$$f_m(x, Q_m) \doteq b_m(Q_m)B - \beta_m(Q_m) + \gamma - C - D. \tag{24}$$

For fixed $x \in (0,1)$, the f_m functions are Lipschitz continuous in the second variable Q_m. For fixed Q_m, $f_m(\cdot, Q_m) \in L^1(0,1)$ so f_m are functions of Caratheodory's type.

Definition 4.5. We say that a function $Q_m \in Q_D + V_Q$ is a weak solution of problem (23) if

$$\int_0^1 Q_m \eta_x + \int_0^1 b_m(Q_m)^S B\eta + \gamma \int_0^1 \eta$$
$$= \int_0^1 C\eta + \int_0^1 D\eta + \int_0^1 \beta_m(Q_m)\eta + Q_m(1)\eta(1), \quad \forall \eta \in V_Q.$$

Lemma 4.6. *Let Q_D be a given no negative real constant. Let $B \in L^1(\Omega)$, $C \in L^1(\Omega)$ and $D \in L^\infty(\Omega)$. Then for each m there exists a unique solution $Q_m \in W^{1,1}(0,1)$ of problem $\mathrm{P}(Q_m)$.*

Proof. We notice, first, that the coefficients B, C and D do not depend on m. Let z, $\bar{z}$ real numbers. Then

$$|b_m(z) - b_m(\bar{z})| \leq C_b|z - \bar{z}|, \qquad |\beta_m(z) - \beta_m(\bar{z})| \leq C_\beta|z - \bar{z}|$$

being C_b and C_β the Lipschitz constants of functions b_m and β_m, respectively. We have

$$\begin{aligned} |f_m(x,z) - f_m(x,\bar{z})| &\doteq |(b_m(z) - b_m(\bar{z}))B - \beta_m(z) + \beta_m(\bar{z})| \\ &\leq |b_m(z) - b_m(\bar{z})|B + |\beta_m(z) - \beta_m(\bar{z})| \\ &\leq C_b|z - \bar{z}|B + C_\beta|z - \bar{z}| = (C_b B(x) + C_\beta)|z - \bar{z}|. \end{aligned}$$

We define $l(x) \doteq (C_b B(x) + C_\beta) \in L^1(\Omega)$, $L(x) = \int_0^x l(s)\,\mathrm{d}s \in C^0(\Omega)$ to introduce, in space $C(\Omega) = C(0,1)$, the new norm $|\phi|^* \doteq \mathrm{Sup}_{(0,1)}|\phi(x)|\,\mathrm{e}^{-L(x)}$, $\forall \phi \in C(0,1)$. Let $\boldsymbol{K}\phi_m$ the integral operator $\boldsymbol{K}\phi_m(x) = \int_0^x f_m(s, Q_m(s))\,\mathrm{d}s$ and consider the integral equation $Q_m = Q_D + \boldsymbol{K}Q_m$. We shall prove that the application $Q_m \to Q_D + \boldsymbol{K}Q_m$ is a contraction in $(C(0,1), |\cdot|^*)$. To this end we observe that

$$|K\phi_m(x) - K\bar{\phi}_m(x)| \le \int_0^x |f_m(s,\phi_m) - f_m(s,\bar{\phi})|\,\mathrm{d}s$$

$$\le \int_0^x l(s)|\phi_m - \bar{\phi}_m|\mathrm{e}^{L(s)}\mathrm{e}^{-L(s)}\,\mathrm{d}s \le |\phi_m - \bar{\phi}_m|^* \int_0^x l(s)\mathrm{e}^{L(s)}\,\mathrm{d}s.$$

Multiplying by $\mathrm{e}^{-L(x)}$ we have finally $|K\phi_m(x) - K\bar{\phi}_m(x)|^* \le k\,|\phi_m - \bar{\phi}_m|^*$ with $k := 1 - \mathrm{e}^{-L(1)} < 1$. □

Notice that the sequence of solutions Q_m is uniformly bounded in $Q_D + V_Q \subset W^{1,1}(\Omega)$ as we can deduce by

Lemma 4.7. *Let Q_m be the solution of problem* $\mathrm{P}(Q_m)$ *in the sense of Definition* (4.5). *Let B, C and D be the coefficients verifying the uniform estimates* (*in k, n and j*) :

$$\|B\|_{L^1(\Omega)} \le C, \quad \|C\|_{L^1(\Omega)} \le C, \quad \|D\|_{L^\infty(\Omega)} \le C.$$

Then

$$(1) \quad \|Q_m\|_{L^\infty(\Omega)} \le C, \qquad (2) \quad \|\partial_x Q_m\|_{L^1(\Omega)} \le C.$$

Proof. (1) We split the proof in two parts: first we bound Q_m^+ and then Q_m^-, both estimates being independent of m. We consider the regularized problems $\mathrm{P}(Q_m)$ and we multiply (23) by the positive part of the sign function: $\mathrm{sgn}_0^+(Q_m) = 1$ if $Q > 0$ and 0 if $Q \le 0$ to get

$$(\partial_x Q_m)\,\mathrm{sgn}_0^+(Q_m) + \beta_m(Q_m)\,\mathrm{sgn}_0^+(Q_m)$$

$$= b_m(Q_m)\,\mathrm{sgn}_0^+(Q_m)B + (\gamma - C - D)\,\mathrm{sgn}_0^+(Q_m).$$

