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Équations aux dérivées partielles/*Partial Differential Equations*

Nonlocal elliptic-parabolic equation arising in the transient regime of a magnetically confined plasma in a Stellarator

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Abstract. We prove the existence and the regularity of solutions of an elliptic-parabolic equation involving the notions of relative rearrangement and monotone rearrangement. These equations were obtained from 3D MHD systems, taking (in particular) into account the Ohm and Faraday's laws and the averaging arguments of Hender and Carreras for obtaining a 2D evolution model. Due to the presence of a strong nonlinearity, we introduce a new notion of solution derived from a new property of the relative rearrangement. This allows us to improve the results obtained in the stationnary case by cancelling the condition of smallness of the parameter λ in the pressure. © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Sur une équation non locale de type elliptique-parabolique apparaissant dans le régime transitoire d'un plasma magnétiquement confiné dans un « Stellarator »

Résumé. *On montre l'existence et la régularité de solution d'une équation du type elliptique-parabolique incluant une non-linéarité qui dépend du réarrangement relatif et monotone de la solution. Ces équations sont obtenues à partir du système de la MHD en trois dimensions, tenant en compte en particulier les lois d'Ohm et de Faraday ainsi que la méthode de moyennisation de Hender et Carreras conduisant ainsi à un modèle d'évolution à deux dimensions. La présence d'une forte non-linéarité et de nouvelles propriétés du réarrangement relatif ont conduit à introduire une nouvelle notion de solutions pour notre problème. Ceci nous permet d'améliorer les résultats obtenus dans le cas stationnaire en supprimant la condition de petitesse sur le paramètre λ lié à la pression.* © 1999 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Note présentée par Jacques-Louis LIONS.

Version française abrégée

On considère un ouvert borné régulier, connexe de $\mathbb{R}^2$. On se donne des fonctions a, b bornées sur Ω, $b \geqslant 0$, $p \in \mathrm{W}^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ (localement lipchitzienne), $0 \leqslant p'(\sigma) \leqslant \lambda \max(\sigma, 0) = \lambda\, \sigma_+$, $\beta(\sigma) = \min(\sigma, 0) + \alpha\sigma_+$, $0 \leqslant \alpha \leqslant 1$, deux réels $F_v > 0$ et $\gamma < 0$ et $\beta(u_0)$ une fonction bornée sur Ω.

Pour un temps $T > 0$, on s'intéresse au problème d'évolution suivant :

$$(\mathcal{P}) \qquad \begin{cases} \dfrac{\partial}{\partial t}\beta(u) - \Delta_x u = a\, G(u) + J(u) & \text{dans} \quad]0,T[\times \Omega = Q, \\ u(t,x) = \gamma & \text{sur} \quad \Sigma_T =]0,T[\times \partial\Omega, \\ \beta(u(0,\,\cdot\,)) = \beta(u_0(\,\cdot\,)) & \text{sur} \quad \Omega, \end{cases}$$

où

$$G(u)(t,x) = a\left[F_v^2 - \int_{|u(t)>0|}^{|u(t)>u_+(t,x)|} \frac{\partial}{\partial\sigma}\left[p(u_*(t,\sigma))\right] b_{*u}(t,\sigma)\,\mathrm{d}\sigma\right]_+^{1/2}$$

et

$$J(u)(t,x) = p'\big(u(t,x)\big)\left[b(x) - b_{*u(t)}\big(|u(t) > u(t,x)|\big)\right].$$

Ici, $|E|$ désigne la mesure d'un ensemble $E \subset \Omega$ (ainsi $|u(t) > \theta| = \mathrm{mes}\,\{y \in \Omega : u(t,y) > \theta\}$), $u_*(t,\,\cdot\,)$ est le réarrangement décroissant de $u(t,\,\cdot\,)$, $b_{*u}(t,\,\cdot\,)$ est le réarrangement relatif de $u(t)$ par rapport à b.

THÉORÈME 1. – *On suppose que* $u_0 \in \mathrm{H}^1(\Omega)$ *et* $\alpha > 0$. *Alors il existe au moins une solution de deuxième catégorie au sens de la définition* 2 *ci-dessous. De plus, on a* $u(t) \in \mathrm{H}^2(\Omega)$ *pour presque tout* $t > 0$ *et vérifie les estimations suivantes (indépendantes de* p*) : pour presque tout* $t > 0$,

1. $\big|\beta(u(t))\big|_\infty \leqslant \big|\beta(\gamma)\big| + t\,|a|_\infty F_v + \big|\beta(u_0) - \beta(\gamma)\big|_\infty$,

2. $$\int_0^t \int_\Omega \big|\nabla u(\sigma,x)\big|^2 \,\mathrm{d}x\,\mathrm{d}\sigma + \int_\Omega \mathrm{d}x \int_0^{u_0(x)} \beta(\sigma+\gamma)\,\mathrm{d}\sigma \leqslant \int_\Omega u_0 \beta(u_0+\gamma)\,\mathrm{d}x + |a|_\infty F_v \int_0^t \mathrm{d}\sigma \int_\Omega |u-\gamma|(\sigma,x)\,\mathrm{d}x.$$

Ce résultat s'étend à une grande classe de fonctions β. Notamment, les estimations ci-dessus sont encore vraies pour le cas $\alpha = 0$.

