



# A Note on Hysteresis in Glaciology

J. I. DÍAZ

Departamento de Matemática Aplicada, Universidad Complutense de Madrid  
28040 Madrid, Spain  
jidiaz@sunma4.mat.ucm.es

E. SCHIAVI

Escuela Superior de Ciencias Experimentales y Tecnología  
Universidad Rey Juan Carlos de Madrid, 28933 Mostoles, Madrid, Spain  
eschiavi@escet.urjc.es

(Received January 1999; accepted February 1999)

Communicated by R. Glowinski

**Abstract**—Recent studies on the mechanism governing the Laurentide ice sheet oscillations of the Last Ice Age focus on the most critical effect of the basal hydraulic processes enhanced when the ice is sliding along soft deformable beds. To understand the import of this, we consider Fowler and Johnson's 0-D hydrological flow model describing the sudden and rapid movements forward (surges) of a till-based 1-D ice sheet sliding on a flat soft bed. The basic idea is that the interplay between the ice sheets dynamics, the basal drainage system, and the sliding law can generate a surging behaviour. Mathematically this means that a multiple valued relationship between the ice flux and the ice thickness arises and the mass conservation equation turns out to be of multivalued type for some special values of the dimensionless parameters involved in the model. Assuming that a multiple valued ice flux law of the Fowler and Johnson type holds, we prove the existence of a weak bounded discontinuous solution to the system which becomes periodic after a suitable time. © 2000 Elsevier Science Ltd. All rights reserved.

**Keywords**—System of nonlinear equations, Ice sheet models.

## 1. THE MODEL EQUATIONS

To show the relationship between multivalued flux laws and the sudden and rapid movements of the ice sheets (*surging*) oscillating between slow and fast flow regimes, it is useful to consider the (simple) parameterized 0-D model proposed by Fowler and Johnson [1]. Neglecting spatial derivatives (by spatial homogenization), the authors obtained a coupled system of five equations in terms of the unknown variables of thickness  $h$ , velocity  $u$ , shear  $\tau$ , water flux  $Q_w$ , and effective pressure  $N$ ,

$$\begin{aligned} \dot{h} &= a - hu, \\ \tau &= h^2, \\ \tau &= \sigma u^r N^s, \end{aligned} \tag{1}$$

$$\begin{aligned} N &= h^{1/6} Q_w^{-1/3}, & \text{if } Q_w > 0, \\ Q_w &= 1 + \tau u - \mu u^{1/2} - \delta h^{-1}, \end{aligned} \tag{1)(cont.)}$$

where  $a$  is the average accumulation rate function and exponents  $r, s$  have typical values  $r \approx 1/4$ ,  $s \approx 1/4$ . The parameters  $\mu, \sigma$ , and  $\delta$  are dimensionless quantities of order  $O(1)$ . Note that all variables are assumed to be just  $t$  dependent and that  $\dot{h} = \frac{dh}{dt}$ .

Let  $q_b \doteq \mu u^{1/2} + \delta h^{-1}$  be the cooling term representing the heating flux released to the colder ice above. System (1) represents, in turn, the mass conservation equation, the shear stress formula, the sliding law, and an equilibrium theory of drainage to describe the effective pressure and the water flux in a hydraulic system composed by canals, see [2]. Using (1)<sub>2,3,4</sub>, it is possible to write

$$\sigma u^r = h^{2-(s/6)} Q_w^{s/3}, \tag{2}$$

and thus we have to solve

$$\begin{aligned} \dot{h} &= a - hu, \\ \left[ 1 + h^2 u - \mu u^{1/2} - \delta h^{-1} \right]^{s/3} &= \sigma u^r h^{(s/6)-2}. \end{aligned} \tag{3}$$

Introducing the new variable  $D = hu$  (describing the ice discharge), we get an implicit relationship between  $h = h(t)$  and  $D = D(t)$ .

$$\left[ 1 + h D - \mu \frac{D^{1/2}}{h^{1/2}} - \frac{\delta}{h} \right]^{s/3} = \sigma D^r h^{(s/6)-2-r}. \tag{4}$$

Equation (4) is valid providing  $0 < q_b < 1 + \tau u = 1 + h^2 u$ , corresponding to a temperate base.

