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On the mathematical analysis of the limit case of a radiative—convective climate model

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1. Introduction

One of the simpler models in the very rich hierarchy of the climate models corresponds to the, so-called, radiative convective models (RCMs). This type of models was initially introduced in [10] in order to describe averaged vertical atmospheric thermal structure when advective sources vanish due to spatial averaging. Since then, several modifications of the pioneer RCMs were introduced by several authors for different purposes (see e.g., the exposition made in [12]).

The main goal of this paper is to carry out the mathematical analysis of the limit case of a RCM introduced by G.L. Stenchikov and A. Robock in 1995 for the investigation of the diurnal cycle. Their model has a forced nature (by prescribed advection) and accounts interactively for boundary layer processes, turbulence, convection, cloudiness, and hydrological cycle, and incorporates a spectral radiative transport in a cloudy and polluted atmosphere. The model was used in [13] to calculate the effects of observed stratospheric aerosol, water vapor and ozone changes after the 1991 Pinatubo eruption.

For a given atmospheric column, let H be the total specific energy, T the temperature, q the water vapor mixing ratio, L the latent heat of vaporization, c_p the specific heat of air at constant pressure, H_s the static energy, g the acceleration due to gravity, and z the altitude. Then $H = c_pT + Lq$, and $H_s = H + gz$. In stable atmosphere regions, $\partial H/\partial t = 0$, and in unstable regions the evolution of specific energy is given by

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$\partial H/\partial t = (\partial/\partial x)D(\partial/\partial x)H_s$, (see [12]), where the variable x represents the normalized pressure.

Here we shall be concerned only with the dry case $q=0$. So $(\partial/\partial x)H_s = \partial H/\partial x - RT/x$ and the proposed model can be formulated in the following terms:

$$\frac{\partial H}{\partial t} = \frac{\partial}{\partial x} k\left(\frac{\partial H}{\partial x} - \frac{RT}{x}\right), \quad x \in (0,1),\ t > 0,$$

$$k\left(\frac{\partial H}{\partial x} - \frac{RT}{x}\right) = 0, \quad x = 0, 1,\ t > 0,$$

$$H(x,0) = H_0(x), \quad x \in (0,1), \tag{1.1}$$

where R is a positive constant (the gas constant for the air), $c_p \in C^0([0,1])$, $c_p(x) > 0$ on $(0,1)$ and the diffusion coefficient degenerates on $(-\infty, 0]$, more precisely,

$$k(s) = \begin{cases} k_0 s & \text{if } s > 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $k_0 = 100\varphi^2 g^2 / P_s^2$, where g is the acceleration due to gravity and φ is the density of the air. For simplicity we shall assume that φ is constant. So, an equivalent formulated is

$$\frac{\partial(c_p T)}{\partial t} = \frac{\partial}{\partial x} k\left(\frac{\partial(c_p T)}{\partial x} - \frac{RT}{x}\right), \quad x \in (0,1),\ t > 0,$$

$$k\left(\frac{\partial(c_p T)}{\partial x} - \frac{RT}{x}\right) = 0, \quad x = 0, 1,\ t > 0,$$

$$T(x,0) = T_0(x), \quad x \in (0,1). \tag{1.2}$$

Problem (1.2) was raised in [12] (see also [8,11]). We recall that x represents the vertical coordinate in terms of the ratio p/p_s between the air pressure to different heights and the surface air pressure (so $x=0$ corresponds to the top of the atmosphere and $x=1$ to the surface). In (1.2) $c_p(x)T(x,t)$ corresponds to the total specific energy in a dry convection situation i.e. $u(x,t) := H(x,t) = c_p(x)T(x,t)$ with $c_p(x) > 0$ the specific heat of air at constant pressure and $T(x,t)$ the temperature (we recall that the water vapor mixing ratio $q(x,t)$ is here assumed to be zero). The convective diffusion function $k(s)$ is assumed to be zero on "stable regions", i.e. when acting on nonpositive balance

$$k\left(\frac{\partial u(x,t)}{\partial x} - \frac{R}{x c_p(x)} u(x,t)\right) \leqslant 0.$$

In fact, problem (1.2) corresponds to the general case $c_p(x)$ but the mathematical difficulties are the same than for $c_p = constant$.

