

Two Problems in Homogenization of Porous Media

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1. INTRODUCTION

The main goal of this work is to present two different problems arising in the Mechanics of *perforated domains* or *porous media*. The first problem concerns the compressible flow of an ideal gas through a porous media and the goal is the mathematical derivation of the Darcy's law. This is relevant in reservoirs, agriculture, soil infiltration, etc. The second problem deals with the incompressible flow of a fluid reacting with the exterior of many packed particles. This is related with absorption and adsorption phenomena in beds or towers, of interest in Chemical Engineering (separation, chemical synthesis, etc.).

A common aspect to both problems is the nature of the spatial domain: porous medium. Some examples arising in different applications can be found in the books by Bear [5], Bensoussan, Lions and Papanicolau [6], Ene and Polisevski [12], Hornung [14], Morris [25], Norman [26], Oleinik, Shmaev and Yosifian [27], and Sanchez-Palencia [28].

We shall assume some periodicity structure on the porous media. More precisely, we start by considering an open bounded set Ω of R^N , with $N = 2$ or 3, and a regular boundary $\partial\Omega$. For any small $\varepsilon > 0$ we consider the perforated domain Ω_ε obtained by intersecting the ε -multiple of a periodic geometry Y : i.e., we define $Y =]0, l_1[\times]0, l_2[\times \dots \times]0, l_N[$, a bounded regular subset of R^N with $\Gamma = \partial\theta - \partial Y$ and $Y^* = Y - \bar{\theta}$. Finally, we define

$$\Omega_\varepsilon = \Omega \cap \varepsilon\theta \text{ and } \Gamma_\varepsilon = \Omega \cap \varepsilon\Gamma.$$

In the first problem the reference open set $\theta = Y_f$ will be the exterior to the solid part Y_s and we will assume that the union of all the solid parts,

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$\Omega - \overline{\Omega}_\varepsilon$, and all the fluid parts, Ω_ε , are connected (i.e. the solid and fluid parts are of one piece) which is possible when $N = 3$. In the second problem the contrary, we shall assume that the solid part $\Omega - \overline{\Omega}_\varepsilon$ is constituted by a sequence of nonconnected obstacles, open subsets of Ω .

In both problems the main goal is the same: to determine the law (system of partial differential equations) satisfied by the homogenized unknowns such as the velocity

$$\mathbf{v} = \lim_{\varepsilon \rightarrow 0} \mathbf{v}_\varepsilon,$$

the density ρ_ε , the pressure p_ε , the concentration of some chemical component u_ε , etc.

Usually, the homogenization method starts by the formal derivation of a limit problem. One starts by the *ansatz* that the unknown functions $\rho_\varepsilon, \mathbf{v}$ have an asymptotic expansion, with respect to ε , of the form

$$\mathbf{v}_\varepsilon(x, t) = \mathbf{v}_\varepsilon(x, y, t)|_{y=\frac{x}{\varepsilon}} = \mathbf{v}_0(x, y, t) + \varepsilon \mathbf{v}_1(x, y, t) + \varepsilon^2 \mathbf{v}_2(x, y, t) + \dots$$

with $\mathbf{v}_i(x, y, t)$ y -periodic with respect to the $y = \frac{x}{\varepsilon}$ variable. And the same for the rest of fluid unknowns (the dependence on the two variables x, y justifies the name of *two-space method* used in the engineering literature: see, Keller [18]). The homogenized laws are obtained by using, as a first element of the analysis, that

$$\nabla = \nabla_x + \frac{1}{\varepsilon} \nabla_y.$$

We point out that some times it is required to assume a special time dependence on \mathbf{v}^ε (see Section 2).

The second part, considerably more difficult, consists in to obtain a rigorous proof of the consequences obtained via formal expansions, but now without any analytical assumption of the type (1). In other words, it must be proved that there exists a \mathbf{v}_0 such that $\mathbf{v}^\varepsilon \rightarrow \mathbf{v}_0$, in some functional space, as $\varepsilon \rightarrow 0$ and the same for the rest of fluid variables ρ_ε and p_ε .