The basic feature of system (3) relies on the possible onset of two different regimes of flux  $D$  (discharges) for the same thickness  $h$ .

In fact, Fowler and Johnson [1] proved that, *fixed*  $h > 0$ , the relationship (4) can be multivalued for some special values of the dimensionless parameters  $\mu, \sigma$ , and  $\delta$  and exponents  $r$  and  $s$ .

Such a behaviour, called *hydraulic run-away*, can provide a mechanism for the thermal control of the oscillatory behaviour of the Laurentide ice sheet during the Last Ice Age.

It is then physically meaningful to analyze mathematically the behaviour of the conservation law (1)<sub>1</sub> when a multivalued flux law is explicitly prescribed. As a preliminary result, this note is devoted to prove the existence of weak solutions to the initial values problem associated with the continuity equation. Moreover, as a by-product of our technique, we can show that, after a finite time, there exists at least a (weak) solution becoming periodic.

### 1.1 Existence of Solutions

We shall consider the following problem. To determine a real valued function  $h(t)$  which is a solution of the initial value problem for ordinary differential equations of multivalued type

$$\begin{aligned} \dot{h} + \mathcal{Q}(h) &\ni a, & \text{in } (0, \infty), \\ h(0) &= h_0, \end{aligned} \tag{5}$$

where  $\mathcal{Q}$  is a continuous graph of  $\mathbb{R}^2$  satisfying

$$\mathbf{H}_{\mathcal{Q}} \equiv \begin{cases} \text{there exist real numbers } h_m \text{ and } h_c \text{ such that } 0 < h_0 < h_m < h_c < \infty \text{ and} \\ \text{there exist } Q_- : [0, h_c] \longrightarrow [0, \infty) \text{ continuous and strictly increasing, } Q_i : \\ [h_c, h_m] \longrightarrow [0, \infty) \text{ continuous and strictly increasing, } Q^+ : [h_m, \infty) \longrightarrow \\ [0, \infty) \text{ continuous and strictly increasing, where } Q_-(h_c) = Q_i(h_c) \text{ and} \\ Q^+(h_m) = Q_i(h_m). \end{cases}$$

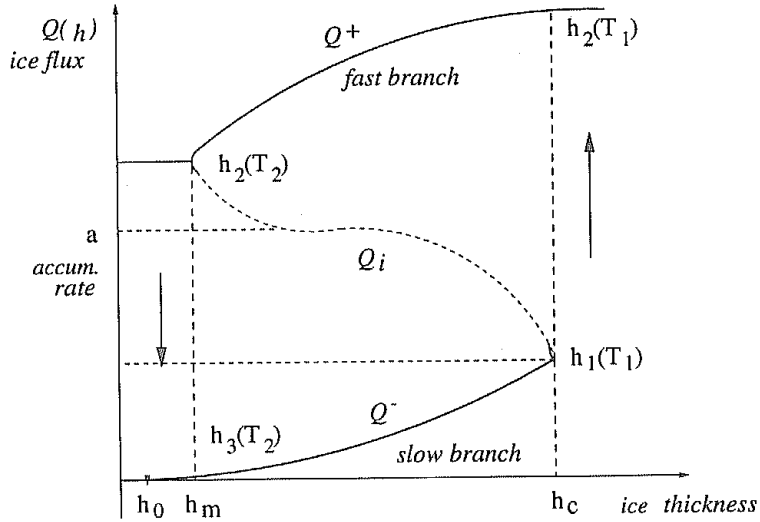


Figure 1. Multiple-valued ice flux as a function of ice thickness. If  $a$  has the intermediate value indicated, then cyclic oscillations will occur between the upper and lower branches. Details on the physics of the mechanism can be found in [1].

We shall assume the following hypothesis on the accumulation rate function  $a$ :

$$\mathbf{H}_a \equiv \{a \text{ is constant and } a \in (Q_i(h_c), Q_i(h_m)) = (Q_-(h_c), Q_+(h_m))\}.$$

Note that the hypothesis  $\mathbf{H}_Q$  allows us to consider a continuous graph of  $\mathbb{R}^2$  of the type shown in Figure 1.

