

## Infinitely many stationary solutions for a simple climate model via a shooting method

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### SUMMARY

In this paper, we study the number of steady solutions of a non-linear model arising in Climatology. By applying a shooting method we show the existence of infinitely many steady solutions for some values of a parameter (the solar constant). This method allows us to determine how many times a solution attains the critical temperature ( $-10^{\circ}\text{C}$ ) at which the coalbedo is assumed to be discontinuous. Copyright © 2002 John Wiley & Sons, Ltd.

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### INTRODUCTION

The model is based on a global energy balance over the Earth surface. The unknown is the mean temperature over each parallel and the time scale is considered relatively large. These models were introduced independently by M. Budyko and W. Sellers in 1969.

The energy balance is obtained when one expresses the *heat variation* in terms of  $R_a - R_e + D$ , where  $R_a, R_e$  represent the absorbed and the emitted energies by the Earth and  $D$  the heat diffusion which is given by a second-order elliptic operator. By expressing each component of the above balance in mathematical terms as function of the temperature  $u$ , we obtain a non-linear parabolic partial differential equation. The general spatial domain for this kind of model is a compact two-dimensional (2-D) Riemannian manifold  $\mathcal{M}$  simulating the Earth. Often, the two-dimensional model is reduced to a one-dimensional (1-D) model by considering the averaged temperature over each parallel. Such simplification considers the

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temperature  $u(x, t)$  only dependent on the latitude  $\lambda$  and  $x = \sin \lambda$ . By assuming North–South symmetry, the obtained model is

$$(P_0) \quad \begin{cases} u_t - (|u_x|^{p-2}u_x)_x + \mathcal{G}(u) + C \in Q\beta(u), & (t, x) \in (0, T) \times (0, 1) \\ u_x(t, 0) = u_x(t, 1) = 0, & t \in (0, T) \\ u(0, x) = u_0(x) & x \in (0, 1) \end{cases}$$

In the pioneer models, the diffusion operator was linear ( $p=2$ ). Later, Stone [1] proposed a non-linear diffusion operator for this kind of model ( $p=3$ ) in order to consider the negative feedback in the eddy flux. The formulation in  $(P_0)$  includes both cases. In this equation there appears a multivalued term  $\beta(u)$  which represents the planetary coalbedo (the fraction of the incoming radiation flux which is absorbed by the surface). This term is multiplied by the Solar constant  $Q$ . From the physical point of view, such a constant can have small variations depending on the obliquity of the terrestrial axis, eccentricity of the Earth orbit, etc.

These climatological models have been studied by different authors. These models are sensitive with respect to variations of the parameter  $Q$ . By using a shooting method we show that there exist infinitely many equilibrium solutions for some values of  $Q$ . This result gives more precision than that obtained for two-dimensional climatological models (see References [2, 3]), where it was proven that there exist at least three stationary solutions for values of  $Q$  in a bounded interval, as well as the uniqueness of solution for  $Q$  big or small enough. As a consequence of such results, we can affirm

- (i) if  $Q < Q_1$  or  $Q > Q_2$  then the stationary problem associated with  $(P_0)$  has a unique solution;
- (ii) if  $Q_1 < Q < Q_2$  then the stationary problem associated with  $(P_0)$  has at least three solutions:  
where

$$Q_1 := \frac{\mathcal{G}(-10) + C}{M} \quad \text{and} \quad Q_2 := \frac{\mathcal{G}(-10) + C}{m} \quad (1)$$

with  $m$  and  $M$  the infimum and the supremum of  $\beta$ , respectively.

The results of this paper improve also a part of Tello [4], where we assume  $p=2$ . We also mention the works [5–9] for other analysis of multiplicity of solutions.

## MULTIPLICITY OF STATIONARY SOLUTIONS

We are concerned with the stationary boundary value problem associated with the model  $(P_0)$

$$(P) \quad \begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \in Q\beta(u) & x \in (0, 1) \\ u'(0) = u'(1) = 0 \end{cases}$$

where  $p \geq 2$  and  $Q > 0$ . We assume the following conditions:

(H<sub>1</sub>)  $\beta$  is a bounded maximal monotone graph of the Heaviside type defined by

$$\beta(u) = \begin{cases} m & \text{if } u < -10 \\ [m, M] & \text{if } u = -10 \\ M & \text{if } u > -10 \end{cases}$$

with  $0 < m < M$ .

