EXTRACTA

MATHEMATICAE

Qualitative Study of Nonlinear Parabolic Equations: an Introduction

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1. INTRODUCTION: THE PROBLEM MODEL

Given Ω, open bounded regular set of $\mathbb{R}$, $N \geq 1$, we consider the *model problem*

$$(P) \begin{cases} b(u)_t - \operatorname{div} \mathbf{A}(x, u, \nabla u) + g(x, u) = f(t, x), & t > 0, \quad x \in \Omega, \\ u = h, & t > 0, \quad x \in \partial\Omega, \\ b(u(0, x)) = b(u_0(x)), & x \in \Omega. \end{cases}$$

ore making explicit the *structural assumptions* on the data b, $\mathbf{A}$, f, h and u_0 us mention some important special examples. Perhaps the simpler example ie *linear heat equation*

$$u_t - \Delta u = f. \tag{1}$$

$b(s) = s$, $\mathbf{A}(x, u, \xi) = \xi$ and $g \equiv 0$. This is a typical example of linear par-differential equation of parabolic type usually considered in undergraduate rses (see, e.g., John [31]). A *modern* treatment starts by introducing the ion of weak solution or by its reformulation as an abstract Cauchy problem a Banach space

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = f(t), \\ u(0) = u_0, \end{cases}$$

(see, e.g., Brezis [17]). It is well known, that one of the main results of the stabilization theory is that if

$$\left. \begin{array}{l} f(t,x) \longrightarrow f_\infty(x) \\ h(t,x) \longrightarrow h_\infty(x) \end{array} \right\} \quad \text{as} \quad t \longrightarrow \infty$$

in some *suitable* sense then the solution of the linear heat equation $u(t,x)$ verifies that

$$u(t,x) \longrightarrow u_\infty(x) \quad \text{as} \quad t \longrightarrow \infty$$

in *some* functional space, with u_∞ satisfying the *associated stationary problem*

$$\left\{ \begin{array}{rcll} -\Delta u_\infty &=& f_\infty(x) & \text{in} \quad \Omega, \\ u_\infty &=& h_\infty & \text{on} \quad \partial\Omega, \end{array} \right. \tag{2}$$

(*the linear diffusion equation*). Notice that problem (2) is also included in the formulation (P) by making $b \equiv 0$, $\mathbf{A}(x,u,\xi) = \xi$, $g \equiv 0$, $f = f_\infty$ and $h = h_\infty$. More in general, *given a choice of $b, \mathbf{A}, g, f, h$ and u_0 leading to a special formulation of (P), the choice of choice of $b \equiv 0, \mathbf{A}$ and g as before leads to the formulation of the associated stationary problem.* In this way (P) include also stationary problems. In order to present some *nonlinear examples*, it is useful *to read (P) as a balance of different phenomena*

$$\underbrace{b(u)_t}_{(I)} \underbrace{-\operatorname{div} \mathbf{A}}_{(II)} + \underbrace{g(x,u) - f(x,t)}_{(III)} = 0.$$

Let us make some comments on the *accumulation term* (I). It arises, for instance, in thermal processes when the *heat capacity* of the medium depends on the temperature. This is the case, e.g., when water and ice are simultaneously present and then $b(u)$ is a strictly increasing function having a discontinuity at $u = 0$. This special case (called *Stefan problem*) requires a delicate mathematical treatment.

In fact, as a general rule, the assumption $b : \mathbb{R} \to \mathbb{R}$ nondecreasing is absolutely fundamental to formulate (P) in the class of problems of parabolic type since otherwise the problem becomes *ill posed* (as, for instance, $-u_t - \Delta u = f$; the backward heat equation).

This type of *accumulation term* (I) also arises in the theory of filtration of a fluid in a porous media. In that case

$$b \in C^0(\mathbb{R}), \ b \text{ nondecreasing},$$

see, e.g., Bear [10]). Now $u(t, x)$ is not a temperature but the *humidity of the soil*. Different choices are possible: in the study of *unsaturated soils* b is assumed to be strictly increasing, as, for example, $b(u) = |u|^{\alpha-1}u$. In the case : *partially saturated soils*, $b(u)$ is not strictly increasing but becomes constant or $u > u^1$, for some $u^1 > 0$. Notice that, in this physical framework, $u \geq 0$ ad so the values of b on $\mathbb{R}^-$ are not relevant. The, so called *dam problem*, orresponds to a limit case in which b is the Heaviside function. This choice ' b also arises in problems of a different physical context, as, for instance, the *ele-Shaw problem* or some problems arising in *lubrication theory* (see, e.g., ayada and Chambat [9] where many other references can be found).

Let us refer now to the *diffusion* and *convection* terms involved in (II). he dependence of $\mathbf{A}(x, u, \nabla u)$ with respect to ∇u (resp. u) leads to diffusion rms (resp. convection terms). Some examples of relevance in the applica- ins are commented in the following. The, so called, *nonlinear heat equation* ises when the *Fourier law* fails and the *thermal conductivity* depends on the mperature (case of many gases, lubricating fluids, etc). Then the diffusion heat leads to the expression

$$\text{div}\,(k(u)\nabla u) = \Delta\beta(u) \quad \text{with} \quad \beta(s) := \int_0^s k(\sigma)d\sigma.$$

most of the cases $\beta(u)$ grows like a power

$$\beta(u) = |u|^{m-1}u \quad \text{with} \quad m > 0.$$

e above second order operator (sometimes written as $-\Delta u^m$) also arises the study of *filtration in porous media* (D'Arcy law) with $m > 1$ and in sma physics when $0 < m < 1$.

A different class of examples of nonlinear terms $\mathbf{A}(x, u, \nabla u)$ arises in the dy of *non-Newtonian fluids*. The study of one-directional flows of some cial fluids (as, for instance, polymer melts, suspensions, paints, animal od, honey, shampoo, etc.) leads to nonlinear diffusion operators of the e

$$\text{div}\left(|\nabla u|^{p-2}\nabla u\right), \text{ (denoted by } \Delta_p u), \quad \text{for some} \quad p > 1.$$

ice that if $p = 2$ then $\Delta_2 = \Delta$ (the linear Laplacian operator, arising in study of Newtonian fluids). The case $1 < p < 2$ corresponds to *pseudo- stic fluids* (as, e.g., gasoline, lubricating oil, etc.) and $p > 2$ arises in the sideration of *dilatant fluids* (as, for instance, the polar ice and glaciers, ano lava, etc.).

The above two operators may become *degenerate* since

$$\Delta u^m = \text{div}\left(m u^{m-1} \nabla u\right) = m u^{m-1} \Delta u + m(m-1) u^{m-2} |\nabla u|^2.$$

So, if $m > 1$ the coefficient of ∇u vanishes on the set $\{(t,x):\ u(t,x) = 0\}$ Analogously,

$$\Delta_p u = \text{div}\left(|\nabla u|^{p-2} \nabla u\right) = |\nabla u|^{p-2} \Delta u + \nabla u \cdot \nabla \left(|\nabla u|^{p-2}\right)$$

and when $p > 2$ the coefficient of ∇u vanishes on the set $\{(t,x):\ \nabla u(t,x) = 0\}$. Due to this reason the *qualitative behavior* of solutions of (P) may be very different (according the assumptions on the data $b, \mathbf{A}(x,u,\nabla u)$ and g) to the one of the solution of the linear heat equation. In fact, to show such kind of differences is one of the main goals of these notes.

We also mention that another relevant choice of nonlinear terms $\mathbf{A}(x, u, \nabla u)$ arises in the study of *transient minimal surfaces*, in which case the second order diffusion operator is given by

$$\text{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right).$$

Concerning the *transport* or *convection* terms, we mention that they arise very often in *Fluid Mechanics*. Usually they appear formulated in terms of an *additive term*, as, for instance, in the case of the temperature in a fluid

$$\underbrace{-\Delta \beta(u)}_{\text{diffusion}} + \underbrace{\mathbf{w} \cdot \nabla u}_{\text{convection}}.$$

If the fluid is *incompressible* (case of *liquids*) then $\text{div}\,\mathbf{w} = 0$ and so we get

$$-\text{div}\left(k(u)\nabla u - u\mathbf{w}\right), \text{ i.e., } \mathbf{A}(x,u,\xi) = k(u)\xi + u\mathbf{w}.$$

Nevertheless, sometimes the convection term is not an additive term but appears in a different form.

$$\text{div}\left(\Phi(\nabla u + K(b(u)\mathbf{e}))\right)$$

where

$$\Phi(\xi) = |\xi|^{p-2}\,\xi, \mathbf{e} \in \mathbb{R}^N \text{ and } K \in C^1(\mathbb{R} : \mathbb{R}).$$

This situation arises, for instance, in the study of *turbulent flow of a fluid through a porous medium* (with $\mathbf{e}$ the vector indicating the *main filtration direction*). For a general exposition on different examples of diffusion-convection

·erators, containing many other references see Díaz [20] and Díaz and de
ιelin [25].

The expression (III) represents the *absorption/forcing* term. The presence
the term $g(x, u) - f(t, x)$ is very typical of many problems arising in reaction-
fusion problems in Biology, Chemistry and other contexts. By writing

$$g(x, u) = g_1(x, u) - g_2(x, u),$$

th g_1 and g_2 nondecreasing functions, we can distinguish the term of *absorp-
n* $g_1(x, u)$ (which contributes to make $|u|$ smaller than if $g_1 = 0$) from the
₂ of *forcing* $g_2(x, u)$ (which contributes to make $|u|$ bigger than if $g_2 = 0$).
In most of the cases

$$g_1(x, u) = \lambda |u|^{q-1} u, \qquad \lambda > 0,$$

h $q > 0$ (the *order of the reaction*). Notice that if $0 < q < 1$, g_1 is *not* a
ıschitz function.