We shall now introduce a notion of weak solution of problem (5) which is coherent with the usual notion of weak solution when  $Q$  is assumed to be a maximal monotone graph of  $\mathbb{R}^2$  (see [3]).

DEFINITION 1. Let  $h_0 > 0$ . We shall say that a function  $h \in W_{\text{Loc}}^{1,1}(0, \infty)$  is a weak solution of problem (5) if we have the following.

1. There exists a function  $\hat{q}(t) \in L^1_{\text{Loc}}(0, \infty)$  such that

$$(a) \quad \hat{q}(t) \in Q(h(t)), \quad \text{a.e. } t \in [0, \infty),$$

and

$$(b) \quad \dot{h}(t) = a - \hat{q}(t), \quad \text{a.e. } t \in (0, \infty).$$

2. The initial condition,  $h(0) = h_0$ , holds.

We can now state the main result of this note about the existence of weak periodic solutions for problem (5).

THEOREM 1. Let the continuous graph  $Q$  and the accumulation rate function  $a$  satisfying the hypotheses  $\mathbf{H}_Q$  and  $\mathbf{H}_a$ , respectively. Let  $h_0 > 0$  be a given initial state. Then there exists (at least) a solution of problem (5) becoming periodic after a finite time  $T_0$  in the sense:

$$h(t + t_p) = h(t), \quad \forall t \geq T_0 := \begin{cases} \int_{h_0}^{h_m} \frac{dh}{a - Q_-(h)}, & \text{if } h_0 \leq h_m, \\ 0, & \text{if } h_0 \in (h_m, h_c), \\ \int_{h_c}^{h_0} \frac{dh}{Q_+(h) - a}, & \text{if } h_0 \geq h_c, \end{cases}$$

where

$$t_p \doteq \int_{h_m}^{h_c} \frac{dh}{a - Q_-(h)} + \int_{h_m}^{h_c} \frac{dh}{Q_+(h) - a}.$$

PROOF. Let us suppose  $0 < h_0 < h_c$  (otherwise the argument can easily be adapted). We consider a function  $h_1(t)$  characterized as a solution of the problem

$$\begin{aligned} \dot{h}_1 &= a - Q_-(h_1), & \text{on } (0, \infty), \\ h_1(0) &= h_0. \end{aligned} \quad (6)$$

By assumption, the right-hand side of (6) is continuous and by Cauchy's theorem, we get the existence of such function  $h_1$ . Also, it is possible to state that  $h_1$  is well defined for  $t \in [0, T_1]$  with  $h_1(T_1) = h_c$  being

$$\int_{h_0}^{h_c} \frac{dh}{a - Q_-(h)} := T_1.$$

Notice that  $\dot{h}_1(t) \geq 0$  because  $a > Q_-(h)$ ,  $\forall h \in [0, h_c]$  by  $\mathbf{H}_a$ . We now define  $h_2(t)$  as a solution of

$$\begin{aligned} \dot{h}_2 &= a - Q^+(h_2), & \text{on } (T_1, T_2), \\ h_2(T_1) &= h_c. \end{aligned} \quad (7)$$

The instant  $T_2$  is inferred by

$$T_2 - T_1 := \int_{h_c}^{h_m} \frac{dh}{a - Q^+(h)} = - \int_{h_m}^{h_c} \frac{dh}{a - Q^+(h)} = \int_{h_m}^{h_c} \frac{dh}{Q^+(h) - a} > 0.$$

