:XTRACTA MATHEMATICAE Vol. 14, Núm. 2, 141 – 155 (1999)

Two Problems in Homogenization of Porous Media

J.I. DÍAZ*

Departamento de Matemática Aplicada, Fac. de Matemáticas, Universidad Complutense, 28040 *Madrid, Spain*

Subject Class. (1991): 35B27, 76S05, 76N10

1. Introduction

he main goal of this work is to present two different problems arising in l Mechanics of *perforated domains* or *porous media*. The first problem :rns the compressible flow of an ideal gas through a porous media and oal is the mathematical derivation of the Darcy's law. This is relevant in servoirs, agriculture, soil infiltration, etc. The second problem deals with ıcompressible flow of a fluid reacting with the exterior of many packed particles. This is related with absorption and adsorption phenomena ds or towers, of interest in Chemical Engineering (separation, chemical :try, etc.,).

common aspect to both problems is the nature of the spatial domain: orous medium. Some examples arising in different applications can be l in the books by Bear [5], Bensoussan, Lions and Papanicolau [6], Ene 'olisevski [12], Hornung [14], Morris [25], Norman [26], Oleinik, Shmaev 'osefian [27], and Sanchez-Palencia [28].

'e shall assume some periodicity structure on the porous media. More :ely, we start by considering an open bounded set Ω of R^N, with $N = 2$ of regular boundary $\partial\Omega$. For any small $\varepsilon > 0$ we consider the perforated in Ω_ε obtained by intersecting the $\varepsilon-$multiple of a periodic geometry Ω: i.e., we define $Y =]0, l_1[\times]0, l_2[\times\cdot\cdot\times]0, l_N[$, a bounded regular subset ' with $\Gamma = \partial\theta - \partial Y$ and $Y^* = Y - \overline{\theta}$. Finally, we define

$$\Omega_\varepsilon = \Omega \cap \varepsilon\theta \text{ and } \Gamma_\varepsilon = \Omega \cap \varepsilon\Gamma.$$

the first problem the reference open set $\theta = Y_f$ will be the exterior to olid part Y_s and we will assume that the union of all the solid parts,

rtially sponsored by the DGES (Spain), Proyect PB96-0583

$\Omega - \overline{\Omega}_\varepsilon$, and all the fluid parts, Ω_ε, are connected (i.e. the solid and fluid] are of one piece) which is possible when $N = 3$. In the second problem the contrary, we shall assume that the solid part $\Omega - \overline{\Omega_\varepsilon}$ is constituted sequence of nonconnected obstacles, open subsets of Ω.

In both problems the main goal is the same: to determine the *law* system of partial differential equations) satisfied by the homogenized unknowns such as the velocity

$$\mathbf{v} = \lim_{\varepsilon \to 0} \mathbf{v}_\varepsilon,$$

the density ρ_ε, the pressure p_ε, the concentration of some chemical compo u_ε, etc.

Usually, the homogenization method starts by the formal derivation o limit problem. One starts by the *ansatz* that the unknowns functions ρ_ε, v have an asymptotic expansion, with respect to ε, of the form

$$\mathbf{v}_\varepsilon(x,t) = \mathbf{v}_\varepsilon(x,y,t)|_{y=\frac{x}{\varepsilon}} = \mathbf{v}_0(x,y,t) + \varepsilon \mathbf{v}_1(x,y,t) + \varepsilon^2 \mathbf{v}_2(x,y,t) + \ldots$$

with $\mathbf{v}_i(x,y,t)$ y-periodic with respect to the $y = \frac{x}{\varepsilon}$ variable. And the sam the rest of fluid unknowns (the dependence on the two variables x, y jus the name of *two-space method* used in the engineering literature: see, Keller [18]). The homogenized laws are obtained by using, as a first ele of the analysis, that

$$\nabla = \nabla_x + \frac{1}{\varepsilon} \nabla_y.$$

We point out that some times it is required to assume a special time de dence on $\mathbf{v}^\varepsilon$ (see Section 2).

The second part, considerably more difficult, consists in to obtain a r ous proof of the consequences obtained via formal expansions, but now wit any analytical assumption of the type (1). In other words, it must be pr that there exists a $\mathbf{v}_0$ such that $\mathbf{v}^\varepsilon \to \mathbf{v}_0$, in some functional space, as ε and the same for the rest of fluid variables ρ_ε and p_ε.