Returning to the *structural assumptions* on the data, in the rest of the
ɔosition, we shall always assume that

$$b : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing, } b(0) = 0, \tag{3}$$

$$\left\{ \begin{array}{c} \mathbf{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \text{ is a Caratheodory function} \\ \text{(i.e., measurable in } x \text{ and continuous in } (u, \xi)), \\ \exists p > 1 \text{ such that } |\mathbf{A}(x, u, \xi)| \leq C(|u|^{\frac{p^*}{p'}} + |\xi|^{p-1}), \quad \forall u \in \mathbb{R}, \\ \forall \xi \in \mathbb{R}^N \text{ with } p' = \frac{p}{p-1}, \quad p^* = \frac{Np}{N-p} \text{ and} \\ (\mathbf{A}(x, u, \xi) - \mathbf{A}(x, u, \xi^*)) \cdot (\xi - \xi^*) > 0, \forall \xi, \xi^* \in \mathbb{R}^N, \xi \neq \xi^*, \end{array} \right. \tag{4}$$

$$\left\{ \begin{array}{c} g \text{ is Caratheodory function and} \\ |g(x, u)| \leq \gamma(|u|)(1 + d(x)), \ d \in L^1(\Omega) \text{ and } \gamma \text{ strictly increasing,} \end{array} \right. \tag{5}$$

$$= f_1 + f_2, \ f_1 \in L^{p'}(0, T : W^{-1,p'}(\Omega)), \ f_2 \in L^1((0, T) \times \Omega), \forall T > 0, \tag{6}$$

$$h \in L^p(0, T : W^{1,p}(\Omega)) \cap L^\infty((0, T) \times \Omega), \forall T > 0, \tag{7}$$

$$u_0 \in L^\infty(\Omega).$$

₮or the sake of simplicity in the exposition, we shall deal merely with
nded *(weak) solutions*

DEFINITION 1. We say that u is a *bounded weak solution* of (P) if $u - h$
$L^p(0, T : W^{1,p}(\Omega)) \cap L^\infty((0, T) \times \Omega)$, $\forall T > 0$, and we have:

$$(i) \begin{cases} b(u)_t \in L^{p'}(0, T : W^{-1,p'}(\Omega)) \text{ and} \\ \int_0^T \langle b(u)_t, v \rangle_{W^{-1,p'} \times W_0^{1,p}} dt + \int_0^T \int_\Omega (b(u) - b(u_0)) v_t dx dt = 0 \\ \forall v \in L^p(0, T: W_0^{1,p}(\Omega)) \cap W^{1,1}(0, T : L^1(\Omega)) \text{ with } v(T, \cdot) \equiv 0, \end{cases}$$

and

$$(ii) \begin{cases} \int_0^T \langle b(u)_t, v \rangle dt + \int_0^T \int_\Omega \mathbf{A}(x, u, \nabla u) \cdot \nabla v dx dt + \int_0^T \int_\Omega g(x, u) v dx dt \\ = \int_0^T \langle f_1, v \rangle dt + \int_0^T \int_\Omega f_2 v dx dt, \\ \forall v \in L^p(0, T: W_0^{1,p}(\Omega)) \cap L^\infty((0, T) \times \Omega), \quad \forall T > 0. \end{cases}$$

The above definition is adapted from Alt and Luckhaus [2].

In the rest of this exposition we shall consider different qualitative proj
erties of solutions of (P) arising according the nature of the nonlinear term
$b(u)$, $\mathbf{A}(x, u, \nabla u)$ and $g(x, u)$. Our plan is the following: Section 2 will be d
voted to two *comparison principles* which will be important tools in our stud
Two qualitative properties are presented in the rest of the exposition: *the i
nite extinction time property* (Section 3) and *the finite speed of propagatio
property* (Section 4). In both of the above sections we shall apply the tw
comparison principles as well as some *energy methods*.

It is clear that the above presentation is far to be exhaustive. Problem
like (P) have attracted the attention of many specialists in the last forty yea
(perhaps the earliest mathematical paper on this subject was [38]). In con
sequence, many other very interesting qualitative properties are today avai
able in the literature. The present notes only pretend to be an elementar
introduction.

2. TWO USEFUL TOOLS

2.1. INTRODUCTION. The study of several qualitative properties for solu
tions of model problem (P) will be carried out thanks to some useful tool: th
comparison principles.

The most popular comparison principle has a pointwise nature and usuall
holds for elliptic and parabolic second order equations (as well as for first orde
hyperbolic equations). A first statement of such a principle is the following:

THEOREM 1. (Pointwise comparison principle) *Let* (f, h, u_0) *and* $(\widehat{f}, \widehat{h}, \widehat{u}_0)$
two set of ordered data, i.e., such that

$$f \leq \widehat{f}, \quad h \leq \widehat{h} \quad and \quad u_0 \leq \widehat{u}_0,$$

their respective domains of definition. Let u *and* $\widehat{u}$ *be (any) solutions of*
) corresponding to (f, h, u_0) *and* $(\widehat{f}, \widehat{h}, \widehat{u}_0)$ *respectively. Then*

$$u(t, x) \leq \widehat{u}(t, x), \quad for \ any \ t > 0 \ and \ a.e. \ x \in \Omega.$$

In the case of *linear* problems, this property is a trivial consequence of the
ximum principle (in fact, it suffices to assume $(\widehat{f}, \widehat{h}, \widehat{u}_0) \equiv (0, 0, 0)$ and so
$\ni 0$). The first (general) result for linear equations seems to be due to Paraf
1892 (later generalizations where due to Picard, Lichtenstein and, finally,
of (in 1927) (see details in the book Gilbarg and Trudinger [30]).
It is clear that for the *nonlinear case* some conditions on b, **A** and g are
ded (notice that the pointwise comparison principle implies the uniqueness
olutions). This topic is still under investigation (see the series of works by
Benilan, J. Carrillo and others). Here we shall *recall* a particular result
a *short* proof) stated in terms of an estimate for a suitable expression.
The second tool refers to another comparison principle, but this time, of a
erent nature. We can call it as the *symmetrized mass comparison principle.*
process of *symmetrization* need to be carefully presented. We start by the
metrization of the domain Ω: Given Ω, an open bounded set of $\mathbb{R}^N$, the
metrized version of Ω is the ball centered at the origin having the same
sure than Ω. Let us call Ω^* to this ball. The condition $m(\Omega) = m(\Omega^*)$ has
lation with the *isoperimetric inequality*

$$L \geq N\omega_N^{\frac{1}{N}} A^{\frac{N-1}{N}} \tag{8}$$

re L is the *lenght* of $\partial\Omega$ (or $m(\partial\Omega)$), A is the *area* of Ω (or $m(\Omega)$) and

ω_N is the area of the unit ball of $\mathbb{R}^N$(i.e., $\omega_N = m(S^{n-1})$).

3) the equality holds if and only if Ω is a ball. This was a first noted by
) de Cartago (850 B.C.) (*in* $\mathbb{R}^2$ *the circles are the domains with fixed area*
ng a longer perimeter). Rigorous proofs of (8) are due to Steiner (1882),
varz (1890) and Schmidt (1939).
The second step of the process of symmetrization consists in the
metrization of data f and u_0. We shall use the notion of the *decreas-*
symmetric rearrangement of a function introduced by H.A. Schwarz in

1890: Given a function $h : \Omega \to \mathbb{R}$, $h \in L^1(\Omega)$, we define the *decreasi:
symmetric rearrangement of* h, h^*, as the (unique) function $h^* : \Omega^* \to$
such that h^* is symmetric (i.e., $h^*(x) = h^*(\widehat{x})$ if $|x| = |\widehat{x}|$), h^* decreas
if $|x|$ decreases and the level sets of h and h^* are equimeasurables (i.
$m(\{x \in \Omega : h(x) > \theta\}) = m(\{x \in \Omega^* : h^*(x) > \theta\}), \forall \theta \in \mathbb{R}$. A mo
systematic definition of h^* can be introduced as follows: we first define t
distribution function of h by

$$\mu : \mathbb{R} \to \mathbb{R}, \ \mu(\theta) := m\{x \in \Omega : h(x) > \theta\}.$$

Then we define the *scalar decreasing rearrangement of* h by

$$\widetilde{h} : (0, m(\Omega)] \to \mathbb{R}, \ \widetilde{h}(s) := \inf\{\theta \in \mathbb{R} : \mu(\theta) \leq s\}$$

(notice that $\widetilde{h}(s) \sim \mu^{-1}(s)$). Finally, we define the *symmetric decreasing*
arrangement of h, by

$$h^* : \Omega^* \to \mathbb{R}, \ h^*(x) := \widetilde{h}(\omega_N |x|^N).$$

Notice that, since h^* is symmetric, we can write $h^*(x) = H(|x|)$ with $H : \mathbb{R}$ ·
$\mathbb{R}$. Nevertheless $H \neq \widetilde{h}$ since $H(r) = \widetilde{h}(\omega_N r^N)$. Notice, also, that assum(
$h \geq 0$, by construction, we have that

$$h \in L^1(\Omega) \text{ implies that } h^* \in L^1(\Omega^*) \text{ and}$$

$$\int_\Omega h(x)dx = \int_{\Omega^*} h^*(x)dx \text{ (the Cavalieri Principle)}$$

and that

$$h \in L^\infty(\Omega) \text{ implies that } h^* \in L^\infty(\Omega^*) \text{ and}$$

$$\operatorname{ess\,sup}_{x \in \Omega} h(x) = \operatorname{ess\,sup}_{x \in \Omega^*} h^*(x).$$

The third step of the process is the *symmetrization of the second ord
operator.* We must replace the diffusion operator div $\mathbf{A}(x, u, \nabla u)$ by anoth
isotropic diffusion operator, i.e., with the same behavior in any direction ʔ
Several possibilities arise. Here we shall consider, merely, a special case. A
sume that condition (4) holds and that, *in addition,*

$$\mathbf{A}(x, u, \xi) \cdot \xi \geq |\xi|^p \quad \forall \xi \in \mathbb{R}^N.$$

en we shall define as *symmetrized operator of* div $\mathbf{A}(x, u, \nabla u)$ the one given

$$\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$$

tice than if we take $\mathbf{A}^*(x, u, \xi) = |\xi|^{p-2}$ then condition (4) holds with the iality sign instead the inequality one).

We also must introduce an *isotropic absorption* by assuming (besides (5)) condition

$$\begin{cases} g(x, u)u \geq \widehat{g}(u)u \quad \text{a.e.} \quad x \in \Omega, \\ \text{for some continuous function} \quad \widehat{g} : \mathbb{R} \to \mathbb{R}. \end{cases} \tag{9}$$

Summarizing, we say that the *symmetrized problem of* (P) is the following :

PROBLEM. (P^*): Find $U : [0, \infty) \times \Omega^* \to \mathbb{R}$ such that

$$(P^*) \begin{cases} b(U)_t - \Delta_p U + \widehat{g}(U) = f^*(t, x), & t > 0, \quad x \in \Omega^*, \\ U = h^*, & t > 0, \quad x \in \partial\Omega^*, \\ b\left(U(0, x)\right) = b\left(u_0^*(x)\right), & x \in \Omega^*. \end{cases}$$

e $f^*(t, \cdot)$ and $u_0^*(\cdot)$ are the decreasing symmetric rearrangements of $f(t, \cdot)$ u_0, respectively. For the sake of simplicity in the exposition we shall ıme *now* that

$$h = h^* = 0. \tag{10}$$

Let us make some remarks on the statement of the *symmetrized mass comison principle*. The first one is that some pioneer authors finding different tions between u and U where Saint-Venant (1856), Poya and Szego (1951) Weimberger (1962). The inequality

$$u^*(x) \leq U(x) , \ x \in \Omega^*, \tag{11}$$

first proved by G. Talenti, in 1976, for the case of the stationary problem ıout absorption term (i.e., $b \equiv 0$ and $g \equiv 0$). Unfortunately, this (point-:) comparison *fails* to be true for parabolic problems (i.e., $b \neq 0$) or/and ɔroblems in presence of absorption terms ($g \neq 0$). In those cases we only compare the *distribution of the mass* of u and U

ΓHEOREM 2. (Symmetrized Mass Comparison Principle (SMCP))

$$\int_{B(0,r)} u^*(t, x)dx \leq \int_{B(0,r)} U(t, x)dx, \forall t > 0, \forall r \in [0, R],$$

med that $\Omega^* = B(0, R)$.