One of the main difficulty, in the mathematical treatment of problem (1.2) comes from the fact that the function k is not "coercive", that is, it does not satisfy the condition

$$\lim_{|s| \to \infty} k(s) = +\infty.$$

In the best of our knowledge, most of the results in the literature on the existence of solutions of quasilinear parabolic problems of the type (1.1) require the above coercivity assumption (see e.g. [1,3,9], etc.). A different problem with a noncoercive formulation was considered in [6]. Our treatment will make a fundamental use of the fact that the spatial variable is one dimensional. This allows to show that the abstract theory of accretive operators can be applied in a suitable Banach space, which yields to the existence and uniqueness of a weak solution of (1.2).

2. On the existence and uniqueness of the weak solution

In order to state the main result of this paper we start by introducing some functional spaces which are motivated by the presence at the equation of the singular term $Ru/xc_p(x)$. Let $\Omega=(0,1)$, given $R>0$ we define the spaces

$$L^2_\rho(\Omega)=\left\{f\in L^1_{\mathrm{loc}}(\Omega);\int_\Omega \rho(x)f^2(x)\,\mathrm{d}x<\infty\right\}$$

and

$$V(\Omega)=\left\{f\in L^2_\rho(\Omega),\frac{\partial f}{\partial x}\in L^2_\rho(\Omega)\right\},$$

where

$$\rho(x)=\mathrm{e}^{\int_x^1 (R/\sigma c_p(\sigma))\,\mathrm{d}\sigma}.$$

Note that $\rho(x)$ goes to infinity when x goes to 0. It is not difficult to show (see, e.g. [14]) that those spaces become the Hilbert spaces with the respective inner product

$$\langle f,g\rangle_{L^2_\rho(\Omega)}=\int_\Omega f(x)g(x)\rho(x)\,\mathrm{d}x$$

and

$$\langle f,g\rangle_{V(\Omega)}=\langle f,g\rangle_{L^2_\rho(\Omega)}+\left\langle\frac{\partial f}{\partial x},\frac{\partial g}{\partial x}\right\rangle_{L^2_\rho(\Omega)}.$$

We have

Theorem 2.1. *For any $u_0\in L^2_\rho(\Omega)$ there exists a unique mild solution $u\in C([0,T]:L^2_\rho(\Omega))$ of Problem (1.2). Moreover, if in addition we assume $u_0\in H^2_{\mathrm{loc}}(\Omega)$ then the mild solution u verifies that $u\in W^{1,\infty}(0,T:L^2_\rho(\Omega))$ and u is a strong solution of Problem (1.2).*

We recall that given a Banach space X and an abstract Cauchy problem

$$\begin{aligned}&\frac{\mathrm{d}u(t)}{\mathrm{d}t}+Au(t)\ni f(t),\quad t\in(0,T),\\&u(0)=u_0,\end{aligned}\tag{2.1}$$

where $A: D(A) \subset X \to 2^X$ and $f \in L^1(0,T;X)$, we say that $u:[0,T] \to X$ is a *mild solution* of problem (2.1) on $[0,T]$ if: $u \in C([0,T];X)$ and for each $\varepsilon > 0$ there exists at least one ε-discretization $D_A(\varepsilon, t_0, \ldots, t_n; f_1, \ldots, f_n)$ of (2.1) on $[0,T]$ (i.e. $0 \leqslant t_0 \leqslant t_1 \leqslant t_2, \ldots, t_n \leqslant T;\ f_1, f_2, \ldots, f_n \in X$),

$$t_i - t_{i-1} \leqslant \varepsilon, \quad \forall i = 1,2,\ldots,n, \text{ and } \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| \, ds \leqslant \varepsilon$$

and a piecewise constant solution $v:[t_0,t_n] \to X$ of the discretized implicit scheme

$$\frac{v(t_i) - v(t_{i-1})}{t_i - t_{i-1}} + A(v(t_i)) \ni f_i$$

associated with $D_A(\varepsilon, t_0, t_1, t_2, \ldots, t_n; f_1, f_2, \ldots, f_n)$ on $[0,T]$ and such that

$$\|u(t) - v(t)\| \leqslant \varepsilon$$

for each $t \in [t_0, t_n]$.

The function u is a *strong solution* if $u(t) \in D(A)$ a.e. $t \in (0,T)$, $u \in W^{1,1}_{\mathrm{loc}}(0,T:X)$ and there exists $g \in L^1_{\mathrm{loc}}(0,T:X)$, $g(t) \in Au(t)$ a.e. $t \in (0,T)$ such that

$$\frac{du}{dt}(t) + g(t) = f(t) \quad \text{a.e. } t \in (0,T).$$

The proof of Theorem 2.1 is based on the application of the abstract Crandall–Ligget Theorem for m-accretive operators, (see [4] and [2]).