In Section 2 of this paper we shall present the formal derivation of Darcy's law [10] (the flow of a liquid through a porous medium the veloci proportional to the gradient of the pressure)

$$\mathbf{v}_0 = \frac{1}{\mu} \mathbb{K}(\rho_0 \mathbf{f} - \nabla_x p_0)$$

ı, jointly with the homogenized density equation and a state assumption $F(\ p_\varepsilon)$, leads to the, so called, *porous media equation*

$$\delta\frac{\partial\rho_0}{\partial t} - \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0\nabla_x F^{-1}(\rho_0)) + \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0^2\mathbf{f}) = 0. \tag{3}$$

hall see later that the derivation of the above equation requires some al arguments: $\mathbf{v}_0$ must be replaced by $\mathbf{v}_2$, i.e. the asymptotic expansion ; with the term of ε^2, and a different macroscopic time scale must be duced. These (unpublished) results were presented in a postgraduate ;e by the author in 1992 (see Díaz [11]). Notice that the above formulation de, as special case, the equation

$$w_t - \Delta w^m = 0, \tag{4}$$

re usually $m > 1$) which is a simpler formulation very studied in the .ematical literature since it is a degenerate equation leading to finite l of propagation properties (see, e.g. the survey Kalashnikov [17] and ferences).

quation (3) is very useful for the study of the flow since instead of several r unknowns (five if $N = 3$) we reduce the problem to the determination ly one, ρ_0. This fact is well known in the literature but a rigorous proof ıot derived until the work by Luc Tartar [29], in 1980, by using homoge- ion techniques. The motivation of our interest comes from the fact that ır's proof deals merely with stationary incompressible fluids and so the ːrvation of the mass is reduced to $\rho_0 = \rho_c$ (a known constant) and

$$\operatorname{div}\mathbf{v}_0 = 0.$$

at case Darcy's law (2) leads to stationary equation

$$\operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_c\nabla_x\ p_0) + \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_c\mathbf{f}) = 0. \tag{5}$$

ːe that equation (5) is now a linear elliptic partial differential equation) and so of a very different nature to the mathematically richer nonlinear ɔolic equation (3). The rigorous proof of the derivation of equation (3) ; to be far to be a mere modification of the Tartar result once that erivation of a priori estimates, for compressible fluids, is a very delicate ion (see P.-L. Lions [21]).

ːction 3 is devoted to a short presentation of some of the results contained ə unpublished manuscript Conca, Díaz and Liñan [8]. It concerns the, already mentioned, second problem in which a stationary reactive fluid fined in Ω_ε, of concentration u_ε, reacts on the boundary of a porous med $\Omega - \overline{\Omega}_\varepsilon$, constituted by a collection of nonconnected open subsets of Ω. A simplified version of the problem is the following

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega_\varepsilon, \\ -\dfrac{\partial u_\varepsilon}{\partial n} = \alpha\varepsilon \left|u_\varepsilon\right|^{p-1} u_\varepsilon & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\alpha > 0$ and the exponent p (called as the *order of the reaction*) ve that $p \in (0,1)$. In that case it is possible to give a rigorous proof of the r obtained via formal expansions and so we shall prove the following

THEOREM 1. *Assume $p \in (0,1)$, $f \in L^2(\Omega)$ and let $V_\varepsilon := \{w \in H^1($ $w = 0$ on $\partial\Omega\}$ and $P_\varepsilon \in L(V_\varepsilon : H_0^1(\Omega))$ be a family of extension operators such that $(P_\varepsilon w)(x) = w(x)$ a.e. $x \in \Omega_\varepsilon)$. Then $P_\varepsilon u_\varepsilon$ converges, (weakl $H_0^1(\Omega)$, as $\varepsilon \to 0$, to a function u_0 characterized as the unique solution o problem*

$$\begin{cases} -\displaystyle\sum_{i,j} q_{i,j} \frac{\partial^2 u_0}{\partial x_i \partial x_j} + \alpha\delta \left|u_0\right|^{p-1} u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\delta = \frac{\mathrm{meas}_{N-1}\,\partial\theta}{\mathrm{meas}(Y^*)}$$

and $q_{i,j}$ are suitable constants depending of θ.

2. A MATHEMATICAL DERIVATION OF THE DARCY'S LAW.

Let $\mathbf{v}_\varepsilon$ be the velocity, ρ_ε the density and p_ε the pressure of a compress fluid occupying the region Ω_ε. The correspondent Navier-Stokes syste formed by the *mass conservation equation*

$$\frac{\partial \rho_\varepsilon}{\partial t} + \mathrm{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0,$$

