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AN ABSTRACT APPROXIMATE CONTROLLABILITY RESULT AND APPLICATIONS TO ELLIPTIC AND PARABOLIC SYSTEMS WITH DYNAMIC BOUNDARY CONDITIONS

IOAN BEJENARU, JESUS ILDEFONSO DÍAZ, & IOAN I. VRABIE

Dedicated to the Memory of Philippe Benilan

Abstract. In this paper we prove an approximate controllability result for an abstract semilinear evolution equation in a Hilbert space and we obtain as consequences the approximate controllability for some classes of elliptic and parabolic problems subjected to nonlinear, possible non monotone, dynamic boundary conditions.

1. Introduction

The goal of the present paper is to prove an approximate controllability result for an abstract evolution equation in a separable Hilbert space and then to obtain some sufficient condition for approximate controllability applying to certain nonlinear parabolic, or elliptic equations subjected to dynamic boundary conditions. More precisely, let V be a separable Hilbert space densely and continuously embedded into a Hilbert space H, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and respectively by $\|\cdot\|$. We denote by V^* the topological dual of V and we identify H with its own dual. So $V \subset H \subset V^*$ with dense and continuous injections. Let $\|\cdot\|_V$ be the norm of V and $(\cdot,\cdot)$ the usual pairing between V and V^*, whose restriction to $H \times H$ coincides with $\langle\cdot,\cdot\rangle$. We consider the following abstract nonlinear evolution equation

$$\begin{gathered} u' + A_H u + F(t,u)u = h(t) \\ u(0) = \xi, \end{gathered} \tag{1.1}$$

where $-A_H : D(A) \subset H \to H$ generates a C_0-semigroup $S(t) : H \to H$, $t \geq 0$, h is fixed in $L^2_{\mathrm{loc}}(\mathbb{R}_+; H)$ and $F : \mathbb{R}_+ \times H \to \mathcal{L}(H)$. As usual, by a *mild solution* of (1.1) on $[0,T]$ we mean a continuous function $u : [0,T] \to H$ which satisfies

$$u(t,\xi) = S(t)\xi + \int_0^t S(t-s)F(s,u(s,\xi))u(s,\xi)\,ds + \int_0^t S(t-s)h(s)\,ds$$

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for each $t \in [0,T]$. We say that (1.1) is *approximate controllable* if for each $T > 0$ the set $\{u(T,\xi);\ u$ is a mild solution of (1.1), $\xi \in H\}$ is dense in H. We emphasize that one of the idea of considering the initial data as a control comes from the *assimilation of data* in Meteorology or in Climatology. See for instance Bayo, Blum, Verron [14] or Le Dimet, Charpentier [48].

The assumptions on A and F are listed below.

(H_1) The operator $A : V \to V^*$ is linear continuous, i.e. $A \in \mathcal{L}(V;V^*)$ and its restriction to H, $A_H : D(A_H) \subset H \to H$ where $D(A_H) = \{u \in V;\ Au \in H\}$ and $A_H u = Au$ for each $u \in D(A_H)$, is self adjoint.

(H_2) There exist $\lambda \in \mathbb{R}$ and $\eta > 0$ such that

$$(Au,u) + \lambda\|u\|^2 \geq \eta\|u\|_V^2$$

for each $u \in V$.

(H_3) The mapping $F : \mathbb{R}_+ \times H \to \mathcal{L}(H)$ satisfies

(i) for each $u \in H$, $F(\cdot,u) : \mathbb{R}_+ \to \mathcal{L}(H)$ is measurable;

(ii) for almost all $t \in \mathbb{R}_+$, $F(t,\cdot) : H \to \mathcal{L}(H)$ is continuous;

(iii) there exists a function $\mu \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}_+)$ such that

$$\|F(t,u)\|_{\mathcal{L}(H)} \leq \mu(t)$$

a.e. for $t \in \mathbb{R}_+$ and for each $u \in H$.

(H_4) The embedding $V \subset H$ is compact.

Our main abstract approximate controllability result is

Theorem 1.1. *If* (H_1), (H_2), (H_3) *and* (H_4) *hold, then* (1.1) *is approximate controllable.*

We denote by $L^2(0,T;D(A_H)) = \{u \in L^2(0,T;V),\ Au(\cdot) \in L^2(0,T;H)\}$ and we recall for easy reference the following specific form of a general backward uniqueness result due to Ghidaglia [38] which is the main ingredient in the proof of Theorem 1.1.

Theorem 1.2. *We assume that* (H_1) *and* (H_2) *hold,* $0 < T < +\infty$ *and* u *satisfies*

$$u \in C([0,T];V) \cap L^2(0,T;D(A_H)),$$
$$u'(t) + Au(t) \in H \quad \text{a.e. for } t \in [0,T],$$
$$\|u'(t) + Au(t)\| \leq \alpha(t)\|u(t)\|_V \quad \text{a.e. for } t \in [0,T],$$

where $\alpha \in \mathcal{L}^2(0,T)$. *If* $u(T) = 0$, *then* $u(t) = 0$ *for each* $t \in [0,T]$.

We note that in Ghidaglia's original result A is allowed to depend on t as well. See Theorem 1.1 in [38]. We also note that, as far as we know, such kind of backward uniqueness results were proved for the first time by Bardos, Tartar [9].

Using Theorem 1.1 we will prove that both the parabolic problem

$$\begin{aligned} u_t - \Delta u + \mathcal{R}(t,u) &\ni 0 \quad \text{in } Q_T \\ u_t + u_\nu + \mathcal{P}(t,u) &\ni 0 \quad \text{on } \Sigma_T \\ u(0) &= u_0^\Omega \quad \text{in } \Omega \\ u(0) &= u_0^\Gamma \quad \text{in } \Gamma \end{aligned} \tag{1.2}$$

and respectively of the semi-dynamic elliptic problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } Q_T \\ u_t + u_\nu + \mathcal{P}(t,u) &\ni 0 \quad \text{on } \Sigma_T \\ u(0) &= u_0^\Gamma \quad \text{in } \Gamma \end{aligned} \tag{1.3}$$

are approximate controllable under suitable assumptions on $\mathcal{R}$ and $\mathcal{P}$. Here Ω is a bounded domain in $\mathbb{R}^n$ whose boundary Γ is of class C^∞ and such that Ω is locally on one side of Γ. See (7.10) and (7.11) in Lions, Magenes [52], p. 38. Moreover, $Q_T = [0,T]\times\Omega$, $\Sigma_T = [0,T]\times\Gamma$, $\mathcal{R},\mathcal{P} : [0,T]\times\mathbb{R}\to 2^{\mathbb{R}}$, $u_0^\Omega \in L^2(\Omega)$, $u_0^\Gamma \in L^2(\Gamma)$ and u_ν is the co normal derivative of u at points of Γ.

The main specific feature of both problems (1.2) and (1.3) is given by the dynamic boundary conditions which, although not too widely considered in the literature, are very natural in many mathematical models as: heat transfer in a solid in contact with a moving fluid (Peddie [59] 1901, March, Weaver [55] 1928, Langer [46] 1932, Bauer [10] 1952), thermoelasticity (Green, Lindsay [39] 1972), diffusion phenomena (Crank [21] 1975), the heat transfer in two phase medium (Stefan problem) (Cannon [18] 1984, Primicerio, Rodrigues [62] 1992, Aiki [2] 1995, Solonnikov, Frolova [68] 1997), thermal energy storage devices (Altman, Ross, Chang [3] 1965), problems in fluid dynamics (Lamb [45] 1916, Friedman, Shinbort [36] 1968, Benjamin, Olver [13] 1982, Okamoto [57] 1983, Lewis, Marsden, Raţiu [49] 1986), diffusion in porous media (Peek [60] 1929, Sun [69] 1996, Filo, Luckhaus [35] 1998), chemical engineering (Lapidus, Amundson [47] 1977, Slinko, Hartmann [67] 1972, Vold, Vold [70] 1983, Baerns, Hofmann, Renken [7] 1987), electronics and semiconductors (long cables: Wagner [72] 1908), semiconductor devices (von Roosbroeck [64] 1950, Mock [56] 1983, Selberher [66] 1984), probability theory and mathematical modelling in Biology (Feller [33] 1952). From a more abstract mathematical point of view see Courant, Hilbert [20] 1965, Lions [50] 1969, Fulton [37] 1977, Kačur [44] 1980, Perriot [61] 1982, Sauer [65] 1982, Díaz, Jimenez [25] 1984, Degiovanni [22] 1985, Grobbelar-Van Dalsen [40] 1987, Gröger [41] 1987, Acquistapace, Terreni [1] 1988, Hintermann [43] 1989, Escher [30] 1993, Amann, Escher [4] 1996 , Amann, Fila [4, 34] 1997, Fila, Quittner [34] 1997, Arrieta, Quittner, Rodriguez-Bernal [6] 2000, among others.