We shall show that the results of Crandall–Ligget Theorem can be applied to the operator A defined on the space $X = L^2_\rho(\Omega)$ by

$$A: D(A) \to L^2_\rho(\Omega),$$

$$D(A) = \left\{ v \in V(\Omega);\ \frac{\partial}{\partial x} k \left(\frac{\partial v}{\partial x} - \frac{Rv}{x c_p(x)} \right) \in L^2_\rho(\Omega), k \left(\frac{\partial v}{\partial x} - \frac{Rv}{x c_p(x)} \right) = 0 \text{ on } \partial\Omega \right\},$$

$$A(v) = -\frac{\partial}{\partial x} k \left(\frac{\partial v}{\partial x} - \frac{Rv}{x c_p(x)} \right) \quad \text{if } v \in D(A).$$

Lemma 2.1. $\overline{D(A)} = L^2_\rho(\Omega)$.

Proof. First, let us observe that $C_c^\infty(\Omega) \subset D(A)$. If $u \in C_c^\infty(\Omega)$ then $\rho(x)\partial u/\partial x \in C_c^\infty(\Omega)$. Thus, $\rho(x)^{-1}(\partial/\partial x)(\rho(x)\partial u/\partial x) \in C_c^\infty(\Omega) \subset L^2_\rho(\Omega)$. Now, let $u \in L^2_\rho(\Omega)$. Then $\rho(x)^{1/2} u \in L^2(\Omega)$, and since $\overline{C_c^\infty(\Omega)} = L^2(\Omega)$ there exists a sequence $u_n \in C_c^\infty(\Omega)$ such that $u_n \to \rho(x)^{1/2} u$ in $L^2(\Omega)$. So we deduce that $\rho(x)^{-1/2} u_n \to u$ in $L^2_\rho(\Omega)$ but $u_n \in C_c^\infty(\Omega)$ implies that $v_n = \rho(x)^{-1/2} u_n \in C_c^\infty(\Omega)$, and so $v_n \to u$ in $L^2_\rho(\Omega)$. $\square$

Lemma 2.2. *A is accretive in $L^2_\rho(\Omega)$.*

Proof. Let $u, v \in D(A)$ and $\lambda \geqslant 0$. Then

$$\|u - v + \lambda(A(u) - A(v))\|_{L^2_\rho(\Omega)}$$

$$= \left(\|u-v\|^2_{L^2_\rho(\Omega)} + \lambda^2\|(A(u)-A(v))\|^2_{L^2_\rho(\Omega)}\right.$$

$$\left. + 2\lambda\int_\Omega \rho(x)(A(u)-A(v))(u-v)\,\mathrm{d}x\right)^{1/2}$$

$$\geqslant \left(\|u-v\|^2_{L^2_\rho(\Omega)} - 2\lambda\int_\Omega\left(\frac{\partial}{\partial x}\left(k\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)\right.\right.\right.$$

$$\left.\left.\left.-k\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)(u-v)\rho(x)\,\mathrm{d}x\right)\right)^{1/2}$$

$$= \left(\|u-v\|^2_{L^2_\rho(\Omega)} + 2\lambda\int_\Omega\left(k\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)-k\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)\right.$$

$$\left.\times\frac{\partial}{\partial x}(\rho(x)(u-v))\,\mathrm{d}x\right)^{1/2}$$

$$= \left(\|u-v\|^2_{L^2_\rho(\Omega)} + 2\lambda\int_\Omega\left(k\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)-k\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)\right.$$

$$\left.\times\left(\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)-\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)\rho(x)\,\mathrm{d}x\right)^{1/2}.$$

Since k is a nonincreasing function and $k(0)=0$ we obtain that

$$\left(\lambda\int_\Omega\left(k\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)-k\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)\right.$$

$$\left.\times\left(\left(\frac{\partial u}{\partial x}-\frac{Ru}{xc_p(x)}\right)-\left(\frac{\partial v}{\partial x}-\frac{Rv}{xc_p(x)}\right)\right)\rho(x)\,\mathrm{d}x\right)\geqslant 0$$

and then

$$\|u-v+\lambda(A(u)-A(v))\|_{L^2_\rho(\Omega)} \geqslant \|u-v\|_{L^2_\rho(\Omega)}$$