:he *momentum conservation equation*

$$\rho_\varepsilon(\frac{\partial \mathbf{v}_\varepsilon}{\partial t} + (\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon) = -\nabla p_\varepsilon + \mu\Delta\mathbf{v}_\varepsilon + \lambda\nabla(\operatorname{div}\mathbf{v}_\varepsilon) + \rho_\varepsilon \mathbf{f}. \tag{9}$$

ssume a constitutive law of the form

$$\rho_\varepsilon = F(p_\varepsilon), \tag{10}$$

e $F : R \to R$ is a strictly increasing function of class C^1. The auxiliary itions are formed by a boundary condition

$$\mathbf{v}_\varepsilon = \mathbf{0}, \quad \text{on } \partial\Omega_\varepsilon \times (0, T)$$

:he initial conditions

$$\rho_\varepsilon(x, 0) = \rho_I(x), \quad \text{on } \Omega_\varepsilon,$$
$$\mathbf{v}_\varepsilon(x, 0) = \mathbf{v}_I(x), \quad \text{on } \Omega_\varepsilon,$$

e ρ_I and $\mathbf{v}_I$ are functions defined on the whole domain Ω, $\rho_I \geq 0, \rho_I \neq 0$. entioned at the introduction we assume a formal expansion in terms of rs of ε. In our case we introduce the variables

$$y = \frac{x}{\varepsilon} \quad \text{and} \quad \tau = \varepsilon^k t$$

ssume the *ansatz*

$$\rho_\varepsilon(x,t) = \rho_0(x,y,\tau) + \varepsilon\rho_1(x,y,\tau) + \varepsilon^2\rho_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t}$$
$$_\varepsilon(x,t) = \varepsilon^n \ (\mathbf{v}_0(x,y,\tau) + \varepsilon\mathbf{v}_1 \ (x,y,\tau) + \varepsilon^2\mathbf{v}_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t}$$
$$p_\varepsilon(x,t) = p_0(x,y,\tau) + \varepsilon p_1(x,y,\tau) + \varepsilon^2 p_2(x,y,\tau) + \cdots |_{y=\frac{x}{\varepsilon}, \tau=\varepsilon^k t}$$

k and n to be determined later (see Remark 2 for a justification of such of expansion). We have

$$\frac{\partial}{\partial t} = \varepsilon^k \frac{\partial}{\partial \tau},$$
$$\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y,$$
$$\operatorname{div} = \operatorname{div}_x + \frac{1}{\varepsilon}\operatorname{div}_y,$$
$$\Delta = \Delta_x + \frac{2}{\varepsilon}\Delta_{xy} + \frac{1}{\varepsilon^2}\Delta_y \ (\Delta_{xy} = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i \partial y_i}).$$

By choosing

$$k = n = 2$$

(see Remark 2 for a justification via Dimensional Analysis) the identifica
of the coefficients of ε^{-1} at the momentum equation leads to the conditi

$$\nabla_y p_0 = 0.$$

So, from (10), "p_0 and ρ_0 are independent of y". The identification o
coefficients of ε at the momentum equation imply that

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 + \lambda \nabla_y (\operatorname{div} \mathbf{v}_0) + \rho_0 \mathbf{f}.$$

Then, using (12) the above equation reduces to

$$\mathbf{0} = -(\nabla_x p_0 + \nabla_y p_1) + \mu \Delta_y \mathbf{v}_0 \ + \rho_0 \mathbf{f}.$$

On the other hand, since $k = n$ we get, through the conservation of the n
by identifying the coefficients of ε^n and ε^{n-1}, that

$$\frac{\partial \rho_0}{\partial t} + \operatorname{div}_x(\rho_0 \mathbf{v}_0) + \operatorname{div}_y(\rho_0 \mathbf{v}_1 + \rho_1 \mathbf{v}_0) = 0,$$

and

$$\operatorname{div}_y(\rho_0 \mathbf{v}_0) = 0.$$

Since ρ_0 is independent of y and obviously we are interested in the case

$$\rho_0(x, \tau) \neq 0,$$

and as $\mathbf{v}_0$ is Y-periodic we conclude that, for fixed x and τ,

$$\operatorname{div}_y \mathbf{v}_0 = 0 \quad \text{in } \theta.$$

So, at the local level the flow is incompressible. Now, we define the n
operator

$$\widetilde{\bullet} = \frac{1}{|Y|} \int_Y \bullet dy$$

and extend by zero all the functions defined on θ. The main result of
section is the following