We say that the problem (1.2) is *approximate controllable* if, for each $T>0$, the set $\big\{(u(T,\cdot), u_{|\Gamma}(T,\cdot);\ (u_0^\Omega,u_0^\Gamma)\in L^2(\Omega)\times L^2(\Gamma)\big\}$ is dense in $L^2(\Omega)\times L^2(\Gamma)$. Similarly, the problem (1.3) is *approximate controllable* if, for each $T>0$, the set $\big\{u_{|\Gamma}(T,\cdot); u_0^\Gamma\in L^2(\Gamma)\big\}$ is dense in $L^2(\Gamma)$.

The hypotheses on $\mathcal{R}$ and $\mathcal{P}$ we shall use are:

(H_5) There exist $g,\beta : \mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$, $a,c,\alpha\in\mathcal{L}^2_{\rm loc}(\mathbb{R}_+;\mathbb{R})$ and $\rho,\phi:\mathbb{R}\to 2^{\mathbb{R}}$ such that:

(i) for each $u,v\in\mathbb{R}$, $g(\cdot,u),\beta(\cdot,v)\in\mathcal{L}^2_{\rm loc}(\mathbb{R}_+;\mathbb{R})$;
(ii) for a.e. $t\in\mathbb{R}_+$, $g(t,\cdot)$ and $\beta(t,\cdot)$ are continuous;
(iii) $\rho,\phi:\mathbb{R}\to 2^{\mathbb{R}}$ are maximal monotone operators;
(iv) for almost all $t\in\mathbb{R}_+$ and for each $u,v\in\mathbb{R}$

$$\begin{gathered}\mathcal{R}(t,u) = g(t,u)+a(t)\rho(u)\\ \mathcal{P}(t,v) = \beta(t,v)+c(t)\phi(v)\,;\end{gathered}$$

(v) there exist and open interval $\mathbb{I}$ and $u_0,v_0\in\mathbb{I}$ such that both ρ and ϕ are single-valued on $\mathbb{I}$ and differentiable at u_0 and respectively at v_0. In addition, for almost all $t\in\mathbb{R}_+$, each $u,v\in\mathbb{R}$, $\rho_u\in\rho(u)$ and $\phi_v\in\phi(v)$, we have

$$\begin{aligned}&|g(t,u)-g(t,u_0)|+|a(t)\rho_u-a(t)\rho(u_0)|\le\alpha(t)|u-u_0|\\&|\beta(t,v)-\beta(t,v_0)|+|c(t)\phi_v-c(t)\phi(v_0)|\le\alpha(t)|v-v_0|.\end{aligned}\tag{1.4}$$

Remark 1.1. We may always assume that in (v) $u_0 = v_0 = \rho(u_0) = \phi(v_0) = 0$. Indeed, let us assume that g, β, ρ and ϕ satisfy (H_5) and set $u = \tilde{u} + u_0$ and $v = \tilde{v} + v_0$. Then $\tilde{g}$, $\tilde{\beta}$, $\tilde{\rho}$ and $\tilde{\phi}$, defined as

$$\tilde{g}(t,\tilde{u}) = g(t,\tilde{u}+u_0) + a(t)\rho(u_0) \quad \tilde{\beta}(t,\tilde{v}) = \beta(t,\tilde{v}+v_0) + c(t)\phi(v_0)$$
$$\tilde{\rho}(\tilde{u}) = \rho(\tilde{u}+u_0) - \rho(u_0) \quad \tilde{\phi}(\tilde{v}) = \phi(\tilde{v}+v_0) - \phi(v_0),$$

satisfy (H_5) too. Obviously u satisfies (1.2) if and only if $\tilde{u}$ satisfies (1.2) with g, β, ρ, ϕ, u_0^Ω and u_0^Γ replaced by $\tilde{g}$, $\tilde{\beta}$, $\tilde{\rho}$, $\tilde{\phi}$, $\tilde{u}_0^\Omega = u_0^\Omega - u_0$ and $\tilde{u}_0^\Gamma = u_0^\Gamma - v_0$, respectively. Moreover, for each $n \in \mathbb{N}^*$[1], the Yosida approximations

$$\tilde{\rho}_n = n\left[I - (I + n^{-1}\tilde{\rho})^{-1}\right] \quad \text{and} \quad \tilde{\phi}_n = n\left[I - (I + n^{-1}\tilde{\phi})^{-1}\right]$$

are differentiable at 0,

$$\tilde{\rho}_n'(0) = \frac{n\tilde{\rho}'(0)}{n + \tilde{\rho}'(0)} \quad \text{and} \quad \tilde{\phi}_n'(0) = \frac{n\tilde{\phi}'(0)}{n + \tilde{\phi}'(0)}.$$

Therefore, $\tilde{g}$, $\tilde{\beta}$, $\tilde{\rho}_n$ and $\tilde{\phi}_n$ satisfy (1.4) uniformly with respect to $n \in \mathbb{N}^*$ and with the very same function $\alpha \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R})$.

The main controllability results referring to (1.2) and (1.3) are:

Theorem 1.3. *If $\mathcal{R}$ and $\mathcal{P}$ satisfy (H_5), then* (1.2) *is approximate controllable.*

Theorem 1.4. *If $\mathcal{P}$ satisfies (H_5), then* (1.3) *is approximate controllable.*

Some controllability results referring to multi valued semilinear problems with Dirichlet or Neumann boundary conditions were first proved in Díaz [23]. See also Díaz, Ramos [28].

2. Proof of Theorem 1.1

The proof consists in two main steps. First, we consider the linear equation

$$\begin{aligned} u' + A_H u + f(t)u &= 0 \\ u(0) &= \xi, \end{aligned} \tag{2.1}$$

where $f \in L^2(0,T;\mathcal{L}(H))$ and we prove that it is approximate controllable. We begin with a question of terminology.

Definition 2.1. We say that the operator $\mathcal{M} : L^2(0,T;\mathcal{L}(H)) \times H \to H$ is *continuous in the weak pointwise convergence topology at* $(f,u^T) \in L^2(0,T;\mathcal{L}(H)) \times H$ if for each bounded sequence $(f_n)_n$ in $L^2(0,T;\mathcal{L}(H))$ which is *weakly pointwise convergent* in $L^2(0,T;\mathcal{L}(H))$ to f, i.e.

$$\lim_n f_n u = fu$$

weakly in $L^2(0,T;H)$ for all $u \in H$, and each sequence $(u_n^T)_n$ with $\lim_n u_n^T = u^T$ strongly in H, we have

$$\lim_n \mathcal{M}(f_n, u_n^T) = \mathcal{M}(f, u^T)$$

strongly H. We say that $\mathcal{M}$ is *continuous in the weak pointwise convergence topology on* $L^2(0,T;\mathcal{L}(H)) \times H$ if it is continuous in the weak pointwise convergence topology at each $(f,u^T) \in L^2(0,T;\mathcal{L}(H)) \times H$.

[1] As usual, $\mathbb{N}^*$ denotes the set of all natural numbers without 0.

Proposition 2.1. *If* (H_1) *and* (H_3) *are satisfied, then, for each* $f \in L^2(0,T\,;\mathcal{L}(H))$ *the system* (2.1) *is approximate controllable. More than this, for each* $T > 0$ *and* $\varepsilon > 0$ *there exists an operator* $\mathcal{M} : L^2(0,T\,;\mathcal{L}(H)) \times H \to H$ *continuous in the weak pointwise convergence topology on* $L^2(0,T\,;\mathcal{L}(H)) \times H$ *such that, for* $\xi = \mathcal{M}(f, u^T)$, *we have*

$$\|u(T,\xi,f) - u^T\| \le \varepsilon, \tag{2.2}$$

where $u(\cdot,\xi,f)$ *denotes the unique mild solution of* (2.1) *corresponding to* ξ *and* f.