Notice that this comparison can be, equivalently, expressed in terms scalar decreasing rearrangement as

$$\int_0^s \widetilde{u}(t,\sigma)d\sigma \leq \int_0^s \widetilde{U}(t,\sigma)d\sigma, \ \forall t > 0, \ \forall s \in [0, m(\Omega)].$$

The SMCP has many applications (as we shall see in other sections). The ma philosophy of the applications is that function U can be easily estimated many cases and thus, thanks to the SMCP, properties for U can be extended similar properties for u. Some books dealing with the symmetrization proce are the ones by Bandle [6], Mossino [35] and Kawohl [33]. The proof we shi present here follows the memoir Díaz [21] (see also Díaz [22]). A different (ai very original) approach is due to Abourjail and Benilan [1]. The first resu in the literature for degenerate parabolic problems was Vázquez [41].

2.2. Proof of the two comparison principles.

On the pointwise comparison principle. We present here a pa ticular version of this principle (more general results will be indicated late for the special case of the diffusion-convection operator arising in the stuc of turbulent flow of a fluid through a porous medium. More precisely, v consider the problem

$$(P_{\phi,K}) \begin{cases} b(u)_t - \text{div} \ (\phi(\nabla u + \mathbf{e}K(b(u)))) + g(x,u) = f(x,t), & t > 0, x \in \Omega, \\ u = h, & t > 0, x \in \partial\Omega \\ b\left(u(0,x)\right) = b(u_0(x)) & x \in \Omega, \end{cases}$$

where $\phi(\xi) = |\xi|^{p-2}\xi$, $p > 1$, $\mathbf{e} \in \mathbb{R}^N$ and $K \in C^0(\mathbb{R}, \mathbb{R})$. Besides the conc tions made explicit in Section 1 we shall made some extra assumptions:

$$(H_{g,b}) \begin{cases} \text{there exists} \ \ C^* \geq 0 \ \ \text{such that} \\ g(\cdot, \eta) - g(\cdot, \widehat{\eta}) \geq -C^* \left(b(\eta) - b(\widehat{\eta})\right), \ \ \ \forall \eta > \widehat{\eta}, \ \eta, \widehat{\eta} \in \mathbb{R}, \end{cases}$$

(notice that $(H_{g,b})$ trivially holds if, for instance, $g(\cdot, \eta)$ is nondecreasing in or if $g(\cdot, \eta) := \widetilde{g}(\cdot, b(\eta))$ with $\widetilde{g}(\cdot, s)$ Lipschitz continuous in s),

$$(H_K) \begin{cases} K(b(\eta)) \text{ is Hölder continuous in } \eta \text{ of exponent } \gamma \geq \frac{1}{p} \text{ if } 1 < p < 2 \\ \text{and } \gamma \geq \frac{1}{p'} \ \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ if } p \geq 2 \ , \\ |K(b(\eta)) - K(b(\widehat{\eta}))| \leq C|\eta - \widehat{\eta}|^\gamma, \ \ \ \forall \eta, \widehat{\eta} \in \mathbb{R}, \end{cases}$$

(notice that condition (4) is now trivially satisfied).

THEOREM 3. *Let* $(f, h, u_0), (\widehat{f}, \widehat{h}, \widehat{u}_0)$ *be such that* $f \leq \widehat{f}, h \leq \widehat{h}$ *and* $u_0 \leq$
on their respective domains. Let $u, \widehat{u}$ *be two bounded weak solutions of*
$_{b,K})$ *associated to* (f, h, u_0) *and* $(\widehat{f}, \widehat{h}, \widehat{u}_0)$, *respectively. Assume, in addition,*
it u *and* $\widehat{u}$ *are strong solutions, i.e.,*

$$b(u)_t, b(\widehat{u})_t \in L^1\left((0, T) \times \Omega\right), \quad \forall T > 0. \tag{12}$$

en $u \leq \widehat{u}$ *on* $(0, T) \times \Omega$. *More in general, if we replace the ordered data*
umption by the simpler condition $h \leq \widehat{h}$ *and* $f_1 \leq \widehat{f}_1$ *then*

$$[b(u(t, \cdot)) - b(\widehat{u}(t, \cdot))]_+ \|_{L^1(\Omega)} \leq e^{C^* t} \|[b(u_0) - b(\widehat{u}_0)]_+\|_{L^1(\Omega)}$$
$$+ \int_0^t e^{C^* \tau} \|[f_2(\tau, \cdot) - \widehat{f}_2(\tau, \cdot)]_+\|_{L^1(\Omega)} d\tau \tag{13}$$

any $t > 0$ $(C^*$ *given in* $(H_{g,b}))$, *where* $\varphi_+ = \max(\varphi, 0)$.

Proof. We take as test function the following approximation of the $sign_0^+ (u$
$)$ function: we start by defining $\Psi_\delta(\eta) := \min(1, \max(0, \frac{\eta}{\delta}))$, for $\delta > 0$
ll. Then we define $v = \Psi_\delta(u - \widehat{u})$. Notice that $v \in L^p(0, T : W_0^{1,p}(\Omega)) \cap$
$((0, T) \times \Omega), \forall T > 0$, and that

$$\nabla v = \begin{cases} \nabla \frac{1}{\delta}(u - \widehat{u}) & \text{if } 0 < u - \widehat{u} < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

n, since $f_1 \leq f_2$, defining the set

$$A_\delta := \{(t, x) \in (0, T) \times \Omega : 0 < u(t, x) - \widehat{u}(t, x) < \delta\}$$

ζet

$$\int_0^T \int_\Omega (b(u)_t - b(\widehat{u})_t) \Psi_\delta(u - \widehat{u}) dx dt + I_1(\delta) + I_2(\delta)$$
$$+ \int_0^T \int_\Omega (g(x, u) - g(x, \widehat{u})) \Psi_\delta(u - \widehat{u}) dx dt$$
$$\leq \int_0^T \int_\Omega \left(f_2 - \widehat{f}_2\right) \Psi_\delta(u - \widehat{u}) dx dt,$$

·e

$$I_1(\delta) = \frac{1}{\delta} \int_0^T \int_{A_\delta} \{\phi\left(\nabla u + K(b(u))\mathbf{e}\right) - \phi\left(\nabla \widehat{u} + K(b(\widehat{u}))\mathbf{e}\right)\} \cdot$$
$$\cdot\{\nabla u + K(b(u))\mathbf{e} - \nabla \widehat{u} - K(b(\widehat{u}))\mathbf{e}\} dx dt,$$

$$I_2(\delta) = \frac{1}{\delta}\int_0^T \int_{A_\delta} \{\phi\left(\nabla u + K(b(u))\mathbf{e}\right) - \phi\left(\nabla\widehat{u} + K(b(\widehat{u}))\mathbf{e}\right)\} \cdot$$
$$\cdot\{-K(b(u))\mathbf{e} + K(b(\widehat{u}))\mathbf{e}\}\, dxdt$$

(here T is arbitrary but fixed, $T > 0$). Applying the Young inequality, $\alpha\beta$ $C(\epsilon)p^{-1}\alpha^p + \epsilon p'^{-1}\beta^{p'}$, we see that

$$|I_2(\delta)| \le \frac{\epsilon}{\delta p'}\int_0^T \int_{A_\delta} |\phi\left(\nabla u + K(b(u))\mathbf{e}\right) - \phi\left(\nabla\widehat{u} + K(b(\widehat{u}))\mathbf{e}\right)|^{p'}\, dxdt$$

$$+\frac{C(\epsilon)}{\delta p}\int_0^T \int_{A_\delta} |K(b(u)) - K(b(\widehat{u}))^p|\, dxdt := I_2^a + I_2^b.$$

We shall only consider the case of $p \in (1,2)$ (the case $p > 2$ is similar a and even, easier). We need an algebraic inequality

LEMMA 1. (see, e.g., Díaz and de Thelin [25]) *Let* $\phi(\xi) := |\xi|^{p-2}\xi$ *w* $p > 1$. *Then, there exists* $C > 0$ *such that*

$$C\left|\phi(\xi) - \phi(\widehat{\xi})\right|^{p'} \le \left\{(\phi(\xi) - \phi(\widehat{\xi}))\cdot(\xi - \widehat{\xi})\right\}^{\frac{\alpha}{2}}\left\{|\phi(\xi)|^{p'} + |\phi(\widehat{\xi})|^{p'}\right\}^{1-\frac{\alpha}{2}}$$

with $\alpha = 2$ *if* $1 < p < 2$ *and* $\alpha = p'$ *if* $p \ge 2$.

Using Lemma 1 we obtain that

$$|I_2^a| \le \epsilon\widetilde{C}I_1(\delta),$$

for some $\widetilde{C}$ independent of δ. Moreover

$$I_2^b \le \frac{C(\epsilon)}{\delta p}\int_{A_\delta} (C|u - \widehat{u}|)^p dxdt \le \widetilde{C}(\epsilon)m(A_\delta)\delta^{\delta p-1}$$

for some $\widetilde{C}(\epsilon) > 0$ independent of δ. Then

$$I_1(\delta) + I_2(\delta) \ge I_1(\delta) - |I_2(\delta)| \ge (1 - \epsilon\widetilde{C})I_1(\delta) - \widetilde{C}(\epsilon)m(A_\delta)\delta^{\delta p-1}.$$

Taking ϵ small enough (so that $1 - \epsilon\widetilde{C} > 0$) and using that $I_1(\delta) \ge 0$ we ha that

$$\lim_{\delta\downarrow 0}(I_1(\delta) + I_2(\delta)) \ge 0$$

and so

$$\int_{u>\widehat{u}} (b(u) - b(\widehat{u}))_t dxdt + \int_{u>\widehat{u}} (g(x,u) - g(x,\widehat{u})) \le 0.$$

om assumption $(H_{g,b})$ we deduce that

$$\int_{u>\widehat{u}} (b(u) - b(\widehat{u}))_t dxdt \leq \int_{u>\widehat{u}} (b(u) - b(\widehat{u})) dxdt,$$

that

$$\int_0^T \int_\Omega \max\{b(u) - b(\widehat{u}), 0\}_t dxdt \leq \int_0^T \int_\Omega \max\{(b(u) - b(\widehat{u})), 0\} dxdt,$$

d, finally

$$\int_\Omega \max\{b(u(T, x)) - b(\widehat{u}(T, x)), 0\} dxdt \leq \int_0^T \int_\Omega \max\{(b(u) - b(\widehat{u})), 0\} dxdt.$$

.en, by Gronwall inequality

$$b(u) \leq b(\widehat{u}) \qquad \text{a.e} \quad (t, x) \in (0, T) \times \Omega.$$

꜀ is strictly increasing this implies that $u \leq \widehat{u}$ and the proof of the first ıclusion ends. In the general case (i.e., when b is merely nondecreasing) it ıains the consideration of the case in which $A_\delta \subset \{b(u) = b(\widehat{u})\}$, for any δ all, (since otherwise the above arguments apply). In that case $I_2(\delta) \equiv 0$ ꝏlies that $I_1(\delta) \equiv 0$. But from Lemma 1

$$I_1(\delta) \geq C\delta \int_0^T \int_\Omega \frac{|\nabla \Psi_\delta(u - \widehat{u})|^2 \quad dxdt}{\{|\nabla u + K(b(u))\mathbf{e}|^p + |\nabla \widehat{u} + K(b(\widehat{u}))\mathbf{e}|^p\}^{\frac{2-p}{p}}} \geq 0.$$

$\Psi(u - \widehat{u}) = 0$ a.e. on $(0, T) \times \Omega$ which implies that $u \leq \widehat{u}$ on this set. The of of the case $p > 2$ and inequality (13) follows the same type of arguments.