(H<sub>2</sub>)  $\mathcal{G}$  is a continuous increasing function with  $\mathcal{G}(0) = 0$  and  $\lim_{s \rightarrow \infty} |\mathcal{G}(s)| = +\infty$ .

(H<sub>3</sub>)  $\mathcal{G}(-10) + C > 0$ .

We will say that  $u$  is a solution of (P) if  $u \in C^1([0, 1])$  and there exists  $z \in L^\infty(0, 1)$ ,  $z(x) \in \beta(u(x))$  a.e.  $x \in (0, 1)$  such that  $u$  verifies the equation  $-(|u'|^{p-2}u')' + \mathcal{G}(u) + C = Qz$  in the weak sense.

The main goal of this paper is to prove that there exists an interval of  $Q$  where the problem has infinitely many solutions.

*Theorem 1.* If  $Q_1 < Q < Q_2$  then (P) has infinitely many solutions. Moreover, there exists  $N_0 \in \mathbb{N}$  such that for every  $K \geq N_0$  there exists at least a solution  $u_K$  which crosses its level  $u_K = -10$ , exactly  $K$  times.

*Proof.* We start by computing the intersections between the graphs  $\mathcal{G}(u) + C$  and  $Q\beta(u)$ . These are constant solutions of problem (P) and the number of intersections depends on the value of the parameter  $Q > 0$ . We compute without difficulty two significant values,  $Q_1$  and  $Q_2$ . If  $Q = Q_1$  or  $Q = Q_2$  then the graphs have two points in common. Moreover, if  $Q < Q_1$  or  $Q > Q_2$  then the intersection is only one point. If  $Q_1 < Q < Q_2$  then the intersection points are the following:

$$u_1 = \mathcal{G}^{-1}(Qm - C) < -10, \quad u_2 = -10 \quad \text{and} \quad u_3 = \mathcal{G}^{-1}(QM - C) > -10 \tag{2}$$

*Step 1:* We study the phase portrait  $(u, u')$  for an auxiliary Cauchy problem. Since the equation (P) is conservative, we obtain the conservation of the total energy

$$\frac{|u'(x)|^p}{p} + V(u(x)) = E, \quad \forall x \in \mathbb{R} \tag{3}$$

for some constant  $E$  and for the following potential function:

$$V(u) = \begin{cases} (QM - C)u - G(u), & u \geq -10 \\ (Qm - C)u - G(u) - 10Q(M - m), & u < -10 \end{cases} \tag{4}$$

where  $G(u) = \int_0^u \mathcal{G}(s) ds$ . This function  $V$  allows us to find the trajectories  $(u, u')$  corresponding to each energy level  $E$ , from Equation (3) (notice that  $V$  is continuous but it is not  $C^1$ ). From the restrictions of  $-V$  on the sets  $x \leq -10$  and  $x \geq -10$  are convex, we see that  $V$  has three relative extrema: two of them are maxima,  $u_1$  and  $u_3$ , and the other  $u_2$  is a

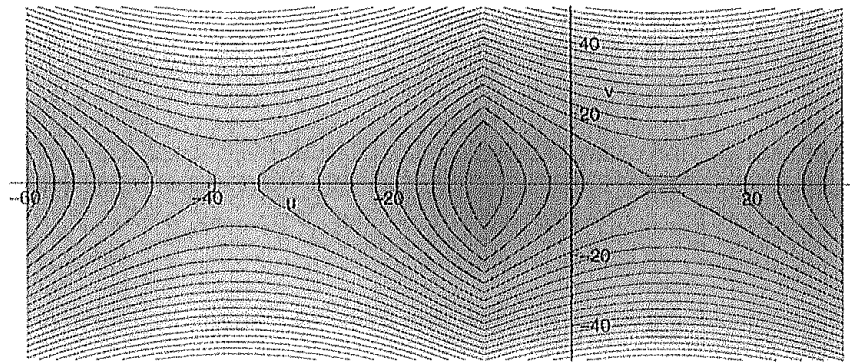


Figure 1. If  $Q_1 < Q < Q_3$  then  $V(u_3) < V(u_1)$ . There exists a homoclinic orbit with  $\omega$ -limit equal to  $u_3$ , which separates a region of the periodic orbits of the others.

