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Équations aux dérivées partielles/*Partial Differential Equations*

Global bifurcation and continua of nonnegative solutions for a quasilinear elliptic problem

Jesús Ildefonso DÍAZ [a], **Jesús HERNÁNDEZ** [b]

[a] Departamento de Matemática Aplicada, Universidad Complutense de Madrid, 28040 Madrid, Espagne
E-mail: jidiaz@sunma4.mat.ucm.es

[b] Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Espagne
E-mail: jesus.hernandez@uam.es

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Abstract. In this Note, we study the existence and multiplicity of solutions, strictly positive or nonnegative having a *dead core* (where the solution vanishes) of a one-dimensional equation of eigenvalue type associated to a quasilinear operator with strong absorption with respect to the diffusion. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Bifurcation globale et continuums de solutions pour un problème quasi-linéaire elliptique

Résumé. *Nous étudions l'existence et la multiplicité de solutions, strictement positives ou positives avec une « zone morte » (où la solution s'annule) d'un problème quasi-linéaire du type valeur propre, associé à un opérateur avec un terme d'absorption qui domine sur la diffusion.* © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Nous étudions l'existence et la multiplicité de solutions, strictement positives ou positives avec une « zone morte » où la solution s'annule, du problème quasi-linéaire :

$$\mathcal{P}_{\lambda,\alpha} \qquad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha\, u^m = \lambda\, u^q \ \text{ sur } (-1,1), \\ u(\pm 1) = 0, \end{cases}$$

avec $p > 1$ et $\alpha,\ \lambda > 0$. Notre hypothèse fondamentale est $0 < m < q \leq p-1$. Cela correspond à une situation où l'absorption est plus forte par rapport à la diffusion (*voir* [4]). Donc ceci comprend aussi bien le cas de problèmes semi-linéaires avec un terme d'absorption non lipschitzien comme $-u'' + \alpha\sqrt{u} = \lambda\, u$, que certaines équations dégénérées comme $-\left(|u'|\, u'\right)' + \alpha\, u = \lambda\, u^2$.

Note présentée par Jacques-Louis Lions.

Nous aurons besoin des notations $f(w) = w^q - w^m$, $F(r) := \int_0^r f(s)\,\mathrm{d}s$ et $r_F = (q/m)^{1/(q-m)}$ pour énoncer nos résultats. Soit aussi $\lambda_1 > 0$ défini par (1). D'après $m < q$, il s'en suit que $\lambda_1 < \infty$. Notre théorème fondamental est le suivant :

THÉORÈME 1. – *Il existe $\lambda_0 \in (0, \lambda_1)$ tel que :*

(a) *si $\lambda \in (0, \lambda_0)$, le problème n'a pas de solution positive ;*

(b) *si $\lambda = \lambda_0$, il y a une solution positive unique $u(\cdot, \mu_+(\lambda_0))$ ($\mu_+(\lambda_0) := \|u\|_\infty$) ;*

(c) *si $\lambda \in (\lambda_0, \lambda_1]$, il y a exactement deux solutions positives $p(\cdot, \lambda) = u(\cdot, \mu_-(\lambda))$ et $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$ ($\mu_-(\lambda) := \|p\|_\infty$, $\mu_+(\lambda) := \|q\|_\infty$). De plus, $p(\cdot, \lambda) < q(\cdot, \lambda)$ sur $(-1, +1)$ et $u'(\pm 1, \mu_-(\lambda_1)) = 0$;*

(d) *si $\lambda \geq \lambda_1$, il existe une solution positive $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$;*

(e) *finalement, si $\lambda > \lambda_1$, nous avons une famille de solutions (non strictement positives) engendrée par $u(\cdot, \mu_-(\lambda_1))$ et si $\lambda > \lambda_1$, nous avons $\mu_-(\lambda) = \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{q-m}} C$ avec $C = \|u(\cdot, \mu_-(\lambda_1))\|_\infty$. D'une façon plus précise, soit v_1 la fonction définie par :*

$$v_1(z) = \left(\frac{\lambda_1}{\alpha}\right)^{1/(q-m)} u\left(zL(\alpha, \lambda_1)^{-1} ;\, \mu_-(\lambda_1)\right)$$

si $|z| \leq L(\alpha, \lambda_1)$, $L(\alpha, \lambda_1) := \alpha^{-\frac{p-1-q}{q-m}} \lambda_1^{\frac{p-1-m}{q-m}}$.