for each $\lambda\geqslant 0$. □

The main difficulty of the proof of Theorem 1 arises in the proof of

Lemma 2.3. *A is m-accretive in $L^2_\rho(\Omega)$.*

We introduce the subspace $V_0(\Omega)\subset V(\Omega)$ given by

$$V_0(\Omega)=\overline{C_0}^{V(\Omega)}.$$

It is easy to see that $V_0(\Omega)$ coincides with the set $\{f\in V(\Omega), f=0 \text{ on } \partial\Omega\}$ once that the trace of a function $f\in V(\Omega)$ is introduced in an usual manner (see e.g. [9] and [7]). We also recall that given $f\in L^2_\rho(\Omega)$, a function $u\in D(A)$ verifies (in a weak sense) the equation $\lambda Au + u = f$, if and only if

$$\int_\Omega \rho(x)\psi\, u\, \mathrm{d}x + \lambda \int_\Omega \rho(x)\psi \left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{Ru}{xc_p(x)}\right)\right)\right) \mathrm{d}x$$

$$- \int_\Omega \psi f \rho(x)\, \mathrm{d}x = 0 \quad \forall \psi \in L^2_\rho(\Omega).$$

Let $\beta : D(\beta) \to \mathscr{P}(\mathbb{R})$ be the maximal monotone graph of $\mathbb{R}^2$ given as $\beta = k^{-1}$, i.e.

$$\beta(s) = \begin{cases} s & \text{if } s > 0, \\ (-\infty, 0) & \text{if } s = 0, \\ \emptyset & \text{if } s < 0. \end{cases}$$

Notice that $s \in \beta(k(s))$ and $r \in k(\beta(r))$, for any $s, r \in \mathbb{R}$. We denote by j the primitive of β, such that $j(0) = 0$, i.e.

$$j(s) = \begin{cases} \frac{1}{2}s^2 & \text{if } s \geqslant 0, \\ +\infty & \text{if } s < 0. \end{cases}$$

Finally, we introduce the function $J : V(\Omega) \to (-\infty, +\infty]$ given by

$$J(v) = \begin{cases} \displaystyle\int_\Omega \rho(x) j(v)\, \mathrm{d}x + \frac{\lambda}{2}\int_\Omega \rho(x)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x & \\ \displaystyle\quad - \int_\Omega \rho(x) f \frac{\mathrm{d}v}{\mathrm{d}x}\, \mathrm{d}x & \text{if } j(v) \in L^2_\rho(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We have

Lemma 2.4. *J is a proper, lower semicontinuous and coercive function on $V(\Omega)$.*

Proof. *J is proper* (i.e. $\exists u \in V(\Omega)$ such that $J(u) < +\infty$). Indeed, we can take $u = 0$ and as $J(0) = 0$ the conclusion is obvious.

J is coercive in $V(\Omega)$: (i.e. $J(u) \to +\infty$ if $\|u\|_{V(\Omega)} \to +\infty$). From the construction of J it results in

$$J(u) \geqslant \tfrac{1}{2}\|u\|_V^2 - \|u\|_V \|f\|_{L^2_\rho}$$

and so we get the conclusion.

J is lower semicontinuous in $V(\Omega)$: Let $u_n \in V$ such that $u_n \to u$ strongly in $V(\Omega)$. If the lower limit of $J(u_n)$ is equal to infinity then, obviously

$$\lim_{n\to\infty} J(u_n) \geqslant J(u).$$

So, let us suppose that $I < \infty$, where $I = \liminf J(u_n)$. Then there exists a subsequence, that we denote again by u_n, such that $J(u_n) \to I$. Moreover, $\rho(x)^{1/2}u_n \geqslant 0$ a.e. and

$$\|\rho(x)^{1/2}u_n\|_{L^2(\Omega)} \leqslant 2I, \ \forall n \geqslant n_0,$$

$$\int_\Omega \rho(x)k(u_n)^2\,\mathrm{d}x = \int_\Omega \rho(x)u_n^2\,\mathrm{d}x.$$

By the Lebesgue dominated convergence theorem we have that

$$\int_\Omega u_n^2\rho(x)\,\mathrm{d}x \geqslant \int_\Omega (u)^2\rho(x)\,\mathrm{d}x = \int_\Omega k(u)^2\rho(x)\,\mathrm{d}x,$$

moreover, since $u_n \rightharpoonup u$ weakly in $V(\Omega)$ we get that

$$\|u_n\|_{V(\Omega)} \geqslant \|u\|_{V(\Omega)}.$$

So, in conclusion

$$I = \liminf J(u_n) \geqslant J(u),$$

which proves the result.