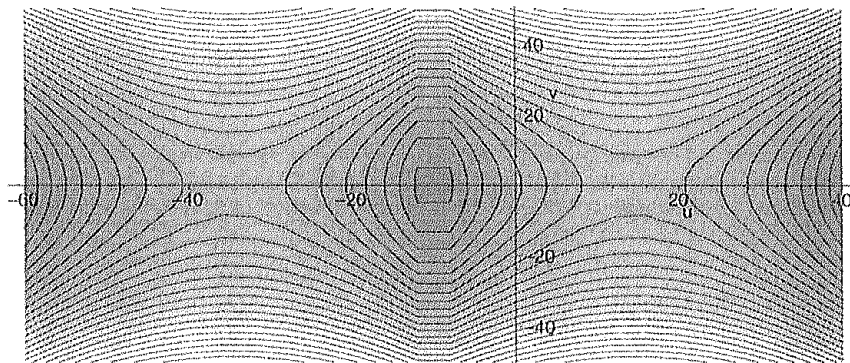


Figure 2. If  $Q = Q_3$  then  $V(u_1) < V(u_3)$ . There exist two heteroclinic orbits with  $\omega$ -limit equal to  $u_1$  and  $u_3$ , respectively, which separate a region of the periodic orbits of the others.

minimum. So, we derive three constant stationary solutions:  $(u_1, 0)$  and  $(u_3, 0)$  are saddle points and  $(u_2, 0)$  is a centre. If  $Q_1 < Q < Q_2$  then  $V(u_2) < V(u_3)$  and  $V(u_2) < V(u_1)$ . We obtain trajectories which do not cross the axis  $u' = 0$ : they correspond to the energy levels  $E > \max\{V(u_1), V(u_3)\}$ . The trajectories corresponding to the energy levels  $E < V(u_2)$  and  $\min\{V(u_1), V(u_3)\} < E < \max\{V(u_1), V(u_3)\}$  cross the axis  $u' = 0$  exactly in one point. Finally, if  $V(u_2) < E < \min\{V(u_1), V(u_3)\}$  we find periodic trajectories (which cut in two different points  $(a, 0)$  and  $(b, 0)$  the axis  $u' = 0$ , where  $u_1 < a < -10 < b < u_3$ ) and the others only in one point.

When we try to compare  $V(u_1)$  and  $V(u_3)$  we find a significant value of  $Q$ , which we call  $Q_3 \in (Q_1, Q_2)$ , verifying  $V(u_1) = V(u_3)$ . We obtain three different cases for the phases portrait  $(u, u')$  (Figs 1–3).

In order to solve the boundary value problem  $(P)$  we shall use a shooting method, which is described in the following step.

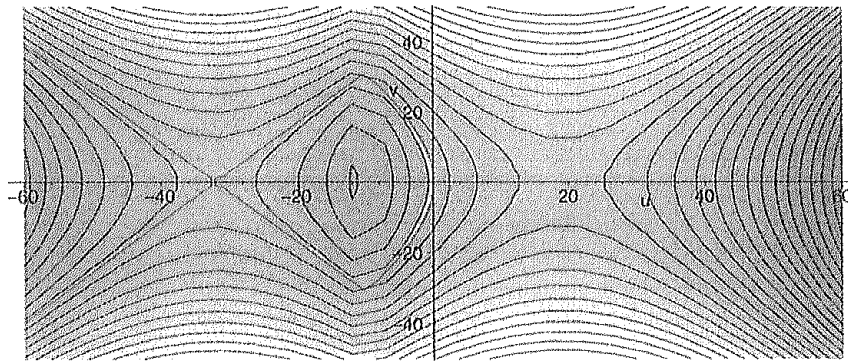


Figure 3. If  $Q_3 < Q < Q_2$  then  $V(u_3) < V(u_1)$ . There exists a homoclinic orbit with  $\omega$ -limit equal to  $u_1$ , which separates a region of the periodic orbits of the others.

*Step 2: Shooting method.* We consider the following Cauchy problem depending on the parameter  $\mu$ :

$$(P_\mu) \begin{cases} -(|u'|^{p-2}u')' + \mathcal{G}(u) + C \in Q\beta(u), & x \in \mathbb{R}^+ \\ u'(0) = 0 \\ u(0) = \mu \end{cases}$$

Our purpose is to determine the values  $\mu$  such that the solution of  $(P_\mu)$  verifies  $u'(1) = 0$ .