Next, as in Henry [42] (see also Fabre, Puel Zuazua [31], Diaz, Henry, Ramos [24] and Ramos [63]) we shall use a fixed point argument as follows. Let $v \in C([0,T]\,;H)$ and let us consider the nonhomogeneous linear problem

$$\begin{gathered} z' + A_H z + F(t,v)z = h \\ z(0) = 0. \end{gathered}$$

In view of (H_3), this problem has a unique mild solution $z(\cdot,v) \in C([0,T]\,;H)$ and so, we can define the operator $\mathcal{K} : C([0,T]\,;H) \to C([0,T]\,;H)$ as

$$\mathcal{K}v = z(\cdot,v) + u(\cdot,\xi,f),$$

where

$$\begin{gathered} f = F(\cdot, v(\cdot)) \\ \xi = \mathcal{M}(f, u^T - z(T,v)). \end{gathered}$$

The second step is contained in:

Proposition 2.2. *If* (H_1), (H_2), (H_3) *and* (H_4) *are satisfied, then the operator* $\mathcal{K}$ *defined as above has at least one fixed point* $v \in C([0,T]\,;H)$.

Once Proposition 2.2 proved, it is clear that the fixed point v of $\mathcal{K}$ is the solution solving the approximate controllability problem for (1.1) and this simply because v is a mild solution of (1.1) and, by virtue of Proposition 2.1, we have

$$\|v(T) - u^T\| = \|u(T,\xi,f) - (u^T - z(T,v))\| \le \varepsilon$$

whenever $f = F(\cdot, v(\cdot))$ and $\xi = \mathcal{M}(f, u^T - z(T,v))$. Before proceeding to the proof of Proposition 2.1, we recall for easy reference the following variant of a compactness result due to Baras, Hassan, Veron [8]. The conclusion of this variant follows from Theorem 2.3.3, p. 47 in Vrabie [71] combined with the simple remark that the graph of a any bounded linear operator is weakly×strongly sequentially closed.

Theorem 2.1. *If* (H_1), (H_2) *and* (H_4) *are satisfied, then the mild solution operator* $\mathcal{S} : H \times L^2(0,T\,;H) \to C([0,T]\,;H)$ *defined by*

$$\mathcal{S}(\xi,g)(t) = S(t)\xi + \int_0^t S(t-s)g(s)\,ds,$$

for each $(\xi,g) \in H \times L^2(0,T\,;H)$ *and* $t \in [0,T]$, *is weakly-strongly sequentially continuous from* $H \times L^2(0,T\,;H)$ *to* $C([\,\delta,T\,]\,;H)$ *for each* $\delta \in (0,T)$. *In addition, if* $\lim_n \xi_n = \xi$ *strongly in* H *and* $\lim_n g_n = g$ *weakly in* $L^2(0,T\,;H)$, *then*

$$\lim_n \mathcal{S}(\xi_n, g_n)(t) = \mathcal{S}(\xi,g)(t)$$

uniformly for $t \in [0,T]$. *Consequently,* $\mathcal{S}$ *maps each* (*relatively compact*)×(*bounded*) *subset in* $H \times L^2(0,T\,;H)$ *into a relatively compact subset in* $C([0,T]\,;H)$.

We continue with the proof of Proposition 2.1. We follow some ideas due to Lions [53], [54] and improved later by Fabre, Puel Zuazua [31], [32].

Proof. Let $T>0$ and $\varepsilon>0$ be fixed, let $f\in L^2(0,T;\mathcal{L}(H))$, $u^T\in H$ and let us define the functional $\mathcal{L}: L^2(0,T;\mathcal{L}(H))\times H\times H\to\mathbb{R}$ by

$$\mathcal{L}(f,u^T,\eta)=\frac{1}{2}\|\varphi(0,\eta,f)\|^2+\varepsilon\|\eta\|-\langle u^T,\eta\rangle,$$

where $\varphi(\cdot,\eta,f)$ denotes the unique solution of the adjoint equation

$$\begin{aligned}\varphi'&=-A_H\varphi-f^*\varphi\\ \varphi(T,\eta,f)&=\eta.\end{aligned}\tag{2.3}$$

One may easily verify that $\mathcal{L}(f,u^T,\cdot)$ is strictly convex, continuous and differentiable at each $\eta\neq 0$. Moreover, for each $(f_n)_n$ in $L^2(0,T;\mathcal{L}(H))$, each $(\eta_n)_n$ and each $(u_n^T)_n$ in H satisfying

$$\begin{gathered}\sup_n\|f_n\|_{L^2(0,T;\mathcal{L}(H))}<\infty\\ \lim_n\|\eta_n\|=+\infty\\ \lim_n\|u_n^T-u^T\|=0,\end{gathered}$$

we have

$$\liminf_n\frac{\mathcal{L}(f_n,u_n^T,\eta_n)}{\|\eta_n\|}\geq\varepsilon.\tag{2.4}$$

Indeed, let us denote by $\xi_n=\|\eta_n\|^{-1}\eta_n$ and let us observe that if

$$\liminf_n\|\varphi(0,\xi_n,f_n)\|>0,$$

then (2.4) is clearly satisfied. So, let us assume that $\liminf_n\|\varphi(0,\xi_n,f_n)\|=0$. We may assume (by extracting some subsequences if necessary) that $\lim_n\xi_n=\xi$ weakly in H and (since H is separable) $\lim_n f_n=f$ weakly pointwise in $L^2(0,T;\mathcal{L}(H))$. See Definition 2.1. Since $\psi_n(t)=\varphi(T-t,\xi_n,f_n(T-\cdot))$ satisfies

$$\begin{aligned}\psi_n'&=A\psi_n+f_n^*(T-\cdot)\\ \psi_n(0)&=\xi_n,\end{aligned}$$

by Gronwall's Lemma, we deduce that $\{f_n^*(T-\cdot)\psi_n\,;\ n\in\mathbb{N}^*\}$ is bounded in $L^2(0,T;H)$. Theorem 2.1 shows that $\lim_n\psi_n(t)=\varphi(T-t,\xi,f(T-t))$ uniformly for t in each compact subset in $(0,T]$, or equivalently that $\lim_n\varphi(t,\xi_n,f_n)=\varphi(t,\xi,f)$ uniformly for t in each compact subset in $[0,T)$. Since $\varphi(0,\xi,f)=0$, by virtue of the Backward Uniqueness Theorem 1.2, we have $\varphi(T,\xi,f)=0$ and thus $\xi=0$. Therefore, $\lim_n\xi_n=0$ weakly in H and thus $\lim_n\langle u_n^T,\xi_n\rangle=0$. Since, for each $n\in\mathbb{N}$, $\varphi(t,\xi_n,f_n)=\|\eta_n\|^{-1}\varphi(t,\eta_n,f_n)$ and $\|\xi_n\|=1$, we have

$$\liminf_n\frac{\mathcal{L}(f_n,u_n^T,\eta_n)}{\|\eta_n\|}=\liminf_n\left\{\frac{\|\eta_n\|}{2}\|\varphi(0,\xi_n,f_n)\|^2+\varepsilon-\langle u_n^T,\xi_n\rangle\right\}\geq\varepsilon$$

and thus (2.4) holds.