We shall analyse each term in turn. We have that

$$(\partial_x Q_m)\,\mathrm{sgn}_0^+(Q_m) = \partial_x Q_m^+ \quad \text{if } Q_m > 0,$$

$$(\partial_x Q_m)\,\mathrm{sgn}_0^+(Q_m) = 0 \quad \text{if } Q_m \le 0.$$

But $\beta_m(Q_m)\,\mathrm{sgn}_0^+(Q_m) = -mQ_m^-\,\mathrm{sgn}_0^+(Q_m) = 0$, moreover $\partial_x Q_m^+ \le |b_m(Q_m^+)|B + \gamma$. From this we deduce that

$$\partial_x Q_m^+ \le Q_m^+ B(x) + (1+\delta)B(x) + \gamma \quad a.e\ x \in \Omega. \tag{25}$$

Let

$$v(x) = \mathrm{e}^{\int_0^x B(s)\,\mathrm{d}s}\left[Q_D + \int_0^x [(1+\delta)B(y) + \gamma]\mathrm{e}^{-\int_0^y B(\tau)\,\mathrm{d}\tau}\,\mathrm{d}y\right]$$

be the solution of the linear problem

$$\begin{aligned} &\partial_x v = vB(x) + (1+\delta)B(x) + \gamma \quad \text{a.e on } \Omega,\\ &v(0) = Q_D \ge 0. \end{aligned} \tag{26}$$

Since Q_m^+ is a subsolution of problem (26) we deduce that

$$Q_m^+(x) \le v(x) \le e^{\|B\|_{L^1(\Omega)}}[Q_D + (1+\delta)\|B\|_{L^1(\Omega)} + \gamma] \le C \quad \text{a.e. } x \in \Omega.$$

Thus

$$Q_m^+ \in L^\infty(\Omega), \quad \|Q_m^+\|_\infty \le C. \tag{27}$$

The boundedness of Q_m^- is similar. The estimates now depend on the cooling coefficients C and D. We multiply (23) by the negative part of the sign function: $\mathrm{sgn}_0^-(Q_m) = -1$ if $Q < 0$ and 0 if $Q \ge 0$. We have that

$$(\partial_x Q_m)\,\mathrm{sgn}_0^-(Q_m) + \beta_m(Q_m)\,\mathrm{sgn}_0^-(Q_m)$$
$$= b_m(Q_m)\,\mathrm{sgn}_0^-(Q_m)B + (\gamma - C - D)\,\mathrm{sgn}_0^-(Q_m). \tag{28}$$

As before we have the inequality

$$\partial_x Q_m^- \le \delta^S B(x) + C(x) + D(x) \quad \text{a.e } x \in \Omega. \tag{29}$$

Then

$$Q_m^- \le \delta^S \|B\|_{L^1(\Omega)} + \|C\|_{L^1(\Omega)} + \|D\|_{L^\infty(\Omega)} \le C. \tag{30}$$

From (27) and (30) we can deduce $Q_m = Q_m^+ - Q_m^- \in L^\infty(\Omega)$, $\|Q_m\|_\infty \le C$ and the uniform boundedness of Q_m is proved.

(2) Since Q_m^+ and Q_m^- (hence Q_m) are bounded in $L^\infty(\Omega)$ it is easy to see, by definition (and continuity) of the approximating sequences $\beta_m(Q_m)$ and $b_m(Q_m)$ that we have $\beta_m(Q_m) \in L^\infty(\Omega)$, $b_m(Q_m) \in L^\infty(\Omega)$. Moreover, the regularity of coefficients B, C and D in turn implies that there exists a function g such that

$$g(x) \doteq b_m(Q_m(x))B(x) - \beta_m(Q_m(x)) + \gamma - C(x) - D(x) \in L^1(\Omega), \quad \|g\|_{L^1(\Omega)} \le C.$$

It can be deduced from $(23)_1$ that $g = \partial_x Q_m \in L^1(\Omega)$. $\square$

Theorem 4.3. *Under the hypothesis of Lemma* 4.6, *let* $\{Q_m\}$ *be the sequence of solutions of the regularizing problems* (23). *Then we have* $Q_m \to Q$ strongly *in* $L^q(\Omega)$, $\forall q \ge 1$ *when* $m \to \infty$ *with* $Q = Q_{k,n}^j$ *the unique solution (at the j-iteration) of the discretized problems* $\mathrm{P}(Q_{k,n}^j)$.

Proof. It is enough to show that $Q_m^- \to 0$ when $m \to \infty$ because in such a case we have that $\beta_m(Q_m) \to \beta(Q)$ in the sense of graphs ($\beta_m(s) \to z \in \beta(s)$, $s \in \mathbb{R}$ when $m \to \infty$). It is worth noting that the measure of the set where β_m and β are different is $|m\,Q_m^-|$ (compare with (21)). By the a priori estimates of Lemma 4.7 we know that Q_m^-, Q_m^+ belong to $W^{1,1}(\Omega)$. Then, there exist Q^+, Q^-, such that: $Q_m^+ \to Q^+$, $Q_m^- \to Q^-$ strongly in $L^q(\Omega)$, $1 \le q < \infty$ and weak $*$ in $L^\infty(\Omega)$. In particular $Q_m^+ \to Q^+$, $Q_m^- \to Q^-$, $Q_m \to Q^+ - Q^- := Q$ strongly in $L^2(\Omega)$. We shall prove that Q is a solution of problem $\mathrm{P}(Q)$ passing to the limit in the weak formulation of the approximating problems $\mathrm{P}(Q_m)$. As usual, we multiply the first relation of (23) by a test function