In Section 2 of this paper we shall present the formal derivation of Darcy's law [10] (the flow of a liquid through a porous medium the velocity is proportional to the gradient of the pressure)

$$\mathbf{v}_0 = \frac{1}{\mu} \mathbf{K}(\rho_0 \mathbf{f} - \nabla_x p_0)$$

1, jointly with the homogenized density equation and a state assumption $F(p_\epsilon)$, leads to the, so called, porous media equation

$$\delta \frac{\partial \rho_0}{\partial t} - \operatorname{div}\left(\frac{1}{\mu} \mathbf{K} \rho_0 \nabla_x F^{-1}(\rho_0)\right) + \operatorname{div}\left(\frac{1}{\mu} \mathbf{K} \rho_0^2 \mathbf{f}\right) = 0. \tag{3}$$

we shall see later that the derivation of the above equation requires some additional arguments: \mathbf{v}_0 must be replaced by \mathbf{v}_2 , i.e. the asymptotic expansion must be taken with the term of ϵ^2 , and a different macroscopic time scale must be deduced. These (unpublished) results were presented in a postgraduate thesis by the author in 1992 (see Díaz [11]). Notice that the above formulation includes, as special case, the equation

$$w_t - \Delta w^m = 0, \tag{4}$$

where $m > 1$ which is a simpler formulation very studied in the mathematical literature since it is a degenerate equation leading to finite speed of propagation properties (see, e.g. the survey Kalashnikov [17] and references).

Equation (3) is very useful for the study of the flow since instead of several unknowns (five if $N = 3$) we reduce the problem to the determination of one, ρ_0 . This fact is well known in the literature but a rigorous proof was not derived until the work by Luc Tartar [29], in 1980, by using homogenization techniques. The motivation of our interest comes from the fact that Tartar's proof deals merely with stationary incompressible fluids and so the derivation of the mass is reduced to $\rho_0 = \rho_c$ (a known constant) and

$$\operatorname{div} \mathbf{v}_0 = 0.$$

In the case Darcy's law (2) leads to stationary equation

$$\operatorname{div}\left(\frac{1}{\mu} \mathbf{K} \rho_c \nabla_x p_0\right) + \operatorname{div}\left(\frac{1}{\mu} \mathbf{K} \rho_c \mathbf{f}\right) = 0. \tag{5}$$

Notice that equation (5) is now a linear elliptic partial differential equation and so of a very different nature to the mathematically richer nonlinear elliptic equation (3). The rigorous proof of the derivation of equation (3) seems to be far to be a mere modification of the Tartar result once that derivation of a priori estimates, for compressible fluids, is a very delicate question (see P.-L. Lions [21]).

Section 3 is devoted to a short presentation of some of the results contained in the unpublished manuscript Conca, Díaz and Liñan [8]. It concerns the,

already mentioned, second problem in which a stationary reactive fluid is confined in Ω_ε , of concentration u_ε , reacts on the boundary of a porous medium $\Omega - \bar{\Omega}_\varepsilon$, constituted by a collection of nonconnected open subsets of Ω . A simplified version of the problem is the following

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ -\frac{\partial u_\varepsilon}{\partial n} = \alpha \varepsilon |u_\varepsilon|^{p-1} u_\varepsilon & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha > 0$ and the exponent p (called as the *order of the reaction*) verifies that $p \in (0, 1)$. In that case it is possible to give a rigorous proof of the results obtained via formal expansions and so we shall prove the following

THEOREM 1. *Assume $p \in (0, 1)$, $f \in L^2(\Omega)$ and let $V_\varepsilon := \{w \in H^1(\Omega) : w = 0 \text{ on } \partial\Omega\}$ and $P_\varepsilon \in L(V_\varepsilon : H_0^1(\Omega))$ be a family of extension operators such that $(P_\varepsilon w)(x) = w(x)$ a.e. $x \in \Omega_\varepsilon$. Then $P_\varepsilon u_\varepsilon$ converges, (weakly in $H_0^1(\Omega)$), as $\varepsilon \rightarrow 0$, to a function u_0 characterized as the unique solution of the problem*

$$\begin{cases} -\sum_{i,j} q_{i,j} \frac{\partial^2 u_0}{\partial x_i \partial x_j} + \alpha \delta |u_0|^{p-1} u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\delta = \frac{\text{meas}_{N-1} \partial\theta}{\text{meas}(Y^*)}$$

and $q_{i,j}$ are suitable constants depending of θ .