In order to prove that for each (f,u^T) $\mathcal{L}(f,u^T,\cdot)$ has exactly one minimum point, we distinguish between two cases: $\|u^T\|>\varepsilon$ and $\|u^T\|\leq\varepsilon$. If $\|u^T\|>\varepsilon$, let us observe that, for each $f\in L^2(0,T;\mathcal{L}(H))$, we have

$$\inf_{\eta\in H}\mathcal{L}(f,u^T,\eta)<0.\tag{2.5}$$

To show (2.5) first let us remark that, by (iii) in (H_3) and Gronwall's Lemma, we have

$$\|\varphi(0,\eta,f)\|\leq k\|\eta\|$$

for each $\eta \in H$, where $k = \exp\left\{\int_0^T \|f(s)\|_{\mathcal{L}(H)} ds\right\}$. Then, since $\|u^T\| > \varepsilon$, for a sufficiently small $\delta > 0$, we have

$$\mathcal{L}(f, u^T, \delta u^T) \le \delta \|u^T\| \left[\left(\frac{k}{2}\delta - 1\right)\|u^T\| + \varepsilon\right] < 0$$

which clearly implies (2.5). Since $\mathcal{L}(f, u^T, \cdot)$ is strictly convex, by virtue of (2.4), it follows that there exists a unique minimum point η^* of $\mathcal{L}(f, u^T, \cdot)$, i.e.

$$\mathcal{L}(f, u^T, \eta^*) = \inf_{\eta \in H} \mathcal{L}(f, u^T, \eta).$$

Moreover, since $\mathcal{L}(f, u^T, 0) = 0$, by (2.5) it follows that $\eta^* \neq 0$, and therefore $\mathcal{L}(f, u^T, \cdot)$ is differentiable at η^* in any direction $\theta \in H$. By Fermat's Necessary Condition for Extremum we have

$$\frac{\partial \mathcal{L}}{\partial \eta}(f, u^T, \eta^*)(\theta) = 0,$$

i.e.

$$\langle \varphi(0, \eta^*, f), \varphi(0, \theta, f)\rangle + \frac{\varepsilon}{\|\eta^*\|}\langle \eta^*, \theta\rangle - \langle u^T, \theta\rangle = 0 \tag{2.6}$$

for each $\theta \in H$. Next, multiplying both sides in (2.1) (with $\xi = \varphi(0, \eta^*, f)$) by $\varphi(t, \theta, f)$, integrating over $[0, T]$ and taking into account of (2.3), we get

$$\langle u(T, \varphi(0, \eta^*, f), f), \varphi(T, \theta, f)\rangle = \langle u(0, \varphi(0, \eta^*, f), f), \varphi(0, \theta, f)\rangle.$$

Since $\varphi(T, \theta, f) = \theta$ and $u(0, \varphi(0, \eta^*, f), f) = \varphi(0, \eta^*, f)$, this relation along with (2.6) yields

$$\langle u(T, \varphi(0, \eta^*, f), f) - u^T + \varepsilon \|\eta^*\|^{-1}\eta^*, \theta\rangle = 0$$

for each $\theta \in H$ and consequently

$$\|u(T, \varphi(0, \eta^*, f), f) - u^T\| = \varepsilon.$$

We may now pass to the analysis of the second case. Namely, if $\|u^T\| \le \varepsilon$, one may easily verify that

$$\mathcal{L}(f, u^T, \eta) \ge \frac{1}{2}\|\varphi(0, \eta, f)\|^2 + (\varepsilon - \|u^T\|)\|\eta\| \ge 0.$$

Since $\mathcal{L}(f, u^T, 0) = 0$ and $\mathcal{L}(f, u^T, \cdot)$ is strictly convex, this implies that 0 is the unique minimum point of $\mathcal{L}(f, u^T, \cdot)$. So, for each $(f, u^T) \in L^2(0, T; \mathcal{L}(H)) \times H$, $\mathcal{L}(f, u^T, \cdot)$ has one and only one minimum point η^*. This enables us to define $\mathcal{M} : L^2(0, T; \mathcal{L}(H)) \times H \to H$ by

$$\mathcal{M}(f, u^T) = \varphi(0, \eta^*, f)$$

where η^* is the unique minimum point of $\mathcal{L}(f, u^T, \cdot)$ and $\varphi(\cdot, \eta^*, f)$ is the unique mild solution of the adjoint equation (2.3) corresponding to f^* and to η^*. Clearly $\mathcal{M}$ satisfies (2.2) and so, to complete the proof, we have merely to show that it is continuous from $L^2(0, T; \mathcal{L}(H)) \times H$ into H in the weak pointwise convergence topology. See Definition 2.1. Thus, let $(f_n)_n$ be a bounded sequence in $L^2(0, T; \mathcal{L}(H))$ such that, for each $u \in H$ we have

$$\lim_n f_n u = f u$$

weakly in $L^2(0, T; H)$ and let $(u_n^T)_n$ with $\lim_n u_n^T = u^T$ strongly in H. By virtue of (2.4) and (2.5) it readily follows that the sequence $(\eta_n^*)_n$, where, for each $n \in \mathbb{N}$, η_n^* is the unique minimum point of $\mathcal{L}(f_n, u_n^T, \cdot)$, is bounded in H. Relabelling if necessary, we may assume that $\lim_n \eta_n^* = \eta$ weakly in H. Reasoning as in the proof of (2.4) we conclude that

$$\lim_n \varphi(0, \eta_n^*, f_n) = \varphi(0, \eta, f) \tag{2.7}$$

strongly in H. So, to conclude, it suffices to show that $\mathcal{M}(f, u^T) = \varphi(0, \eta, f)$, or equivalently that η is the unique minimum point η^* of $\mathcal{L}(f, u^T, \cdot)$. To this aim let us observe first that

$$\liminf_n \mathcal{L}(f_n, u_n^T, \psi) = \mathcal{L}(f, u^T, \psi)$$

for each $\psi \in H$. Indeed, since $(f_n)_n$ is bounded in $L^2(0, T\,; \mathcal{L}(H))$, by virtue of Gronwall's Lemma, we deduce that $(\varphi(t, \psi, f_n))_n$ is uniformly bounded on $[0, T]$. Then, a simple argument involving Theorem 2.1 and the fact that, for each $u \in H$, $\lim_n f_n u = fu$ weakly in $L^2(0, T\,; H)$ completes the proof of the equality above. Next, by this relation, (2.7), the weak lower semicontinuity of the norm in H, the weak convergence of $(\eta_n^*)_n$ to η, and the weak pointwise convergence of $(f_n)_n$ to f, we have

$$\begin{aligned} \mathcal{L}(f, u^T, \eta) = \liminf_n \mathcal{L}(f_n, u_n^T, \eta) \le \liminf_n \mathcal{L}(f_n, u_n^T, \eta_n^*) \le \\ \le \liminf_n \mathcal{L}(f_n, u_n^T, \psi) = \mathcal{L}(f, u^T, \psi) \end{aligned}$$

for each $\psi \in H$. Consequently $\eta = \eta^*$. Finally, let us observe that each subsequence of the initial sequence $(\eta_n^*)_n$ has in its turn a weakly convergent subsequence to the very same limit η^* and this achieves the proof. □

We may now proceed to the proof of Proposition 2.2.

Proof. The idea of proof consists in showing that $\mathcal{K}$ satisfies the hypotheses of Schauder's Fixed Point Theorem. More precisely, we will prove that $\mathcal{K}$ is continuous and has compact range. To this aim, let us observe that $\mathcal{K}$ can be decomposed as $\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1$, where

$$\begin{gathered} \mathcal{K}_0 v = z(\cdot, v) \\ \mathcal{K}_1 v = u(\cdot, \mathcal{M}(F(\cdot, v(\cdot)), u^T - z(T, v)), F(\cdot, v(\cdot))). \end{gathered}$$

One may easily see that $\mathcal{K}_0$ is continuous and has relatively compact range. See (H_3) and Theorem 2.1. So, it suffices to show that $\mathcal{K}_1$ enjoys the same properties. To this aim let $(v_n)_n$ be a sequence in $C([0, T]\,; H)$ which is uniformly convergent on $[0, T]$ to some function v. By virtue of (H_3) and Lebesgue Dominated Convergence Theorem, it follows that

$$\lim_n F(\cdot, v_n(\cdot)) = F(\cdot, v(\cdot))$$

strongly in $L^2(0, T\,; \mathcal{L}(H))$. An appeal to Proposition 2.1 shows that

$$\lim_n \mathcal{M}(F(\cdot, v_n(\cdot)), u^T - z(T, v_n))) = \mathcal{M}(F(\cdot, v(\cdot)), u^T - z(T, v)))$$

strongly in H and therefore we have

$$\begin{aligned} \lim_n u(\cdot, \mathcal{M}(F(\cdot, v_n(\cdot)), u^T - z(T, v_n)), F(\cdot, v_n(\cdot))) = \\ = u(\cdot, \mathcal{M}(F(\cdot, v(\cdot)) u^T - z(T, v)), F(\cdot, v(\cdot))) \end{aligned}$$

strongly in $C([0, T]\,; H)$. Recalling the definition of $\mathcal{K}_1$ this last relation is equivalent to