Observe also that  $\dot{h}_2(t) \leq 0$  because  $a < Q^+(h)$ ,  $\forall h \in [h_m, \infty)$ . Finally, we shall introduce  $h_3(t)$  as a solution of the problem

$$\begin{aligned} \dot{h}_3 &= a - Q_-(h_3), & \text{on } (T_2, T_3), \\ h_3(T_2) &= h_m, \end{aligned} \quad (8)$$

whereas  $T_3$  is given by

$$T_3 - T_2 := \int_{h_m}^{h_c} \frac{dh}{a - Q_-(h)}.$$

Following inductively this process and renaming  $h_2 = h_{2,1}$ ,  $h_3 = h_{3,1}$ , we build up a solution  $h(t)$  of problem (5):

$$h(t) = \begin{cases} h_1(t), & \text{on } (0, T_1), \\ h_{2,1}(t), & \text{on } [T_1, T_2), \\ h_{3,1}(t), & \text{on } [T_2, T_3), \\ \dots & \\ h_{2,n}(t), & \text{on } [T_{2n-1}, T_{2n}), \\ h_{3,n}(t), & \text{on } [T_{2n}, T_{2n+1}), \\ \dots & \end{cases} \quad (9)$$

where the functions  $h_{2,n}$  and  $h_{3,n}$  are defined recursively using  $h_{2,1}$  and  $h_{3,1}$  as

$$\begin{aligned} h_{2,n} &\equiv h_{2,1}, & \text{on } [T_{1+2(n-1)}, T_{2+2(n-1)}], & \quad \forall n \in \mathbb{N}, \\ h_{3,n} &\equiv h_{3,1}, & \text{on } [T_{2+2(n-1)}, T_{3+2(n-1)}], & \quad \forall n \in \mathbb{N}, \end{aligned}$$

being  $T_{2n-1} \doteq T_1$  and  $T_{2n} = T_2$ ,  $\forall n \in \mathbb{N}$ . Finally, we observe that, by construction,

$$h_1(T_1) = h_c = h_2(T_1), \quad h_2(T_2) = h_m = h_3(T_2),$$

where  $t_p$  is the period defined by

$$t_p \doteq T_3 - T_1 = (T_3 - T_2) + (T_2 - T_1) = \int_{h_m}^{h_c} \frac{dh}{a - Q_-(h)} + \int_{h_m}^{h_c} \frac{dh}{Q^+(h) - a}. \quad \blacksquare$$

REMARK 1. The functions  $h(t)$  and  $\hat{q}(t)$  have the following regularity properties:

$$\begin{aligned} h &\in C^1(\mathbb{R} - \{T_1 + (n-1)t_p, T_2 + (n-1)t_p, \forall n \in \mathbb{N}\}), \\ \hat{q} &\in C^0(\mathbb{R} - \{T_1 + (n-1)t_p, T_2 + (n-1)t_p, \forall n \in \mathbb{N}\}). \end{aligned}$$

Moreover, denoting by  $[[\cdot]]$  the jump of a function, we have

$$\begin{aligned} [[\dot{h}(T_1 + (n-1)t_p)]] &= -[[\hat{q}(T_1 + (n-1)t_p)]] = -Q^+(h_c + 0) + Q_-(h_c - 0) = Q_m^+ - Q_c^- \leq 0, \\ [[\dot{h}(T_2 + (n-1)t_p)]] &= -[[\hat{q}(T_2 + (n-1)t_p)]] = Q^+(h_c + 0) + Q_-(h_c - 0) = Q_m^- - Q_c^+ \geq 0. \end{aligned}$$

REMARK 2. Theorem 1 provides a rigorous proof of the time periodicity corresponding to relaxation oscillations suggested by Fowler and Johnson [1]. For a more realistic model, see [4], while its mathematical analysis can be found in [5].

## REFERENCES

1. A.C. Fowler and C. Johnson, Hydraulic runaway: A mechanism for thermally regulated surges of ice sheets, *J. Glaciol.* **41**, 554–561, (1995).
2. A.C. Fowler, *Mathematical Models in the Applied Sciences*, Cambridge University Press, (1997).
3. H. Brezis, *Operateur Maximaux Monotones et Semigroupes de Contractions Dans les Espaces de Hilbert*, North-Holland, Amsterdam, (1973).
4. A.C. Fowler and E. Schiavi, A theory of ice-sheet surges, *Journal of Glaciology* **44** (146), 104–118, (1998).
5. J.I. Díaz and E. Schiavi, On a degenerate parabolic/hyperbolic system giving rise to a free boundary, *Nonlinear Analysis: Real World Applications Series B* **38**, 649–673, (1999).