$\eta \in W^{1,1}(\Omega)$. Integrating by parts we get

$$\int_0^1 Q_m\eta_x + \int_0^1 b_m(Q_m)B\eta - \int_0^1 \beta_m(Q_m)\eta - Q_m(1)\eta(1)$$

$$= \int_0^1 C\eta + \int_0^1 D\eta - \gamma\int_0^1 \eta. \tag{31}$$

As $Q_m \to Q$ *weakly* $*$ in $L^\infty(\Omega)$ we have

$$\int_0^1 Q_m\eta_x \to \int_0^1 Q\eta_x.$$

From the strong convergence of $Q_m \to Q$ in $L^2(\Omega)$ we deduce

$$\int_0^1 Q_m\eta \to \int_0^1 Q\eta$$

and by continuity of the sequence b_m we get

$$\int_0^1 b_m(Q_m)B\eta \to \int_0^1 (\delta+Q)^S B\eta.$$

The continuous imbedding $W^{1,1}(\Omega) \hookrightarrow C([0,1])$ justifies that $Q_m(1)\eta(1) \to Q(1)\eta(1)$. We can pass to the limit $(m \to \infty)$ in the weak formulation (31) of problem $\mathrm{P}(Q_m)$ obtaining

$$\int_0^1 Q_m\eta_x + \int_0^1 b_m(Q_m)^S B\eta - \int_0^1 \beta_m(Q_m)\eta - Q_m(1)\eta(1)$$

$$\to \int_0^1 Q\eta_x + \int_0^1 (\delta+Q)^S B\eta - \int_0^1 z\eta - Q(1)\eta(1)$$

for each $\eta \in V_\xi$ and some function $z \in L^1(\Omega)$, $z \in \beta(Q)$. In fact

$$\lim_{m\to\infty}\int_0^1 \beta_m(Q_m)\eta = \int_0^1 Q\eta_x + \int_0^1 (\delta+Q)^S B\eta$$

$$+ \int_0^1 (\gamma - C - D)\eta - Q(1)\eta(1) = l \in \mathbb{R}.$$

Taking $\eta = -mQ_m^-$ and applying Corollary 1.4 of Brezís [5] we have

$$\|mQ_m^-\|_{L^2(\Omega)} \doteq \operatorname*{Max}_{\theta\in L^2(\Omega)} \left| \frac{\int_0^1 m\,Q_m^-\theta}{\|\theta\|_{L^2}} \right| \le L$$

for each $\theta \in L^2(\Omega)$ such that $\|\theta\|_{L^2(\Omega)} \le 1$. Hence $\|mQ_m^-\|^2_{L^2(\Omega)} \le L^2$ and we can pass to the limit when $m \to \infty$ to get

$$\|Q_m^-\|^2_{L^2(\Omega)} \le \frac{L^2}{m^2} \to 0.$$

Finally, we observe that the sequence $\{\beta_m(Q_m)\}$ is bounded (uniformly in m) in $L^2(\Omega)$ then there exists $z \in L^2(\Omega)$ such that

$$\beta_m(Q_m) \rightharpoonup z \quad \text{in } L^2(\Omega) - \text{weakly}.$$

Since β is a maximal monotone graph it generates a strong-weakly closed operator on $L^2(\Omega)$ and so we have $z \in \beta(Q)$ (see [2]). □

5. Passing to the limit

We already obtain the a priori estimates

$$\|(h^j_{k,n})_x\|_{L^p(\Omega)} \le C, \qquad \|(\xi^j_{k,n})_x\|_{L^1(\Omega)} \le C,$$

$$\|Q^j_{k,n}\|_{L^\infty(\Omega)} \le C, \qquad \|(Q^j_{k,n})_x\|_{L^1(\Omega)} \le C \tag{32}$$

uniformly in j, $\forall\, k,n$ fixed and $\forall t \in I_{k,n}$. Hence

$$\|A^{j-1}_{k,n}\|_{L^\infty(\Omega)} \le C, \qquad \|B^j_{k,n}\|_{L^1(\Omega)} \le C, \qquad \|C^j_{k,n}\|_{L^1(\Omega)} \le C,$$

$$\|D^j_{k,n}\|_{L^\infty(\Omega)} \le C, \qquad \|E^j_{k,n}\|_{L^{p'}(\Omega)} \le C. \tag{33}$$

By applying Poincaré inequality, Sobolev and Lebesgue theorems we get the following result:

Lemma 5.1. *Let $\{(h^j_{k,n})^m\}$, $\{\xi^j_{k,n}\}$ and $\{Q^j_{k,n}\}$ be the sequences of solutions of problems* $\mathrm{P}(h^j_{k,n})$, $\mathrm{P}(\xi^j_{k,n})$ *and* $\mathrm{P}(Q^j_{k,n})$ *respectively. Then, passing to the limit when $j \to \infty$, $\forall\, k,n$ fixed and $\forall\, t \in I_{k,n}$ we have $h^j_{k,n} \to h_{k,n}$, $\xi^j_{k,n} \to \xi_{k,n}$, $Q^j_{k,n} \to Q_{k,n}$ strongly in $L^q(\Omega)$, $\forall\, q \ge 1$. Moreover $h^j_{k,n} \to h_{k,n}$, $\xi^j_{k,n} \to \xi_{k,n}$ strongly in $C^0([0,1])$.*

Proof. We first notice that the Poincaré inequality holds on spaces V_h, V_ξ, V_Q. Moreover, by the classical Sobolev theorem we have the compact imbedding $\{h^m_D\} + V_h \hookrightarrow L^p(\Omega)$. Hence $h_j \to h$ *strongly* in $L^p(\Omega)$. In particular $h_j(x) \to h(x)$ a.e. $x \in \Omega$. By the Lebesgue theorem (notice that $\int_\Omega |h_j(x)|^q \le C$, $\forall q \ge 1$ because h_j is a priori bounded Lemma 4.2 we get the strong convergence in $L^q(\Omega)$. Being $p > N = 1$ the convergence in the class of continuous functions is also obtained by the compact Sobolev imbedding $\{h^m_D\} + V_h \hookrightarrow C^0([0,1])$ and we deduce that $h^j \to h$ in $C^0([0,1])$ because the energy $\|(h^m)^j_x\|_{L^p(\Omega)}$ is uniformly bounded. Also $\xi^j \to \xi$ in $C^0([0,1])$ because $\|\xi^j_x\|_{L^{p'}(\Omega)}$ is uniformly bounded. In the case of $\{Q^j_{k,n}\}$, which is a uniformly bounded sequence in $Q_{D_{k,n}} + V_Q \subset W^{1,1}(\Omega)$ we can argue as before. Consequently, we apply the Poincaré inequality, the Sobolev theorem and the Lebesgue theorem to get $Q^j_{k,n} \to Q_{k,n}$ as $j \to \infty$ in $L^q(\Omega)$, $\forall\, q \ge 1$, $\forall\, k,n$ *fixed* and $\forall t \in I_{k,n}$. □