From the phase portraits studied in the first step we deduce that the solutions which attain at least two times the value  $u' = 0$  are the solutions given by the periodic trajectories, that is, associated with energy level  $V(u_2) \leq E \leq \min\{V(u_1), V(u_3)\}$ . The idea is to choose the periodic trajectories which start in  $(\mu, 0)$  and arrives at  $(\lambda, 0)$  at the time  $x = 1$ . That is, integrating the conservation law equation (3), we obtain

$$\int_{u(0)}^{u(1)} \frac{ds}{\pm(p(E - V(s)))^{1/p}} = \int_0^1 d\sigma$$

where the sign of  $(p(E - V(s)))^{1/p}$  is the same as that of  $u'$ . The period of the periodic orbit of the phase portrait is given by the expression,

$$\tau = 2 \int_a^{-10} \frac{ds}{(p(E - V(s)))^{1/p}} + 2 \int_{-10}^b \frac{ds}{(p(E - V(s)))^{1/p}} \tag{5}$$

where  $(a, 0), (b, 0)$  are the two different points in which the trajectories pass by the axis  $u' = 0$ . This is equivalent to say that  $a$  and  $b$  verify:  $u_1 < a < -10 < b < u_3$  and  $V(b) = V(a) < \min\{V(u_1), V(u_3)\}$ . Consequently, there exists  $b^* > -10$  such that  $V(b^*) = \min\{V(u_1), V(u_3)\}$ , then, the below condition can be written as  $-10 < b < b^*$ .

If  $p=2$  and  $\mathcal{G}(u) = Bu$ , where  $B$  is a positive constant, it is possible to obtain the explicit expression for  $\tau$ ,

$$\tau = \frac{2}{\sqrt{B}} \operatorname{Ln} \left( \frac{((QM - C)/\sqrt{B} + 10\sqrt{B} + \sqrt{2(E - V(-10))})(C - Qm)/\sqrt{B} - 10\sqrt{B} + \sqrt{2(E - V(-10))})}{((QM - C)/\sqrt{B} - b\sqrt{B})(C - Qm)/\sqrt{B} + a\sqrt{B}} \right) \tag{6}$$

where  $(a, 0)$  and  $(b, 0)$  are two different points of the periodic orbit.

If  $p \geq 2$ , under the hypothesis  $(H_3)$ , we have obtained the following estimates for the period  $\tau$  of a periodic trajectory which contains the points  $(a, 0)$  and  $(b, 0)$  with  $a < -10 < b$ ,

$$\tau_1 \leq \tau \leq \tau_2$$

where

$$\tau_1 = \frac{2p^{1-1/p}}{p-1} ((b+10)^{1-1/p}(\mathcal{G}(u_3) - \mathcal{G}(-10))^{-1/p} + (-10-a)^{1-1/p}(\mathcal{G}(-10) - \mathcal{G}(a))^{-1/p}) \tag{7}$$

$$\tau_2 = \frac{2p^{1-1/p}}{p-1} ((b+10)^{1-1/p}(\mathcal{G}(u_3) - \mathcal{G}(b))^{-1/p} + (-10-a)^{1-1/p}(\mathcal{G}(a) - \mathcal{G}(u_1))^{-1/p}) \tag{8}$$

We have used the Mean Value Theorem for  $\mathcal{G}$  to produce these estimates.

The solution  $u$  of  $(P)$  satisfies one of these four equalities,

- (I)  $u(0) = \mu = b, \quad u(1) = a$
- (II)  $u(0) = \mu = a, \quad u(1) = b$
- (III)  $u(0) = \mu = b = u(1)$
- (IV)  $u(0) = \mu = a = u(1)$

The solutions of types I and II cross the level  $-10$  an odd number of times, while for the solutions of types III and IV, it is in an even number of times. We only analyse the case I, the others can be analysed in an analogous way. Moreover, case I is the unique case verifying that  $u(0) > u(1)$ , which makes this more realistic because  $x=0$  represents the Equator and  $x=1$  the North pole. So, we assume

$$u(0) = \mu = b > -10$$

$$u(1) = a < -10$$

We notice that every  $\mu$  determines  $a_\mu$  as the unique solution of  $V(s) = V(\mu)$  on  $(u_1, -10)$ . Now, the problem is to study whether there exists or not  $\mu$  such that the time to arrive from  $\mu$  to  $a_\mu$  is exactly  $x=1$ , after  $N$  complete turns and a half, that is,

$$N\tau + \int_{-10}^{u(0)} \frac{ds}{(p(E - V(s)))^{1/p}} + \int_{u(1)}^{-10} \frac{ds}{(p(E - V(s)))^{1/p}} = 1 \tag{9}$$