Remark 1. It can be proved (see Díaz and de Thelin [25]) that if b is a ꞩchitz *function* and u_0 is regular enough then any bounded weak solution strong solution (i.e., $b(u)_t \in L^1(Q_T)$, $Q_T := (0, T) \times \Omega$). The proof of the tence of strong solutions under more general conditions on b is a delicate ꞇ (see the *recent* results by Benilan and Gariepy [13]).

Remark 2. The (pointwise) comparison principle can be obtained for ker solutions by using more complicated arguments and other selected ons of solutions (*entropy solutions, renormalized solutions, good solu-*s,...). See the works by Benilan and Touré, Benilan and Wittbold, Carrillo, ꝺ,...

Remark 3. The *quantitative* inequality (13) is a typical consequence
the application of abstract results (*the T-accretiveness of the operator*). A
illustration of how this theory can be applied to the concrete case of proble
$(P_{\phi,K})$ (when $h \equiv 0$) is due to Bouhsiss [16]

ON THE SYMMETRIZED MASS COMPARISON PRINCIPLE.   We recall th
this time we assume the additional conditions

$$\mathbf{A}(x, u, \xi) \cdot \xi \geq |\xi|^p, \tag{1}$$

$$g(x, u)u \geq \widehat{g}(u)u \quad \text{for some} \quad \widehat{g} \in C(\mathbb{R} : \mathbb{R}), \tag{1}$$

and, for simplicity, (10). Here we also assume that

$$f = f_2 \in L^1_{loc}(0, \infty : L^1(\Omega)).$$

We shall only consider (for simplicity) the case in which u and U are nonne
ative functions.

THEOREM 4. *Assume that $\widehat{g}$ is nondecreasing or locally Lipschitz and th
the function*

$$\varphi(\eta) := \widehat{g}(b^{-1}(\eta))$$

is well defined and can be decomposed as

$$\varphi = \varphi_1 + \varphi_2 \tag{1}$$

with φ_1 convex and φ_2 concave. Then

$$\int_0^s b(\widetilde{u}(t, \sigma))d\sigma \leq \int_0^s b(\widetilde{U}(t, \sigma))d\sigma \quad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty). \tag{1}$$

Idea of the proof. First of all we point out that conclusion (17) is stat
by approximations of the data $(f, u_0, b \text{ and } \mathbf{A})$ leading to the convergence
solutions in $L^1(0, T : L^1(\Omega))$. Due to that, we can assume the data regul
enough (and, in particular, that u and U are strong solutions $b(u)_t \in L^1(Q_T$
$b(U)_t \in L^1(Q_T^*), Q_T^* := (0, T) \times \Omega$ and that b is strictly increasing.
    Step 1. The radially symmetric problem. We define

$$K(t, s) = \int_0^s b(\widetilde{U}(t, \sigma))d\sigma$$

here $\widetilde{U}(t,\cdot)$ is the scalar decreasing rearrangement of $U(t,\cdot)$. First of all, let
; prove that $U(t,x)$ decreases when $|x|$ increases. By the symmetry of the
ita (and the uniqueness of solutions, implicitly assumed) we deduce that
$(t,x) = U(t,|x|)$. Moreover $U_r := \frac{\partial}{\partial r}U(t,r)$, $r = |x|$ verifies that

$$
\begin{cases}
\frac{\partial}{\partial t}\left(b'(U)U_r\right) - \frac{\partial^2}{\partial r^2}\left(|U_r|^{p-2}U_r\right) + \widehat{g}'(U)U_r = F_r & \text{in } (0,T) \times (0,R), \\
U_r(t,0) = 0, \quad U_r(t,R) \leq 0, & t \in (0,T), \\
U_r(0,r) = U_{0,r}(r) & r \in (0,R),
\end{cases}
$$

iere $\Omega^* = B(0,R)$, $U_0(r) = \widetilde{u}_0(\omega_N r^N)$ and $F(t,r) = \widetilde{f}(t,\omega_N r^N)$. Then by
e maximum principle (here is possible to apply classical results since U_r can
assumed to be smooth), as $F_r(t,\cdot) \leq 0$ and $U_{0,r}(\cdot) \leq 0$, we deduce that
$(t,\cdot) \leq 0$, i.e., $U(t,r)$ decreases when r increases. In consequence, $U(t,\cdot) =$
$'(t,\cdot)$ (the function coincides with its decreasing symmetric rearrangement),
d so
$$
U(t,x) = \widetilde{U}(t,\omega_N r^N), \quad r = |x|.
$$
aking
$$
s = \omega_N r^N \qquad (s \in (0,m(\Omega)))
$$
get that
$$
\frac{\partial K}{\partial s}(t,s) = b(\widetilde{U}(t,s)), \qquad \frac{\partial U}{\partial r} = N\omega_N^{\frac{1}{N}}s^{\frac{N-1}{N}}\frac{\partial \widetilde{U}}{\partial s}.
$$
: deduce that K satisfies the parabolic (fully non-linear) problem
$$
(N^*)\begin{cases}
\frac{\partial K}{\partial t} - a(s)\left|\frac{\partial}{\partial s}b^{-1}(\frac{\partial K}{\partial s})\right|^{p-2}\frac{\partial}{\partial s}b^{-1}(\frac{\partial K}{\partial s}) + \\
\displaystyle\int_0^s \widehat{g}\left(b^{-1}\left(\frac{\partial K}{\partial s}(t,\sigma)\right)\right)d\sigma = \int_0^s \widetilde{f}(t,\sigma)d\sigma, & \begin{array}{l} s \in (0,m(\Omega)) \\ t \in (0,T), \end{array} \\
K(t,0) = 0, \quad K(t,m(\Omega)) = 0, & t \in (0,T), \\
\displaystyle K(0,s) = \int_0^s b(\widetilde{u}_0(\sigma))d\sigma & s \in (0,m(\Omega)),
\end{cases}
$$
ere
$$
a(s) := \left[N\omega_N^{1/N}s^{(n-1)/n}\right]^p.
$$

Step 2. Study of the rearrangement of u. Given $\widetilde{u}(t,\cdot)$ (the scalar decreasing
rrangement of the solution u of (P)), we define
$$
k(t,s) = \int_0^s b(\widetilde{u}(t,\sigma))d\sigma.
$$

The main goal of this second step is to prove that $k(t,s)$ is a subsolution (
(FN^*) in the sense that it verifies all the conditions but replacing the full
nonlinear equation by the inequality

$$
\frac{\partial k}{\partial t} - a(s)\left|\frac{\partial}{\partial s}b^{-1}\left(\frac{\partial k}{\partial s}\right)\right|^{p-2}\frac{\partial}{\partial s}b^{-1}\left(\frac{\partial k}{\partial s}\right)
$$
$$
+ \int_0^s \widehat{g}\left(b^{-1}\left(\frac{\partial k}{\partial s}(t,\sigma)\right)\right)d\sigma \le \int_0^s \widetilde{f}(t,\sigma)d\sigma, \tag{1ξ}
$$

$s \in (0, m(\Omega)), t \in (0,T)$. The proof of this inequality is quite long and tecl
nical. This process can be also divided in several steps:

(i) Define the function $T_{\tau,h} : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$
\begin{aligned}
T_{\tau,h}(s) &= 0 && \text{if } 0 \le s \le t, \\
T_{\tau,h}(s) &= s - t && \text{if } t < s \le t + h, \\
T_{\tau,h}(s) &= h && \text{if } s > t + h.
\end{aligned}
$$

We take $v = T_{\tau,h}(u)$, as test function. Passing to the limit, as $h \downarrow 0$, we deduc
that

$$
-\frac{\partial}{\partial\theta}\int_{u>\theta}|\nabla u|^p dx \le \int_0^{\mu(\theta)}\widetilde{f}(t,s)ds - \int_0^{\mu(\theta)}\widehat{g}(\widetilde{u}(t,s))ds - \int_{u>\theta}\frac{\partial b(u)}{\partial t}dx
$$

where we used the assumptions (14) and (9) and where $\mu(\theta)$ denotes tl
distribution function of $u(t,\cdot)$.

(ii) We have that

$$
N\omega_N^{1/N}\mu(\theta)^{(N-1)/N} \le \left(-\mu'(\theta)\right)^{1/p'}\left(-\frac{\partial}{\partial\theta}\int_{u>\theta}|\nabla u|^p dx\right)^{1/p}
$$

(this is a classical result in the rearrangement theory: the proof uses the, s
called, *Fleming-Rishel formula*, the *isoperimetric inequality* and the notion (
perimeter in the Giorgi sense).

(iii) the following identity holds

$$
\int_{u>\theta}\frac{\partial b(u)}{\partial t}dx = \int_0^{\mu(\theta)}\frac{\partial b(\widetilde{u}(t,\sigma))}{\partial t}d\sigma = \frac{\partial k}{\partial t}(t,\mu(\theta))
$$

(although a first proof of this formula already appears in the book by Band
[6] a more general, and rigorous, proof is due to Mossino and Rakotoson [36]
An easy manipulation of (i), (ii), (iii) leads to the wanted inequality for k.

Step 3. Comparison using the fully nonlinear equation. First of all, notice at the comparison

$$k(t, s) \leq K(t, s) \qquad \forall t \in [0, T], \quad \forall s \in (0, m(\Omega)),$$

incides with the conclusion of the theorem. The main difficulty now is not sociated to the very *complicated* diffusion operator but with the *nonlocal* ture of the zero order perturbation term. The key idea to obtain the result that, by assumption (16),

$$\varphi(r) - \varphi(\widehat{r}) \leq \left(\varphi_1'(r) + \varphi_2'(\widehat{r}) \right) (r - r) \qquad \forall r, \widehat{r} \in \mathbb{R}$$

se for instance, Taylor formula, the convexity of φ_1 and the concavity of). Then

$$
\begin{aligned}
\int_0^s \left[\widehat{g}(\widetilde{U}(t, \sigma)) - \widehat{g}(\widetilde{u}(t, \sigma)) \right] d\sigma \quad &\leq \quad \int_0^s \left[\varphi_1'(b(\widetilde{U}(t, \sigma))) + \varphi_2'(b(\widetilde{u}(t, \sigma))) \right] \cdot \\
&\qquad \cdot \left[b(\widetilde{U}(t, \sigma)) + b(\widetilde{u}(t, \sigma)) \right] d\sigma \\
&\leq \quad C_1 \left| k(t, s) - K(t, s) \right| \\
&\qquad + C_2 \max_{\tau \in [0, T], \sigma \in [0, s]} \left| k(\tau, \sigma) - K(\tau, \sigma) \right|,
\end{aligned}
$$

some positive constants C_1 and C_2. The comparison is now a consequence of classical pointwise comparison principle also related to the T-accretiveness he *complicated* operator, but this time in the space $C^0(\overline{\Omega})$, (details can be nd in Díaz [21]: see also other references indicated at the Introduction of section).