We notice that the imbedding $W^{1,1}(\Omega) \hookrightarrow C^0([0,1])$ is continuous but not compact and so the previous estimates (32), (33) do not suffice to get the strong convergence of $Q^j_{k,n}$ in $C([0,1])$. In fact the analysis of system $(S^j_{k,n})$ reveals that the difficult term, passing to the limit in the weak formulation, is the product $(\delta+Q^j)^S|h^j_x|^p$ representing the frictional heating due to viscous dissipation. Nevertheless, we have

Lemma 5.2. *$(\delta + Q^{j-1})^S|h^j_x|^p \to (\delta + Q)^S|h_x|^p$, strongly in $L^1(\Omega)$, when $j \to \infty$. In particular $\xi^j_x/(\xi^j)^{1/2} \to \xi_x/\xi^{1/2}$ in $L^1(\Omega)$ as $j \to \infty$.*

Proof. We consider $w^j=(h^j_{k,n})^m$ being $h^j_{k,n}$ a weak bounded solution of problem $P(h^j_{k,n})$. Without loss of generality, we can suppose that $w_D \equiv 0$. Multiplying by w^j and integrating by parts we have

$$\int_\Omega (\delta + Q^{j-1})^S|w^j_x|^p \,\mathrm{d}x = -\frac{1}{k}\int_\Omega (w^j)^{(1/m)+1} + \int_\Omega f^j w^j \,\mathrm{d}x$$

with $f^j = a_{k,n} + 1/k(w^j_{k,n-1})^{1/m} \in L^\infty(\Omega)$, $\|f^j\|_{L^\infty(\Omega)} \le C$, uniformly in j. Using Lebesgue's dominated convergence theorem

$$\int_\Omega (w^j)^{(1/m)+1} \,\mathrm{d}x \to \int_\Omega w^{(1/m)+1} \,\mathrm{d}x, \tag{34}$$

when $j \to \infty$ and we deduce that

$$\int_\Omega f^j w^j \,\mathrm{d}x \to \int_\Omega f w \,\mathrm{d}x \quad \text{when } j \to \infty. \tag{35}$$

Then

$$\int_\Omega (\delta + Q^{j-1})^S|w^j_x|^p \,\mathrm{d}x \to -\frac{1}{k}\int_\Omega w^{(1/m)+1} \,\mathrm{d}x + \int_\Omega f w \,\mathrm{d}x \quad \text{when } j \to \infty.$$

Multiplying in $P(h^j_{k,n})$ by w and integrating on Ω we get

$$\int_\Omega (\delta + Q)^S|w_x|^p \,\mathrm{d}x = -\frac{1}{k}\int_\Omega w^{(1/m)+1} + \int_\Omega f w \,\mathrm{d}x.$$

Hence

$$\int_\Omega (\delta + Q^{j-1})^S|w^j_x|^p \,\mathrm{d}x \to \int_\Omega (\delta + Q)^S|w_x|^p \,\mathrm{d}x.$$

inally, since $\xi^j_x \to \xi_x$ in $L^1(\Omega)$ we have that $\xi^j_x/(\xi^j)^{1/2} \to \xi_x/\xi^{1/2}$ a.e. $x \in \Omega$ and from the a priori estimate and the Lebesgue theorem we get the last convergence of the statement. □

As a consequence, we get the strong convergence of $(h^m_j)_x$ to $(h^m)_x$ in $L^p(\Omega)$.

Lemma 5.3. *We have that $w^j_x \to w_x$* strongly *in $L^p(\Omega)$* and *$(\delta + Q^j)^S|h^j_x|^p \to (\delta + Q)^S|h_x|^p$* strongly *in $L^1(\Omega)$.*

Proof. Subtracting the equations verified by w^j and w, and multiplying by $w^j - w$ we get

$$
\begin{aligned}
I_j \doteq & \int_\Omega (\delta + Q^{j-1})^S [|w_x^j|^{p-2} w_x^j - |w_x|^{p-2} w_x](w_x^j - w_x)\,\mathrm{d}x \\
= & \int_\Omega [(\delta + Q)^S - (\delta + Q^{j-1})^S](|w_x|^{p-2} w_x)(w_x^j - w_x)\,\mathrm{d}x \\
& + \int_0^1 (f^j - f)(w^j - w) - \frac{1}{k}\int_\Omega ((w^j)^{1/m} - w^{1/m})(w^j - w)\,\mathrm{d}x.
\end{aligned}
$$

By the previous Lemma, (34), (35), the Hölder inequality and the Lebesgue's dominated convergence theorem we deduce that $I_j \to 0$ if $j \to \infty$. Finally, as $p > 2$ and using the fact that the sequence Q^j is uniformly bounded in $L^\infty(\Omega)$, it is well known (see, e.g. [8, Lemma 4.10]) that there exists $C > 0$ (independently of j) such that

$$
C\int_\Omega |w_x^j - w_x|^p \le \int_\Omega (\delta + Q^{j-1})^S [|w_x^j|^{p-2} w_x^j - |w_x|^{p-2} w_x](w_x^j - w_x)\,\mathrm{d}x = I_j.
$$

So, $w_x^j \to w_x$, *strongly* in $L^p(\Omega)$. Moreover, since $\{(\delta + Q^j)^S\}$ is uniformly bounded in $L^\infty(\Omega)$ and $|h_x^j|^p \to |h_x|^p$ *strongly* in $L^1(\Omega)$ we obtain the second conclusion. □

The question of the convergence of the above implicit scheme seems very delicate due to the lack of information on the time dependence of the function $Q(x,t)$ and so on the coefficient of the nonlinear diffusion operator. Nevertheless, it is possible to obtain a partial result on the mentioned convergence by assuming some extra information on Q_t.