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1. Introduction

In this Note we consider the nonlocal elliptic-parabolic problem

$$(\mathcal{P}) \qquad \begin{cases} \dfrac{\partial}{\partial t}\beta(u) - \Delta_x u = a\, G(u) + J(u) & \text{in} \quad]0,T[\times \Omega, \\ u = \gamma & \text{on} \quad]0,T[\times \partial\Omega, \\ \beta\big(u(0,\,\cdot\,)\big) = \beta(u_0(\,\cdot\,)) & \text{on} \quad \Omega, \end{cases}$$

where Ω is a regular open connected set of $\mathbb{R}^2$, $T > 0$, $\beta(\sigma) = -\sigma_- + \alpha\sigma_+$, $0 \leqslant \alpha \leqslant 1$, $\sigma_- := -\min(0, \sigma)$, $\sigma \in \mathbb{R}$, $a \in \mathrm{L}^\infty(\Omega)$, the functions G and J are defined by

$$G(u)(t,x) = \left[F_v^2 - 2 \int_{|u(t)>0|}^{|u(t)>u_+(t,x)|} \frac{\partial}{\partial\sigma} \left[p(u_*(t,\sigma)) \right] b_{*u}(t,\sigma)\,\mathrm{d}\sigma \right]_+^{1/2},$$

where $F_v > 0$, $0 \leqslant p'(\sigma) \leqslant \lambda \max(\sigma, 0) = \lambda\sigma_+$, $\lambda > 0$, $b \in \mathrm{L}^\infty(\Omega)$, $b > 0$ a.e. in Ω and

$$J(u)(t,x) = p'(u(t,x)) \left[b(x) - b_{*u(t)}\left(|u(t) > u(t,x)| \right) \right].$$

The monotone and relative rearrangements $u(t)_*$ and $b_{*u(t)}$ will be defined later. The boundary datum is a known constant $\gamma < 0$ and finally $u_0 \in \mathrm{H}^1(\Omega)$ with $\beta(u_0) \in \mathrm{L}^\infty(\Omega)$. Problem $(\mathcal{P})$ arises in the study of the transient regime of the magnetic confinement of a fusion plasma in a Stellarator device. Starting from the ideal MHD system and using some scaling arguments on the characteristic times for the involved phenomena (*see* [11]) and the approach already followed for equilibrium regime in [7] and [6], it is shown that the *averaged poloidal flux function* $u(t,x)$ of the magnetic field $\mathbf{B}(t,x)$ satisfies $(\mathcal{P})$. We point out that a similar elliptic-parabolic model but without the nonlocal terms G and J appears in the context of partially saturated flows in porous media (*see* e.g. [3] and [2]). In this Note we present an existence result for a suitable notion of weak solution of problem $(\mathcal{P})$.

2. First and second category weak solutions

We start by recalling some useful notations. For a measurable function $u :]0,T[\times \Omega \to \mathbb{R}$, and for a fixed $t \in]0,T[$, we set $u(t) : \Omega \to \mathbb{R}$, $u(t)(x) = u(t,x)$. Given $\theta \in \mathbb{R}$, let

$$|u(t) > \theta| = \text{measure}\,\{ y \in \Omega : u(t)(y) > \theta \}$$

and, analogously,

$$|u(t) = \theta| = \text{measure}\,\{ y \in \Omega : u(t)(y) = \theta \}.$$

For a fixed $\sigma \in \Omega_* =]0, |\Omega|[$, the *monotone rearrangement* of $u(t)$ at σ is

$$u(t)_*(\sigma) = u_*(t,\sigma) = \inf\{ \theta \in \mathbb{R},\ |u(t) > \theta| \leqslant \sigma \}.$$

Given a function $b \in \mathrm{L}^1(Q)$ ($Q :=]0,T[\times \Omega$), we set for $\sigma \in]0, |\Omega|[$, $t \in]0,T[$:

$$w(t,\sigma) = \int_{\{x : u(t)(x) > u_*(t,\sigma)\}} b(t,x)\,\mathrm{d}x + \int_0^{\sigma - |u(t) > u_*(t,\sigma)|} \left(b(t)|_{\{u(t) = u_*(t,\sigma)\}} \right)_* (s)\,\mathrm{d}s,$$

where $b(t)|_{\{u(t)=u_*(t,\sigma)\}}$ is the restriction of $b(t)$ to the set $\{x : u(t)(x) = u_*(t,\sigma)\}$. The *relative rearrangement of* $b(t)$ *with respect to* $u(t)$ is the weak derivative $\dfrac{\partial w}{\partial\sigma}(t,\sigma)$, we set

$$b(t)_{*u(t)}(\sigma) = b_{*u}(t,\sigma) = \frac{\partial w}{\partial\sigma}(t,\sigma).$$

Some properties on monotone rearrangement can be found in [8], [1], [14]. On the other hand, it is known (*see* [8], [9], [10]), that $w(t) \in \mathrm{W}^{1,p}(\Omega_*)$, provided that

$$b(t) \in \mathrm{L}^p(\Omega) \text{ and } \left|\frac{\partial w}{\partial \sigma}(t)\right|_{\mathrm{L}^p(\Omega_*)} \leqslant |b(t)|_{\mathrm{L}^p(\Omega)}, \quad 1 \leqslant p \leqslant +\infty.$$

Due to presence of nonlocal terms in $(\mathcal{P})$, two different notions of weak solutions can be introduced.