Remark 4. Thanks to a result due to Hardy, Littlewood and Polya in 1929 :, e.g., [6]), the comparison

$$\int_0^s b(\widetilde{u}(t, \sigma)) \, d\sigma \leq \int_0^s b(\widetilde{U}(t, \sigma)) d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty),$$

lies that

$$\int_0^s \Phi\left((\widetilde{u}(t, \sigma))\right) d\sigma \leq \int_0^s (\Phi(b(\widetilde{U}(t, \sigma))) d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty)$$

any convex nondecreasing function Φ. In particular, if

$$b \text{ is a concave function}$$

we get

$$\int_0^s \widetilde{u}(t,\sigma)d\sigma \leq \int_0^s \widetilde{U}(t,\sigma)d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0, \infty),$$

which is the conclusion presented at the Introduction of this section. Notic that a different application of the above result by Hardy, Littlewood and Poly is that

$$||b(u(t,\cdot))||_{L^q(\Omega)} \leq ||b(U(t,\cdot))||_{L^q(\Omega^*)}$$

for any $q \in [1, \infty)$. Indeed, it suffices to use $\Phi(r) = |r|^q$ and that

$$\int_0^{m(\Omega)} |b(\widetilde{u}(t,\sigma))|^q d\sigma = \int_{\Omega^*} |b(u^*(t,x))|^q dx = \int_\Omega |b(u(t,x))|^q dx.$$

3. THE FINITE EXTINCTION TIME PROPERTY

3.1. INTRODUCTION.   One of the most natural questions concerning pro lem (P) is the *stabilization of solutions*: Assumed that

$$f(t,\cdot) \longrightarrow f_\infty(\cdot) \quad \text{and} \quad h(t,\cdot) \longrightarrow h_\infty(\cdot) \quad \text{as} \quad t \to +\infty$$

in *suitable* functional spaces then $u(t,\cdot) \longrightarrow u_\infty(\cdot)$ as $t \to +\infty$ (in some *sui* able sense) with $u_\infty(\cdot)$ solution of the associated stationary problem

$$(P_\infty) \begin{cases} -\text{div}\, \mathbf{A}(x, u_\infty, \nabla u_\infty) + g(x, u_\infty) = f_\infty(x), & x \in \Omega, \\ u_\infty = h_\infty, & \text{on } \partial\Omega. \end{cases}$$

A general result, stated in terms of the *omega limit set*

$$\omega(u) := \{u_\infty \in W^{1,p}(\Omega) : \exists\, t_n \to \infty \text{ such that}$$
$$u(t_n, \cdot) \to u_\infty \text{ in } L^p(\Omega), \text{ as } n \to \infty\}$$

jointly with stronger convergence results (but for different particular case can be found in Díaz and de Thelin [25]. For stronger convergence resul for one-dimensional particular equations see Feireisel and Simondon [28] ar their references.

Very often $f_\infty \equiv 0$, $h_\infty \equiv 0$ and $\mathbf{A}$ and g are such that $u_\infty \equiv 0$ is the uniqu solution to problem (P_∞). In several applications (case of models in plasn physics and also in some chemical reactions) it is observed that there is a vei strong stabilization in the following sense: *there exists a finite time* $T_0 >$ *such that* $u(t,x) \equiv 0$, $\forall\, t \geq T_0$ *and a.e.* $x \in \Omega$. This property is called ;

ιe *finite extinction time property* and has been considered by many authors
. the literature. The main goal of this section is to illustrate the application
˙ the above two comparison principles to the study of this property. A third
.ethod (using *energy arguments* and so applicable to higher order parabolic
·oblems and systems) will be also presented.

3.2. The finite extinction time via the pointwise comparison
.INCIPLE.   A first result proving the occurrence of this property for some
.ecial formulation of problem (P) is the following

THEOREM 5. *Let u satisfying*

$$
(P_{\alpha,p}) \begin{cases} \left(|u|^{\alpha-1} u\right)_t - \triangle_p u = 0, & t \in (0,\infty), x \in \Omega, \\ u = 0, & t \in (0,\infty), x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}
$$

.th

$$
u_0 \in C_c(\Omega), \ i.e., \ with \ \operatorname{supp} u_0 \ a \ compact \ subset \ of \ \Omega. \tag{19}
$$

·sume that

$$
(p-1) < \alpha. \tag{20}
$$

ιen *the finite extinction time property holds.*

Proof. We assume u in the class of solutions in which the pointwise com-
·ison principle holds (due to the special formulation of $(P_{\alpha,p})$ it can be
.wn (Benilan [11]) that this is our case for any $\alpha > 0$ and $p > 1$). Then if
.resp. $\underline{u}$) is a supersolution of problem $(P_{\alpha,p})$ (resp. subsolution) then

$$
\underline{u} \leq u \leq \overline{u}. \tag{21}
$$

if we are able to construct $\overline{u}$ (resp. $\underline{u}$) vanishing after a finite time this
·perty also holds for u. Inspired in a pioneering paper (Sabinina [39]) we
.ll construct $\overline{u}$ as a *separable* supersolution, i.e., $\overline{u}(t,x) = \Phi(t)w(x)$. Since
want to have $\Phi \geq 0$ and $w \geq 0$, we define

$$
N\overline{u} := \left(|\overline{u}|^{\alpha-1} \overline{u}\right)_t - \triangle_p \overline{u} = (\Phi^\alpha)_t w^\alpha - \Phi^{p-1} \triangle_p w.
$$

take Φ such that

$$
\begin{cases} (\Phi^\alpha)_t = -\lambda\Phi^{p-1}, \ t \in (0,\infty), \\ \Phi(0) = M, \end{cases} \tag{22}
$$

with $\lambda > 0$ and $M > 0$ to be determined. Due to the crucial assumption (2
the solution of (22) vanishes after a finite $T_\Phi > 0$ (notice that $\Psi := \Phi^\alpha$ verifi
an ODE with a term which is not Lipschitz $\Psi_t + \lambda\Psi^{\frac{p-1}{\alpha}} = 0$). Notice al
that (22) is integrable since it is a first order ordinary equation of separab
variables. Then

$$N\overline{u} = \Phi^{(p-1)}\left(-\lambda w^\alpha - \Delta_p w\right).$$

In consequence we choose, as w, the solution of the *first eigenvalue proble
for the* Δ_p operator, i.e., $\lambda = \lambda_1 > 0$ and

$$\begin{cases} -\Delta w = \lambda_1 w^{p-1} & \text{on} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

(the existence of a unique function w satisfying that $w > 0$ on Ω ar
$\|w\|_{L^\infty(\Omega)} = 1$ was due to Anane [3] and Barles [8]). Then

$$\begin{aligned} N\overline{u} &= \Phi^{p-1}\left(-\lambda_1 w^\alpha + \lambda_1 w^{p-1}\right) \\ &= \lambda_1\Phi^{p-1}w^{p-1}\left(1 - w^{\alpha-(p-1)}\right) \geq 0 \end{aligned}$$

since $0 \leq w \leq 1$ and $\alpha > (p-1)$.

The boundary condition holds

$$\overline{u}(t,x)\big|_{(0,\infty)\times\partial\Omega} = \Phi(t)w\big|_{\partial\Omega} = 0.$$

The comparison between the initial data

$$u_0(x) \leq Mw(x), \quad x \in \Omega$$

trivially holds by taking M big enough (recall the assumption (19) on u_0
The construction of $\underline{u} \leq 0$ is similar. ∎

Remark 5. The above statement can be improved in many different di
ections (but with longer proofs). For instance, in the case of $p = 2$ tl
homogeneity assumed on b is not needed. More precisely, in G. Díaz and J.
Díaz [18], the finite extinction time property was established for the proble

$$\begin{cases} b(u)_t - \Delta u = f(x,t), & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{2}$$

by assuming

$$\int_{0+} \frac{ds}{b^{-1}(s)} < +\infty \tag{2}$$

d the existence of T_f such that $f(t, x) \equiv 0$, for $t > T_f$ and $x \in \Omega$. Notice at now $p = 2$ and that if $b(s) = |s|^{\alpha-1}s$ then (25) if and only if $\alpha > 1$, ., the same condition than (20). In fact, in this paper it is also shown that ndition (25) is also necessary for the existence of a finite extinction time.

Remark 6. Notice that the *finite extinction time* can not be satisfied (in se of the general formulation of (P)) each time that the *strong maximum inciple* holds (see, e.g., Nirenberg [37]) or the *unique continuation property* verified (see, e.g., Ghidaglia [29] and its references).

When condition (20) holds, it is said that we have a *fast diffusion* (in t, this term is more appropriate when talking on the balance between the :umulation and the diffusion terms). It is very easy to see that if we assume ') then the conclusion of the above theorem remains true under the presence a nondecreasing absorption term as, for instance,

$$\left(|u|^{\alpha-1} u \right)_t - \triangle_p u + |u|^{q-1} u = 0$$

any $q > 0$. The finite extinction time property also occurs due to suitable ance between the accumulation and absorption terms. It is the so called ong *absorption case*.

THEOREM 6. *Let u satisfying*

$$(P_{\alpha,p,q}) \begin{cases} \left(|u|^{\alpha-1} u \right)_t - \triangle_p u + |u|^{q-1} u = 0, & t \in (0, \infty), \, x \in \Omega, \\ u = 0, & t \in (0, \infty), \, x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

h

$$u_0 \in L^\infty (\Omega). \tag{26}$$

ume

$$\mu > 0 \quad \text{and} \quad 0 < q < \alpha \quad \text{with } p > 1 \quad \text{arbitrary.} \tag{27}$$

n *the finite extinction time property holds.*

Proof. It is easy to see that the function $\overline{u}(x, t) = \Phi(t)$, with Φ the que) solution of the ODE

$$\begin{cases} (\Phi^\alpha)_t + \mu\Phi^q = 0, & t \in (0, \infty), \\ \Phi(0) = M, \end{cases} \tag{28}$$

1pare it with (22)) is a supersolution once that $M \geq \|u_0\|_{L^\infty(\Omega)}$. The mption (27) implies that Φ vanishes after some finite time T_Φ. ∎

Remark 7. A general survey containing many references on this proper
is due to Kalashnikov [32].

3.3. THE FINITE EXTINCTION TIME VIA THE MASS SYMMETRIZED CO[?]
PARISON PRINCIPLE.   Thanks to the *mass symmetrized comparison princip*
it is possible to extend the last two theorems to more general equations f
which the construction of super and subsolutions can be very difficult (sp
cially in the case of the first of the theorems).

THEOREM 7. *Let u be the solution of (P) with $f \equiv 0, h \equiv 0, u_0 \in C_c\,(\Omega$*
$u_0 \geq 0$ and assume $b(u) = |u|^{\alpha-1}\,u$, (14) and (9). We also suppose that o[?]
of the two following conditions holds:

$$\left\{ \begin{array}{l} (p-1) < \alpha \text{ and} \\ \varphi(\eta) := \widehat{g}(|\eta|^{\frac{1}{\alpha}-1}\eta) = \varphi_1\,(\eta) + \varphi_2\,(\eta)\,, \eta \in \mathbb{R} \\ \text{with } \varphi_1 \text{ (resp. } \varphi_2\text{) nondecreasing and convex} \\ \text{(resp. nondecreasing and concave),} \end{array} \right. \qquad (2[?]$$

or

$$\left\{ \begin{array}{l} \widehat{g}(\eta) = \mu\,|\eta|^{q-1}\,\eta \text{ with } \mu > 0 \text{ and} \\ \quad q < \alpha. \end{array} \right. \qquad (3[?]$$

Then the finite extinction time property is verified. More precisely, if we defi[?]
as $T_{0,\Omega}$ the first extinction time (in which $\|u\,(T_0, \cdot)\|_{L^1(\Omega)} \equiv 0$) then

$$T_{0,\Omega} \leq T_{0,\Omega^*},$$

where T_{0,Ω^*} is the first extinction time for the symmetrized problem (P^*).