$$\lim_n \mathcal{K}_1(v_n) = \mathcal{K}_1(v)$$

and thus $\mathcal{K}_1$ is continuous from $C([0,T];H)$ into itself. Next, since by (iii) in (H_3) we have $\|F(t,v(t))\|_{\mathcal{L}(H)} \leq \mu(t)$ for almost all $t \in [0,T]$, in view of Proposition 2.1, the mapping $v \mapsto \mathcal{M}(F(\cdot,v(\cdot)), u^T - z(T,v)))$ has relatively compact range. Indeed, if $(v_n)_n$ is a given sequence in $C([0,T];H)$, then $(f_n)_n$ defined by $f_n = F(\cdot, v_n(\cdot))$ is bounded in $L^2(0,T;\mathcal{L}(H))$ and since H is separable, it has at least one subsequence which is weakly pointwise convergent. Furthermore, by (iii) in (H_3) and Theorem 2.1, it follows that, on a subsequence at least, $(z(T,v_n))_n$ is strongly convergent in H. Let us denote for simplicity these subsequences again by $(f_n)_n$ and respectively by $(z(T,v_n))_n$, and let us observe that, by Proposition 2.1, it follows that there exists $\lim_n \mathcal{M}(f_n, u^T - z(T,v_n))$ in the norm topology of H. Thus $\{\mathcal{M}(F(\cdot,v(\cdot)), u^T - z(T,v))\,;\ v \in C([0,T];H)\}$ is relatively compact in H. An appeal to Theorem 2.1 shows that $\mathcal{K}_1(C([0,T];H))$ is relatively compact in $C([0,T];H)$ and this completes the proof. $\square$

Remark 2.1. Let $T > 0$, $u^T \in H$ and $\varepsilon > 0$ be fixed and let us consider the sequence of problems

$$\begin{gathered} u_n' + A_H u_n + F_n(t,u_n)u_n = h_n(t) \\ \|u_n(T) - u^T\| \leq \varepsilon \end{gathered} \tag{2.8}$$

where, for each $n \in \mathbb{N}^*$, $h_n \in L^2(0,T;H)$ and F_n satisfies (H_3). Assume that (H_1), (H_2) and (H_4) hold, and for each $n \in \mathbb{N}^*$, u_n is a solution of (2.8). If there exists $\lim_n F_n(t,v)v = B(t,v)$ uniformly for v in compact subsets in H and a.e. for t in $[0,T]$, $\lim_n u_n = u$ in $C([0,T];H)$ and $\lim_n h_n = h$ weakly in $L^2(0,T;H)$, then, by virtue of Theorem 2.1, u is a solution of the limiting problem

$$\begin{gathered} u' + A_H u + B(t,u) = h(t) \\ \|u(T) - u^T\| \leq \varepsilon. \end{gathered} \tag{2.9}$$

Remark 2.2. Assume that (H_1), (H_2), (H_3) and (H_4) hold. If the solution u_n of (2.8) is given by the previous fixed point device, F_n satisfies (iii) in (H_3) with μ independent of n and $(h_n)_n$ is bounded in $L^2(0,T;H)$, then $(u_n)_n$ has at least one subsequence which converges in $C([0,T];H)$ to some function u. If, in addition, there exist $\lim_n F_n(t,v)v = B(t,v)$ uniformly for v in compact subsets in H and a.e. for t in $[0,T]$ and $\lim_n h_n = h$ weakly in $L^2(0,T;H)$, then u is a solution of (2.9). Indeed, let $\mathcal{K}_n$ be the operator defined just before Proposition 2.2 and associated with the approximate problem (2.8), i.e.

$$\mathcal{K}_n v = z_n(\cdot,v) + u(\cdot,\xi_n,f_n),$$

where z_n is the unique mild solution of

$$\begin{gathered} z_n' + A_H z_n + F_n(t,v)z_n = h_n \\ z_n(0) = 0, \end{gathered}$$

and $u(\cdot,\xi_n,f_n)$ is the unique mild solution of (2.1) corresponding to ξ_n and f_n defined by

$$\begin{gathered} f_n = F_n(\cdot,v(\cdot)) \\ \xi_n = \mathcal{M}(f_n, u^T - z_n(T,v)). \end{gathered}$$

Let u_n be the any fixed point of $\mathcal{K}_n$. By (iii) in (H_3) we have that $(F_n(\cdot,u_n))_n$ is bounded in $L^2(0,T;\mathcal{L}(H))$ and so, by Gronwall's Lemma and Theorem 2.1,

$\{z(T,u_n)\,;\ n\in\mathbb{N}^*\}$ is relatively compact in H. By (iii) in (H_3) and the separability of H it follows that $(F_n(\cdot,u_n))_n$ has at least one weakly pointwise convergent subsequence (see Definition 2.1). Then, by Proposition 2.1, we conclude that $\{\mathcal{M}(F_n(\cdot,u_n),u^T-z(T,u_n))\,;\ n\in\mathbb{N}^*\}$ is relatively compact in H too. Again by Theorem 2.1, we deduce that $(u_n)_n$ has at least one subsequence which converges in $C([0,T]\,;H)$ to some function u. If there exist $\lim_n F_n(t,v)v = B(t,v)$ uniformly for v in compact subsets in H and a.e. for t in $[0,T]$ and $\lim_n h_n = h$ weakly in $L^2(0,T\,;H)$, by Remark 2.1 we conclude that u is a solution of (2.9) and this achieves the proof.

3. Proof of Theorem 1.3

We shall prove that (1.2) can be equivalently rewritten as an abstract problem of the form (1.1) and then we shall use Theorem 1.1. We begin by showing how to choose the Hilbert spaces V and H and how to define the operator A. Namely, take $V=\{(u,v)\in H^1(\Omega)\times H^{1/2}(\Gamma);\ u_{|\Gamma}=v\}$ which is a real separable Hilbert space isomorphic to $H^1(\Omega)$ where the latter is endowed with the equivalent norm

$$\|u\|_{H^1(\Omega)}=\Big(\|\nabla u\|^2_{L^2(\Omega)}+\|u_{|\Gamma}\|^2_{L^2(\Gamma)}\Big)^{1/2}.$$

Let us define $A:V\to V^*$ by

$$(A(u,v),(\varphi,\psi))=\int_\Omega \nabla u\cdot\nabla\varphi\,dx,$$

where $(\cdot,\cdot)$ is the usual pairing between V and V^*. Take $H=L^2(\Omega)\times L^2(\Gamma)$ which endowed with the usual inner product

$$\langle (u,v),(\tilde u,\tilde v)\rangle=\langle u,\tilde u\rangle_{L^2(\Omega)}+\langle v,\tilde v\rangle_{L^2(\Gamma)},$$

is a real Hilbert space. We define the restriction $A_H : D(A_H)\subset H\to H$ of A to H by $D(A_H)=\{(u,v)\in V;\ A(u,v)\in H\}$ and $A_H(u,v)=A(u,v)$, for each $(u,v)\in D(A_H)$. It is easy to see that

$$D(A_H)=\{(u,v)\in L^2(\Omega)\times L^2(\Gamma);\ \Delta u\in L^2(\Omega),\ u_\nu\in L^2(\Gamma),\ u_{|\Gamma}=v\}$$

and $A_H(u,v)=(-\Delta u,u_\nu)$.

Lemma 3.1. *The operator A defined above satisfies the hypotheses (H_1) and (H_2).*

Proof. Clearly $A\in\mathcal{L}(V,V^*)$. Moreover, for each $(u,v)\in V$ we have

$$(A(u,v),(u,v))=\|\nabla u\|^2_{L^2(\Omega)}.$$

So A satisfies (H_2) with $\lambda=\eta=1$. In addition, from the relation above it readily follows that A_H is accretive. Since, by a classical result on linear elliptic problems $I+A_H$ is surjective, we deduce that A_H is m-accretive. Next, let us observe that

$$\langle A_H(u,v),(\varphi,\psi)\rangle=-\langle\Delta u,\varphi\rangle+\langle u_\nu,\psi\rangle_{L^2(\Gamma)}=\langle\nabla u,\nabla\varphi\rangle=$$
$$=-\langle u,\Delta\varphi\rangle+\langle u,\varphi_\nu\rangle_{L^2(\Gamma)}=\langle (u,v),A_H(\varphi,\psi)\rangle,$$

for each $(u,v),(\varphi,\psi)\in D(A_H)$. Consequently A_H is symmetric. Since A_H is m-accretive, by virtue of Corollary 1.1.45 in Brezis, Cazenave [17], p. 13, it follows that A_H is self adjoint. Thus A satisfies (H_1) and this completes the proof.