DEFINITION 1 (First category weak solution). – We will say that a function u is a *first category weak solution* of $(\mathcal{P})$ if the function $v = u - \gamma$ satisfies:

1. $v \in \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))$, $\dfrac{\partial}{\partial t}\beta(v+\gamma) \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$;
2. for a.e. $t \in]0,T[$, $|\nabla v(t,\cdot) = 0| = 0$;
3. $\dfrac{\partial}{\partial t}\beta(v+\gamma) - \Delta_x v = a\,G(v+\gamma) + J(v+\gamma)$ in $\mathcal{D}'(\Omega)$ for a.e. t and $\beta(v+\gamma)|_{t=0} = \beta(u_0)$.

DEFINITION 2. – We will say that a function u is a *second category weak solution* of $(\mathcal{P})$ if the function $v = u - \gamma$ satisfies:

1. $v \in \mathrm{L}^2(0,T;\mathrm{H}_0^1(\Omega))$, $\dfrac{\partial}{\partial t}\beta(v+\gamma) \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega))$;
2. (relative rearrangement condition): there exists a bounded function $b^v \in \mathrm{L}^\infty(Q)$, satisfying for a.e. t, for all $\theta \in \mathbb{R}$, for all $\varphi \in \mathrm{C}(\mathbb{R})$, with $\varphi(v)(t) \in \mathrm{L}^1(\Omega)$,

$$\int_{\{x:v(t)(x)>\theta\}} b^v\varphi(v)(t,x)\,\mathrm{d}x = \int_{\{x:v(t)(x)>\theta\}} b\varphi(v)(t,x)\,\mathrm{d}x;$$

3. $\dfrac{\partial}{\partial t}\beta(v+\gamma) - \Delta_x v = a\,G(v+\gamma) + p'(v+\gamma)[b-b^v]$ in $\mathcal{D}'(\Omega)$ for a.e. t, and $\beta(v+\gamma)|_{t=0} = \beta(u_0)$.

PROPOSITION 1. – *Any solution u of the first category is a solution of a second category.*

PROPOSITION 2. – *If u is a solution of the second category satisfying:*

1. $|\nabla u(t,\cdot) = 0| = 0$ *for a.e. t;*
2. *there exists a Borel map $g^u : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$g^u \circ u = b^v, \quad v = u - \gamma;$$

Then u is a solution of the first category.

3. Main results

THEOREM 1. – *Let $\alpha > 0$. Then there exists, at least, a second category weak solution u of $(\mathcal{P})$. Furthermore, $u(t) \in \mathrm{H}^2(\Omega)$ for a.e. $t > 0$.*

The main idea of the proof of the above result is to approach the problem $(\mathcal{P})$. We use a Galerkin argument for a uniformly parabolic approached problem obtained by replacing β by β_ε with β_ε a C^1 function such that $\beta_\varepsilon(0) = 0$, $0 < \alpha \leqslant \beta_\varepsilon' \leqslant 2$ and $\beta_\varepsilon \to \beta$ as $\varepsilon \to 0$. We use some properties of the relative rearrangement (*see* [6], [12], [13] and [5]) in order to pass to the limit as $\varepsilon \to 0$ and as $h \to 0$ thanks to some a-priori estimates which are collected in the following results:

PROPOSITION 3. – *Any second category weak solution u satisfies:*

$$\left|\beta(u(t))\right|_\infty \leqslant |\beta(\gamma)| + t\,|a|_\infty F_v + \left|\beta(u_0) - \beta(\gamma)\right|_\infty.$$

PROPOSITION 4. – *Any second category weak solution u satisfies the estimate, for all $t \geqslant 0$:*

$$\int_0^t \int_\Omega \left|\nabla u(\sigma, x)\right|^2 \mathrm{d}x \, \mathrm{d}\sigma + \int_\Omega \mathrm{d}x \int_0^{u_0(x)} \beta(\sigma + \gamma)\, \mathrm{d}\sigma \leqslant \int_\Omega u_0 \beta(u_0 + \gamma)\, \mathrm{d}x + |a|_\infty F_v \int_0^t \mathrm{d}\sigma \int_\Omega |u - \gamma|(\sigma, x)\, \mathrm{d}x.$$

Some additional qualitative properties will be also given in the detailed paper [4] where the case $\alpha = 0$ will be discussed.

Remark. – Some of the results of this Note also hold for more general monotone functions β.

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