J is convex: It suffices to remark that J is the addition of two convex functions $\int_\Omega \rho(x)j(v)\,\mathrm{d}x$ and $\frac{1}{2}\int_\Omega \rho(x)(\mathrm{d}v/\mathrm{d}x)^2\,\mathrm{d}x$ and a linear function $\int_\Omega \rho(x)f(\mathrm{d}v/\mathrm{d}x)\,\mathrm{d}x$. □

Proof of Lemma 2.3. Since J is a coercive, l.s.c. and convex function, there exists a minimum $w \in V(\Omega)$ of J. Moreover, $0 \in \partial J(w)$. So we obtain that there exists $b \in L^2_\rho(\Omega)$ with $b(x) \in \beta(u(x))$ a.e. $x \in (0,1)$, such that

$$\int_\Omega \rho(x)bv\,\mathrm{d}x + \lambda\int_\Omega \rho(x)\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x + \int_\Omega \rho(x)f\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = 0, \quad \forall v \in V_0(\Omega).$$

Now, if we define

$$u(x) = \rho(x)^{-1}\int_0^x \rho(\sigma)\beta(w(\sigma))\,\mathrm{d}\sigma + c\rho(x)^{-1}$$

with

$$c = -\int_\Omega \left(f + \rho(x)^{-1}\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma - \lambda\frac{\mathrm{d}w}{\mathrm{d}x}\right)\mathrm{d}x,$$

it results in

$$\int_\Omega \rho(x)b(x)v(x)\,\mathrm{d}x = \int_\Omega v(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x$$

and integrating by parts we get

$$\int_\Omega v(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma + c\right)\mathrm{d}x$$

$$= -\int_\Omega \frac{\mathrm{d}v}{\mathrm{d}x}\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma + c\right)\mathrm{d}x + v\int_1^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma\Big|_{x=0}^{x=1} = I.$$

Since $v=0$ on $\partial\Omega$ we obtain that

$$I=-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\rho(x)^{-1}\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma+c\right)\,\mathrm{d}x=-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}u\,\mathrm{d}x$$

and

$$\begin{aligned}\int_\Omega \frac{\mathrm{d}w}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}x}\rho(x)\,\mathrm{d}x&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(k(b))\,\mathrm{d}x\\&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}(k(\rho(x)^{-1}\rho(x)b))\,\mathrm{d}x\\&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\rho(x)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma\right)\right)\right)\,\mathrm{d}x\\&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\rho(x)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}\rho(x)\rho(x)^{-1}\right.\right.\\&\qquad\left.\left.\times\left(\int_0^x \rho(\sigma)b(\sigma)\,\mathrm{d}\sigma\right)\right)\right)\,\mathrm{d}x\\&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\rho(x)^{-1}\frac{\mathrm{d}}{\mathrm{d}x}(\rho(x)u)\right)\right)\,\mathrm{d}x\\&=\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\,\mathrm{d}x\end{aligned}$$

and so

$$\begin{aligned}&-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}u\,\mathrm{d}x+\lambda\int_\Omega \rho(x)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\,\mathrm{d}x\\&\quad-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}f\,\mathrm{d}x=0\end{aligned}$$

for any $v\in V_0$. It is easy to prove that any $\psi\in L^2_\rho(\Omega)$ can be expressed as $\mathrm{d}v/\mathrm{d}x$ for some $v\in V(\Omega)$, ($v=\int_0^x\psi\,\mathrm{d}x$), and $v\in V\cap C^\infty(\bar\Omega)$ can be written as $v_0+m\int_0^x\rho(x)^{-1}\,\mathrm{d}x$, where $v_0\in V_0(\Omega)$ and $m\in\mathbb{R}$. Then

$$\begin{aligned}&-\int_\Omega mu\,\mathrm{d}x+\lambda\int_\Omega m\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\,\mathrm{d}x-\int_\Omega mf\,\mathrm{d}x\\&\quad=m\left(-\int_\Omega u\,\mathrm{d}x+\lambda\int_\Omega\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\,\mathrm{d}x-\int_\Omega f\,\mathrm{d}x\right)=0\end{aligned}$$

as consequence of the choice of the constant c. Adding both expressions we obtain

$$-\int_\Omega \rho(x)\frac{\mathrm{d}(v_0+m\int_0^x \rho(\sigma)^{-1}\,\mathrm{d}\sigma)}{\mathrm{d}x}u\,\mathrm{d}x$$