Alors, pour tout y vérifiant $|y| \leq 1 - \ell(\lambda)$, $\ell(\lambda) := (\lambda_1/\lambda)^{\frac{p-1-m}{(q-m)(p-1)}}$, la fonction

$$r(x; y) = \begin{cases} \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} v_1((x-y)L(\alpha, \lambda_1)) & \text{pour } |x-y| \leq \ell(\lambda), \\ 0 & \text{pour } |x-y| > \ell(\lambda), \end{cases}$$

est une solution de $P_{\lambda,\alpha}$. En fait, si N est un entier et $\lambda \geq \lambda_1 N^{\frac{q-m}{p-1-m}}$, $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ est un vecteur donné avec

$$-1 \leq y_i - \ell(\lambda),\ y_i + \ell(\lambda) \leq y_{i+1} - \ell(\lambda),\ i = 1, \ldots, N-1,\ y_N + \ell(\lambda) \leq 1,$$

et si on définit l'ensemble $S_N(\lambda)$ de solutions comme

$$r(x, \mathbf{y}) = \begin{cases} \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} v_1((x-y_i)L(\alpha, \lambda_1)) & \text{pour } |x-y_i| \leq \ell(\lambda), \\ 0 & \text{pour } |x-y_i| > \ell(\lambda), \end{cases}$$

alors l'ensemble des solutions positives non triviales $P(\lambda)$ est formé par $S(\lambda)$ plus la solution $q(\cdot, \lambda)$, où $S(\lambda)$ est donné par $\mathcal{S}(\lambda) = \bigcup_{j=1}^{N} \mathcal{S}_j(\lambda)$.

La démonstration utilise des considérations concernant le diagramme de bifurcation pour le problème auxiliaire :

$$\mathcal{P}(L) \qquad \begin{cases} -\left(|v'|^{p-2}v'\right)' = v^q - v^m & \text{sur } (-L, L), \\ v(\pm L) = 0, \end{cases}$$

avec le paramètre L. Elle fait intervenir plusieurs changements de variables, plus une étude fine des propriétés des intégrales qui apparaissent quand on étudie dans le plan des phases un problème de Cauchy associé.

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1. Introduction

In this work we shall study the existence and multiplicity of nonnegative solutions u of the quasilinear problem:

$$\mathcal{P}_{\lambda,\alpha} \qquad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha\, u^m = \lambda\, u^q & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases}$$

where $p > 1$ and α, λ are positive numbers. Our main interest is on the range of values $0 < m < q \leq p-1$ which corresponds to a strong absorption with respect to the diffusion (*see* [4]). So, the framework of this work includes semilinear equations with a non-Lipschitz absorption term as $-u'' + \alpha\sqrt{u} = \lambda u$, as well as degenerate quasilinear equations as, for instance, $-\big(|u'|\,u'\big)' + \alpha\, u = \lambda\, u^2$. Some comments on previous work are collected in Remark 1 below.

To state our main result it is useful to introduce the notation $f(w) = w^q - w^m$ and $F(r) := \int_0^r f(s)\,\mathrm{d}s$. We also introduce $r_F = (q/m)^{1/(q-m)}$ (the unique zero of $F(r)$). Let $\lambda_1 > 0$ be given by:

$$\lambda_1 := \alpha^{(p-1-q)/(p-1-m)} \left(\frac{(p-1)^{1/p}}{p^{1/p}} \int_0^{r_F} \frac{\mathrm{d}r}{(-F(r))^{1/p}} \right)^{(p-1)(q-m)/(p-1-m)}. \tag{1}$$

Notice that $\lambda_1 < \infty$ thanks to the assumption $m < q$. Our main result is the following:

THEOREM 1. – *There exists a $\lambda_0 \in (0, \lambda_1)$ such that:*

(a) *if $\lambda \in (0, \lambda_0)$, there is no positive solution;*

(b) *if $\lambda = \lambda_0$, there is a unique positive solution $u(\,\cdot\,, \mu_+(\lambda_0))$ $(\mu_+(\lambda_0) := \|u\|_\infty)$;*