Proof. By the mass symmetrized comparison principle and the result [?]
Hardy, Littlewood and Polya mentioned in the above Section we have that

$$\|b\,(u\,(t, \cdot))\|_{L^1(\Omega)} \leq \|b\,(U\,(t, \cdot))\|_{L^1(\Omega)}$$

for any $t > 0$. Assumption (29) (resp (30)) allows to apply Theorem 5 (re[?]
Theorem 8) which proves the result. ∎

Remark 8. Notice that the general structure of $\mathbf{A}(x, u, \xi)$ may be the orig[?]
of very complicated behaviors of the solution of the associated eigenvalu[?]
problem

$$\left\{ \begin{array}{ll} -\mathrm{div}\,\mathbf{A}\,(x, w, \nabla w) = \lambda w^{p-1} & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{array} \right.$$

So that the arguments of the proof of Theorem 5 do not apply directly [?]
problem (P).

3.4. THE FINITE EXTINCTION TIME VIA AN ENERGY METHOD.   A method
ιich do not use any comparison principle can be applied to the study of this
ρperty. The following is merely a special version of the method:

THEOREM 8. *Let u be the solution of (P) with $h \equiv 0$,*

$$\begin{cases} f \in L^\infty \left((0,\infty) \times \Omega\right) \text{ such that } \exists T_f > 0 \text{ with} \\ f(t,x) \equiv 0 \text{ a.e. } t \geq T_f \text{ and a.e. } x \in \Omega, \end{cases} \tag{31}$$

$\in L^\infty(\Omega)$, $b(u) = |u|^{\alpha-1} u$, $\alpha > 0$, $\mathbf{A}$ *satisfying (14) and*

$$g(x,\eta) \geq 0 \quad \forall \eta \in \mathbb{R}. \tag{32}$$

ιume *that (20) holds (i.e., $p - 1 < \alpha$). Then the finite extinction property*
ds.

Proof. We take as *test function* $v = |u|^{k-1} u$ (which we shall write, for
.plicity, as $v = u^k$) with $k > 0$ to be determined later. We also write u^α
;ead of $|u|^{\alpha-1} u$ by simplicity in the notation (nevertheless, it is *not* required
t $u \geq 0$). Integrating on the open (bounded) set Ω in each term of the
.ation we get :

$$\begin{aligned} \int_\Omega \frac{\partial u^\alpha}{\partial t} u^k dx &= \int_\Omega \alpha u^{(\alpha-1)+k} u_t dx \\ &= \frac{\alpha}{(\alpha+k)} \frac{d}{dt} \left(\int_\Omega u^{\alpha+k} dx \right) \end{aligned}$$

> justification of the final formula for u weak solution of (P), i.e., without
condition $(u^\alpha)_t \in L^1(\Omega)$, is due to Alt and Luckhaus [2]),

$$\begin{aligned} - \int_\Omega \operatorname{div} \mathbf{A}(x,u,\nabla u) u^k dx &= k \int_\Omega \mathbf{A}(x,u,\nabla u) \cdot \nabla u u^{k-1} dx \\ &\geq k \int_\Omega |\nabla u|^p u^{k-1} dx. \end{aligned}$$

using (31) and (32) we get that, if $t > T_f$, then

$$\frac{\alpha}{(\alpha+k)} \frac{d}{dt} \int_\Omega u^{\alpha+k}(x,t) dx + k \int_\Omega |\nabla u|^p u^{k-1} dx \leq 0.$$

need the following *interpolation* result

LEMMA 2. *Let $p \geq 1$ and $k \geq 1$. There exists a constant $C = C(m(\Omega),$ $N, k)$ such that if $w \in W_0^{1,1}(\Omega)$ and $\int_\Omega |\nabla w|^p |w|^{k-1} dx < +\infty$ we have that*

$$\left(\int_\Omega |w|^s dx \right)^{\frac{p+k-1}{s}} \leq C k^p \int_\Omega |\nabla w|^p |w|^{k-1} dx$$

with

$$
\begin{array}{ll}
1 \leq s \leq \frac{N(p+k-1)}{N-p} & \text{if} \ \ p < N, \\
1 \leq s \leq \infty & \text{if} \ \ p = N, \\
s = \infty & \text{if} \ \ p > N.
\end{array}
$$

Idea of the proof of the Lemma. Define $z(x) = |w(x)|^{\frac{p+k-1}{p}} \, sign(w(x))$ Then

$$\int_\Omega |\nabla z|^p dx = \left(\frac{p+k-1}{p} \right)^p \int_\Omega |\nabla w|^p |w|^{k-1} dx$$

and the conclusion follows from the application of the Poincaré-Sobolev an Hölder inequalities. ∎

Continuation of the proof of Theorem 8. By the above lemma we have

$$\frac{\alpha}{(\alpha+k)} \frac{d}{dt} \left(\int_\Omega u^{\alpha+k}(t,x) dx \right) + C \left(\int_\Omega u^s(t,x) dx \right)^{\frac{p+k-1}{s}} \leq 0$$

for $t > T_f$. Applying Hölder inequality we get

$$\left(\int_\Omega u^{\alpha+k}(t,x) dx \right)^{\frac{1}{\alpha+k}} \leq C(n(\Omega)) \left(\int_\Omega u^s(t,x) dx \right)^{\frac{1}{s}}$$

(take $k = 1$ if $p \geq N$ and $k \geq \frac{N}{p}(\alpha - (p-1)) - \alpha$ if $p < N$). Then if we defin

$$Y(t) := \int_\Omega u^{\alpha+k}(t,x) dx$$

we have that

$$
\begin{cases}
Y'(t) + C Y(t)^\gamma \leq 0 \quad \text{on} \quad (T_f, \infty), \quad \gamma = \frac{p+k-1}{\alpha+k} \in (0,1), \\
Y(T_f) = Y_f > 0.
\end{cases}
$$

So, again, $\exists T_0 > T_f$ such that $Y(t) \equiv 0$ if $t \geq T_0$ and the conclusion holds.

Remark 9. Some similar energy method can be applied to the case strong absorption (see, e.g., Tsutsumi [40]).

Remark 10. Under some extra decay assumptions on $f(t, \cdot)$, near T_f, it
ossible to show something unexpected: $T_0 = T_f$ (see Antontsev and Díaz

.

Remark 11. Similar energy methods applied to *higher order quasilinear
abolic equations* can be found in Bernis [14], [15].

Remark 12. One of pioneering applications of this type of energy methods
concerning the case $p = 2$ and $\Omega = \mathbb{R}^N$. In that case the condition for
existence of a finite extinction time is

$$\alpha > \frac{N}{N - 2},$$

nger than $\alpha > 1$ correspondent to bounded domains (see Benilan and
ndall [12]).

As a final and global remark we point out that the three methods used
iis section can be also applied to the study of other different qualitative
erties, as for instance, the existence of a *finite blow-up time* T_∞ (such
$\|b(u(t, \cdot))\|_{L^r(\Omega)} \to +\infty$ as $t \to +\infty$, for some $r \in [1, +\infty]$). Obviously,
property requires completely different assumptions on $\mathbf{A}, b$ and g. The
iection between the finite extinction time and the finite blow-up time
erties for a couple of different nonlinear equations has been considered in
ohl and Peletier [34].

4. THE FINITE SPEED OF PROPAGATION PROPERTY

.1. INTRODUCTION. The formulation of problem (P) is very general. It
des not only the linear heat equation

$$u_t - \Delta u = 0 \tag{33}$$

nany other cases in which the behavior of the correspondent solutions is
different to the one of the solution of the linear heat equation (remember
emarks concerning the finite extinction time property as peculiar of fast
sion or strong absorption and opposite to properties as the strong max-
ı principle or the unique continuation property which holds for the linear
tion).
nother qualitative property typical of some suitable *nonlinear* models
rns the *finite speed of propagation of disturbances*: if the initial datum

u_0 vanishes on a positively measured set of Ω (i.e., $\operatorname{supp}(u_0) \subset \Omega$) th
$\operatorname{supp} u(t,\cdot) \subset \Omega$, for any $t \in (0, t^*)$, for some $t^* > 0$.

This behavior (typical of the linear *wave* equation) fails for the linear h
equation (this can be illustrated in many ways: the strong maximum princip
the explicit representation formula for $\Omega = \mathbb{R}^N$, etc). It is said that the lin
heat equation has an *infinite* speed of propagation.

When the finite speed of propagation holds then

$$\operatorname{supp}(u(t,\cdot)) := \overline{\{x \in \Omega: \quad u(t,x) \neq 0\}} \subset \Omega$$

(at least for some small times t) and so some hypersurfaces $(0,\infty) \times \mathbb{R}^N)$

$$\mathcal{F} = \bigcup_{t>0} \mathcal{F}(t), \quad \mathcal{F}(t) = \partial(\operatorname{supp} u(t,\cdot)) - \partial\Omega$$

are formed. Those hypersurfaces are called as *free boundaries* (since they
not a priori determined) and play a very important role in the study of t
model (usually is in those free boundaries where are located the singularit
of the gradient and/or the second derivatives of the solutions).

The main goal of this section is to illustrate how the two comparis
principles can be applied to the study of the occurrence of this property.
in the previous section, a third method (involving different *energy argumen*
will be also presented.

4.2. THE FINITE SPEED OF PROPAGATION VIA THE POINTWISE COMP/
ISON PRINCIPLE.   As in the Subsection 3.2, the main idea will be to constr
suitable super and subsolutions (now vanishing locally in some subdomain
In fact, those functions use to be constructed by modifying *special soluti*
of the equation (so this task is closer to an quantitative study of pde's th
the usual approach to pde's by methods of *functional analysis*).