HEOREM. *Assume* (14) *and* (11). *Then*

$$\delta \frac{\partial \rho_0}{\partial \tau} + \operatorname{div}_x(\rho_0 \widetilde{\mathbf{v}_0}) = 0, \tag{15}$$

e

$$\delta = \frac{|\theta|}{|Y|} \text{ (the porosity of the medium).} \tag{16}$$

over, there exists a constant symmetric and positively defined matrix $\mathbf{K}$ *that*

$$\widetilde{\mathbf{v}}_0(x,\tau) = \frac{1}{\mu}\mathbf{K}[\rho_0(x,\tau)\mathbf{f}(x,\tau) - \nabla_x p_0(x,\tau)]. \tag{17}$$

roof. It is clear that

$$\widetilde{\rho_0}(x,\tau) = \delta\rho_0(x,\tau).$$

we apply the mean operator to the equation (12). We have

$$\widetilde{\operatorname{div}_y \mathbf{v}_1} = \frac{1}{|Y|}\int_Y \operatorname{div}_y \mathbf{v}_1 dy = \frac{1}{|Y|}\int_{\partial Y} \mathbf{v}_1 \cdot \mathbf{n} d\sigma = 0$$

$\mathbf{v}_1 = \mathbf{0}$ on Γ and $\mathbf{v}_1$ is Y-periodic. Moreover

$$\begin{aligned} \widetilde{\ \ } \mathbf{v}_0 &= \frac{1}{|Y|}\int_Y \nabla_y \rho_1 \mathbf{v}_0 dy \\ &= \frac{1}{|Y|}[\int_Y \operatorname{div}_y(\rho_1 \mathbf{v}_0)dy - \int_Y \rho_1 \operatorname{div}_y \mathbf{v}_0 dy] = \frac{1}{|Y|}\int_{\partial Y} \rho_1 \mathbf{v}_0 \cdot \mathbf{n} d\sigma = 0 \end{aligned}$$

o we get equation (15) which is the macroscopic mass conservation of
ɔmogenized fluid. In order to show (17) we point out that $\mathbf{v}_0$ solves the
s problem

$$\begin{cases} -\mu\Delta_y \mathbf{v}_0 = -\nabla_y p_1 + \mathbf{f}^* & \text{in } \theta \times (0,\infty), \\ \operatorname{div}_y \mathbf{v}_0 = 0 & \text{in } \theta \times (0,\infty), \\ \mathbf{v}_0 = \mathbf{0} & \text{on } \Gamma \times (0,\infty), \\ \mathbf{v}_0 \text{ is } Y\text{-periodic,} & \end{cases}$$

$\mathbf{f}^* = \rho_0 \mathbf{f} - \nabla_x p_0$. So, $\mathbf{v}_0$ coincides with the unique weak solution in the
of Leray (see, for instance [30]), i.e. $\mathbf{v}_0 \in V_\theta$ and

$$\mu \int_Y \nabla_y \mathbf{v}_0 \cdot \nabla_y \mathbf{w} dy = \int_\theta \mathbf{f}^* \cdot \mathbf{w} dy$$

for any $\mathbf{w} \in \dot{V_\theta}$ where

$$V_\theta := \{\mathbf{w} \in \mathbf{H}^1(\theta) : \operatorname{div}_y \mathbf{w} = 0, \mathbf{w} \text{ is } Y\text{-periodic and } \mathbf{w} = \mathbf{0} \text{ on } \Gamma\}.$$

As in [28, Proposition 2.1], if for $1 \leq i \leq N$ we define $\mathbf{v}^i\ (y)$, $\mathbf{v}^i \in V_\theta$, a solutions of the auxiliary problems

$$\int_Y \nabla_y \mathbf{v}^i \cdot \nabla_y \mathbf{w} dy = \int_\theta w_i dy$$

assumed $\mathbf{w} = \sum w_i \mathbf{e}_i$, then, by linearity, we get that

$$\mathbf{v}_0 = \frac{1}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i})\mathbf{v}^i.$$

Thus, applying the mean operator we get that

$$v_{0j} = \frac{K_{ij}}{\mu}(\rho_0 f_i - \frac{\partial p_0}{\partial x_i}).$$

The fact that the permeability matrix $\mathbf{K} = (K_{ij})$ is symmetric and posit defined follows as Proposition 2.2 of [28]. ∎

COROLLARY 1. *Under the assumptions of the above theorem and the law* (10) *we have that* ρ_0 *satisfies the quasilinear parabolic equation*

$$\delta\frac{\partial \rho_0}{\partial \tau} - \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0 \nabla F^{-1}(\rho_0)) + \operatorname{div}(\frac{1}{\mu}\mathbf{K}\rho_0^2 \mathbf{f}) = 0.$$