(c) *if $\lambda \in (\lambda_0, \lambda_1]$, there are two positive solutions $p(\,\cdot\,,\lambda) = u(\,\cdot\,, \mu_-(\lambda))$ and $q(\,\cdot\,,\lambda) = u(\,\cdot\,,\mu_+(\lambda))$ $(\mu_-(\lambda) := \|p\|_\infty, \mu_+(\lambda) := \|q\|_\infty)$. Moreover, $p(\,\cdot\,,\lambda) < q(\,\cdot\,,\lambda)$ on $(-1,+1)$ and $u'(\pm 1, \mu_-(\lambda_1)) = 0$;*

(d) *if $\lambda \geq \lambda_1$, there is one positive solution $q(\,\cdot\,,\lambda) = u(\,\cdot\,,\mu_+(\lambda))$;*

(e) *finally, if $\lambda > \lambda_1$, there is a family of nonnegative solutions which are generated by $u(\,\cdot\,,\mu_-(\lambda_1))$ and for $\lambda > \lambda_1$, we have $\mu_-(\lambda) = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} C$ with $C = \|u(\,\cdot\,,\mu_-(\lambda_1))\|_\infty$. More precisely, let v_1 be the function defined on $|z| \leq L(\alpha,\lambda_1)$, $L(\alpha,\lambda_1) := \alpha^{-(p-1-q)/(q-m)}\lambda_1^{(p-1-m)/(q-m)}$ by the identity*

$$v_1(z) = \left(\tfrac{\lambda_1}{\alpha}\right)^{1/(q-m)} u\big(zL(\alpha,\lambda_1)^{-1}\,;\, \mu_-(\lambda_1)\big).$$

Then, for any y, $|y| \leq 1 - \ell(\lambda)$, $\ell(\lambda) := (\lambda_1/\lambda)^{(p-1-m)/(q-m)(p-1)}$, the function

$$r(x;y) = \begin{cases} \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} v_1((x-y)L(\alpha,\lambda_1)) & \text{for } |x-y| \leq \ell(\lambda), \\ 0 & \text{for } |x-y| > \ell(\lambda), \end{cases}$$

is a solution of $\mathcal{P}_{\lambda,\alpha}$. In fact, if N is a positive integer and $\lambda \geq \lambda_1 N^{\frac{q-m}{p-1-m}}$, given a vector $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ with

$$-1 \leq y_i - \ell(\lambda),\ y_i + \ell(\lambda) \leq y_{i+1} - \ell(\lambda),\ i = 1, \ldots, N-1,\ y_N + \ell(\lambda) \leq 1,$$

and if we define the set of solutions of $\mathcal{S}_N(\lambda)$ as the one given by functions of the form

$$r(x,\mathbf{y}) = \begin{cases} \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} v_1((x-y_i)L(\alpha,\lambda_1)) & \text{for } |x-y_i| \leq \ell(\lambda), \\ 0 & \text{for } |x-y_i| > \ell(\lambda), \end{cases}$$

then the set of nontrivial and nonnegative solutions of $\mathcal{P}(\lambda)$ is formed by $\mathcal{S}(\lambda)$ jointly with $q(\,\cdot\,,\lambda)$, where $\mathcal{S}(\lambda)$ is the set defined by $\mathcal{S}(\lambda) = \bigcup_{j=1}^N \mathcal{S}_j(\lambda)$.

The proof will be obtained as a consequence of the study of the bifurcation diagram for the auxiliary problem:

$$\mathcal{P}(L) \qquad \begin{cases} -\big(|v'|^{p-2}v'\big)' = v^q - v^m & \text{in } (-L, L), \\ v(\pm L) = 0, \end{cases}$$

in which we consider the equation on a general interval $(-L, L)$ and take L as variable parameter.We shall prove that:

THEOREM 2. – *We define*

$$\gamma(\mu) := \frac{1}{[p/(p-1)]^{1/p}} \int_0^{\mu} \frac{\mathrm{d}r}{(F(\mu)-F(r))^{1/p}}. \tag{2}$$