The next lemma, which perhaps is not new, is another ingredient in the proof of Theorem 1.3.

Lemma 3.2. *Let $(K, \mathcal{A}, \lambda)$ be a finite measure space and let $b_n : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a sequence of functions satisfying*

(i) *for each $n \in \mathbb{N}^*$ and each $u \in \mathbb{R}$, the mapping $t \mapsto b_n(t, u)$ is measurable;*
(ii) *for each $n \in \mathbb{R}$ and a.e. for $t \in \mathbb{R}_+$, the mapping $u \mapsto b_n(t, u)$ is continuous;*
(iii) *there exist $\alpha, \gamma \in \mathcal{L}^2_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ such that*
$$|b_n(t, u)| \le \alpha(t)|u| + \gamma(t)$$
for each $u \in \mathbb{R}$ and a.e. for $t \in \mathbb{R}_+$.

Let $B_n : \mathbb{R}_+ \times L^2(K, \lambda; \mathbb{R}) \to L^2(K, \lambda; \mathbb{R})$ be the realization of b_n in $L^2(K, \lambda; \mathbb{R})$, i.e.
$$B_n(t, u)(\kappa) = b_n(t, u(\kappa))$$
for each $u \in L^2(K, \lambda; \mathbb{R})$ and a.e. for $t \in \mathbb{R}_+$ and $\kappa \in K$. If $\lim_n b_n(t, v) = b(t, v)$ uniformly for v in bounded subsets in $\mathbb{R}$ and a.e. for $t \in \mathbb{R}_+$, then
$$\lim_n B_n(t, v) = B(t, v)$$
uniformly for v in compact subsets in $L^2(K, \lambda; \mathbb{R})$ and a.e. for $t \in \mathbb{R}_+$, where $B : \mathbb{R}_+ \times L^2(K, \lambda; \mathbb{R}) \to L^2(K, \lambda; \mathbb{R})$ is the realization of b in $L^2(K, \lambda; \mathbb{R})$, i.e.
$$B(t, u)(\kappa) = b(t, u(\kappa))$$
for each $u \in L^2(K, \lambda; \mathbb{R})$ and a.e. for $t \in \mathbb{R}_+$ and $\kappa \in K$.

Proof. Clearly b satisfies (i), (ii) and (iii). Consequently, for each $u \in L^2(K, \lambda; \mathbb{R})$ $t \mapsto B(t, u)$ is measurable; for almost all $t \in \mathbb{R}_+$ $u \mapsto B(t, u)$ is continuous and $\|B(t, u)\|_{L^2(K,\lambda;\mathbb{R})} \le \alpha(t)\|u\|_{L^2(K,\lambda;\mathbb{R})} + \gamma(t)$ for each $u \in L^2(K, \lambda; \mathbb{R})$ and a.e. for $t \in \mathbb{R}_+$. Therefore, to complete the proof it suffices to show that for a.e. $t \in \mathbb{R}_+$ and for each convergent sequence $(u_n)_n$ in $L^2(K, \lambda; \mathbb{R})$, we have
$$\lim_n B_n(t, u_n) = B(t, u)$$
in $L^2(K, \lambda; \mathbb{R})$, where $u = \lim_n u_n$. In order to show this, fix $t \in \mathbb{R}_+$ for which $\lim_n b_n(t, v) = b(t, v)$ uniformly for v in bounded subsets in $\mathbb{R}$ and let $(u_n)_n$ in $L^2(K, \lambda; \mathbb{R})$ with $\lim_n u_n = u$. By Lebesgue Theorem, on a subsequence at least, $(B_n(t, u_n))_n$ is a.e. convergent on K to $B(t, u)$. Furthermore, since $\{u_n ;\ n \in \mathbb{N}^*\}$ is relatively compact in $L^2(K, \lambda; \mathbb{R})$, it is uniformly integrable and thus, by (iii), $\{B_n(t, u_n) ;\ n \in \mathbb{N}^*\}$ enjoys the same property. So, by Vitali's Theorem (see Dunford, Schwartz [29], Theorem 15, p. 150) we have that
$$\lim_n \|B_n(t, u_n) - B(t, u)\|_{L^2(K,\lambda;\mathbb{R})} = 0$$
on that subsequence. To complete the proof we have merely to observe that, if we assume by contradiction that there exists another subsequence $(u_k)_k$ of $(u_n)_n$ and $\varepsilon > 0$ such that $\|B_k(t, u_k) - B(t, u)\|_{L^2(K,\lambda;\mathbb{R})} \ge \varepsilon$ for each k, then by the very same arguments we conclude that, on a sub-subsequence, $\lim_p B_p(t, u_p) = B(t, u)$, relation which contradicts the preceding one. This contradiction can be eliminated only if the conclusion of lemma holds and this achieves the proof. □

Now we come back to the proof of Theorem 1.3. By Remark 1.1, we may assume without loss of generality that (v) in (H_5) holds with $u_0 = v_0 = \rho(u_0) = \phi(v_0) = 0$. First, we will assume the extra-condition that both g and β are differentiable at 0 with respect to their second argument and both ρ and ϕ are identically 0. So, let us define $F : \mathbb{R}_+ \times H \to \mathcal{L}(H)$ by

$$\{[F(t,u,v)](\varphi,\psi)\}(x,\sigma) = (\varphi(x),\psi(\sigma)) \begin{pmatrix} f_1(t,u(x)) & 0 \\ 0 & f_2(t,v(\sigma)) \end{pmatrix}, \tag{3.1}$$

for each $(u,v),(\varphi,\psi) \in L^2(\Omega) \times L^2(\Gamma)$ and a.e. for $x \in \Omega$ and $\sigma \in \Gamma$, where $f_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are defined by

$$f_1(t,u) = \begin{cases} \dfrac{g(t,u) - g(t,0)}{u} & \text{if } u \neq 0 \\ \dfrac{\partial g}{\partial u}(t,0) & \text{if } u = 0 \end{cases} \tag{3.2}$$

and

$$f_2(t,v) = \begin{cases} \dfrac{\beta(t,v) - \beta(t,0)}{v} & \text{if } v \neq 0 \\ \dfrac{\partial \beta}{\partial v}(t,0) & \text{if } v = 0. \end{cases} \tag{3.3}$$

At this point let us observe that (1.2) can be equivalently rewritten as an ordinary differential equation in H of the form

$$\begin{gathered} w' + A_H w + F(t,w)w = h(t) \\ w(0) = w_0, \end{gathered}$$

where A_H and F are as above, $h = -(g(\cdot,0),\beta(\cdot,0))$, $w = (u,v)$ and $w_0 = (u_0^\Omega, u_0^\Gamma)$. Inasmuch as both $g(t,\cdot)$ and $\beta(t,\cdot)$ satisfy (H_5) and are differentiable at 0, the mapping F defined as above satisfies (H_3). Since (H_4) obviously holds, the conclusion follows from Theorem 1.1. By the very same arguments it follows that, whenever, in addition to (H_5), $g(t,\cdot)$, $\beta(t,\cdot)$ are differentiable at 0 and ρ, ϕ are single-valued and differentiable at 0, the conclusion of Theorem 1.3 still holds true. Next we consider the general case and we show that the conclusion follows by Remark 2.2 along with a standard approximation technique. Namely, let $\theta : \mathbb{R} \to \mathbb{R}$ be a mollifier, i.e. a C^∞ function with $\theta(x) \geq 0$ for each $x \in \mathbb{R}$, $\operatorname{supp}\theta \subseteq [-1,1]$ and $\int_{\mathbb{R}} \theta(x)\,dx = 1$. For $n \in \mathbb{N}^*$, let us define $g_n(t,u) = g(t,u) * \theta_n(u)$, $\beta_n(t,v) = \beta(t,v) * \theta_n(v)$, where $\theta_n(x) = n\theta(nx)$ and ρ_n, ϕ_n are the Yosida approximations of ρ and respectively ϕ, i.e.

$$\rho_n = n\left[I - (I + n^{-1}\rho)^{-1}\right] \quad \text{and} \quad \phi_n = n\left[I - (I + n^{-1}\phi)^{-1}\right].$$