2. A MATHEMATICAL DERIVATION OF THE DARCY'S LAW.

Let \mathbf{v}_ε be the velocity, ρ_ε the density and p_ε the pressure of a compressible fluid occupying the region Ω_ε . The correspondent Navier-Stokes system formed by the *mass conservation equation*

$$\frac{\partial \rho_\varepsilon}{\partial t} + \text{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0,$$

the momentum conservation equation

$$\rho_\varepsilon \left(\frac{\partial \mathbf{v}_\varepsilon}{\partial t} + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon \right) = -\nabla p_\varepsilon + \mu \Delta \mathbf{v}_\varepsilon + \lambda \nabla (\operatorname{div} \mathbf{v}_\varepsilon) + \rho_\varepsilon \mathbf{f}. \quad (9)$$

assume a constitutive law of the form

$$\rho_\varepsilon = F(p_\varepsilon), \quad (10)$$

where $F : R \rightarrow R$ is a strictly increasing function of class C^1 . The auxiliary conditions are formed by a boundary condition

$$\mathbf{v}_\varepsilon = \mathbf{0}, \quad \text{on } \partial\Omega_\varepsilon \times (0, T)$$

the initial conditions

$$\begin{aligned} \rho_\varepsilon(x, 0) &= \rho_I(x), & \text{on } \Omega_\varepsilon, \\ \mathbf{v}_\varepsilon(x, 0) &= \mathbf{v}_I(x), & \text{on } \Omega_\varepsilon, \end{aligned}$$

where ρ_I and \mathbf{v}_I are functions defined on the whole domain Ω , $\rho_I \geq 0, \rho_I \neq 0$. Mentioned at the introduction we assume a formal expansion in terms of powers of ε . In our case we introduce the variables

$$y = \frac{x}{\varepsilon} \quad \text{and} \quad \tau = \varepsilon^k t$$

assume the ansatz

$$\begin{aligned} \rho_\varepsilon(x, t) &= \rho_0(x, y, \tau) + \varepsilon \rho_1(x, y, \tau) + \varepsilon^2 \rho_2(x, y, \tau) + \dots \Big|_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t} \\ \mathbf{v}_\varepsilon(x, t) &= \varepsilon^n (\mathbf{v}_0(x, y, \tau) + \varepsilon \mathbf{v}_1(x, y, \tau) + \varepsilon^2 \mathbf{v}_2(x, y, \tau) + \dots \Big|_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t} \\ p_\varepsilon(x, t) &= p_0(x, y, \tau) + \varepsilon p_1(x, y, \tau) + \varepsilon^2 p_2(x, y, \tau) + \dots \Big|_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t} \end{aligned}$$

where k and n to be determined later (see Remark 2 for a justification of such an expansion). We have

$$\begin{aligned} \frac{\partial}{\partial t} &= \varepsilon^k \frac{\partial}{\partial \tau}, \\ \nabla &= \nabla_x + \frac{1}{\varepsilon} \nabla_y, \\ \operatorname{div} &= \operatorname{div}_x + \frac{1}{\varepsilon} \operatorname{div}_y, \\ \Delta &= \Delta_x + \frac{2}{\varepsilon} \Delta_{xy} + \frac{1}{\varepsilon^2} \Delta_y \quad (\Delta_{xy} = \sum_{i=1}^N \frac{\partial^2}{\partial x_i \partial y_i}). \end{aligned}$$