Then $\gamma'(\mu) = 0$ *has a unique root* $\mu_0 \in (r_F, \infty)$. *We introduce the numbers* $L_0 = \gamma(\mu_0)$ *and* $L_1 = \gamma(r_F)$. *For* $L \geq L_0$ *we denote by* $\mu_+(L)$ *the largest solution of the nonlinear equation* $L = \gamma(\mu)$, *and for* $L_1 \geq L \geq L_0$ *let* $\mu_-(L)$ *be the smallest solution. Then we have the following cases:* (i) *if* $L \in (0, L_0)$ *there is no positive solution;* (ii) *if* $L = L_0$, *there is a unique positive solution* $v(\,\cdot\,, \mu_+(L_0))$; (iii) *if* $L \in (L_0, L_1]$, *there are two positive solutions* $P(\,\cdot\,, L) = v(\,\cdot\,, \mu_-(L))$ *and* $Q(\,\cdot\,, L) = v(\,\cdot\,, \mu_+(L))$; *moreover,* $P(\,\cdot\,, L) < Q(\,\cdot\,, L)$ *on* $(-L, L)$ *and* $v'(\pm 1, \mu_-(L_1)) = 0$; (iv) *if* $L \geq L_1$, *there is one positive solution* $Q(\,\cdot\,, L) = v(\,\cdot\,, \mu_+(L))$; (v) *for any* $L > L_1$, *there is a family of nonnegative solutions which is generated by* $v(\,\cdot\,, \mu_-(L_1))$. *In fact, for any h,* $|h| \leq L - L_1$, *the function*

$$s(x,h) = \begin{cases} v(x-h, \mu_-(L_1)) & \textit{for } |x-h| \leq L_1, \\ 0 & \textit{for } |x-h| > L_1, \end{cases}$$

is also a nonnegative solution. If N *is a positive integer and* $L \geq NL_1$, *given a vector* $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ *with* $-L \leq y_i - L_1$, $y_i + L_1 \leq y_{i+1} - L_1$, $i = 1, \ldots, N-1$, $y_N + L_1 \leq L$ *the function*

$$s(x,\mathbf{y}) = \begin{cases} v(x-y_i, \mu_-(L_1)) & \textit{for } |x-y_i| \leq L_1, \\ 0 & \textit{for } |x-y_i| > L_1, \end{cases} \qquad \textit{for } i = 1, \ldots, N,$$

is a nonnegative solution. We call $\mathcal{S}_N(L)$ *the collection of such solutions* $r(x, \mathbf{y})$. *Finally, for* $L > L_1$ *let* N *be the integral part of* L/L_1 *and let* $\mathcal{S}(L) = \bigcup_{j=1}^{N} \mathcal{S}_j(L)$. *Then the set of nontrivial solutions of* $\mathcal{P}(L)$ *is formed by* $\mathcal{S}(L)$ *jointly with* $Q(\,\cdot\,, L)$.

2. Proof of Theorem 1 from Theorem 2

Let $u_{\lambda,\alpha}$ be a solution of $\mathcal{P}_{\lambda,\alpha}$. Then the change of variables

$$u_{\lambda,\alpha}(x) = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} v\big(x\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)}\big) \tag{3}$$

transforms $u_{\lambda,\alpha}$ into a solution v of the problem $\mathcal{P}(L)$ with $L := \alpha^{-\frac{p-1-q}{q-m}}\lambda^{\frac{p-1-m}{q-m}}$ and, conversely, any solution v of problem $\mathcal{P}(L)$ into a solution of $\mathcal{P}_{\lambda,\alpha}$. We define $\lambda_0 := \big[\gamma(\mu_0)\,\alpha^{\frac{p-1-q}{q-m}}\big]^{\frac{q-m}{p-1-m}}$. From Theorem 2 we get that the bifurcation equation $L = \gamma(\mu)$ for the solutions of $\mathcal{P}(L)$ leads to the equivalent bifurcation equation:

$$\alpha^{-(p-1-q)/(q-m)}\lambda^{(p-1-m)/(q-m)} = \gamma\left(\|u_{\lambda,\alpha}\|_\infty \left(\frac{\lambda}{\alpha}\right)^{1/(q-m)}\right) \tag{4}$$

for the solutions of $\mathcal{P}_{\lambda,\alpha}$. Since $\gamma(\mu_0)$ is the minimum value of γ we deduce that if $\lambda < \lambda_0$, equation (4) has no solution and for $\lambda = \lambda_0$, there is only one solution. This proves (a) and (b). Since the range of the branch γ_+ is $[\gamma(\mu_0), +\infty)$ we deduce that for any $\lambda > \lambda_0$, the equation (4) has, at least, a solution which implies the existence of a solution of $\mathcal{P}_{\lambda,\alpha}$, $q(\cdot,\lambda)$. If $\lambda \in (\lambda_0, \lambda_1]$, from the continuity of the branch γ_-, we deduce the existence of a second solution of the equation (4) which implies the existence of the solution $p(\cdot,\lambda)$. Both roots correspond to the two solutions $\mu_- < \mu_+$ of the equation $L = \gamma(\mu)$ and then