Since, by Remark 1.1, $g_n(t,\cdot)$, $\beta_n(t,\cdot)$, ρ_n and ϕ_n all are differentiable at 0 and satisfy (H_5), by the preceding proof, we know that, for each $n \in \mathbb{N}^*$, each $T > 0$, each $w^T \in H$ and each $\varepsilon > 0$, there exists at least one function $w_n = (u_n, v_n)$ satisfying

$$\begin{gathered} w_n' + A_H w_n + F_n(w_n)w_n + G_n(w_n)w_n = p_n \\ \|w(T) - w^T\| \leq \varepsilon, \end{gathered} \tag{3.4}$$

where the mappings F_n and G_n correspond to g_n, β_n and respectively to ρ_n and ϕ_n in a similar manner as F corresponds to g and β (see (3.1), (3.2) and (3.3)) and

$$p_n(t)(x,\sigma) = -(g_n(t,0) + a(t)\rho_n(0), \beta_n(t,0) + c(t)\phi_n(0))$$

a.e. for $t \in \mathbb{R}_+$, $x \in \Omega$ and $\sigma \in \Gamma$. By Remark 1.1 we know that g_n, β_n, ρ_n and ϕ_n satisfy the growth condition (v) in (H_5) uniformly with respect to n and with the very same function α. So, it follows that $F_n + G_n$ satisfies (H_3) with μ independent of n. Therefore, by Remark 2.2, we conclude that, on a subsequence at least, $(w_n)_n$

converges strongly in $C([0,T]\,;H)$ to some element $w^* = (u^*, v^*)$. On the other hand, let us observe that the system (3.4) can be equivalently rewritten as

$$\begin{gathered} w_n' + A_H w_n + F_n(w_n)w_n = h_n \\ \|w(T) - w^T\| \le \varepsilon, \end{gathered} \tag{3.5}$$

where

$$h_n(t)(x,\sigma) = -(a(t)\rho_n(u_n(t,x)), c(t)\phi_n(v_n(t,\sigma))) - (g_n(t,0), \beta_n(t,0))$$

a.e. for $t \in \mathbb{R}_+$, $x \in \Omega$ and $\sigma \in \Gamma$. At this point, we notice that for almost all $t \in \mathbb{R}_+$,

$$\lim_n g_n(t,u) = g(t,u) \quad \text{and} \quad \lim_n \beta_n(t,v) = \beta(t,v)$$

uniformly for u, v on compact subsets in $\mathbb{R}$. So, by Lemma 3.2, it follows that

$$\lim_n F_n(t,u,v)(u,v) = B(t,u,v)$$

uniformly on compact subsets in $L^2(\Omega) \times L^2(\Gamma)$ and a.e. for $t \in \mathbb{R}_+$, where

$$B(t,u,v)(x,\sigma) = (g(t,u(x)), \beta(t,v(\sigma))) - (g(t,0), \beta(t,0)).$$

Furthermore, since by (v) in (H_5) both $(\rho_n(u_n))_n$ and $(\phi_n(v_n))_n$ are bounded in $L^\infty(0,T\,;L^2(\Omega))$ and respectively in $L^\infty(0,T\,;L^2(\Gamma))$ we may assume without loss of generality that $\lim_n a\rho_n(u_n) = a\rho^*$ and $\lim_n c\phi_n(v_n) = c\phi^*$ weakly in $L^2(0,T\,L^2(\Omega))$ and respectively in $L^2(0,T\,;L^2(\Gamma))$, where a and c are the functions in (H_5). As a consequence $\lim_n h_n = h$ weakly in $L^2(0,T\,;L^2(\Omega) \times L^2(\Gamma))$, where

$$h(t)(x,\sigma) = -(a(t)\rho^*(t,x), c(t)\phi^*(t,\sigma)) - (g(t,0), \beta(t,0))$$

a.e. for $t \in \mathbb{R}_+$, $x \in \Omega$ and $\sigma \in \Gamma$. Since the realizations of ρ and ϕ in $L^2(0,T\,;L^2(\Omega))$ and respectively in $L^2(0,T\,;L^2(\Gamma))$ are both demiclosed (see Brezis [16], Corollaire 2.5, p. 33) we have $\rho^*(t,x) \in \rho(u^*(t,x))$ and $\phi^*(t,\sigma) \in \phi(v^*(t,\sigma))$ a.e. for $t \in [0,T]$, $x \in \Omega$ and $\sigma \in \Gamma$. So, the conclusion of Theorem 1.3 follows by passing to the limit in (3.5) and taking into account of Remark 2.1 and this completes the proof. $\square$

Remark 3.1. Let us consider $g \equiv \beta \equiv 0$ and $a(t) \equiv c(t) \equiv 1$ and let us define the operator $\mathcal{N} : D(\mathcal{N}) \subset H \to H$ by $\mathcal{N}(u,v) = (-\Delta u + \rho_u, u_\nu + \phi_v)$ for each $(u,v) \in D(\mathcal{N})$, where $D(\mathcal{N})$ consists of all (u,v) in $L^2(\Omega) \times L^2(\Gamma)$ for which there exists $(\rho_u, \phi_v) \in L^2(\Omega) \times L^2(\Gamma)$ with $\rho_u(x) \in \rho(u(x))$, $\phi_v(\sigma) \in \phi(v(\sigma))$ a.e. for $x \in \Omega$ and $\sigma \in \Gamma$, and satisfying $-\Delta u + \rho_u \in L^2(\Omega)$, $u_{|\Gamma} = v$, $u_\nu + \phi_v \in L^2(\Gamma)$. Then, $\mathcal{N}$ is the subdifferential of the following l.s.c., convex and proper function $\Theta : H \to \overline{\mathbb{R}}$

$$\Theta(u,v) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega \|\nabla u(x)\|^2 dx + \int_\Omega G(u(x))\,dx + \int_\Gamma B(v(\sigma))\,d\sigma \\ \qquad \text{if } u \in H^1(\Omega),\ u_{|\Gamma} = v,\ G(u) \in L^1(\Omega),\ B(v) \in L^1(\Gamma) \\ +\infty \quad \text{otherwise}, \end{cases}$$

where $\partial G = \rho$ and $\partial B = \phi$. The fact that the operator above is m-accretive (even in an L^1-setting) was observed for the first time by Benilan [12].

Remark 3.2. If at least one of the operators ρ or ϕ are not everywhere defined the system (1.2) is not approximate controllable. This follows from the simple remark that each solution of (1.2) neccesarily satisfies $u(T,x) \in \overline{D(\rho)}$ and $u_{|\Gamma}(T,\sigma) \in \overline{D(\phi)}$ a.e. for $x \in \Omega$ and $\sigma \in \Gamma$. So, no element $(u^T, u^T_{|\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ which does not satisfy the above necessary condition, i.e. $(u^T, u^T_{|\Gamma}) \notin \overline{D(\rho)} \times \overline{D(\phi)}$ a.e. for $x \in \Omega$ and $\sigma \in \Gamma$, can be approximated by final states of the system (1.2). However in the case in which either $\overline{D(\rho)} \neq \mathbb{R}$, or $\overline{D(\phi)} \neq \mathbb{R}$, one may ask whether or not we have an approximate controllability result of the type: for each $T > 0$ the set of all T-final states of (1.2) is dense in the set

$$\left\{(u,v) \in L^2(\Omega) \times L^2(\Gamma)\,;\ (u(x), v(\sigma)) \in \overline{D(\rho)} \times \overline{D(\phi)},\ \text{a.e. for } x \in \Omega \text{ and } \sigma \in \Gamma\right\}.$$

This simple remark shows that, in the case of a nonlinear possible multivalued m-accretive operator $A : D(A) \subseteq H \to 2^H$, it is natural to say that the system

$$u' + Au \ni 0$$
$$u(0) = \xi$$

is *approximate controllable* if for each $T > 0$ the set of all T-final states of the system, i.e. $\{u(T,\xi)\,;\ \xi \in \overline{D(A)}\}$ is dense in $\overline{D(A)}$.