Proposition 5.1. *Assume that* $ta_t \in L^{p'}(0,T : V_h')$ *and that there exists* $(t_0,t_1) \subset [0,T]$ *such that* $Q \in L^\infty((t_0,t_1) \times \Omega)$ *and* $Q_t(x,t)$ *is bounded from above in measure, i.e.,* $\exists C \ge 0$ *such that*

$$
Q_t \le C \quad in\ \mathcal{M}((t_0,t_1) \times \Omega). \tag{36}
$$

Then the implicit semidiscretized scheme is convergent. More exactly, there exists a function $h \in L^\infty(t_0,t_1 : L^1(\Omega))$ *with* $\partial_t h \in L^{p'}(t_0,t_1 : V_h')$ *and there exists a subsequence of the functions*

$$
h_k(x,t) \doteq h_{k,n}(x,t) \quad if\ t \in ((n-1)k, nk) \cup (t_0,t_1)
$$

such that $h_k \to h$ *in* $L^1((t_0,t_1) \times \Omega)$.

Proof. We follow closely the arguments of the proof of the existence Theorem 1.7 of Alt and Luckhaus [17]. The key idea is the use of their compactness lemma (Lemma 1.9) assuring exactly that $h_k \to h$ in $L^1((t_0,t_1) \times \Omega)$. In order to apply such a result we have to check that

$$k\int_{t_0}^{t_1-k}\int_{\Omega}(\partial_t^{-k}h_k)\partial_t^{-k}((h_k)^m)\leq C, \tag{37}$$

$$\int_{t_0}^{t_1}\int_{\Omega}(\partial_t^{-k}h_k)(h_k)^m\leq C \tag{38}$$

hold. It is easy to see that condition (38) can be derived from the energy estimate obtained previously in Lemma 4.3. In order to show (37) we merely indicate here how to obtain something similar for the nondiscretized solution (suitable straightforward arguments allow to extend the proof for the backward difference quotients). Multiplying the equation by $(t-t_0)(h^m)_t$ and integrating by parts we get

$$\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}h_t(h^m)_t+\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}(\delta+Q)^S|h_x^m|^{p-2}h_x^m(h^m)_{xt}$$
$$=\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}a(h^m)_t$$

but

$$\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}(\delta+Q)^S|h_x^m|^{p-2}h_x^m(h_x^m)_t$$
$$=\frac{1}{p}\int_{t_0}^{t_1}(t-t_0)\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega}(\delta+Q)^S|h_x^m|^p\right)$$
$$-\frac{S}{p}\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}(\delta+Q)^{S-1}|h_x^m|^pQ_t$$
$$=\frac{1}{p}(t-t_0)\int_{\Omega}(\delta+Q(x,t))^S|h_x^m(x,t)|^p\Big|_{t=t_0}^{t=t_1}$$
$$-\frac{1}{p}\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}(\delta+Q)^S|h_x^m|^p$$
$$-\frac{S}{p}\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}(\delta+Q)^{S-1}|h_x^m|^pQ_t.$$

Analogously,

$$\int_{t_0}^{t_1}\int_{\Omega}(t-t_0)a(h^m)_t=(t-t_0)\int_{\Omega}ah^m\Big|_{t=t_1}^{t=t_2}-\int_{t_0}^{t_1}\int_{\Omega}(a+(t-t_0)a_t)h^m.$$

Then, using the energy estimate, assumption (36) and that $Q\in L^\infty((t_0,t_1)\times\Omega)$ we find that

$$\int_{t_0}^{t_1}(t-t_0)\int_{\Omega}h_t(h^m)_t\leq C$$

for some positive constant C and Lemma 1.9 of Alt and Luckhaus [1] can be applied. □

6. On the free boundary

In this section we shall analyze the existence and the behaviour of the free boundary associated to the Q function which represents the amount of water flux produced by frictional heating in the basal drainage system (for a similar obstacle problem see [9]).

The right-hand side in the Q equation is a *forcing* term which represents the net heat balance at the bed. It splits into two basic components: viscous dissipation minus convective cooling and geothermal heat flux minus conductive cooling. By means of some comparison properties of the solutions of the water flux equation we shall find sufficient conditions to have a (fast) moving interface at the base of the ice sheet, separating cold regions ($Q=0$) from the temperate ones ($Q>0$). We shall describe in detail the backward propagation of this interface.