By choosing

$$k = n = 2$$

(see Remark 2 for a justification via Dimensional Analysis) the identification of the coefficients of ε^{-1} at the momentum equation leads to the condition

$$\nabla_y p_0 = 0.$$

So, from (10), “ p_0 and ρ_0 are independent of y ”. The identification of coefficients of ε at the momentum equation imply that

$$0 = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 + \lambda \nabla_y (\operatorname{div} \mathbf{v}_0) + \rho_0 \mathbf{f}.$$

Then, using (12) the above equation reduces to

$$0 = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 + \rho_0 \mathbf{f}.$$

On the other hand, since $k = n$ we get, through the conservation of the mass by identifying the coefficients of ε^n and ε^{n-1} , that

$$\frac{\partial \rho_0}{\partial t} + \operatorname{div}_x (\rho_0 \mathbf{v}_0) + \operatorname{div}_y (\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0,$$

and

$$\operatorname{div}_y (\rho_0 \mathbf{v}_0) = 0.$$

Since ρ_0 is independent of y and obviously we are interested in the case

$$\rho_0(x, \tau) \neq 0,$$

and as \mathbf{v}_0 is Y -periodic we conclude that, for fixed x and τ ,

$$\operatorname{div}_y \mathbf{v}_0 = 0 \quad \text{in } \theta.$$

So, at the local level the flow is incompressible. Now, we define the n operator

$$\tilde{\circ} = \frac{1}{|Y|} \int_Y \circ dy$$

and extend by zero all the functions defined on θ . The main result of section is the following

THEOREM. Assume (14) and (11). Then

$$\delta \frac{\partial \rho_0}{\partial \tau} + \operatorname{div}_x(\rho_0 \widetilde{\mathbf{v}}_0) = 0, \tag{15}$$

e

$$\delta = \frac{|\theta|}{|Y|} \text{ (the porosity of the medium).} \tag{16}$$

over, there exists a constant symmetric and positively defined matrix \mathbf{K} that

$$\widetilde{\mathbf{v}}_0(x, \tau) = \frac{1}{\mu} \mathbf{K}[\rho_0(x, \tau)\mathbf{f}(x, \tau) - \nabla_x p_0(x, \tau)]. \tag{17}$$

roof. It is clear that

$$\widetilde{\rho}_0(x, \tau) = \delta \rho_0(x, \tau).$$

we apply the mean operator to the equation (12). We have

$$\widetilde{\operatorname{div}_y \mathbf{v}_1} = \frac{1}{|Y|} \int_Y \operatorname{div}_y \mathbf{v}_1 dy = \frac{1}{|Y|} \int_{\partial Y} \mathbf{v}_1 \cdot \mathbf{n} d\sigma = 0$$

$\mathbf{v}_1 = \mathbf{0}$ on Γ and \mathbf{v}_1 is Y -periodic. Moreover

$$\begin{aligned} \overline{\mathbf{v}_0} &= \frac{1}{|Y|} \int_Y \nabla_y \rho_1 \mathbf{v}_0 dy \\ &= \frac{1}{|Y|} \left[\int_Y \operatorname{div}_y(\rho_1 \mathbf{v}_0) dy - \int_Y \rho_1 \operatorname{div}_y \mathbf{v}_0 dy \right] = \frac{1}{|Y|} \int_{\partial Y} \rho_1 \mathbf{v}_0 \cdot \mathbf{n} d\sigma = 0 \end{aligned}$$

o we get equation (15) which is the macroscopic mass conservation of homogenized fluid. In order to show (17) we point out that \mathbf{v}_0 solves the problem

$$\begin{cases} -\mu \Delta_y \mathbf{v}_0 = -\nabla_y p_1 + \mathbf{f}^* & \text{in } \theta \times (0, \infty), \\ \operatorname{div}_y \mathbf{v}_0 = 0 & \text{in } \theta \times (0, \infty), \\ \mathbf{v}_0 = \mathbf{0} & \text{on } \Gamma \times (0, \infty), \\ \mathbf{v}_0 \text{ is } Y\text{-periodic,} \end{cases}$$