$$+\lambda\int_\Omega \rho(x)\left(\frac{\mathrm{d}(v_0+m\int_0^x \rho(\sigma)^{-1}\,\mathrm{d}\sigma)}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\mathrm{d}x$$

$$-\int_\Omega \rho(x)\frac{\mathrm{d}(v_0+m\int_0^x \rho(\sigma)^{-1}\,\mathrm{d}\sigma)}{\mathrm{d}x}f\,\mathrm{d}x=0$$

for any $v\in V_0$, and any $m\in\mathbb{R}$. Then

$$-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}u\,\mathrm{d}x+\lambda\int_\Omega \rho(x)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\mathrm{d}x$$

$$-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}f\,\mathrm{d}x=0$$

for any $v\in V\cap C^\infty$ and since $\overline{V\cap C^\infty}=V$, by density, we get that

$$-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}u\,\mathrm{d}x+\lambda\int_\Omega \rho(x)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\mathrm{d}x$$

$$-\int_\Omega \rho(x)\frac{\mathrm{d}v}{\mathrm{d}x}f\,\mathrm{d}x=0$$

for any $v\in V$. In conclusion:

$$-\int_\Omega \rho(x)\psi\,u\,\mathrm{d}x+\lambda\int_\Omega \rho(x)\psi\left(\frac{\mathrm{d}}{\mathrm{d}x}\left(k\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}\right)\right)\right)\mathrm{d}x$$

$$-\int_\Omega \rho(x)\psi f\,\mathrm{d}x=0$$

for any $\psi\in L^2_\rho(\Omega)$. Moreover, since $k(\mathrm{d}u/\mathrm{d}x-Ru/xc_p(x))=w\in V_0$ we get that $k(\mathrm{d}u/\mathrm{d}x-Ru/xc_p(x))=0$ on $\partial\Omega$ which ends the proof. □

3. On the nonincreasing number of stable and unstable regions for regular solutions

In this section, we study some qualitative results on the solution of the problem. Our main goal is the study of the stable and unstable regions. We will work with the function

$$w=\frac{\mathrm{d}u}{\mathrm{d}x}-\frac{Ru}{xc_p(x)}.$$

Assume u is regular enough, then w satisfies the problem

$$\frac{\partial w}{\partial t}=\rho^{-1}\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}k(w)\right),\quad x\in(0,1),\ t>0,$$

$$k(w)=0, \quad x=0, \text{ and } x=1, \ t>0,$$

$$w(x,0)=w_0(x), \quad x\in(0,1), \tag{3.1}$$

at least in the weak sense: that is

$$\int_\Omega \rho\frac{\partial w}{\partial t}\eta\,\mathrm{d}x+\int_\Omega \rho\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial}{\partial x}(k(w))\right)\mathrm{d}x=0$$

for any $\eta\in V_0$.

From the point of view of the physics of the model, the function w, represents the differential of the total static energy (in the dry case), and gives us information about the stable and unstable regions. The region where $w\leqslant 0$, corresponds to a stable vertical distribution of static energy with the absence of convection (remember that in this region $k\equiv 0$). The stable regions are formed by the set of points of the vertical column of air ($x\in[0,1]$) such that $w\leqslant 0$, and the unstable regions where $w>0$. Notice that in this last case convection occurs mixing the air.

The main goal of this section is to prove that the number of stable and unstable regions is not increasing in time for regular solutions. First, we shall prove that the boundary between stable and unstable regions is a semicontinuous graph. In this section we assume, for simplicity, the extra regularity, $w\in C^1(\Omega\times(0,\infty))$.

Definition 3.1. Let $\phi_i(t)$ be defined by

$$\phi_1(t):=\inf\{x\in[0,1], w(x,t)>0,\},$$

$$\phi_{2n}(t):=\sup\{x>\phi_{2n-1}(t), \text{ such that } w(s,t)>0, \ \forall s\in(\phi_{2n-1}(t),x)\},$$

$$\phi_{2n+1}(t):=\sup\{x>\phi_{2n}, \text{ such that } w(s,t)\leqslant 0, \ \forall s\in(\phi_{2n}(t),x)\}.$$

As in [5], we prove, in the next proposition, the lower semicontinuity of ϕ_1. The case of ϕ_{2i+1} and the upper semicontinuity of ϕ_{2i}, is proved in the same way.