$$\mu_- = \|P(\cdot,\lambda)\|_\infty \left(\tfrac{\lambda}{\alpha}\right)^{1/(q-m)}, \quad \mu_+ = \|Q(\cdot,\lambda)\|_\infty \left(\tfrac{\lambda}{\alpha}\right)^{1/(q-m)}$$

for a suitable λ which proves that $\|p(\cdot,\lambda)\|_\infty < \|q(\cdot,\lambda)\|_\infty$. Moreover, since $p(\cdot,\lambda) = \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{q-m}} P\left(x\alpha^{-\frac{p-1-q}{q-m}}\lambda^{\frac{p-1-m}{q-m}} : L\right)$ and $q(\cdot,\lambda) = \left(\frac{\alpha}{\lambda}\right)^{\frac{1}{q-m}} Q\left(x\alpha^{-\frac{p-1-q}{q-m}}\lambda^{\frac{p-1-m}{q-m}} : L\right)$, using (iii) of Theorem 2, we get (c). Part (d) is proven in a similar way.

Remark 1. – Some related results in the literature are the following: the case of $m > 1$ was studied in [7] in the larger class of possible changing sign solutions. Their results are of a completely different nature to our Theorem 1. A closer result can be found in Section 2 of [2] where the authors consider the same equation for $p = q = 2$ but searching for 2π-periodic solutions on R. We point out that several points of the description made in Theorem 1 remain true for the same type of problems in higher dimensions (*see* [5]). In this last direction it is interesting to mention the paper [3] where the authors consider the case $p = 2$, $0 < m < 1 < q < (N+2)/(N-2)$ in $\mathbb{R}^N$, $N \geq 3$. Nevertheless, no multiplicity study is made there.

3. Proof of Theorem 2

LEMMA 1. – *A function v is a positive solution of problem $\mathcal{P}(L)$ if and only if*

$$\frac{1}{[p/(p-1)]^{1/p}} \int_{v(x)}^{\mu} \frac{\mathrm{d}r}{(F(\mu) - F(r))^{1/p}} = |x| \text{ for } |x| \leq L,$$

where $\mu \in [r_F, \infty)$ and $L > 0$ are related by the equation $\gamma(\mu) = L$.

Proof. – If a positive solution exists, then necessarily it will have a maximum $\mu > 0$ in some point $\zeta \in (-L, L)$. So, let us consider

$$\mathcal{CP} \qquad \begin{cases} -\left(|v'|^{p-2}v'\right)' = f(v), \\ v(\zeta) = \mu, \quad v'(\zeta) = 0. \end{cases}$$

If $\mu < r_f$ (the zero of f) no solution of $\mathcal{CP}$ may satisfy $\mathcal{P}(L)$. Multiplying by v', integrating by parts and using the initial conditions and that $v' \leq 0$ near $x = \zeta$ we find

$$-v'(x) = A^{-1}\left(F(\mu) - F(v(x))\right), \tag{5}$$

where $A(r) := [(p-1)/p]r^p$. It is easy to see that if r_F is the (unique) positive number such that $F(r_F) = 0$, then if $\mu \in (r_f, r_F)$ no solution of $\mathcal{CP}$ may satisfy $\mathcal{P}$. So, let $\mu \in [r_F, \infty)$. When $\mu = r_F$ the integral of the function γ may have a second singularity at $r = 0$ which is integrable. For a *positive solution* v of problem $\mathcal{CP}$, $v = 0$ only at $r = \pm L$. Therefore $\zeta = 0$ and the proof holds. □

The next result shows some general qualitative behavior of the graph of $\gamma(\mu)$.

PROPOSITION 1. – *We have:* (i) $\gamma \in \mathrm{C}[r_F, \infty) \cap \mathrm{C}^1(r_F, \infty)$; (ii) $\gamma(\mu) \to +\infty$ *and* $\gamma'(\mu) \to +\infty$ *as* $\mu \to +\infty$; (iii) $\gamma'(\mu) \to -\infty$ *as* $\mu \downarrow r_F$.