$\mathbf{f}^* = \rho_0 \mathbf{f} - \nabla_x p_0$. So, \mathbf{v}_0 coincides with the unique weak solution in the of Leray (see, for instance [30]), i.e. $\mathbf{v}_0 \in V_\theta$ and

$$\mu \int_Y \nabla_y \mathbf{v}_0 \cdot \nabla_y \mathbf{w} dy = \int_\theta \mathbf{f}^* \cdot \mathbf{w} dy$$

for any $\mathbf{w} \in \dot{V}_\theta$ where

$$V_\theta := \{\mathbf{w} \in \mathbf{H}^1(\theta) : \operatorname{div}_y \mathbf{w} = 0, \mathbf{w} \text{ is } Y\text{-periodic and } \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}.$$

As in [28, Proposition 2.1], if for $1 \leq i \leq N$ we define $\mathbf{v}^i(y)$, $\mathbf{v}^i \in V_\theta$, as solutions of the auxiliary problems

$$\int_Y \nabla_y \mathbf{v}^i \cdot \nabla_y \mathbf{w} dy = \int_\theta w_i dy$$

assumed $\mathbf{w} = \sum w_i \mathbf{e}_i$, then, by linearity, we get that

$$\mathbf{v}_0 = \frac{1}{\mu} \left(\rho_0 f_i - \frac{\partial p_0}{\partial x_i} \right) \mathbf{v}^i.$$

Thus, applying the mean operator we get that

$$v_{0j} = \frac{K_{ij}}{\mu} \left(\rho_0 f_i - \frac{\partial p_0}{\partial x_i} \right).$$

The fact that the permeability matrix $\mathbf{K} = (K_{ij})$ is symmetric and positive defined follows as Proposition 2.2 of [28]. ■

COROLLARY 1. *Under the assumptions of the above theorem and the law (10) we have that ρ_0 satisfies the quasilinear parabolic equation*

$$\delta \frac{\partial \rho_0}{\partial \tau} - \operatorname{div} \left(\frac{1}{\mu} \mathbf{K} \rho_0 \nabla F^{-1}(\rho_0) \right) + \operatorname{div} \left(\frac{1}{\mu} \mathbf{K} \rho_0^2 \mathbf{f} \right) = 0.$$

In particular, if $\mathbf{f} = \mathbf{0}$, $\delta = 1/\mu$ and $\mathbf{K} = \mathbf{I}$ (the identity matrix) then

$$\frac{\partial \rho_0}{\partial \tau} - \Delta \varphi(\rho_0) = 0,$$

where φ is the increasing function defined as

$$\varphi(s) := \int_0^s \frac{\sigma}{F'(F^{-1}(\sigma))} d\sigma.$$

Remark 1. The special expansion could be replaced by a standard one with terms in ε and ε^0 for the velocity and without a macroscopic time scale by including physical parameters suitably scaled at the microscopic equation, for instance

$$\rho_\varepsilon \left(\varepsilon^k \frac{\partial \mathbf{v}_\varepsilon}{\partial t} + \varepsilon^k (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon \right) = -\nabla p_\varepsilon + \mu \varepsilon^n \Delta \mathbf{v}_\varepsilon + \lambda \varepsilon^n \nabla (\operatorname{div} \mathbf{v}_\varepsilon) + \rho_\varepsilon \mathbf{f}.$$

condition (11) means that (very small) viscosities $\mu\epsilon^2$ and $\lambda\epsilon^2$ are in dimensional balance with the rest of the terms of the equation. If we do not assume the condition (11) then the Darcy's law may become integro-differential (see Lions [19] and Allaire [1]), nonlinear, or it may disappear and become a deterministic law (see Section 7.4 of Sanchez-Palencia [28] and Mikelić [23]). Nonlinear Darcy's laws appear in a natural way in the study of Non-Newtonian flows in porous media (see Lions and Sanchez-Palencia [20] and survey Mikelić [24]).