6.1. Complementary formulation

Let $T>0$, $\Omega=(0,1)$ and $Q_T \doteq \Omega\times(0,T)$. System (S) admits the following complementary formulation: find three functions h, Q and ξ satisfying

$$\partial_t h - \partial_x[(\delta+Q)^S h^{R+1}|h_x|^{R-1}h_x] = a \quad \text{in } Q_T, \tag{39}$$

$$\begin{aligned} &\partial_x Q - (\delta+Q)^S h^{R+1}|h_x|^{R+1} + \mu\xi_x\xi^{-1/2} - \gamma + \lambda h^{-1} \geq 0, \quad Q\geq 0 \quad \text{in } Q_T, \\ &(\partial_x Q - (\delta+Q)^S h^{R+1}|h_x|^{R+1} + \mu\xi_x\xi^{-1/2} - \gamma + \lambda h^{-1})Q = 0 \quad \text{in } Q_T, \end{aligned} \tag{40}$$

$$\partial_x \xi = (\delta+Q)^S h^R |h_x|^R \quad \text{in } Q_T, \tag{41}$$

$$\begin{aligned} &h(t,1) = h_D(t,1), \quad t\in(0,T), \\ &h_x(t,0) = 0, \quad t\in(0,T), \\ &Q(t,0) = Q_D(t), \quad t\in(0,T), \\ &\xi(t,0) = 0, \quad t\in(0,T), \\ &h(0,x) = h_0(x) \quad \text{on } \Omega. \end{aligned} \tag{42}$$

The coefficient $\mu\xi_x\xi^{-1/2}$ can be deduced from the $(\delta+Q)^S h^{p-1}|h_x|^{p-1}\xi^{-1/2}$ term when substituted in the sliding law.

6.2. Comparison properties

We introduce a function $\alpha(x,t)$ (which represents a reduced heat balance between the advective cooling term and the frictional heating term) defined by

$$\begin{aligned} \alpha(x,t) &\doteq -[B(x,t) - C(x,t)] \\ &= -[h^{R+1}|h_x|^{R+1} - \beta h^R|h_x|^R\xi^{-1/2}] \quad a.e.\,(x,t)\in Q_T. \end{aligned} \tag{43}$$

Notice that $\alpha = C - B$ and then $-B \leq \alpha \leq C$ as $B, C \geq 0$. By energy estimates we also deduce that α is bounded in $L^1(Q_T)$. Using (43) we write the *forcing* term in the water flux equation in form

$$f(x,t) = -\alpha(x,t)(\delta + Q)^S + \gamma - \lambda h^{-1} \in L^1(Q_T). \tag{44}$$

Let f^* be the component of f due to the geothermal heat flux minus conductive cooling

$$f^*(x,t) \doteq \gamma - D(x,t) = \gamma - \lambda h^{-1} \in L^\infty(Q_T), \tag{45}$$

where the smoothness of f and f^* can be deduced from Lemma 4.2 and the energy estimates (32), (33) which remain true for solutions of the non-discretized problem. The water flux conservation equation can be written as

$$\partial_x Q + \alpha(x,t)(\delta + Q)^S + \beta(Q) \ni f^*(x,t) \quad \text{a.e } (x,t) \in Q_T. \tag{46}$$

Given $\bar{\alpha}$, $\hat{\alpha} \in L^1(Q_T)$ such that $\bar{\alpha}(x,t) \le \alpha(x,t) \le \hat{\alpha}(x,t)$ a.e $(x,t) \in Q_T$, let Q, $Q_{\bar{\alpha}}$, $Q_{\hat{\alpha}}$ be the solutions (they are unique because of the results of Section 4 of Eq. (46) associated to the data α, $\bar{\alpha}$, $\hat{\alpha}$ and verifying the (same) initial condition $Q = Q_{\bar{\alpha}} = Q_{\hat{\alpha}} = Q_D$. We consider also the same reduced forcing term f^* in the respective problems.

Notice that the function α defined by (43) is not a priori positive and it is useful to compare it with functions $\bar{\alpha}$ and $\hat{\alpha}$ such that $\bar{\alpha} < 0 < \hat{\alpha}$. The greatest absolute values of $\bar{\alpha}$ correspond to temperate, heating conditions while the greater the value of $\hat{\alpha}$ the higher the cooling rate.

We have the following comparison principle:

Lemma 6.1. *Let us suppose $\bar{\alpha}, \hat{\alpha} \in L^1(Q_T)$ such that*

$$\bar{\alpha}(x,t) \le \alpha(x,t) \le \hat{\alpha}(x,t) \quad a.e\ (x,t) \in Q_T. \tag{47}$$

Then

$$Q_{\bar{\alpha}}(x,t) \ge Q(x,t) \ge Q_{\hat{\alpha}}(x,t) \quad a.e\ (x,t) \in Q_T. \tag{48}$$

Proof. We shall prove that $Q_{\bar{\alpha}} \ge Q$. The second inequality in (48) can be shown in the same way. Let Q and $Q_{\bar{\alpha}}$ be the solutions associated with the functions $\alpha(x,t)$, $\bar{\alpha}(x,t)$, being $\bar{\alpha}(x,t) \le \alpha(x,t)$ and to the equations

$$\partial_x Q + \alpha(x,t)(\delta + Q)^S + \beta(Q) \ni f^*(x,t) \quad \text{a.e } (x,t) \in Q_T, \tag{49}$$

$$\partial_x Q_{\bar{\alpha}} + \bar{\alpha}(\delta + Q_{\bar{\alpha}})^S + \beta(Q_{\bar{\alpha}}) \ni f^*(x,t) \quad \text{a.e } (x,t) \in Q_T \tag{50}$$

verifying, moreover, the same initial condition $Q = Q_{\bar{\alpha}} = Q_D$. Subtracting (49) and (50) and multiplying by $\mathrm{sgn}_0^+(Q - Q_{\bar{\alpha}})$ (or by a regularization of it) we get

$$\partial_x (Q - Q_{\bar{\alpha}})^+ \le [\bar{\alpha}(\delta + Q_{\bar{\alpha}})^S - \alpha(\delta + Q_{\bar{\alpha}})^S]\, \mathrm{sgn}_0^+(Q - Q_{\bar{\alpha}}) \quad \text{a.e } (x,t) \in Q_T, \tag{51}$$

where we used

$$(\beta(Q) - \beta(Q_{\bar{\alpha}}))\, \mathrm{sgn}_0^+(Q - Q_{\bar{\alpha}}) \ge 0 \quad \text{a.e } (x,t) \in Q_T.$$