In particular, if $\mathbf{f} = \mathbf{0}, \delta = 1/\mu$ *and* $\mathbf{K} = \mathbf{I}$ *(the identity matrix) then*

$$\frac{\partial \rho_0}{\partial \tau} - \Delta\varphi(\rho_0) = 0,$$

where φ *is the increasing function defined as*

$$\varphi(s) := \int_0^s \frac{\sigma}{F'(F^{-1}(\sigma))} d\sigma.$$

Remark 1. The special expansion could be replaced by a standard one with terms in ε and ε^0 for the velocity and without a macroscopic time s by including physical parameters suitably scaled at the microscopic equat as, for instance

$$\rho_\varepsilon(\varepsilon^k \frac{\partial \mathbf{v}_\varepsilon}{\partial t} + \varepsilon^k(\mathbf{v}_\varepsilon \cdot \nabla)\mathbf{v}_\varepsilon) = -\nabla p_\varepsilon + \mu\varepsilon^n \Delta \mathbf{v}_\varepsilon + \lambda\varepsilon^n \nabla(\operatorname{div} \mathbf{v}_\varepsilon) + \rho_\varepsilon \mathbf{f}.$$

condition (11) means that (very small) viscosities $\mu\varepsilon^2$ and $\lambda\varepsilon^2$ are in dimensional balance with the rest of the terms of the equation. If we ɔt assume the condition (11) then the Darcy's law may become integro-ːential (see Lions [19] and Allaire [1]), nonlinear, or it may disappear deterministic law (see Section 7.4 of Sanchez-Palencia [28] and Mikelic Nonlinear Darcy's laws appears in a natural way in the study of Non-;onian flows in porous media (see Lions and Sanchez-Palencia [20] and urvey Mikelic [24]).

:emark 2. The above special expansion and the condition (11) may be ìed by using Dimensional Analysis. In order to do that let us introduce characteristics units L, t_c, T_c, p_c, ρ_c, v_c, V_c for the macroscopic length, ime in the microscopic and macroscopic flow, the pressure, the density the velocity in the microscopic and macroscopic flow respectively. We ntroduce the dimensionless variables

$$\overline{x} = \frac{x}{L},\ \overline{t} = \frac{t}{t_c},\ \overline{\tau} = \frac{\tau}{T_c},\ \overline{p} = \frac{p}{p_c},\ \overline{\rho} = \frac{\rho}{\rho_c},\ \overline{\mathbf{v}_\varepsilon} = \frac{\mathbf{v}_\varepsilon}{v_c},\ \overline{\mathbf{v}_0} = \frac{\mathbf{v}_0}{V_c}.$$

:e that the microscopic characteristic length is then given by εL. Thus ;e that the microscopic momentum conservation equation becomes

$$\begin{aligned}(\rho_c \frac{v_c}{t_c}\overline{\rho}_\varepsilon)\frac{\partial \overline{\mathbf{v}}_\varepsilon}{\partial \overline{t}} + (\rho_c \frac{v_c^2}{\varepsilon L})\overline{\rho}_\varepsilon(\overline{\mathbf{v}}_\varepsilon \cdot \nabla)\overline{\mathbf{v}}_\varepsilon) \\ = -(\frac{\delta_c p}{\varepsilon L})\nabla\overline{p}_\varepsilon + (\mu \frac{v_c}{\varepsilon^2 L^2})\Delta\overline{\mathbf{v}}_\varepsilon + (\lambda \frac{v_c}{\varepsilon^2 L^2})\nabla(\operatorname{div}\overline{\mathbf{v}}_\varepsilon) + \rho_c\overline{\rho}_\varepsilon \mathbf{f},\end{aligned} \tag{20}$$

ɜ $\delta_c p$ denotes the characteristic pressure changes. Since the Reynolds and olds-Strouhal of the microscopic flow

$$Re = \frac{\rho_c v_c \varepsilon L}{\mu}, \quad ReSt = \frac{\rho_c v_c \varepsilon^2 L^2}{t_c}$$

ɜry small (remember that $\varepsilon << 1$) the material time derivative terms of quation (20) can be neglected and we get that

$$-(\mu \frac{v_c}{\varepsilon^2 L^2})\Delta\overline{\mathbf{v}}_\varepsilon - (\lambda \frac{v_c}{\varepsilon^2 L^2})\nabla(\operatorname{div}\overline{\mathbf{v}}_\varepsilon) = -(\frac{\delta_c p}{\varepsilon L})\nabla\overline{p}_\varepsilon + \rho_c\overline{\rho}_\varepsilon \mathbf{f}. \tag{21}$$

ng

$$\frac{\delta_c p}{\varepsilon L} = \frac{p_c}{L}$$

we get, identifying the parameters of (21), that

$$v_c = \frac{p_c L}{(\mu+\lambda)}\varepsilon^2$$

and so the significant terms of the microscopic velocity are of order two such as is implied by the special expansion and the assumption (11). Or other hand, from (22) and the expansion for $\mathbf{v}_\varepsilon$we deduce that necessarily $\frac{c}{\varepsilon^2}$ for some constant c. Then arguing as before but now for the macrosc mass conservation equation

$$\delta\frac{\partial\rho_0}{\partial\tau} + \mathrm{div}_x(\rho_0\widetilde{\mathbf{v}_0}) = 0$$

we deduce that

$$\frac{\rho_c}{T_c} = \frac{\rho_c V_c}{L}.$$

So, we get

$$T_c = \frac{c/L}{\varepsilon^2}$$

which justifies the change of scale $\tau = \varepsilon^2 t$ previously assumed.