Remark 2. The above special expansion and the condition (11) may be justified by using Dimensional Analysis. In order to do that let us introduce the characteristic units $L, t_c, T_c, p_c, \rho_c, v_c, V_c$ for the macroscopic length, time in the microscopic and macroscopic flow, the pressure, the density, the velocity in the microscopic and macroscopic flow respectively. We introduce the dimensionless variables

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{\tau} = \frac{\tau}{T_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{\rho} = \frac{\rho}{\rho_c}, \quad \bar{\mathbf{v}}_\epsilon = \frac{\mathbf{v}_\epsilon}{v_c}, \quad \bar{\mathbf{v}}_0 = \frac{\mathbf{v}_0}{V_c}.$$

Notice that the microscopic characteristic length is then given by ϵL . Thus we see that the microscopic momentum conservation equation becomes

$$\begin{aligned} & (\rho_c \frac{v_c}{t_c} \bar{\rho}_\epsilon) \frac{\partial \bar{\mathbf{v}}_\epsilon}{\partial \bar{t}} + (\rho_c \frac{v_c^2}{\epsilon L}) \bar{\rho}_\epsilon (\bar{\mathbf{v}}_\epsilon \cdot \nabla) \bar{\mathbf{v}}_\epsilon \\ &= -(\frac{\delta_c p}{\epsilon L}) \nabla \bar{p}_\epsilon + (\mu \frac{v_c}{\epsilon^2 L^2}) \Delta \bar{\mathbf{v}}_\epsilon + (\lambda \frac{v_c}{\epsilon^2 L^2}) \nabla (\operatorname{div} \bar{\mathbf{v}}_\epsilon) + \rho_c \bar{\rho}_\epsilon \mathbf{f}, \end{aligned} \tag{20}$$

where $\delta_c p$ denotes the characteristic pressure changes. Since the Reynolds and Péclet-Strouhal of the microscopic flow

$$Re = \frac{\rho_c v_c \epsilon L}{\mu}, \quad ReSt = \frac{\rho_c v_c \epsilon^2 L^2}{t_c}$$

are very small (remember that $\epsilon \ll 1$) the material time derivative terms of equation (20) can be neglected and we get that

$$-(\mu \frac{v_c}{\epsilon^2 L^2}) \Delta \bar{\mathbf{v}}_\epsilon - (\lambda \frac{v_c}{\epsilon^2 L^2}) \nabla (\operatorname{div} \bar{\mathbf{v}}_\epsilon) = -(\frac{\delta_c p}{\epsilon L}) \nabla \bar{p}_\epsilon + \rho_c \bar{\rho}_\epsilon \mathbf{f}. \tag{21}$$

Therefore

$$\frac{\delta_c p}{\epsilon L} = \frac{p_c}{L}$$

we get, identifying the parameters of (21), that

$$v_c = \frac{p_c L}{(\mu + \lambda)} \varepsilon^2$$

and so the significant terms of the microscopic velocity are of order two such as is implied by the special expansion and the assumption (11). On other hand, from (22) and the expansion for \mathbf{v}_ε we deduce that necessarily $\frac{c}{\varepsilon^2}$ for some constant c . Then arguing as before but now for the macroscopic mass conservation equation

$$\delta \frac{\partial \rho_0}{\partial \tau} + \operatorname{div}_x(\rho_0 \widetilde{\mathbf{v}}_0) = 0$$

we deduce that

$$\frac{\rho_c}{T_c} = \frac{\rho_c V_c}{L}.$$

So, we get

$$T_c = \frac{c/L}{\varepsilon^2}$$

which justifies the change of scale $\tau = \varepsilon^2 t$ previously assumed.

Remark 3. Notice that the macroscopic scaling $\tau = \varepsilon^2 t$ means, equivalently, that the homogenized equation is obtained for times asymptotically large in the microscopic time scale (for instance, $\tau = 1$ corresponds to $t = 1/\varepsilon^2$).