We introduce the function $\chi_{\{Q > Q_{\bar{\alpha}}\}} \doteq 1$ if $Q > Q_{\bar{\alpha}}$ and $\doteq 0$ if $Q \le Q_{\bar{\alpha}}$. Notice that if $Q \le Q_{\bar{\alpha}}$ then $\mathrm{sgn}_0^+(Q - Q_{\bar{\alpha}}) = 0$ while if $Q > Q_{\bar{\alpha}}$ then $(\delta + Q_{\bar{\alpha}})^S < (\delta + Q)^S$ so (51) and (47) imply

$$\partial_x (Q - Q_{\bar{\alpha}})^+ \le (\bar{\alpha} - \alpha)(\delta + Q)^S \chi_{\{Q > Q_{\bar{\alpha}}\}} \le 0 \quad \text{a.e } (x,t) \in Q_T. \tag{52}$$

Being $(Q - Q_{\bar{\alpha}})^+(0) = 0$ we deduce from (52) that $(Q - Q_{\bar{\alpha}})^+(x) \leq 0$ because $\bar{\alpha} \leq \alpha$; moreover we have (by definition) $(Q - Q_{\bar{\alpha}})^+(x) \geq 0$ then $(Q - Q_{\bar{\alpha}})^+(x) \equiv 0$ and, finally $Q \leq Q_{\bar{\alpha}}$ on $(0,1)$. □

Remark 6.1. *In the physical meaningful case we can take $\bar{\alpha} = -B \in L^1(Q_T)$ and $\hat{\alpha} = C \in L^1(Q_T)$.*

Corollary 6.1. *In hypothesis of Lemma 6.1 we define by $x_Q(t) \in [0,1]$ the cold/temperate transition points:*

$$x_Q(t) \doteq \inf\{x \in [0,1] : Q(x,t) > 0\}.$$

Then

$$x_{Q_{\bar{\alpha}}}(t) \leq x_Q(t) \leq x_{Q_{\hat{\alpha}}}(t), \quad \forall t \in [0,T].$$

Proof. By contradiction. Let us suppose that there exists an instant $t^* \in (0,T]$ such that $x_Q(t^*) < x_{Q_{\bar{\alpha}}}(t^*)$. Let $\varepsilon \in (0,1)$ such that $x_{Q_{\bar{\alpha}}}(t^*) = x_Q(t^*) + \varepsilon < 1$. Then

$$Q(x_{Q_{\bar{\alpha}}}(t^*), t^*) = Q(x_Q(t^*) + \varepsilon, t^*) > 0, \qquad Q_{\bar{\alpha}}(x_{Q_{\bar{\alpha}}}(t^*), t^*) = 0 \tag{53}$$

because $Q(x,t^*)$ is a continuous function in $x \in [0,1]$. But this shows that (53) is an absurd because it is contrary to the comparison Lemma 6.1. □

As an application we shall give sufficient conditions in order that $Q_{\hat{\alpha}}$ has a spatially backwards moving boundary.

Lemma 6.2. *Let $\hat{\alpha}$ verifying hypothesis of Lemma 6.1. Let us suppose $Q_{\hat{\alpha}}(\cdot,t) \in C^1([0,t])$ and let $t_0, t_1 \in (0,T)$, $t_0 < t_1$, such that $y(t) : [t_0,t_1] \to (0,1)$ is a continuous function. Moreover, we shall assume that $\partial_x f^* \in C^0(Q_T)$ and*

$$f^*(x,t) > \hat{\alpha}(x,t)\delta^S \quad \text{c.t.} \quad t \in (t_0,t_1) \quad \text{c.t.} \quad x \in]y(t),1[. \tag{54}$$

Then, if $Q_{\hat{\alpha}}(y(t),t) > 0$ we have that

$$Q_{\hat{\alpha}}(x,t) > 0, \quad \forall x \in]y(t),1[, \quad \forall t \in (t_0,t_1).$$

Proof. By contradiction. We assume that there exists $y^*(t) \in (0,1)$ such that $Q_{\hat{\alpha}}(y^*(t),t) = 0$, $Q_{\hat{\alpha}} > 0$ in $(y(t), y^*(t))$. We consider the equation verified by $Q_{\hat{\alpha}}$ in $(y(t), y^*(t))$. We have $\beta(Q) \equiv 0$ in $(y(t), y^*(t))$ then $Q_{\hat{\alpha}}$ satisfies

$$\partial_x Q_{\hat{\alpha}} + \hat{\alpha}(\delta + Q_{\hat{\alpha}})^S = f^* \quad \text{in } (y(t), y^*(t)) \subset (0,1). \tag{55}$$

From (55) and by definition of $y^*(t)$ we have

$$\partial_x^- Q_{\hat{\alpha}}|_{(y^*(t),t)} \leq 0 \quad \text{in } \mathscr{D}'(0,1),$$

where ∂_x^- denotes the left lateral partial derivative evaluated at $(y^*(t),t)$. By substituting in (55) we get

$$f^*(y^*(t),t) \leq \hat{\alpha}\delta^S$$

and this is a contradiction with (54). □

Remark 6.2. *Notice that hypothesis on f^* is natural in the sense that it implies the existence of a infimum thickness at which the melting mechanism is activated:*

$$\gamma - \frac{\lambda}{h(x,t)} > \hat{\alpha}\delta^S \quad \Leftrightarrow \quad h > \frac{\lambda}{\gamma - \hat{\alpha}\delta^S} \doteq h_{\inf}$$

and this is realistic because if h increases then the shear stress increases and a larger quantity of heat is produced by viscous dissipation.