Proposition 3.1. *$\phi:=\phi_1:[0,T]\to(0,1)$, is lower semicontinuous* (*i.e.* $\liminf_{t\to t_0}\phi=\phi(t_0)$).

Proof. Following [5], assuming $\|w_t\|_{L^\infty}$ is bounded, taking $t_2>t_1$, $t_2-t_1\leqslant 1$, first we shall prove

$$\phi(t)\geqslant\min\{\phi(t_1),\phi(t_2)\}-C(t_2-t_1)^{1/2}, \quad \forall t\in[t_1,t_2]. \tag{3.2}$$

We define

$$b=(t_2-t_1)^{1/2}C, \quad a=\min\{\phi(t_1),\phi(t_2)\}-b.$$

In the case $a\leqslant 0$ the conclusion holds trivially, when $a>0$ the function

$$\bar{w}(x)=\begin{cases} c\displaystyle\int_a^{a+x}(\sigma-a)^{-1/3}\rho^{-1}(\sigma)\,\mathrm{d}\sigma & \text{if } a\leqslant x\leqslant a+b,\\ 0 & \text{if } x\leqslant a,\end{cases}$$

s a supersolution of problem (3.1). By the comparison principle (or T-accretiveness n L^1 of the operator), it results in

$$\bar{w} \geqslant w, \quad \text{in } (t_1, t_2) \times (a, a+b).$$

Consequently, $k(w) = 0$ in $x \leqslant a$, $t_1 < t < t_2$, which proves (3.2).

Now

$$a_1 = \liminf_{t \to t_0^-} \phi(t), \qquad a_2 = \limsup_{t \to t_0^-} \phi(t).$$

As in [5], if $a_1 < a_2$, there exists an increasing subsequence t_n such that

$$\phi(t_{2k}) \to a_1 \quad \text{and} \quad \phi(t_{2k+1}) \to a_2. \tag{3.3}$$

By (3.2) we know that

$$\phi(t_{2i}) \geqslant \min\{\phi(t_{2i-1}), \phi(t_{2i+1})\} - C(t_{2i+2} - t_{2i})^{1/2}$$

and taking limits when $i \to \infty$ we obtain $a_1 \geqslant a_2$, which is a contradiction, and then $a_1 = a_2$. In the same way, by taking a decreasing sequence t_k, we get that the existence of

$$\lim_{t \to t_0^+} \phi(t).$$

If $\lim_{t \to t_0^+} \phi(t) > \lim_{t \to t_0^-} \phi(t)$, calling

$$a_3 := \lim_{t \to t_0^+} \phi(t) \quad \text{and} \quad a_1 = \lim_{t \to t_0^-} \phi(t),$$

it results $w(t_0, x) = 0$ in $x \in [a_1, a_3]$. Let $x_0 \in (a_1, a_3)$ and

$$Q_1 = \{(x,t), a_1 < x_0 - \delta \leqslant x \leqslant x_0 + \delta < a_3,\ t_0 - \delta \leqslant t < t_0\}$$

for small positive δ, such that $w > 0$ in Q. Let Σ be the parabolic boundary of Q, by the choice of Q, w satisfies

$$\frac{\partial w}{\partial t} - \rho^{-1} \frac{\partial}{\partial x}\left(\rho \frac{\partial}{\partial x} w\right) = 0 \quad \text{in } Q,$$

$$w(x,t) = w(x,t)|_{\chi_\Sigma} > 0 \quad \text{on } (x,t) \in \Sigma, \tag{3.4}$$

by minimum principle we obtain $w(x_0, t_0) > 0$, and then $a_3 = a_1$, and furthermore the lower semicontinuity of ϕ is proved. In the same way, we prove the upper semicontinuity of ϕ_{2i}. $\square$

Now we prove the main result of this section.

Theorem 3.1. *The solution u of the problem* (1.1) *does not increase the number of stable* (*respectively unstable*) *regions.*