To start with, let us consider the nonlinear equation

$$\left(|u|^{\alpha-1} u\right)_t - \Delta_p u = 0, \quad \alpha > 0, \quad p > 1. \tag{(}$$

Although we remain interested in the Cauchy-Dirichlet problem $(P_{\alpha,q})$, it
useful to start by considering the pure Cauchy problem (i.e., $\Omega = \mathbb{R}^N$).
very important family of *exact solutions* is the one given by

$$U_M(t,x) = \frac{1}{t^\lambda} \left[C - k \frac{\sum_{i=1}^N |x_i|^{p'}}{t^{\beta p'}} \right]_+^{(p-1)/(p-1-\alpha)} \tag{(}$$

ch arises when

$$(p - 1) > \alpha \tag{36}$$

tice that the *fast diffusion* was $(p - 1) < \alpha)$, where

$$p' = \frac{p}{p - 1}, \quad \beta = \frac{\alpha}{(\alpha + 1)(p - N) + (N - 1)p},$$

$$\lambda = \frac{N\beta}{\alpha} \quad \text{and} \quad k = \beta^{(p'-1)}\frac{p - 1 - \alpha}{p},$$

> 0 arbitrary). Such solutions were obtained, by first time, by G. I. enblatt in 1952 for the case $p = 2$ (also in the case, they were refound by Σ. Pattle in 1959). The case $p \neq 2$ was found by A. Bamberger in 1975. point out that when $p \neq 2$ the solution U_M is not radially symmetric with ect to the usual Euclidean norm of $\mathbb{R}^N$. Nevertheless, it is possible to find er exact solutions with free boundaries and symmetry (although they are so explicit as U_M). Many references on this topic can be found in the eys by Kalashnikov [32] and [42]. We also point out that:

$$\int_{\mathbb{R}^N} U_M(t, x)dx = M, \qquad M = M(C, \alpha, p, N),$$

$$U(t, \cdot) \to M\delta_0(x),$$

that the free boundary generated by U_M is explicitly given by the equation

$$\sum_{i=1}^{N} |x_i|^{p'} = \frac{C}{k}t^{\beta p'}.$$

ı simple result is the following.

THEOREM 9. *Let u satisfying*

$$\begin{cases} (|u|^{\alpha-1} u)_t - \Delta_p u = 0, & t \in (0, \infty), x \in \Omega, \\ u = 0, & t \in (0, \infty), x \in \partial\Omega, \\ u(0, x) = u_0(x), & x \in \Omega, \end{cases}$$

$$\begin{cases} u_0 \in C_c(\Omega) \quad \text{such that} \\ \operatorname{supp} u_0 \subset B(x_0, R_0) \subset \Omega. \end{cases} \tag{37}$$

me that

$$(p - 1) > \alpha. \tag{38}$$

ı *the finite speed of propagation holds.*

Proof. As in Theorem 5, we can apply the pointwise comparison princi
thanks to the result by Benilan [11]. By choosing M big enough and that
to the assumption (37) we have that

$$u_0(x) \leq U_M(\tau, x - \widehat{x}_0) \qquad \forall x \in \Omega,$$

for some $\tau > 0$. Since the function $\overline{u}(t, x) := U_M(t + \tau, x - \widehat{x}_0)$ satisfies th

$$\begin{cases} \left(|\overline{u}|^{\alpha-1}\overline{u}\right)_t - \Delta_p \overline{u} = 0, & t \in (0, \infty), x \in \Omega, \\ \overline{u} \geq 0, & t \in (0, \infty), x \in \partial\Omega, \\ \overline{u}(0, x) \geq u_0(x) & x \in \Omega, \end{cases}$$

we conclude that

$$u(t, x) \leq \overline{u}(t, x) \quad t > 0, x \in \Omega.$$

By taking (if needed) different values of M and τ we get, similarly that

$$-U_{M'}(t + \tau', x - \widehat{x}_0) \leq u(t, x) \quad x \in \Omega, t > 0.$$

Thus, at least for $t \in [0, t^*)$ with t^* small enough, we conclude that

$$u(t, x) \equiv 0 \qquad \text{a.e.} \quad x \in \Omega - B(\widehat{x}_0, R(t))$$

for some function $R(t)$ and the result follows.   ∎

Remark 13. Again, the above statement can be improved in many differ
directions. For instance, in the case $p = 2$ we can replace $b(u) = |u|^{\alpha-1}u$ b
general nondecreasing function satisfying that

$$\int_{0+} \frac{ds}{b(s)} < +\infty \tag{(:}$$

and the finite speed of propagation holds (see Díaz [19]). Notice that if $p =$
and $b(u) = |u|^{\alpha-1}u$ then (39) holds if and only if $\alpha < 1$, i.e., same condit:
than (38). If $N = 1$ (and $p = 2$) it was proved by A.S. Kalashnikov (a
independently by L. A. Peletier) in 1974, that condition (39) is also necessa

Remark 14. Once that the free boundary exists it becomes interest:
to study its dynamics: how fast it starts near $t = 0$ (in some cases th
is a *waiting time*), how it behaves for $t \rightarrow +\infty$), the regularity of the f
boundary,etc.). Many of those questions remain still open (see the sur\
Kalashnikov [32]).

When assumption (38) holds it is said that we have a *slow diffusion*. It easy to see that if (38) holds then the finite speed of propagation remains ue under the presence of nondecreasing absorption term as, for instance,

$$\left(|u|^{\alpha-1} u\right)_t - \Delta_p u + \mu |u|^{q-1} u = 0, \quad \mu > 0,$$

r any $q > 0$. The finite speed of propagation also occurs when the balance tween the diffusion and absorption is suitable (called again as the *strong ab- rption case*). We can consider, even, the case of nonhomogeneous boundary nditions.

THEOREM 10. *Let u satisfying*

$$(P_{\alpha,p,q}) \begin{cases} \left(|u|^{\alpha-1} u\right)_t - \Delta_p u + \mu |u|^{q-1} = 0, & t \in (0,\infty), x \in \Omega, \\ u = h, & t \in (0,\infty), x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

th

$$\in L^\infty \left((0,\infty) \times \Omega\right) \cap L^p_{loc}\left(0,\infty : W^{1,p}(\Omega)\right), \quad h \geq 0 \ on \ (0,\infty) \times \partial\Omega, \quad (40)$$

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \ on \ \Omega. \tag{41}$$

sume

$$\mu > 0 \quad and \quad 0 < q < p - 1. \tag{42}$$

en the finite speed of propagation holds. More precisely: a) There exists a sitive constant $L > 0$ such that the null set of $u(t,\cdot)$ is not empty assumed t the set

$$\Omega - \left(\mathrm{supp}\,(u_0) \bigcup \left(\cup_{\tau>0} \mathrm{supp}\,(h(\tau,\cdot))\right)\right)$$

ig enough, i.e.,

$$I\left(u\left(t,\cdot\right)\right) := \left\{x \in \Omega : u(t,x) = 0\right\} \supset$$

$$\left\{x \in \Omega; d(x, \mathrm{supp}\,(u_0) \bigcup \left(\cup_{\tau>0} \mathrm{supp}\,(h(\tau,\cdot))\right) \geq L\right\}$$

any $t > 0$. b) If we assume, in addition, that

$$q < \alpha \leq 1 \tag{43}$$

n there exists $t_0 \geq 0$ such that for every $t \geq t_0$

$$N\left(u\left(t,\cdot\right)\right) \supset \left\{x \in \Omega : d\left(x, \cup_{\tau>0} \mathrm{supp}\,(h(\tau,\cdot)\right) \geq \widehat{L}\right\}$$

some $\widehat{L} > 0$.

Proof. We recall a result of Díaz [20] proving that the function

$$w_\lambda(x) = C_\lambda^* |x - x_0|^{\frac{1}{p-1-q}},$$

$$C_\lambda^* = \left[\frac{\lambda(p-1-q)^p}{p^{(p-1)}(pq + N(p-1-q))} \right]^{\frac{1}{p-1-q}}$$

satisfies that

$$-\Delta_p w_\lambda + \lambda |w_\lambda|^{q-1} w_\lambda = 0,$$

assumed that (42) holds, i.e., $\lambda > 0$ and $q < p-1$. Let us prove a). Let $x_0 \in \Omega$ (supp $u_0 \bigcup \bigcup_{\tau>0}$ supp $h(\tau, \cdot)$), and let $R = d(x, ($ supp $u_0 \bigcup \bigcup_{\tau>0}$ supp $h(\tau, \cdot))$ Consider $\Omega(x_0) := B(x_0, R) \cap \Omega$. Then $\overline{u}(t, x) := W_\mu(x)$ is a *local supersol tion*, i.e., a supersolution on $\Omega(x_0)$ since

$$
\begin{aligned}
(|\overline{u}|^{\alpha-1}\overline{u})_t - \Delta_p \overline{u} + \mu|\overline{u}|^{q-1}\overline{u} = 0 \quad &\text{on} \quad (0, \infty) \times \Omega(x_0), \\
\overline{u}(0, x) \geq 0 = u_0(x), \quad &\text{on} \quad \Omega(x_0) \\
\overline{u}(t, x) \geq 0 = h(t, x) \quad &\text{on} \quad (0, \infty) \times \Omega(x_0) \cap \partial\Omega,
\end{aligned}
$$

and the condition

$$\overline{u}(t, x) \geq u(t, x) \quad \text{on} \quad (0, \infty) \times \partial\Omega(x_0) - \partial\Omega,$$

is satisfied if, for instance,

$$C_\mu R^{\frac{p}{p-1-q}} \geq ||u||_{L^\infty((0,\infty)\times\Omega)} \quad (\geq u(t, x) \quad \text{a.e.}(t, x)),$$

i.e., if

$$R \geq \left[\frac{||u||_{L^\infty((0,\infty)\times\Omega)}}{C_\mu} \right]^{\frac{p-1-q}{p}}$$

(notice that $||u||_{L^\infty((0,\infty)\times\Omega)} < \infty$ thanks to the assumptions on h and u_0, we can prove in many ways: for instance by using a suitable global supe solution). Then by the pointwise comparison principle on $(0, \infty) \times \Omega(x_0)$ obtain that

$$0 \leq u(t, x) \leq C_\mu^* |x - x_0|^{\frac{p}{p-1-q}}$$

and so $u(t, x_0) = 0$ (even if u is not necessarily continuous).

To prove part b) we take as *local supersolution* the function

$$\overline{u}(t, x) := w_{\mu/2} + V(t)$$

.th $V(t)$ satisfying

$$\begin{cases} \frac{d}{dt}\left(|V|^{\alpha-1} V\right) + \frac{\mu}{2} |V|^{q-1} V = 0, \\ V(0) = ||u_0||_{L^\infty(\Omega)}, \end{cases} \tag{44}$$

·,

$$V(t) = \left[||u_0||_{L^\infty(\Omega)}^{\alpha-q} - \frac{\mu(\alpha-q)}{2\alpha} t \right]_+^{\frac{1}{\alpha-q}} . \tag{45}$$

ien

$$\left(|\overline{u}|^{\alpha-1}\,\overline{u}\right)_t = \alpha\left(w_{\mu/2}(x) + V(t)\right)^{\alpha-1} \dot{V} \geq \frac{d}{dt}\left(|V|^{\alpha-1} V\right),$$

$$\Delta_p \overline{u} = \Delta_p w_{\mu/2},$$

$$\mu|\overline{u}|^{q-1}\overline{u} \geq \tfrac{\mu}{2}|w_{\mu/2}|^{q-1} w_{\mu/2} + \tfrac{\mu}{2}|V|^{q-1}V ,$$

.d so

$$\left(|\overline{u}|^{\alpha-1}\,\overline{u}\right)_t - \Delta_p \overline{u} + \mu\,|\overline{u}|^{q-1}\,\overline{u} \geq 0.$$

)reover

$$\overline{u}(0,x) = w_{\mu/2} + V(0) \geq ||u_0||_{L^\infty(\Omega)}^{\alpha-q} \geq u_0(x).$$

.ally, taking

$$t_0 = \frac{2\alpha}{\mu(\alpha-q)}||u_0||_{L^\infty(\Omega)}^{\alpha-q}$$

get that $V(t) \equiv 0 \quad \forall t \geq t_0$ and the conclusion follows as in part a). ∎

Remark 15. The above result is taken from Díaz and Hernández [23] where .er, and more general, results can be found.