Remark 3. Notice that the macroscopic scaling $\tau = \varepsilon^2 t$ means, q itatively, that the homogenized equation is obtained for times asymp cally large in the microscopic time scale (for instance, $\tau = 1$ corresp to $t = 1/\varepsilon^2$).

Remark 4. In the mathematical literature on the case of nonstationary compressible flows in porous media it also used a quasilinear parabolic e tion of the type (19) nevertheless its justification via homogenization theo not clear. In the case of two miscible fluids in a porous medium the presen nonlinear terms is due to capillary effects leading to jump pressure empi relations of the type

$$p^1 - p^2 = G(c^1)$$

where p^i is the pressure of the $i-$fluid and c^1 is the concentration of one of fluids (see Antontsev, Kazhikhov and Monhakhov [2], Auriault and Sanc Palencia [4], Gagneux and Madaune-Tort [13] and the survey Bourgat [7]). also point out that our point of view is different to the one considered by o authors in which the homogenization process is applied to some microsc quasilinear parabolic equations (see, e.g. Artola [3] and Damlamian [9]).

ℓemark 5. As already mentioned, a rigorous proof of the convergence, for :ase of stationary incompressible fluids, was given in Tartar [29] (see also survey Allaire [1] indicating some improvements but always for incomsible fluids). The main difficulty in this treatment is that is not enough otain a priori estimates for $(\mathbf{v}_\varepsilon/\varepsilon^2, p_\varepsilon)$ (which are independent of ε) since concerns with a functional space, $\mathbf{H}_0^1(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon)$, which varies with ε. efore, $(\mathbf{v}_\varepsilon, p_\varepsilon)$ needs to be extended to the whole homogenized set Ω. The nsion of $\mathbf{v}_\varepsilon$ is obvious (we take the value $\mathbf{0}$ outside Ω_ε). In the case of the sure p_ε we take the value

$$\frac{1}{|\theta|}\int_{\theta} p_\varepsilon.$$

onjecture that this type of extension and the recent results of P.L. Lions (see also P.L. Lions and Masmoudi [22]) will allow to get a rigorous proof e above Theorem.

ON THE HOMOGENIZED REACTION BETWEEN A FLUID AND A SOLID CHEMICAL SPECIE ON THE WALLS OF A POROUS MEDIUM

lea of the proof of Theorem 1. By multiplying by u_ε, integrating by parts using the monotonicity of the function $u \to |u|^{p-1}u$ we get that there s $M > 0$, independent of ε, such that

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq M, \ \forall \varepsilon > 0. \tag{23}$$

ssumption the extension operators are continuous i.e.,

$$\int_\Omega |\nabla P_\varepsilon(v)|^2 \, dx \leq C \int_{\Omega_\varepsilon} |\nabla v|^2 \, dx, \ \forall v \in V_\varepsilon, \ \forall \varepsilon.$$

nere exists a subsequence (labeled as $P_{\varepsilon'}u_{\varepsilon'}$) such that $P_{\varepsilon'}u_{\varepsilon'}$ converges, dy) in $H_0^1(\Omega)$, as $\varepsilon' \to 0$, to a function $u_0 \in H_0^1(\Omega)$ (in fact, since ems (6) and (7) have a unique solution the above convergence holds for omplete sequence $P_\varepsilon u_\varepsilon$). It remains to show that u_0 is a weak solution . To do that we define

$$\widetilde{\xi}_\varepsilon = \begin{cases} \nabla u_\varepsilon, & \text{in } \Omega_\varepsilon, \\ 0, & \text{in } \Omega - \Omega_\varepsilon. \end{cases}$$