4. Proof of Theorem 1.4

Take $V = H^{1/2}(\Gamma)$, $H = L^2(\Gamma)$ and let us define the operator $A : V \to V^*$ by

$$Ay = u_\nu,$$

for each $y \in V$, where u is the unique $H^1(\Omega)$-solution of the nonhomogeneous elliptic problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= y \quad \text{on } \Gamma. \end{aligned} \tag{4.1}$$

Next, let us define $A_H : D(A_H) \subset H \to H$ by $D(A_H) = \{y \in V;\ Ay \in H\}$ and $A_H u = Au$ for each $u \in D(A_H)$.

Lemma 4.1. *The operator A satisfies the hypothesis (H_1) and (H_2).*

Proof. Obviously $A \in \mathcal{L}(V, V^*)$. Moreover, for each $y, z \in D(A_H)$, we have

$$\langle A_H y, z\rangle_{L^2(\Gamma)} = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} + (\Delta u, v)_{(H^1(\Omega),(H^1(\Omega))^*)}$$

and

$$\langle y, A_H z\rangle_{L^2(\Gamma)} = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} + (u, \Delta v)_{(H^1(\Omega),(H^1(\Omega))^*)},$$

where u satisfies (4.1) while v satisfies a similar equation with y replaced with z. Here $(\cdot,\cdot)_{(H^1(\Omega),(H^1(\Omega))^*)}$ is the usual pairing between $H^1(\Omega)$ and its topological dual $(H^1(\Omega))^*$. Since $\Delta u = \Delta v = 0$, it readily follows that

$$\langle A_H y, z\rangle_{L^2(\Gamma)} = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} = \langle y, A_H z\rangle_{L^2(\Gamma)}$$

for each $y, z \in D(A_H)$. So A_H is symmetric and monotone. Now recalling that an equivalent norm on $H^1(\Omega)$ is defined by $\|u\|^2_{H^1(\Omega)} = \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Gamma)}$ and observing that $(Ay, y) = \|\nabla u\|^2_{L^2(\Omega)}$ for each $y \in V$, we get

$$(Ay, y) = \|u\|^2_{H^1(\Omega)} - \|u\|^2_{L^2(\Gamma)}.$$

But the trace operator $\mathcal{T} : H^1(\Omega) \to H^{1/2}(\Gamma)$, $\mathcal{T}u = u_{|\Gamma}$, is continuous and therefore there exists $k > 0$ such that $\|\mathcal{T}u\|_{H^{1/2}(\Gamma)} \leq k\|u\|_{H^1(\Omega)}$ for each $u \in H^1(\Omega)$. As a consequence we get (H_2) with $\lambda = 1$ and $\eta = k^{-2}$. To complete the proof we have merely to show that A_H is maximal, i.e. that $I + A_H$ is surjective. Obviously $y + A_H y = f$ if and only if $y = u_{|\Gamma}$ where u is the unique $H^1(\Omega)$ solution of the elliptic problem

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u + u_\nu &= f \quad \text{on } \Gamma. \end{aligned}$$

So, $(I + A_H)$ is surjective if and only if, for each $f \in L^2(\Gamma)$, the unique $H^1(\Omega)$ solution of the problem above satisfies $u \in L^2(\Gamma)$ and $u_\nu \in L^2(\Gamma)$. But this follows from Lemma 5.1 in Appendix and this completes the proof. $\square$

As in the case of Theorem 1.3, we will consider first the case in which β is differentiable at 0 with respect to its second argument and $\phi \equiv 0$. So, let us define the mapping $F : \mathbb{R}_+ \times H \to \mathcal{L}(H)$ by

$$\{[F(t,u)]\varphi\}(\sigma) = f(t, u(\sigma))\varphi(\sigma)$$

for each $(t,u) \in \mathbb{R}_+ \times H$, each $\varphi \in H$ and a.e. for $\sigma \in \Gamma$, where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t,u) = \begin{cases} \dfrac{\beta(t,u) - \beta(t,0)}{u} & \text{if } u \neq 0 \\ \dfrac{\partial \beta}{\partial u}(t,0) & \text{if } u = 0. \end{cases}$$

At this point let us observe that the problem (1.3) can be equivalently rewritten in an abstract form as

$$\begin{aligned} u' + A_H u + F(t,u)u &= h(t) \\ u(0) &= \xi, \end{aligned}$$

where A and F are as above, $h = -\beta(\cdot, 0)$ and $\xi = u_0^\Gamma$. Obviously F satisfies (H_3). Observing that the embedding $V \subset H$ is compact, the conclusion is a direct consequence of Lemma 4.1 and Theorem 1.1. Since the general case follows by the very same arguments as those in the proof of Theorem 1.3 we do not enter into details. The proof is complete.

Remark 4.1. Some approximate controllability results for (1.2) and (1.3) in the sense of Díaz, Lions [26], [27], i.e. in the case in which these systems are explosive, may be found in Bejenaru, Díaz, Vrabie [11]. For other controllability results (with the initial datum as control) referring to systems driven by other types of partial differential equations see, for instance, Lions [53] and Constantin, Foiaş, Kukavica, Majda [19] and the references therein.

5. Appendix

The following result is “essentially” known. Nevertheless, the proofs in the literature for related results (see, e.g., Brezis [15]) require an additional coercivity term at the equation (as, for instance: $-\Delta u + \alpha u = 0$ with $\alpha > 0$).

Lemma 5.1. *For each $f \in L^2(\Gamma)$, the problem*

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u + u_\nu &= f \quad \text{on } \Gamma \end{aligned} \tag{5.1}$$

has a unique solution $u \in H^1(\Omega)$ satisfying $u \in H^{1/2}(\Gamma)$ and $u_\nu \in L^2(\Gamma)$.

Proof. We begin by observing that for $f \in C^\infty(\Gamma)$ the the problem (5.1) has a classical solution which is of the class C^∞ on Ω. So, let $f \in L^2(\Gamma)$ and let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $C^\infty(\Gamma)$ such that $f_k \overset{L^2(\Gamma)}{\longrightarrow} f$. We then have

$$0 = \int_\Omega \|\nabla(u_k - u_p)\|^2\,dx - \int_\Gamma \left(\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right)(u_k - u_p)\,d\sigma = \|\nabla(u_k-u_p)\|^2_{L^2(\Omega)} -$$

$$-\frac{1}{2}\left\langle \frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}, u_k - u_p\right\rangle_{L^2(\Gamma)} - \frac{1}{2}\left\langle \frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}, u_k - u_p\right\rangle_{L^2(\Gamma)}.$$

Using the boundary conditions to substitute $\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}$ in the first inner product in the right hand-side by $-u_k + u_p + f_k - f_p$ and $u_k - u_p$ in the second one by $-\frac{\partial u_k}{\partial\nu} + \frac{\partial u_p}{\partial\nu} + f_k - f_p$, we get

$$\|\nabla(u_k-u_p)\|^2_{L^2(\Omega)} + \frac{1}{2}\|u_k-u_p\|^2_{L^2(\Gamma)} - \frac{1}{2}\langle f_k - f_p, u_k - u_p\rangle_{L^2(\Gamma)} +$$

$$+\frac{1}{2}\left\|\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right\|^2_{L^2(\Gamma)} - \frac{1}{2}\left\langle f_k - f_p, \frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right\rangle_{L^2(\Gamma)} = 0$$

and thus

$$\|\nabla(u_k-u_p)\|^2_{L^2(\Omega)} + \|u_k-u_p\|^2_{L^2(\Gamma)} + \left\|\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right\|^2_{L^2(\Gamma)} \le$$

$$\le \|f_k - f_p\|_{L^2(\Gamma)}\|u_k - u_p\|_{L^2(\Gamma)} + \|f_k - f_p\|_{L^2(\Gamma)}\left\|\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right\|_{L^2(\Gamma)}.$$

Hence

$$\|u_k - u_p\|_{H^1(\Omega)} + \left\|\frac{\partial u_k}{\partial\nu} - \frac{\partial u_p}{\partial\nu}\right\|_{L^2(\Gamma)} \le \sqrt{2}\|f_k - f_p\|_{L^2(\Gamma)}$$

for each $k, p \in \mathbb{N}$. This inequality shows that $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $H^1(\Omega)$ and both $(u_{|\Gamma_k})_{k\in\mathbb{N}}$ and $\left(\frac{\partial u_k}{\partial\nu}\right)_{k\in\mathbb{N}}$ are Cauchy sequences in $L^2(\Gamma)$. Since the uniqueness is obvious, the proof is complete. □

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