The first part of the proof is related to stable regions. Let Ω_0 be a connected stable region of the interval $[0, 1]$. This region changes with the time. Let $\Omega_0(t)$ be this region at time t, let $a(t)$ and $b(t)$ be the evolution of his boundary (they can be 0, or 1). By the regularity of the solution they are continuous functions. Then $\Omega_0(t)$ is connected for all $t > 0$. We take $k(w)$ as test function. Integrating by parts in $\Omega_0(t)$ we obtain

$$\frac{1}{2k_0}\int_{\Omega_0(t)} \rho(x)\frac{\partial k^2(w)}{\partial t}\,\mathrm{d}x + \int_{\Omega_0(t)} \rho(x)\left(\frac{\partial}{\partial x}(k(w))\right)^2 \mathrm{d}x = 0,$$

since $w(a(t)) = w(b(t)) = 0$, and so

$$\frac{1}{2k_0}\int_0^{t^*}\int_{b(t)}^{a(t)} \rho(x)\frac{\partial(k^2(w))}{\partial t}\,\mathrm{d}x\,\mathrm{d}t + \int_0^{t^*}\int_{b(t)}^{a(t)} \rho(x)\left(\frac{\partial}{\partial x}(k(w))\right)^2 \mathrm{d}x\,\mathrm{d}t = 0.$$

Applying the Leibnitz Theorem,

$$\frac{1}{2k_0}\int_0^{t^*}\frac{\partial}{\partial t}\left(\int_{b(t)}^{a(t)} \rho(x)k^2(w)\,\mathrm{d}x\right)\mathrm{d}t + \int_0^{t^*}\int_{b(t)}^{a(t)} \rho(x)\left(\frac{\partial}{\partial x}(k(w))\right)^2 \mathrm{d}x\,\mathrm{d}t = 0.$$

Since $w(a(t)) = w(b(t)) = 0$ and integrating by parts in t, it results that

$$\frac{1}{2k_0}\int_{\Omega_0(t^+)} \rho(x)k^2(w)\,\mathrm{d}x - \frac{1}{2k_0}\int_{\Omega_0} \rho(x)k^2(w)\,\mathrm{d}x$$
$$+\int_0^{t^*}\int_{b(t)}^{a(t)} \rho(x)\left(\frac{\partial}{\partial x}(k(w))\right)^2 \mathrm{d}x\,\mathrm{d}t = 0.$$

Since Ω_0 is a stable region $\int_{\Omega_0(t^*)} \rho(x)k^2(w)\,\mathrm{d}x = 0$ for any $t^* > 0$. Then, new unstable regions in his interior do not appear and $\Omega_0(t^*)$ is connected. In the second part of the proof, let Ω_1 a connected unstable region, caused by the boundary conditions, Ω_1 is in the inner part of the interval, like in the stable regions. We denote by $\Omega_1(t)$ the evolution of this region, we proceed by contradiction. Let us suppose there exists a $t^* > 0$ such that $\Omega_1(t^*)$ is not connected. We take an interval $(c,d) \subset (\min\{x \in \Omega_1(t^*)\}, \max\{x \in \Omega_1(t^*)\})$, such that there is a connected stable region enclosed in this. Let $t_1 < t^*$ be such that $(c,d) \subset \Omega_1(t_1)$ and $c \in \Omega_1(t)$, $d \in \Omega_1(t)$ for all $t \in [t_1, t^*]$ let $k_1 = \min\{w(x,t_1), w(c,t), w(d,t)$, with $x \in (c,d)$ and $t \in [t_1,t^*]\}$. It is clear that k_1 is a positive constant. Let $s : \mathbb{R} \to \mathbb{R}$ be the positive part function. Let s_ε a convex regularization of s such that $s(\cdot) > s_\varepsilon(\cdot)$. Let $g_\varepsilon(t) = \int_c^d \rho(x)s_\varepsilon(k_1 - w(x,t))\,\mathrm{d}x$. By the regularity of w and s_ε, it results that g_ε is a C^1 function. Differentiating the function we get that

$$\frac{\mathrm{d}g_\varepsilon(t)}{\mathrm{d}t} = -\int_c^d \rho(x)\frac{\partial w}{\partial t}s'_\varepsilon(k_1 - w(x,t))\,\mathrm{d}x$$
$$= -\int_c^d \rho(x)s''_\varepsilon(k_1 - w(x,t))\left(\frac{\partial}{\partial x}(k(w))\right)^2 \mathrm{d}x.$$

From the convexity of the function it results in $\mathrm{d}g_\varepsilon(t)/\mathrm{d}t \leqslant 0$, since $g_\varepsilon(0) = 0$, and g_ε is a positive function we get $g_\varepsilon(t) = 0$, that implies $w(x,t) \geqslant k_1$ for all $(x,t) \in (c,d) \times (t_1,t^*)$. But then there would not exists a stable region in the interior of (c,d), which is a contradiction with the assumption of the existence. In conclusion a stable (unstable) region can disappear for a certain $t > 0$, but it can never appear for any stable (unstable) region. □

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