It is clear that $\widetilde{\xi}_{\varepsilon'} \rightharpoonup \xi$, in $L^2(\Omega)$-weakly, as $\varepsilon' \to 0$, and that

$$\int_\Omega \widetilde{\xi}_{\varepsilon'} \cdot \nabla\varphi dx + \alpha\varepsilon' \int_{\Gamma_\varepsilon} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'}\varphi ds = \int_\Omega \mathcal{X}_{\Omega_{\varepsilon'}} f\varphi dx,$$

where $\mathcal{X}_{\Omega_{\varepsilon'}}$ is the characteristic function of $\Omega_{\varepsilon'}$. By standard homogeniza techniques, it is possible to show that

$$\begin{cases} \mathcal{X}_\Omega \rightharpoonup \frac{\text{meas}\,(Y^*)}{\text{meas}\,(Y)} & \text{in } L^q(\Omega) - \text{weakly, } \forall\ 1 \le q < \infty, \text{ as } \varepsilon' \to 0 \\ \mathcal{X}_\Omega \rightharpoonup \frac{\text{meas}\,(Y^*)}{\text{meas}\,(Y)} & \text{in } L^\infty(\Omega) - \text{weakly-star, as } \varepsilon' \to 0. \end{cases}$$

On the other hand, thanks to the assumption $p < 1$ we get that

$$\lim_{\varepsilon' \to 0} \int_{\Gamma_\varepsilon} |u_{\varepsilon'}|^{p-1} u_{\varepsilon'}\varphi ds = \frac{\text{meas}(\partial\theta)}{\text{meas}(Y)} \int_\Omega |u_0|^{p-1} u_0\varphi dx.$$

So, at the limit

$$-\operatorname{div}\xi + \alpha\delta\, |u_0|^{p-1} u_0 = f, \text{ in } \Omega.$$

In order to obtain an expression of ξ in terms of ∇u_0, for $i = 1,..,N$, introduce the auxiliary cellular problems

$$\begin{cases} -\Delta_y \mathcal{X}_i = 0 & \text{in } Y^*, \\ -\dfrac{\partial \mathcal{X}_i}{\partial n_y} = n_i & \text{on } \partial\theta, \\ \mathcal{X}_i \ Y^*\text{-periodic in } y \end{cases}$$

and then the functions

$$\phi_{i\varepsilon}(x) = \varepsilon[\mathcal{X}_i(\frac{x}{\varepsilon}) + y_i], \ \forall x \in \Omega_\varepsilon$$

and

$$\eta_i^\varepsilon = \nabla\phi_{i\varepsilon}.$$

It is not difficult to show that if $\widetilde{\eta_i^\varepsilon}$ denotes the corresponding extensior zero to $\Omega - \Omega_\varepsilon$, then

$$\begin{cases} \left(\widetilde{\eta_i^\varepsilon}\right)_j \rightharpoonup \frac{1}{\text{meas}(Y)}\left(\int_{Y^*} \frac{\partial \mathcal{X}_i}{\partial y_i} dy + \text{meas}(Y^*)\delta_{ij}\right) \\ \qquad := \frac{\text{meas}(Y^*)}{\text{meas}(Y)} q_{ij} \text{ in } L^2(\Omega) - \text{weakly.} \end{cases}$$

lly, using that (after some technical arguments)

$$-\int_\Omega \xi \cdot \nabla x_i \varphi dx + \frac{\text{meas}(Y^*)}{\text{meas}(Y)} \int_\Omega q_i \cdot \nabla u_0 \varphi dx = 0$$

onclusion follows. ∎

emark 6. After proving the above result (as consequence of the visit of onca to the Universidad Complutense de Madrid, in November of 1986), ecome aware of some related results in the literature, mainly the papers ornung and W. Jäger [15], [16]. In those papers, the authors consider y general formulation which contain problem (6) as an special case but r the structural assumption $p \geq 1$. The case $p < 1$ is left there as an problem and so the above theorem is not covered by their results.

emark 7. The reaction term on the boundary of the particles

$$-\frac{\partial u_\varepsilon}{\partial n} = \alpha\varepsilon \left|u_\varepsilon\right|^{p-1} u_\varepsilon \quad \text{on } \Gamma_\varepsilon,$$

mplification of a more complicated situation. In fact, each of the particles a porous medium and so there is a diffusion and reaction at its interior. e case of a spherically symmetric isolated particle it is possible to express ∂n in terms of the value of u_ε on Γ_ε once that one assumes that at the or of the particle there is a chemical reaction of the type

$$-\Delta u_\varepsilon + \alpha\varepsilon \left|u_\varepsilon\right|^{m-1} u_\varepsilon = 0, \text{ in } \theta$$

/ega and Liñán [31]).

References

ALLAIRE, G., One-phase newtonian flows, in "Hornung [14] ", 1997, 45–76.

ANTONTSEV, S.N., KAZHIKHOV, A.V., MONHAKHOV, V.N., "Boundary Value Problems in Mechanics of Nonhomogeneous Fluids", North-Holland, Amsterdam, 1990.

ARTOLA, M., Nonlinear diffusion equations in heterogeneous media, *Math. Comput. Modelling,* **11** (1988), 747–750.