Then, for every  $\mu \in (-10, b^*)$ , we have  $\tau(\mu)$  and we are interested in the number of solutions  $(N, \mu) \in \mathbb{N} \times (-10, b^*)$  of the equation  $(N + \frac{1}{2})\tau(\mu) = 1$ , that is,

$$\tau(\mu) = \frac{2}{2N + 1}$$

In order to see this, we study the functions  $\tau_1$  and  $\tau_2$ . We observe that  $\tau_1(\mu)$  and  $\tau_2(\mu)$  are continuous and increasing functions on the interval  $(-10, b^*)$ , where

$$\begin{aligned} b^* &= u_3 & \text{if } Q \in (Q_1, Q_3] \\ b^* &< u_3, \quad V(u_1) &= V(b^*) & \text{if } Q \in (Q_3, Q_2) \end{aligned}$$

Moreover, the function has a vertical asymptote  $\mu = b^*$ . From  $\tau_1(-10) = \tau_2(-10) = 0$  and the properties of  $\tau_1$  and  $\tau_2$  we see that there exists  $N_0$  such that for all  $N \geq N_0$  there exist  $\mu_1$  and  $\mu_2$  such that

$$\tau_1(\mu_1) = \frac{2}{2N + 1} = \tau_2(\mu_2) \tag{10}$$

Thus, we can conclude that there exists  $\mu \in (\mu_1, \mu_2)$  such that  $(N + \frac{1}{2})\tau(\mu) = 1$ . So, we obtain that for each  $N \geq N_0$  there exists a solution of  $(P)$  which crosses  $2N + 1$  times the level  $-10$ .

Thus, we have proved the existence of infinitely many solutions of  $(P)$  for  $Q \in (Q_1, Q_2)$ . On the other hand, the construction allows us to deduce that the family of solutions obtained is uniformly bounded because every solution  $u$  of  $(P)$  verifies  $u_1 \leq u(x) \leq u_3$  (in fact  $u'$  is also bounded since  $(u, u')$  is a periodic trajectory). □

### FINAL COMMENTS

Theorem 1 shows that the simple model studied has a complex behaviour under small and critical variations of  $Q$ . The results of Theorem 1 open some new problems:

We know that the solutions  $u(t, x)$  of the evolution problem (1-D and 2-D models) go to a stationary solution when  $t \rightarrow \infty$  in the following sense: we define the  $\omega$ -limit set as

$$\omega(u) = \{u_\infty \in W^{1,p}(\Omega) \cap L^\infty(\Omega) : \exists t_n \rightarrow +\infty \text{ such that } u(t_n, \cdot) \rightarrow u_\infty \text{ in } L^2(\Omega)\}$$

In Reference [2], we have proved:

- Let  $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$  (the initial data) and let  $u$  be the solution. Then (i)  $\omega(u) \neq \emptyset$ ,
- (ii) if  $u_\infty \in \omega(u)$  then  $\exists t_n \rightarrow +\infty$  such that  $u(t_n + s, \cdot) \rightarrow u_\infty$  in  $L^2(-1, 1; L^2(\Omega))$  and  $u_\infty \in W^{1,p}(\Omega)$  is a weak solution of the stationary problem. (iii) Actually, if  $u_\infty \in \omega(u)$  then  $\exists \{\hat{t}_n\} \rightarrow +\infty$  such that  $u(\hat{t}_n, \cdot) \rightarrow u_\infty$  strongly in  $W^{1,p}(\Omega)$ .

Now we know that the model has infinitely many stationary solutions. Which is the limit of  $u(t, x)$  when we consider  $\forall t$ ? How can we distinguish such a limit  $u_\infty$  in terms of the initial data and  $f(t, \cdot)$  among the infinitely many stationary solutions? Is it true that  $\omega(u)$  is formed by a single element  $u_\infty$  or it is formed by a multiple set of stationary solutions.

The model studied in Reference [2] includes the *insolation function*  $S(x)$  and the multiplicity of at least three solutions for  $Q \in (Q_1, Q_2)$  was proved. It would be interesting to extend the

conclusion of Theorem 1 to the 1-D model including the insolation effect. A related work for a 1-D EBM, with linear diffusion ( $p=2$ ), is due to Hetzer [10]. So, the second open problem is to analyse the number of solutions for the 1-D model with  $R_a(x,u) = QS(x)\beta(u)$ ,  $p \geq 2$  and  $\beta$  multivalued.

The results of multiplicity of stationary solutions for the 2-D climate energy balance model say that there exist *at least three* stationary solutions for  $Q \in (Q_1, Q_2)$ . An extension of Theorem 1 for these models will be very interesting.

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