Remark 4. In the mathematical literature on the case of nonstationary compressible flows in porous media it also used a quasilinear parabolic equation of the type (19) nevertheless its justification via homogenization theory is not clear. In the case of two miscible fluids in a porous medium the present nonlinear terms is due to capillary effects leading to jump pressure empirical relations of the type

$$p^1 - p^2 = G(c^1)$$

where p^i is the pressure of the i -fluid and c^1 is the concentration of one of fluids (see Antontsev, Kazhikhov and Monakhov [2], Auriault and Sanchez-Palencia [4], Gagneux and Madaune-Tort [13] and the survey Bourgat [7]). We also point out that our point of view is different to the one considered by other authors in which the homogenization process is applied to some microscopic quasilinear parabolic equations (see, e.g. Artola [3] and Damlamian [9]).

Remark 5. As already mentioned, a rigorous proof of the convergence, for the case of stationary incompressible fluids, was given in Tartar [29] (see also survey Allaire [1] indicating some improvements but always for incompressible fluids). The main difficulty in this treatment is that is not enough to obtain a priori estimates for $(\mathbf{v}_\varepsilon/\varepsilon^2, p_\varepsilon)$ (which are independent of ε) since this concerns with a functional space, $\mathbf{H}_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, which varies with ε . Therefore, $(\mathbf{v}_\varepsilon, p_\varepsilon)$ needs to be extended to the whole homogenized set Ω . The extension of \mathbf{v}_ε is obvious (we take the value $\mathbf{0}$ outside Ω_ε). In the case of the pressure p_ε we take the value

$$\frac{1}{|\theta|} \int_\theta p_\varepsilon.$$

We conjecture that this type of extension and the recent results of P.L. Lions (see also P.L. Lions and Masmoudi [22]) will allow to get a rigorous proof of the above Theorem.

ON THE HOMOGENIZED REACTION BETWEEN A FLUID AND A SOLID
CHEMICAL SPECIE ON THE WALLS OF A POROUS MEDIUM

Idea of the proof of Theorem 1. By multiplying by u_ε , integrating by parts using the monotonicity of the function $u \rightarrow |u|^{p-1}u$ we get that there exists $M > 0$, independent of ε , such that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq M, \quad \forall \varepsilon > 0. \tag{23}$$

Assumption the extension operators are continuous i.e.,

$$\int_\Omega |\nabla P_\varepsilon(v)|^2 dx \leq C \int_{\Omega_\varepsilon} |\nabla v|^2 dx, \quad \forall v \in V_\varepsilon, \quad \forall \varepsilon.$$

There exists a subsequence (labeled as $P_{\varepsilon'}u_{\varepsilon'}$) such that $P_{\varepsilon'}u_{\varepsilon'}$ converges, (weakly) in $H_0^1(\Omega)$, as $\varepsilon' \rightarrow 0$, to a function $u_0 \in H_0^1(\Omega)$ (in fact, since problems (6) and (7) have a unique solution the above convergence holds for the complete sequence $P_\varepsilon u_\varepsilon$). It remains to show that u_0 is a weak solution. To do that we define

$$\tilde{\xi}_\varepsilon = \begin{cases} \nabla u_\varepsilon, & \text{in } \Omega_\varepsilon, \\ 0, & \text{in } \Omega - \Omega_\varepsilon. \end{cases}$$

It is clear that $\tilde{\xi}_{\varepsilon'} \rightharpoonup \xi$, in $L^2(\Omega)$ -weakly, as $\varepsilon' \rightarrow 0$, and that

$$\int_{\Omega} \tilde{\xi}_{\varepsilon'} \cdot \nabla \varphi dx + \alpha \varepsilon' \int_{\Gamma_{\varepsilon}} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'} \varphi ds = \int_{\Omega} \chi_{\Omega_{\varepsilon'}} f \varphi dx,$$

where $\chi_{\Omega_{\varepsilon'}}$ is the characteristic function of $\Omega_{\varepsilon'}$. By standard homogeniza techniques, it is possible to show that

$$\begin{cases} \chi_{\Omega} \rightharpoonup \frac{\text{meas}(Y^*)}{\text{meas}(Y)} & \text{in } L^q(\Omega) - \text{weakly, } \forall 1 \leq q < \infty, \text{ as } \varepsilon' \rightarrow 0 \\ \chi_{\Omega} \rightharpoonup \frac{\text{meas}(Y^*)}{\text{meas}(Y)} & \text{in } L^\infty(\Omega) - \text{weakly-star, as } \varepsilon' \rightarrow 0. \end{cases}$$