Remark 16. In the model of chemical reactions, the *null set* $N(u(t,\cdot))$ is .ed as *dead core*. In that model usually $h(t,x) \equiv 1$ and so $N(u(t,\cdot))$ only urs at the interior of Ω.

Remark 17. Notice that if $h \equiv 0$ part b) shows the *extinction in finite* e. Notice also that assumptions (42) (in addition to (43)) implies the nation of dead core for t large even for $h \equiv 1$ and $u_0 > 0$. This property a similar nature to the so called *instantaneous shrinking of the support* ιblished by Brezis and Friedman in 1976, or by Evans and Knerr in 1979, h for the case of $\Omega = \mathbb{R}^N$ and $u_0 > 0$ such that $\lim_{|x|\to\infty} u_0(x) = 0$ (see rences in the survey Kalashnikov [32]).

4.3. THE FINITE SPEED OF PROPAGATION VIA THE MASS SYMMETRIZE
COMPARISON PRINCIPLE.   The above method requires the construction
sophisticated supersolutions. This is possible only for simple nonlinear ope
ators. The application of the mass symmetrized comparison principle sho
us how important is to have symmetry conditions on the partial differenti
equation in order to have solutions with *small* support.

THEOREM 11. *Let u be the solution of (P) with $f \equiv 0$, $h \equiv 0$, $u_0 \in C_c(\Omega$*
$u_0 \geq 0$ and assume $b(u) = |u|^{\alpha-1}u$, (14) and (9). We also suppose the followin
conditions

$$\begin{cases} (p-1) > \alpha, \\ \phi(\eta) := \widetilde{g}\left(|\eta|^{\frac{1}{\alpha}-1}\eta\right) = \varphi_1(\eta) + \varphi_2(\eta), \quad \eta \in \mathbb{R} \\ \text{with } \varphi_1 \text{ (resp. } \varphi_2) \text{ nondecreasing convex} \\ \text{(resp. nondecreasing concave)}, \end{cases}$$

and

$$\int_\Omega b(u(t,x))dx = \int_{\Omega^*} b(U(t,x))dx, \quad \forall t \geq 0, \tag{4}$$

where U denotes the solution of the symmetrized problem. Then the suppo
of $u(t,\cdot)$ satisfy

$$m\left(\text{supp}\, u(t,\cdot)\right) \geq m\left(\text{supp}\, U(t,\cdot)\right) \tag{4'}$$

for any $t > 0$.

Proof. By using the mass symmetrized comparison principle, (46) and th

$$\int_\Omega b(u(t,x)dx = \int_0^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma$$

we have

$$\int_s^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma = \int_0^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma - \int_0^s b(\widetilde{u}(t,\sigma))d\sigma$$

$$\geq \int_0^{m(\Omega)} b(\widetilde{U}(t,\sigma))d\sigma - \int_0^s b(\widetilde{U}(t,\sigma))d\sigma.$$

Let

$$\text{support of } \widetilde{u} = [0, R_u(t)], \quad 0 < R_u(t) \leq m(\Omega)$$
$$\text{support of } \widetilde{U} = [0, R_U(t)], \quad 0 < R_U(t) \leq m(\Omega)$$

ecall that $\widetilde{u}$ and $\widetilde{U}$ are nondecreasing functions). Then, necessarily $R_u(t) \geq$
$U(t)$ since otherwise we would deduce that

$$\int_{R_u(t)}^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma \geq \int_{R_u(t)}^{R_U(t)} b(\widetilde{U}(t,\sigma))d\sigma > 0$$

rich is a contradiction. Finally, it suffices to remark that

$$\widetilde{u}(t,\cdot) = [0, m(\operatorname{supp} u(t,\cdot))]$$

nalogously for U) and the conclusion holds. ∎

Remark 18. Notice that by (47) if $\operatorname{supp} U(t^*,\cdot) = \Omega$, for some $t^* > 0$, then
pp $u(t^*,\cdot) = \Omega$.

Remark 19. Assumption (46) is satisfied, for instance, when the *conserva-*
n of the mass holds, i.e.,

$$\int_{\Omega} b(u(t,x))dx = \int_{\Omega} b(u_0(x))dx, \quad \forall t > 0.$$

that case $\int_{\Omega} b(u_0(x))dx = \int_{\Omega^*} b(U_0(x))dx = \int_{\Omega^*} b(U(t,x))dx$ and (46) is
ified. The *conservation of the mass* is typical of pure diffusion processes
., when $g = \widetilde{g}$). It can be shown (see Díaz [21]) that assumption (46) is
verified when, besides the Dirichlet condition $u(t,x) = 0$, $t > 0$, $x \in \partial\Omega$,
have the additional information that

$$\frac{\partial u}{\partial \mathbf{n}}(t,x) = 0 \quad \text{for } t \in (0,\widehat{T}), \quad x \in \partial\Omega,$$

some $\widehat{T} > 0$ (in that case the conclusion (47) holds at least for $t \in [0,\widehat{T})$).

In the case of strong absorption we can allow a nonzero Dirichlet condition

THEOREM 12. *Let u be the solution of (P) with $f \equiv 0$ and*

$$h(t,x) \equiv h, \quad \text{a positive constant.} \tag{48}$$

$u_0 \in L^\infty(\Omega)$ *with*

$$0 \leq u_0(x) \leq h \quad \text{a.e. } x \in \Omega. \tag{49}$$

Assume $b(u) = |u|^{\alpha-1}u$, (14), (9) *and*

$$\widehat{g}(\eta) = \mu|\eta|^{q-1}\eta \quad \text{with } \mu > 0 \text{ and } q < (p-1).$$

Then the supports of $u(t, \cdot)$ *and* $U(t, \cdot)$ *satisfy that*

$$m\left(\operatorname{supp} u(t, \cdot)\right) \geq m\left(\operatorname{supp} U(t, \cdot)\right) \quad \text{for } t > 0. \tag{5(}$$

Idea of the proof. By introducing the change of variables $v(t, x) = h$ $u(t, x)$ and $V(t, x) = h - U(t, x)$ we can apply the mass symmetrized con parison principle to v and V. Finally, it suffices to apply the result by Hard Littlewood and Polya for an appropriate choice of convex function Φ (see Día [21]). ∎

Remark 20. Estimates (47) and (50) allows to compare the *waiting tim* (when arising) for u and U.

Remark 21. Estimate (50) shows that the *dead core* has a bigger measu under radially symmetric conditions. That was first observed in Bandle an Stakgold [7].

4.4. THE FINITE SPEED OF PROPAGATION VIA AN ENERGY METHOI
The study of the finite speed of propagation (and other qualitative propertie can be carried out by using some energy arguments which, in contrast wit the ones of Section 3, now have a local character.

THEOREM 13. *Let* **A** *satisfying* (4) *and*

$$|\mathbf{A}(x, u, \xi)| \leq C|\xi|^{p-1}.$$

Let $g(x, u)$ *such that*
$$g(x, \eta)\eta \geq 0 \qquad \forall \eta \in \mathbb{R}.$$

Assume
$$\alpha < (p-1)$$

and let u *be a local solution of the equation*

$$\left(|u|^{\alpha-1}u\right)_t - \operatorname{div}\mathbf{A}(x, u, \nabla u) + g(x, u) = 0 \quad \text{on} \quad (0, \infty) \times B(x_0, R)$$

ɔr some $x_0 \in \mathbb{R}^N$, $R > 0$) such that

$$u(0, x) = 0 \quad a.e. \quad x \in B(x_0, \rho_0), \qquad \rho_0 < R.$$

ɪen there exists $t^* > 0$ and $\rho : [0, t^*] \longrightarrow [0, \rho_0]$ nondecreasing such that

$$u(t, x) = 0 \quad a.e. \quad x \in B(x_0, \rho(t)).$$

Idea of the proof. By multiplying by u and integrating by parts we get

$$\frac{\alpha}{\alpha + 1} \int_{B_\rho} |u(t, x)|^{\alpha+1} dx + \int_0^t \int_{B_\rho} \mathbf{A}(x, u, \nabla u) \cdot \nabla u \, dx \, ds$$
$$\leq \int_0^t \int_{\partial B_\rho} u \mathbf{A}(x, u, \nabla u) \cdot \mathbf{n} \, d\Gamma \, ds$$

ɪis can be rigorously justified from the notion of bounded weak *local* solu-
ɴ). Here $B_\rho = B(x_0, \rho)$. We introduce the *local energies*

$$E(t, \rho) := \int_0^t \int_{B_\rho} \mathbf{A}(x, u, \nabla u) \cdot \nabla u \, dx \, ds$$

ɪ

$$b(t, \rho) := \operatorname{ess\,sup}_{s \in (0, t)} \left(\frac{\alpha}{\alpha + 1} \int_{B_\rho} |u(s, x)|^{\alpha+1} dx \right).$$

ɪng Hölder inequality we get that

$$b + E \leq \frac{1}{pc} \left(\int_0^t \int_{B_\rho} |u|^p dx \, ds \right) \left(\frac{\partial E}{\partial \rho} \right)^{\frac{p-1}{p}}$$

ere we used that

$$\frac{\partial E}{\partial \rho}(t, \rho) = \int_0^t \int_{\partial B_\rho} \mathbf{A}(x, u, \nabla u) \cdot \nabla u \, d\Gamma \, ds.$$

need the following.

LEMMA 3. (Interpolation-trace inequality) *For any* $\sigma \in [0, p-1]$ *there*
ɪt $C > 0$ *and* $\theta \in [0, 1]$ *such that for any* $w \in W^{1,p}(G)$, G *open bounded*
of $\mathbb{R}^N$, *we have*

$$\|w\|_{L^p(\partial\Omega)} \leq C \left(\|\nabla w\|_{L^p(G)} + \|w\|_{L^{\sigma+1}(G)} \right)^\theta \left(\|w\|_{L^{\sigma+1}(G)} \right)^{1-\theta}.$$

Applying the Lemma and Young inequality we obtain that

$$E^{\gamma} \leq Ct^{\frac{1-\theta}{p-1}} \left(\frac{\partial E}{\partial \rho} \right)$$

for some exponent $\gamma \in (0,1)$. This implies the result. ∎

Remark 22. Notice that the result holds without making explicit the boundary conditions. It has a *local* nature.

Remark 23. The first local energy method was due to S.N. Antontsev in 1981. A rigorous justification of his arguments, containing also several improvements, was made in Díaz and Veron [27].

Remark 24. Other qualitative properties (as the *formation of dead core*, *the instantaneous shrinking of the support*, etc) can be proved by this type of local energy arguments. See, e.g., Antontsev, Díaz and Shmarev [5]. Those authors are preparing a book containing many other applications.

Remark 25. For the application of this type of arguments to *higher order equations* see Bernis [14], [15] and their references.

As a global, and final, remark we mention that the finite speed of propagation, the finite extinction time and other qualitative properties can be analyzed for hyperbolic first order equations of the type

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Phi_i(u) + g(x,u) = f(t,x)$$

see Díaz and Veron [26] and Díaz and Kruhzkov [24].

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