AURIAULT, J.L., SANCHEZ-PALENCIA, E., Remarques sur la loi de Darcy pour les écoulements biphasiques en milieu poreux, *Journal de Mécanique Théorique et Appliquée,* Special issue (1986), 141–156.

BEAR, J., "Dynamics of Fluids in Porous Media", Elsevier, New York, 1972.

BENSOUSSAN, A., LIONS, J.-L., PAPANICOLAU, G., "Asymptotic Analysis for Periodic Structures", North Holland-Elsevier, Amsterdam, 1978.

[7] BOURGAT, A., Two-phase flow, in "Hornung [14]", 1997, 95 – 127.

[8] CONCA, C., DÍAZ, J.I., LIÑAN, A., Homogenization in reactive through porous media, *(unpublished manuscript).*, (1987).

[9] DAMLAMIAN, A., How to homogenize a nonlinear diffusion equation: Ste problem, *SIAM J. Math. Anal.,* **12** No.3 (1981), 306 – 313.

[10] DARCY, H., "Les Fontaines Publiques de la Ville de Dijon ", Dalmont, I 1856.

[11] DÍAZ, J.I., "Nonlinear Parabolic Problems in Fluid Mechanics", Le Notes of a Postgraduate Seminar at the Universidad de Oviedo (Sp notes (in Spanish) taken by A. Mateos and R. Sarandeses, 1992.

[12] ENE, H.I., POLISEVSKI, D., "Thermal Flow in Porous Media", D. R Dordrecht, 1987.

[13] GAGNEUX, G., MADAUNE-TORT, M., "Analyse Mathématique de Mo non Linéaires de l'Ingénierie Pétrolière", Springer, Paris, 1996.

[14] HORNUNG, U., "Homogenization and Porous Media", Springer, New 1997.

[15] HORNUNG, U., JÄGER, W., A model for chemical reactions in porous dia, in "Complex Chemical Reaction Systems: Mathematical Modeling Simulation", W. Jäger and J. Warnatz eds, Volume 47 of Chemical Ph 1987, Springer-Verlag, Berlin, 318 – 334.

[16] HORNUNG, U., JÄGER, W., Diffusion, convection, adsorption and rea of chemicals in porous media, *J. Differential Equations,* **92** (1991), 225.

[17] KALASHNIKHOV, A.S., Some problems of the qualitative theory of no ear degenerate second-order parabolic equations, *Russ. Math. Survs.* (1987), 169 – 222.

[18] KELLER, J.B., Darcy's law for flow in porous media and the two-s method, in "Nonlinear Partial Differential Equations in Engineering Applied Sciences", R.L. Sternberg ed., Marcel Dekker, New York, 1 429 – 443.

[19] LIONS, J.-L., "Some Methods in the Mathematical Analysis of Systems Their Control", Science Press and Gordon and Breach, Beijing and York, 1981.

[20] LIONS, J.-L., SANCHEZ-PALENCIA, E., Écoulement d'un fluid visco tique de Bingham dans un milieu poreux, *J. Math. Pures et Appl.* (1981), 341 – 360.

[21] LIONS, P.-L., "Mathematical Topics in Fluid Mechanics. Vol.2. Compres Models ", Clarendon Press, Oxford, 1998.

[22] LIONS, P.-L., MASMOUDI, N., On a free boundary barotropic m *Cahiers du CEREMADE,* 1998.

[23] MIKELIC, A., Homogenization of nonstationary Navier-Stokes equations domain with a grained boundary, *Ann. Mat. Pura et App.,* **IV 158** (1 167 – 179.

[24] MIKELIC, A., Non-newtonian flow, in "Hornung [14]", 1997, 77 – 94.

[25] MORRIS, G.A., JACKSON, J., "Absorption Towers", Butterworths S tific Publications, London, 1953.

[26] NORMAN, W.S., "Absorption, Distillation and Cooling Towers", Long London, 1961.

[27] OLEINIK, O.A., SHMAEV, A.S., YOSEFIAN, G.A. "Mathematical P

lems in Elasticity and Homogenization", Elsevier, Amsterdam, 1992.

SANCHEZ-PALENCIA, E., "Non-Homogeneous Media and Vibration Theory", Springer-Verlag, Berlin, 1980.

TARTAR, L., Incompressible fluid flow in a porous medium: convergence of the homogenization process, in "[28]", 1980, 368–377.

TEMAM, R., "Navier-Stokes Equations", North-Holland- Elsevier, Amsterdam, 1979.

VEGA, J.M., LIÑÁN, A., Isothermal nth order reaction in catalytc pellets: effect of external mass transfer resistance, *Chemical Engineering Science,* **34** (1979), 1319–1322.