Remark 6.3. *The energy estimates and the Lemma 4.2 assure that*

$$f^* = (\gamma - \lambda/h) \in L^\infty(Q_T), \qquad \partial_x f^* = (h_x/h^2) \in L^p(Q_T)$$

then $f^* \in L^p(0,T;W^{1,p}(\Omega)) \cap L^\infty(Q_T)$.

Theorem 6.1. *Let $f^*(x,t)$ such that*

1. *there exists a curve $X_h(t) \in [0,1]$ such that X_h is decreasing in time: $X_h(t_1) < X_h(t_2)$ if $t_1 > t_2$),*
2. *$\exists t^* > t_0$ such that $X_h(t^*) = 0$,*
3. *$f^*(x,t) > \hat{\alpha}\delta^S$, a.e. $t \in (t_0,t^*)$, a.e. $x \in]X_h(t),1[$.*

Then the moving boundary $x_{\hat{Q}}(t)$ associated to $\hat{Q}$ propagates backwards and reaches $x = 0$ when $t = t^*$ i.e., $x_{\hat{Q}}(t^*) = 0$.

Proof. Let $\psi(x,t)$ be the (unique) solution of the Cauchy problem

$$\begin{aligned} &\partial_x\psi + \hat{\alpha}(\delta + \psi)^S = f^*(x,t), \\ &\psi(X_h(t),t) = 0. \end{aligned} \tag{56}$$

Then (by the initial condition and the lower bound on f^*) we have that

$$\partial_x\psi_{|X_h(t)} = f^*(X_h(t),t) - \hat{\alpha}(\delta + \psi(X_h(t),t))^S = f^*(X_h(t),t) - \hat{\alpha}\delta^S > 0.$$

By continuity $\psi(x,t) > 0$ in a right neighborhood of $(X_h(t),t)$. Moreover, ψ verifies (it is a sub-solution) $\psi(x,t) \le \hat{Q}(x,t)$ because $f^* - \beta(\hat{Q}) \ge f^*$ (by definition $\beta(Q) \le 0$) then $\hat{Q}$ verifies

$$\begin{aligned} &\partial_x\hat{Q} + \hat{\alpha}(\delta + \hat{Q})^S \ge f^*(x,t) \quad \text{in } (0,T) \times \Omega, \\ &\hat{Q}(X_h(t),t) \ge 0 \end{aligned} \tag{57}$$

and we apply the comparison principle for ordinary differential equations. Finally, by Lemma 6.2, $\hat{Q} > 0$, $\forall x \in]X_h,1[$. As $X_h(t) \to 0$ if $t \to t^*$ we apply Corollary 6.1 to $\psi \le \hat{Q}$ to get $x_{\hat{Q}}(t) \le x_\psi(t)$ from where we deduce the last part of the theorem. $\square$

By formulating the equation of Q as

$$\partial_x Q + \beta(Q) \ni f \quad \text{in } (0,T) \times \Omega \tag{58}$$

with the m.m.g. β defined by (2) and $f \in L^1(Q_T)$ given by (44) we can obtain a sufficient condition for the existence of the free boundary (given as the boundary of the set of points $(x,t) \in Q_T$ where $Q > 0$).

Proposition 6.1. *Assume* $\|Q\|_{L^\infty(Q_T)} \leq C$ *and, for* $t \in (0,T)$ *fixed, let*

$$N_t(f,\varepsilon) \doteq \{x \in (0,1) : f(x,t) < -\varepsilon\}, \tag{59}$$

a positive measured set for some $\varepsilon > 0$: $|N_t| > 0$. *Assume that the interior set to* $N_t(f,\varepsilon)$ *has* $k \in \mathbb{N}$ *connexe components*: *i.e.*

$$\operatorname{int}(N_t(f,\varepsilon)) = \bigcup_{i=1}^{k} (a_i(t), b_i(t)), \qquad (a_i(t), b_i(t)) \subset (0,1).$$

Then

$$Q(x,t) = 0 \qquad \text{for any } x \in (a_i(t), b_i(t)) \text{ such that}$$
$$b_i(t) > x \geq a_i(t) + \tfrac{C}{\varepsilon}, \ \text{for some } i = 1,\dots,k.$$

Proof. It suffices to notice that the function

$$\bar{Q} = [C - \varepsilon(x - a_i)]_+, \quad x \in (a_i(t), b_i(t))$$

is a "local" supersolution of the problem in the sense that

$$\partial_x \bar{Q} + \beta(\bar{Q}) = -\varepsilon > f \qquad \text{on } (a_i(t), b_i(t)),$$
$$\bar{Q}(a_i) = C \geq Q(a_i).$$

Then, by the comparison principle (applied on $(a_i(t), b_i(t))$) we have $0 \leq Q(x,t) \leq \bar{Q}(x)$, for any $x \in (a_i(t), b_i(t))$. As $\bar{Q} \equiv 0$ on $[a_i(t) + C/\varepsilon, b_i(t))$ we deduce the conclusion. □

Remark 6.4. *The assumption* $|N_t| > 0$ *for some* $\varepsilon > 0$ *corresponds to a cooling effect and can generate a cold basal thermal regime where* $Q = 0$. *It allows for a partially cold/temperate bed and generalizes the previous Fowler and Johnson* [11] *treatment on fully temperate beds. If* $|N_t| = 1$ *we have cold based dynamics all along the till, where the (slow) flow is due to the small quantity of shear (*δ *parameter) in the ice. If* $0 < |N_t| < 1$ *the bed is partially frozen (possibly in patches). Details on the physics and the numerical analysis of the mechanism whereby the ice sheet can switch between the slow* ($Q = 0$)/*fast* ($Q > 0$) *flow regimes can be found in Fowler and Schiavi* [13].

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