On the other hand, thanks to the assumption $p < 1$ we get that

$$\lim_{\varepsilon' \rightarrow 0} \int_{\Gamma_{\varepsilon}} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'} \varphi ds = \frac{\text{meas}(\partial\theta)}{\text{meas}(Y)} \int_{\Omega} |u_0|^{p-1} u_0 \varphi dx.$$

So, at the limit

$$-\text{div } \xi + \alpha \delta |u_0|^{p-1} u_0 = f, \text{ in } \Omega.$$

In order to obtain an expression of ξ in terms of ∇u_0 , for $i = 1, \dots, N$, introduce the auxiliary cellular problems

$$\begin{cases} -\Delta_y \mathcal{X}_i = 0 & \text{in } Y^*, \\ -\frac{\partial \mathcal{X}_i}{\partial n_y} = n_i & \text{on } \partial\theta, \\ \mathcal{X}_i \text{ } Y^*\text{-periodic in } y \end{cases}$$

and then the functions

$$\phi_{i\varepsilon}(x) = \varepsilon[\mathcal{X}_i(\frac{x}{\varepsilon}) + y_i], \forall x \in \Omega_{\varepsilon}$$

and

$$\eta_i^{\varepsilon} = \nabla \phi_{i\varepsilon}.$$

It is not difficult to show that if $\tilde{\eta}_i^{\varepsilon}$ denotes the corresponding extensor zero to $\Omega - \Omega_{\varepsilon}$, then

$$\begin{cases} (\tilde{\eta}_i^{\varepsilon})_j \rightharpoonup \frac{1}{\text{meas}(Y)} (\int_{Y^*} \frac{\partial \mathcal{X}_i}{\partial y_j} dy + \text{meas}(Y^*) \delta_{ij}) \\ := \frac{\text{meas}(Y^*)}{\text{meas}(Y)} q_{ij} \text{ in } L^2(\Omega) - \text{weakly.} \end{cases}$$

lly, using that (after some technical arguments)

$$-\int_{\Omega} \xi \cdot \nabla x_i \varphi dx + \frac{\text{meas}(Y^*)}{\text{meas}(Y)} \int_{\Omega} q_i \cdot \nabla u_0 \varphi dx = 0$$

conclusion follows. ■

Remark 6. After proving the above result (as consequence of the visit of once to the Universidad Complutense de Madrid, in November of 1986), become aware of some related results in the literature, mainly the papers Hornung and W. Jäger [15], [16]. In those papers, the authors consider a more general formulation which contain problem (6) as an special case but under the structural assumption $p \geq 1$. The case $p < 1$ is left there as an open problem and so the above theorem is not covered by their results.

Remark 7. The reaction term on the boundary of the particles

$$-\frac{\partial u_\varepsilon}{\partial n} = \alpha \varepsilon |u_\varepsilon|^{p-1} u_\varepsilon \quad \text{on } \Gamma_\varepsilon,$$

is a simplification of a more complicated situation. In fact, each of the particles is part of a porous medium and so there is a diffusion and reaction at its interior. In the case of a spherically symmetric isolated particle it is possible to express $\frac{\partial u}{\partial n}$ in terms of the value of u_ε on Γ_ε once that one assumes that at the interior of the particle there is a chemical reaction of the type

$$-\Delta u_\varepsilon + \alpha \varepsilon |u_\varepsilon|^{m-1} u_\varepsilon = 0, \quad \text{in } \theta$$

(see Vega and Liñán [31]).

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