Proof. – It is useful to introduce the function $\Lambda(\mu) = \dfrac{p^{1/p}}{[p-1]^{1/p}}\gamma(\mu) = \mu \displaystyle\int_0^1 \frac{\mathrm{d}\tau}{(F(\mu) - F(\tau\mu))^{1/p}}$. Then

$$\Lambda'(\mu) = \frac{\Lambda(\mu)}{\mu} - \frac{\mu}{p}\int_0^1 \frac{F'(\mu) - \tau F'(\tau\mu)\,\mathrm{d}\tau}{(F(\mu) - F(\tau\mu))^{(p+1)/p}}. \tag{6}$$

For $\mu \in (r_F, \infty)$, we have that $F'(\mu) \neq 0$ and it is not difficult to verify that the integral in (6) is convergent and that $\Lambda'(\mu) \in C(r_F, \infty)$, $\Lambda \in C([r_F, \infty))$. For the rest of the proof it is useful to introduce the auxiliary function $\theta(t) := p\,F(t) - t\,f(t)$. Then we get

$$\Lambda'(\mu) = \frac{1}{\mu p} \int_0^{\mu} \frac{(\theta(\mu) - \theta(r))\,\mathrm{d}r}{(F(\mu) - F(r))^{(p+1)/p}}. \tag{7}$$

The proof that $\Lambda'(\mu) \to -\infty$ as $\mu \downarrow r_F$ uses Fatou's lemma and the fact that $\int_0^{\delta} \frac{\mathrm{d}r}{(F(\mu)-F(r))^{(p+1)/p}}$ is not integrable. Property (iii) is proven in a similar way. □

In order to get a more precise information on the number of zeros of function $\gamma'(\mu)$ we need some different arguments.

PROPOSITION 2. – *The equation $\gamma'(\mu) = 0$ has a unique root $\mu_0 \in (r_F, \infty)$.*

Proof. – We follow closely the proof given in the nondegenerate case [8]. The following properties hold: (A) there is a $\mu_1 \in (r_F, \infty)$ such that $\theta(r) < 0$ on $(0, \mu_1)$ and $\theta(r) > 0$ on (μ_1, ∞); (B) there is a $\mu_2 \in (0, \mu_1)$ such that $\theta'(r) < 0$ on $(0, \mu_2)$ and $\theta'(r) > 0$ on (μ_2, ∞); and (C) there exists a $\mu_3 \in (0, \mu_2)$ such that $(r\theta(r))' < 0$ on $(0, \mu_3)$ and $(r\theta(r))' > 0$ on (μ_3, ∞). It follows from properties (A) and (B) that $\Lambda'(\mu) > 0$ on (μ_1, ∞) and, if $r_F < \mu_2$, $\Lambda'(\mu) < 0$ on (r_F, μ_2). It is clear that necessarily $\Lambda'(\mu)$ has at least one zero in the interval $J := [\max(r_F, \mu_2), \mu_1]$. In fact, there can be at most one by proving that

$$\Lambda''(\mu) + C\Lambda'(\mu) > 0 \tag{8}$$

on this interval J, for some $C > 0$ (notice that then $\Lambda''(\mu) > 0$ on any of such zero). The proof uses the formula

$$\Lambda''(\mu) = \frac{1}{\mu^2 p} \int_0^{\mu} \frac{\{(\delta_2\theta')(\delta_1 F) - (p + 1/p)(\delta_1\theta)(\delta_2 f)\}}{(\delta_1 F)^{(2p+1)/p}}\,\mathrm{d}r,$$

where $(\delta_1 h)(r) = h(\mu) - h(r)$ and $(\delta_2 h)(r) = \mu h(\mu) - r\,h(r)$, $0 \le r < \mu$. □

The crucial point in the rest of the proof of Theorem 2 is that $v'(\pm 1, \mu_-(L_1)) = 0$. This follows from (5). Similar ideas can be found in Proposition 3 of [1].

Remark 2. – Theorem 2 holds for a larger class of functions f (*see* [5]). A very interesting situation occurs when $p > 2$ and, for instance $f(v) = v(1-v)(v-a)$, for some $a < 1$. In that case it is possible to show the existence of nontrivial solutions taking its maximum on a positive measured subset (*see* [6]).

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